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Weight Modules and Their Extensions over a Class of Algebras Similar to the Enveloping Algebra of sl(2, $\mathbb{C}$)

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INTRODUCTION

The paper is concerned with the representational properties of certain algebras, studied by Smith in [Sm], $R_f = \mathbb{C}[A, B, H]$ with defining relations $HA - AH = A$, $HB - BH = -B$, $AB - BA = f(H)$, where $f(x)$ is a polynomial in $\mathbb{C}[x]$. When $f(x)$ has degree one, $R_f \cong U(\mathrm{sl}(2, \mathbb{C}))$. In loc. cit. it has been investigated how these algebras, for different $f(x)$, are similar to, as well as different from, $U(\mathrm{sl}(2, \mathbb{C}))$. It has been established there that the algebras R_f are Noetherian domains of Gelfand–Kirillov dimension 3 and their finite dimensional modules may be studied by using a theory of Verma modules, highest weight modules, and a BGG (Bernstein–Gelfand–Gelfand) category $\mathscr{O}$. For general $f(x)$ the category of finite dimensional R_f-modules is not semisimple. In this paper we further the investigation of the BGG-category $\mathscr{O}$. The new phenomena in the study of finite dimensional representations, compared to the case of $U(\mathrm{sl}(2, \mathbb{C}))$, stem from the following facts: (a) the length of a Verma module $V(\lambda)$ can be larger than 2, (b) there may exist non-split self-extensions of finite dimensional simple modules. From [Sm] we recall that the connected component of $\mathscr{O}$ is equivalent to the category of all finite dimensional A-modules, mod-A, for some finite dimensional BGG-algebra A in the sense of [I] (or a quasi-hereditary algebra in the sense of [CPS]). Here we will provide the precise structure of A by giving a quiver with

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relations. Contrary to the $U(\mathrm{sl}(2, \mathbb{C}))$-case, the representation type of A is infinite in general. Considering the category $\mathscr{O}_1$ of finite dimensional R_f-modules in $\mathscr{O}$, we see that the connected components of $\mathscr{O}_1$ may be of infinite representation type too; however, we may obtain the corresponding algebras by quivers with relations and the algebras are finite dimensional self-injective $\mathbb{C}$-algebras. Furthermore, we determine when a nonsplit extension of a finite dimensional simple module by itself does exist and we provide a detailed investigation of the extensions between different highest weight modules. In the final section we establish that there are "enough" projective objects in the category mod-R_f of all finite dimensional R_f-modules; i.e., there exists for every $M \in$ mod-R_f a projective cover $P \to M$ in mod-R_f.

1. PRELIMINARIES

Fix $f(x) \in \mathbb{C}[x]$ and write R for R_f as defined above. We use notation as in [Sm]. A lot of the structure of the finite dimensional simple R-modules may be expressed in terms of $U(x)$ determined up to a constant by the following relation.

1.1. $U(x+1) - U(x-j+1) = f(x) + f(x-1) + \cdots + f(x-j+1)$, for $j \in \mathbb{N}$. If M is a left R-module then for $\nu \in \mathbb{C}$ we define the ν weight-space of M to be $M_\nu = \{m \in M,\ Hm = \nu m\}$. We say that M is a *highest* weight module if there exists $\nu \in \mathbb{C}$ such that we have

1. $\dim_{\mathbb{C}} M_\nu = 1$ (HW)
2. $M = RM_\nu$
3. If $M_\mu \neq 0$ then $\nu - \mu \in \mathbb{N} \cup \{0\}$.

If $\nu \in \mathbb{C}$ satisfies (HW) for M then ν is unique as such and we call ν the *highest weight of* M. When we write $\nu \geq \mu$ we will mean $\nu - \mu \in \mathbb{N} \cup \{0\}$. The subalgebra of R generated by H and A is isomorphic to $U(\mathscr{B})$, the enveloping algebra of the Borel subalgebra $\mathscr{B}$ of $\mathrm{sl}(2, \mathbb{C})$ (we shall write $\mathrm{sl}(2)$ for $\mathrm{sl}(2, \mathbb{C})$ from here on). For $\lambda \in \mathbb{C}$ we write $\mathbb{C}_\lambda$ for the one-dimensional $U(\mathscr{B})$-module annihilated by $H - \lambda$ and A. The *Verma-module of highest weight* λ is $V(\lambda) = R \otimes_{U(\mathscr{B})} \mathbb{C}_\lambda$. We write $1 \otimes \mathbb{C}_\lambda = \mathbb{C}_{v_\lambda}$, where v_λ is the highest weight vector of $V(\lambda)$. Clearly, each $B^j v_\lambda$ has weight $\lambda - j$ and $V(\lambda) = \bigoplus \{V(\lambda)_{\lambda - j},\ j \in \mathbb{N} \cup \{0\}\}$ with $\dim_{\mathbb{C}} V(\lambda)_{\lambda - j} = 1$ for all $j \in \mathbb{N} \cup \{0\}$. The action of A on a weight vector increases its weight by 1.

1.2. The submodules of $V(\lambda)$ are precisely

$$\left\{\mathbb{C}[B]B^j v_\lambda, U(\lambda+1) - U(\lambda - j + 1) = 0 \text{ for } j \in \mathbb{N}\right\}.$$

Therefore we obtain the following properties in a rather straightforward way.

1.3. The module $V(\lambda)$ is universal and the length of $V(\lambda)$ equals the number of distinct $j \in \mathbb{N}\{0\}$ such that $U(\lambda + 1) = U(\lambda - j + 1)$.

1.4. For λ, $\nu \in \mathbb{C}$ we have

1. $\dim_{\mathbb{C}} \operatorname{Hom}_R(V(\nu), V(\lambda)) \leq 1$
2. $\operatorname{Hom}_R(V(\nu), V(\lambda)) = \mathbb{C}$ if and only if $\nu = \lambda - j$ with $j \in \mathbb{N}U\{0\}$ and $U(\lambda + 1) - U(\lambda - j + 1) = 0$, i.e., $U(\lambda + 1) = U(\nu + 1)$.
3. Every submodule of $V(\lambda)$ has the form $V(\nu)$ for some ν.

1.5. If λ, ν, $\mu \in \mathbb{C}$ and $V(\mu)$ is a submodule of $V(\nu)$ and $V(\lambda)$ then:

1. When $\lambda \geq \nu$ then $U(\lambda + 1) = U(\nu + 1)$ and $V(\nu)$ is a submodule of $V(\lambda)$.
2. When $\nu \geq \lambda$ then $U(\nu + 1) = U(\lambda + 1)$ and $V(\lambda)$ is a submodule of $V(\nu)$.

It follows from the foregoing that top $V(\lambda)$ is a simple R-module, say $L(\lambda) =$ top $V(\lambda)$. We arrive at the following.

1.6. Any finite dimensional simple R-module is isomorphic to one of the $L(\lambda)$ and $L(\lambda) \cong V(\lambda)/B^j V(\lambda)$, where $j \in \mathbb{N}$ is minimal such that $U(\lambda + 1) = U(\lambda - j + 1)$.

We define the BGG-category $\mathcal{O}$ as the category consisting of the objects that are the R-modules M satisfying:

1. The module M is the sum of its H-weight spaces (BGG)
2. For all $m \in M$, $\dim_{\mathbb{C}} (\mathbb{C}[A]m) < \infty$
3. M is a finitely generated R-module.

It is clear that the Verma $V(\lambda)$, as well as the simple modules $L(\lambda)$, are in $\mathcal{O}$. The category $\mathcal{O}$ can be decomposed in its connected components and the simple objects in a minimal connected component are given as follows.

1.7. Let $\lambda \neq \nu$ in $\mathbb{C}$, then $\operatorname{Ext}^1_R(L(\nu), L(\lambda)) \neq 0$ if and only if $U(\lambda + 1) = U(\nu + 1)$ and either:

(a) $\lambda - \nu \in \mathbb{N}$ or

(b) $\nu - \lambda \in \mathbb{N}$.

We write $\nu \uparrow \lambda$ if $\operatorname{Ext}^1_R(L(\nu), L(\lambda)) \neq 0$, and $\lambda - \nu \in \mathbb{N}$. We write $\nu \sim \lambda$ if there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $\operatorname{Ext}^n_R(L(\nu), L(\lambda_1)) \neq 0$, $\operatorname{Ext}^1_R(L(\lambda_1), L(\lambda_2)) \neq 0, \ldots, \operatorname{Ext}^1_R(L(\lambda_n), L(\lambda)) \neq 0$. Let λ be a maximal element with respect to the relation $\uparrow$ and let $\mathcal{O}_\lambda$ be the full subcategory of $\mathcal{O}$ consisting of those objects having for their composition factors exactly those $L(\nu)$ for which $\nu \sim \lambda$. It is not hard to see that the categories $\mathcal{O}_\lambda$ are nothing but the connected components of $\mathcal{O}$.

1.8. The number of simple R-modules in $\mathcal{O}_\lambda$ cannot exceed $\deg_x U(x)$. We use $\operatorname{Ext}_{\mathcal{O}}(-, -)$ to denote the extensions that are again in $\mathcal{O}$.

2. THE STRUCTURE OF $\mathcal{O}$ AND $\mathcal{O}_1$

The study of the BGG-category $\mathcal{O}$ is now reduced to the study of $U(x)$, in view of (1.2) and (1.7), and the connected components $\mathcal{O}_\lambda = \mathcal{O}(\lambda_1, \ldots, \lambda_n)$. Proposition 4.7. of [Sm] states that $\mathcal{O}_\lambda$ is equivalent to the category mod-A of finite dimensional modules over a finite dimensional $\mathbb{C}$-algebra A. We aim to describe A by giving a quiver with relations defining A. From the representation theory of quivers with relations we know that this task comes down to realizing all projective objects in $\mathcal{O}_\lambda$. Let us first look at extensions of Verma modules in $\mathcal{O}_\lambda$.

2.1. LEMMA. *Suppose that $V(\lambda_1)$ and $V(\lambda_2)$ are Verma modules in $\mathcal{O}_\lambda$, then* $\dim_{\mathbb{C}} \operatorname{Ext}_{\mathcal{O}}(V(\lambda_1), V(\lambda_2)) = 1$ *if and only if* $\lambda_1 < \lambda_2$; *if not, then we have* $\dim_{\mathbb{C}} \operatorname{Ext}_{\mathcal{O}}(V(\lambda_1), V(\lambda_2)) = 0$.

Proof. Let the exact sequence

$$(M): 0 \to V(\lambda_2) \overset{\varphi}{\to} M \overset{\psi}{\to} V(\lambda_1) \to 0$$

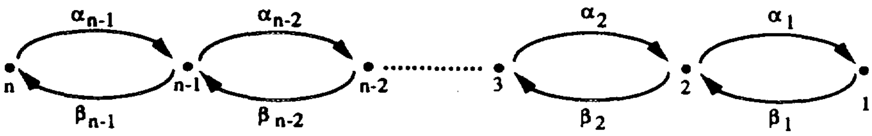
determine an element of $\operatorname{Ext}_{\mathcal{O}}(V(\lambda_1), V(\lambda_2))$ and let v, w be the highest weight vector of $V(\lambda_1)$, $V(\lambda_2)$ respectively. Select non-zero $x, y \in M$ such that $\psi(x) = v$, $\varphi(w) = y$. Since the action of A on M maps M_i to M_{i+1}, $\lambda_1 \geq \lambda_2$ would lead to $Ax = 0$. Moreover, since M is semisimple as a $\mathbb{C}[H]$-module we have $AB^i x = (BA + f(H))B^{i-1}x = \cdots = (U(\lambda_1 + 1) - U(\lambda_1 - i + 1))B^{i-1}x$, $AB^i y = (U(\lambda_2 + 1) - U(\lambda_2 - j + 1))B^{j-1}y$, for any $i, j \in \mathbb{N}$. Hence (M) splits and $\dim_{\mathbb{C}} \operatorname{End}_{\mathcal{O}}(V(\lambda_1), V(\lambda_2)) = 0$. On the other hand if $\lambda_1 < \lambda_2$ then we put $j = \lambda_2 - \lambda_1 \in \mathbb{N}$ (from (1.7)). Let $Ax = B^{j-1}y$; then the following rules define a non-split (M): $HB^i x = (\lambda_1 - i)B^i x$; $HB^i y = (\lambda_2 - i)B^i y$; $AB^i x = (U(\lambda_1 + 1) - U(\lambda_1 - i + 1)$ $B^{i-1}x + B^{i+j-1}y$; $AB^i y = (U(\lambda_2 + 1) - U(\lambda_2 - i + 1))B^{i-1}y$. But if (M) is non-split then the extension defined by it is unique since we must have $Ax = cB^{j-1}y$ for some $c \in \mathbb{C}^*$. ∎

Now we go on to construct a sequence of cyclic R-modules M_i in $\mathcal{O}_\lambda = \mathcal{O}(\lambda_1, \ldots, \lambda_n)$ such that for $i = 1, \ldots, n$ we have

$$M_{n+1} = 0, M_i/M_{i+1} \simeq V(\lambda_i). \tag{2.2}$$

We may start by $M_{n+1} = 0$ and $M_n = V(\lambda_n)$. Let x_n (resp. x_{n-1}) be the non-zero highest weight vector of $M_n = V(\lambda_n)$ resp. $(V(\lambda_{n-1}))$. We put $M_{n-1} = M_n \oplus V(\lambda_{n-1})$ as a $\mathbb{C}$-vector space and define an R-module structure on it as follows. The action of B and H on M_{n-1} is naturally induced by their action on M_n and $V(\lambda_{n-1})$; the action of A on M_{n-1} is the natural one on M_n so we only have to define $Ax_{n-1} = x_n$. One easily checks that M_{n-1} becomes an R-module in the above way. So suppose we have defined M_{i+1} and that M_{i+1} has a vector of weight λ_{i+1} for a generator; then set $M_i = M_{i+1} \oplus V(\lambda_i)$ as a $\mathbb{C}$-vectorspace and let x_i be the non-zero highest weight vector of $V(\lambda_i)$. Putting $Ax_i = x_{i+1}$ and defining the action of B and H on M_i as those induced by the R-module structures of M_{i+1} and $V(\lambda_i)$ yields an R-module structure on M_i satisfying the requirements of (2.2.). Since M_i is cyclic, (2.2.) yields: top M_i = top $V(\lambda_i) = L(\lambda_i)$ and there exists a surjective R-morphism $\alpha_i : P(\lambda_i) \to M_i$. In view of the BGG-reciprocity principle, $P(\lambda_i)$ has a Verma-module filtration and one has: $(P(\lambda_i): V(\lambda_j)) = [V(\lambda_j): L(\lambda_i)]$. From this it follows that the length of $P(\lambda_i)$ equals the length of M_i and therefore α_i must be bijective. From $P(\lambda_i) \cong M_i$ we then derive that $P(\lambda_i)$ is isomorphic to the non-split extension of $P(\lambda_{i+1})$ by $V(\lambda_i)$.

2.3. THEOREM. *Suppose $\lambda \in \mathbb{C}$ is maximal with respect to $\uparrow$, then the connected component $\mathscr{O}_\lambda = \mathscr{O}(\lambda_1, \ldots, \lambda_n)$ is equivalent to the category* mod-A *of finite dimensional A-modules, where the algebra A is defined by the following quiver with relations*:



$I = \{\alpha_{n-1}\beta_{n-1}, \beta_{n-1}\alpha_{n-1} - \alpha_{n-2}\beta_{n-2}, \ldots, \beta_2\alpha_2 - \alpha_1\beta_1\}$, *where n is the cardinality of* $\{\nu \in \mathbb{C}, U(\lambda + 1) = U(\nu + 1), \lambda - \nu \in \mathbb{N}U\{0\}\}$.

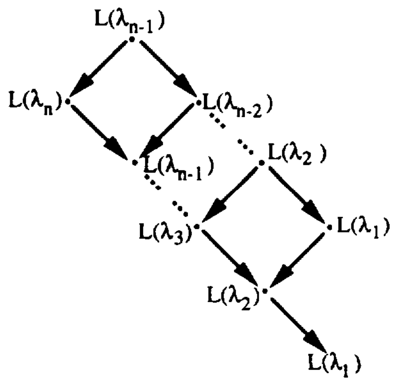
Proof. Let $P(\lambda_n), \ldots, P(\lambda_1)$ be the indecomposable projective objects in $\mathscr{O}_\lambda$, where $\lambda_n = \lambda$, $\lambda_n > \lambda_{n-1} > \cdots > \lambda_1$. Proposition 4.7 of [S] entails that $P(\lambda_i)$ has a Verma module filtration and satisfies the BGG reciprocity principle: $(P(\lambda_i): V(\lambda_j)) = (V(\lambda_j): L(\lambda_i))$. Therefore we obtain

$$(P(\lambda_n): V(\lambda_n)) = (V(\lambda_n): L(\lambda_n)) = 1$$
$$[P(\lambda_n): V(\lambda_i)] = [V(\lambda_i): L(\lambda_n)] = 0 \quad \text{for } i \le n - 1.$$

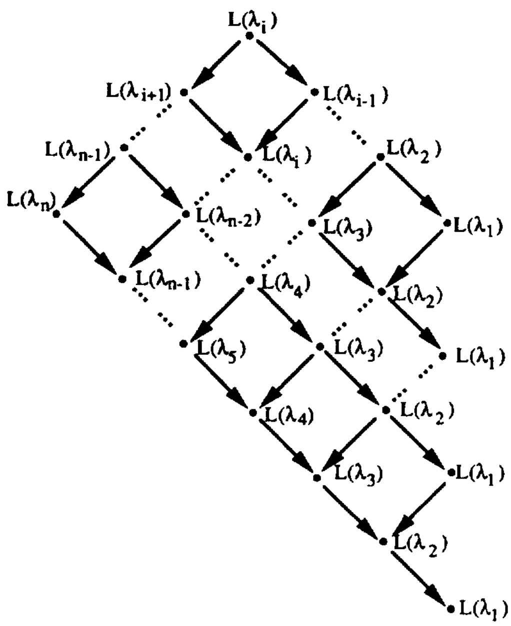
So $P(\lambda_n) = V(\lambda_n)$. The structure of the composition factor of $P(\lambda_n)$ is indicated by the diagram

$$L(\lambda_n) \to L(\lambda_{n-1}) \cdots L(\lambda_2) \to L(\lambda_1)$$
$$(P(\lambda_{n-1}): V(\lambda_n)) = (V(\lambda_n): L(\lambda_{n-1})) = 1$$
$$(P(\lambda_{n-1}): V(\lambda_{n-1})) = [V(\lambda_{n-1}): L(\lambda_{n-1})] = 1$$
$$[P(\lambda_{n-1}): V(\lambda_i)] = [V(\lambda_i): L(\lambda_{n-1})] = 0 \quad \text{for } i < n - 1.$$

Hence $P(\lambda_{n-1})$ is nothing but the non-split extension of $V(\lambda_n)$ by $V(\lambda_{n-1})$ in view of Lemma 2.1. The structure of the composition factors of $P(\lambda_{n-1})$ is described by:

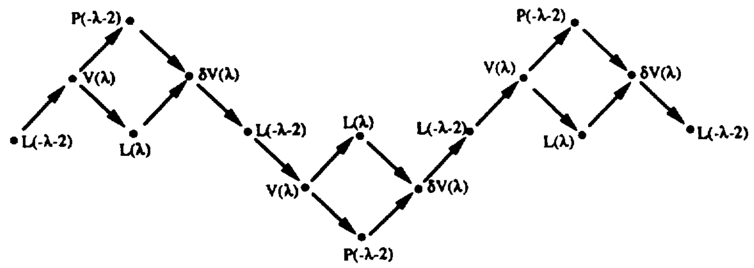


In general $P(\lambda_i)$ is isomorphic to the non-split extension of $P(\lambda_{i+1})$ by $V(\lambda_i)$. In view of the construction of $M_i \cong P(\lambda_i)$, the structure of the composition factors of $P(\lambda_i)$ is visualized as



By the usual quiver-techniques one may derive that A is given by the quiver with relations as in Theorem 2.3. ∎

For $\lambda \in \mathbb{N}$, the connected component $\mathcal{O}_\lambda$ of $\mathcal{O}$ in the case of $U(\mathrm{sl}(2))$ corresponds to the case $n = 2$ in Theorem 2.3. We may describe the category $\mathcal{O}_\lambda$ by its Auslander–Reiten quiver:

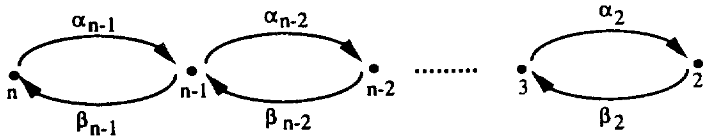


For $n = 3$ in Theorem 2.3 the connected component $\mathcal{O}_\lambda$ of $\mathcal{O}$ is again of finite type; that is, there is only a finite number of isomorphism classes of indecomposable R-modules in $\mathcal{O}_\lambda$. However, when $n \geq 4$ the category $\mathcal{O}_\lambda$ will be of infinite type.

2.4. EXAMPLE. Take $f(x) = x(x-1)(x-2)\cdots(x-n+1) - (x-1)(x-2)\cdots(x-n)$ and $U(x) = (x-1)(x-2)\cdots(x-n)$, then the Verma modules of R_f which are in $\mathcal{O}_{n-1}$ are $V(0), V(1), \ldots, V(n-1)$ and so the category $\mathcal{O}_{n-1}$ is equivalent to mod-A as given in Theorem 2.3. This shows that the situation of Theorem 2.3 can be realized by different $f(x)$ for any large n.

The final part of this section is devoted to the study of the full subcategory $\mathcal{O}_1$ of $\mathcal{O}$, consisting of the finite dimensional R-modules in $\mathcal{O}$. Again $\mathcal{O}_1$ decomposes into a direct sum of components. The full subcategory of $\mathcal{O}_\lambda$ generated by the finite dimensional objects is denoted by $\mathcal{O}_1^\lambda$ and this is nothing but a connected component of $\mathcal{O}_1$. Moreover, every connected component of $\mathcal{O}_1$ is clearly of the form $\mathcal{O}_1^\lambda$.

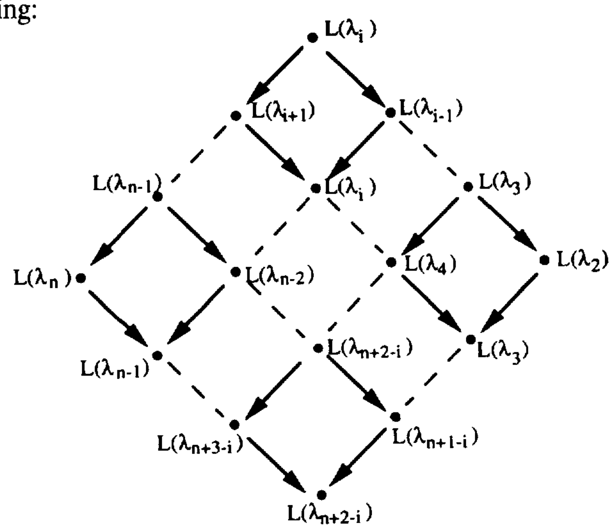
2.5. COROLLARY. *With notation as above, $\mathcal{O}_1^\lambda$ is equivalent to the category mod-B of finite dimensional B-modules where B is a finite dimensional self-injective algebra given by the quiver*:



and $I = \{\alpha_{n-1}\beta_{n-1}, \beta_{n-1}\alpha_{n-1} - \alpha_{n-2}\beta_{n-2}, \ldots, \beta_3\alpha_3 - \alpha_2\beta_2, \beta_2\alpha_2\}$.

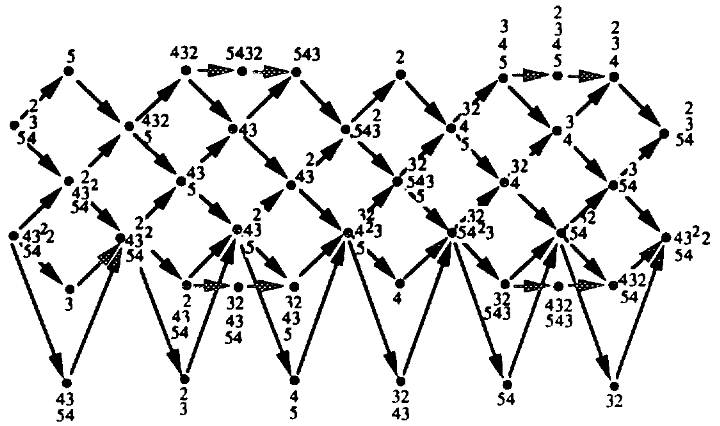
Proof. In $\mathcal{O}_\lambda$ = mod-A the only infinite dimensional simple R-module is $V(\lambda_1) = L(\lambda_1)$. Hence it follows that $\mathcal{O}_1^\lambda \cong$ mod-B, where B is the quotient of A modulo the vertex 1 that corresponds to $V(\lambda_1)$. Therefore B

is indeed given by the quiver with relations I as above. Now we may visualize the structure of the projective B-modules $P_B(\lambda_i)$, $i = 2, \ldots, n$, by the following:



This is exactly the injective B-module $Q(\lambda_{n+2-i})$. ∎

In Corollary 2.5, the number of simple modules in $\mathscr{O}_1^\lambda$ is equal to $n - 1$. When $n = 2, 3, 4, 5$, all $\mathscr{O}_1^\lambda$ are of infinite type; when $n \geq 6$, $\mathscr{O}_1^\lambda$ becomes of infinite type. The following is the Auslander–Reiten quiver of $\mathscr{O}_1^\lambda$ when $n = s$ (the extreme left-hand side agrees with the extreme right one):



By (1.8), the number n of simple R-modules in $\mathscr{O}_\lambda$ is equal to or less than deg $U(x)$, so, if deg $f(x) \leq 4$, every connected component $\mathscr{O}_1^\lambda$ of $\mathscr{O}_1$ for R_f is of finite type. This means that we may obtain a thorough understanding about the finite dimensional R_f-modules in the BGG-category $\mathscr{O}$ in the cases where deg $f(x) \leq 4$.

3. SELF-EXTENSIONS OF FINITE DIMENSIONAL SIMPLE MODULES OVER R

As pointed out in the Introduction the essential difficulty in classifying the finite dimensional R-modules, rather than the $U(\mathrm{sl}(2))$-case, resides in the fact that non-split self-extensions of finite dimensional simple R-modules may exist in this case. Nevertheless a slightly more accurate investigation of these new phenomena allows us to obtain a fairly complete description.

3.1. THEOREM. *With notation as in Section* 2, *let* $L(\lambda)$ *be a finite dimensional simple R-module and write* $\dim_{\mathbb{C}} L(\lambda) = d + 1$:

(i) *If* $x = \lambda$ *is a simple root of* $U(x+1) - U(x-d)$ *then we have* $\dim_{\mathbb{C}} \operatorname{Ext}_R(L(\lambda), L(\lambda)) = 0$.

(ii) *If* $x = \lambda$ *is a multiple root of* $U(x+1) - U(x-d)$ *then we have* $\dim_{\mathbb{C}} \operatorname{Ext}_R(L(\lambda), L(\lambda)) = 1$.

Remark. (ii) Cf. Proposition 5.11. of [S]. (i) is a sharpened form of Theorem 5.7. of [S]. We do include a proof of this fundamental result for the reader's convenience.

Proof of Theorem 3.1. (i) Let us start from the assumption that $\operatorname{Ext}_R(L(\lambda), L(\lambda))$ is non-zero and establish first that the non-split extension of $L(\lambda)$ by $L(\lambda)$ is necessarily a unique one. Let

$$(e): 0 \to L(\lambda) \overset{\varphi}{\to} M \overset{\psi}{\to} L(\lambda) \to 0$$

be a non-split exact sequence. Because $L(\lambda)$ has a decomposition into H-weight spaces, the extension M has an H-primary decomposition. Set $M_{\lambda-i} = \{m \in M \mid (H-(\lambda-i))^k m = 0 \text{ for some } k > 0\}$. Since the H-weight space $L(\lambda)_\lambda$ is one-dimensional, $\dim_{\mathbb{C}} M_\lambda = 2$, and it is easily seen that $(H-\lambda)^2 M_\lambda = 0$. Therefore we may suppose that M_λ is generated by two vectors x_1 and x_2 as a $\mathbb{C}$-space basis, where $x_2 = \varphi(w)$, $\psi(x_1) = v$, such that v, w are the highest weight vectors of $L(\lambda)$. One easily obtains that the actions of A (resp. B) on M map M_j to M_{j+1} (resp. M_j to M_{j-1}). Since the exactness of (e) is compatible with H primary decomposition, we obtain that $Ax_1 = 0$, $Ax_2 = 0$, $B^{d+1}x_1 = 0$, and $B^{d+1}x_2 = 0$. If $Hx_1 = \lambda x_1$, of course, $Hx_2 = \lambda x_2$, then any $M_{\lambda-i}$ is generated by $B^i x_1$ and $B^i x_2$ as a $\mathbb{C}$-space basis. Moreover, $HB^i x_j = (\lambda - i)B^i x_j$, $j = 1, 2$, and

$$\begin{aligned} AB^i x_j &= (BA + f(H))B^{i-1}x_j = (f(\lambda) + f(\lambda-1) + \cdots \\ &\qquad + f(\lambda - i + 1))B^{i-1}x_j \\ &= ((U(\lambda+1) - U(\lambda - i + 1))B^{i-1}x_j \end{aligned}$$

for $j = 1, 2$ by induction. This implies that the sequence (e) splits. So $Hx_1 \neq \lambda x_1$. However, $\psi((H-\lambda)x_i) = (H-\lambda)v = 0$; hence $(H-\lambda)x_1 = cx_2$ for some $c \in \mathbb{C}^*$. Up to replacing x_2 by a scalar multiple we may assume that $(H-\lambda)x_1 = x_2$. Now it is easily seen that the extension (e) is "unique" and $M \simeq R/I$, where $I = RA + RB^{d+1} + R(H-\lambda)^2$.

The key observation is that $g(H)x_1 = g(\lambda)x_1 + g'(\lambda)x_2$ for any polynomial $g(x) \in \mathbb{C}[x]$ now (where g' denotes the derivative of g). The H-primary spaces $M_{\lambda-i}$ of M are generated by $B^i x_i$ and $B^i x_2$ as a $\mathbb{C}$-space basis, when $0 \leq i \leq d$; and $M_{\lambda-i} = 0$ when $i \geq d+1$. We see that $B^{d+1}x_1 = 0$. However, $AB^{d+1}x_1 = (BA + f(H))B^d x_1 + (BA + f(\lambda-d))B^d x_1 + f'(\lambda-d)B^d x_2$, by induction:

$$
\begin{aligned}
AB^{d+1}x_i &= (f(\lambda) + f(\lambda-1) + \cdots \\
&\qquad + f(\lambda-d))B^d x_1 + (f'(\lambda) \\
&\qquad\qquad + f'(\lambda-1) + \cdots + f'(\lambda-d))B^d x_2 \\
&= (U(\lambda+1) - U(\lambda-d))B^d x_1 \\
&\qquad + (U'(\lambda+1) - U'(\lambda-d))B^d x_2.
\end{aligned}
$$

Therefore $U(\lambda+1) - U(\lambda-d) = 0$ and $U'(\lambda+1) - U'(\lambda-d) = 0$; i.e., $x = \lambda$ is a root of $U(x+1) - U(x-d)$ of multiplicity larger than 1, so (i) is proved.

(ii) If $x = \lambda$ is a root of $U(x+1) - U(x-d)$ of multiplicity larger than 1, then it is straightforward to check that the non-split extension M is well defined by the rules $HB^i x_1 = (\lambda - i)B^i x_1 + B^i x_2$, $HB^i x_2 = (\lambda - i)B^i x_2$, and $AB^i x_i = (U(\lambda+1) - U(\lambda-i+1))B^{i-1}x_i + (U'(\lambda+1) - U'(\lambda-i+1))B^{i-1}x_2$, $AB^i x_2 = (U(\lambda+1) - U(\lambda-i+1))B^{i-1}x_2$; for instance,

$$
\begin{aligned}
&(HA - AH)B^i x_1 \\
&\quad = H\big((U(\lambda+1) - U(\lambda-i+1))B^{i-1}x_1 + (U'(\lambda+1) \\
&\qquad - U'(\lambda-i+1))B^{i-1}x_2\big) - A\big((\lambda-i)B^i x_1 + B^i x_2\big) \\
&\quad = (U(\lambda+1) - U(\lambda-i+1))\big((\lambda-i+1)B^{i-1}x_1 + B^{i-1}x_2\big) \\
&\qquad + (U'(\lambda+1) - U'(\lambda-i+1))(\lambda-i+1)B^{i-1}x_2 - (\lambda-i) \\
&\qquad \times\big((U(\lambda+1) - U(\lambda-i+1))B^{i-1}x_1 \\
&\qquad + (U'(\lambda+1) - U'(\lambda-i+1))B^{i-1}x_2\big)
\end{aligned}
$$

$$-(U(\lambda+1)-U(\lambda-i+1))B^{i-1}x_2$$
$$=(U(\lambda+1)-U(\lambda-i+1))B^{i-1}x_1$$
$$+(U'(\lambda+1)-U'(\lambda-i+1))B^{i-1}x_2$$
$$=AB^i x_1.$$

The proof of (i) just shows that $\dim_{\mathbb{C}} \operatorname{Ext}_R(L(\lambda), L(\lambda)) = 1$. ∎

Let $L(\lambda)$ be a simple R-module with $\dim_{\mathbb{C}} L(\lambda) = 1$. Let $\mathscr{M}_{L(\lambda)}$ denote the full subcategory of mod-R whose objects are those finite dimensional R-modules M such that any composition factor of M is isomorphic to $L(\lambda)$. Let $\gamma(\lambda)$ be the multiplicity of λ as a root of $U(x+1) - U(x-d)$.

3.2. LEMMA. *Let $N_r = R/I_r$. $I_r = RA + RB^{d+1} + R(H-\lambda)^r$:*

(i) *If $0 < r < r(\lambda)$, then there exists a unique non-split extension*

$$(e_r): 0 \to N_r \xrightarrow{\varphi_r} M_{r+1} \xrightarrow{\psi_r} L(\lambda) \to 0$$

Moreover, $M_{r+1} \simeq N_{r+1} = R/I_{r+1}$, $I_{r+1} = RA + RB^{d+1} + R(H-\lambda)^{r+1}$.

(ii) *If $r = r(\lambda)$, then any extension $0 \to N_r \to M \to L(\lambda) \to 0$ splits.*

Proof. The structure of N_r may be interpreted as follows: the $(H - (\lambda - i))$-primary space has $\{B^i w_1, \ldots, B^i w_r\}$ for $\mathbb{C}$-vector space basis and the action of R on N_r is defined by $B^{d+1}w_i = 0$, $Aw_i = 0$, and $g(H)w_i = g(\lambda)w_i + g'(\lambda)w_{i+1} + \cdots + g^{(r-1)}(\lambda)w_r$ for $i = 1, \ldots, r$ and any $g(x) \in \mathbb{C}[x]$. We may assume that the $(H - (\lambda - i))$-weight space of $L(\lambda)$ is generated by $\{B^i v\}$. If we have the non-split extension (e_r), then the $(H - (\lambda - i))$-primary space of M_{r+1} has $\{B^i x_1, B^i x_2, \ldots, B^i x_r, B^i y\}$ for a $\mathbb{C}$-space basis such that $x_i = \varphi_r(w_i) \cdot \psi_r(y) = v$. We observe now the following properties of the R-module M_{r+1}:

1. Since the exact sequence (e_r) is exact on any $(H - (\lambda - i))$-primary space, and A maps $(H - (\lambda - i))$-primary spaces to $(H - (\lambda - i) + 1)$-ones, B maps $(H - (\lambda - i))$-primary spaces to $(H - (\lambda - i) - 1)$-ones, it follows that $Ay = 0$, $B^{d+1}y = 0$.

2. Assume that $Hy = \lambda y + t$, $t = b_1x_1 + b_2x_2 + \cdots + b_rx_r$, $b_i \in \mathbb{C}$. The action of A on $B^i y$ is determined by that of H on $B^i y$ since $AB - BA = f(H)$. Therefore we conclude that $t \neq 0$ since the extension (e_r) is non-split. Moreover, up to applying a sequence of elementary linear transformations, we may conclude that the action of H on the space generator by $\{y, x_1, \ldots, x_r\}$ has the Jordan form:

$$\begin{pmatrix} \lambda & -1 & & 0 \\ & \lambda & -1 & \\ & & \ddots & -1 \\ 0 & & & \lambda \end{pmatrix}.$$

Here we may assume that $Hy = y + x_1$, $Hx_1 = \lambda x_1 + x_2, \ldots, Hx_{r-1} = \lambda x_{r-1} + x_r$, $Hx_r = \lambda x_r$.

3. Since $B^{d+1}y = 0$ and, because of (2), we obtain (by induction): $AB^{d+1}y = (f(\lambda) + f(\lambda - 1) + \cdots + f(\lambda - d))B^d y + (f'(\lambda) + f'(\lambda - 1) + \cdots + f'(\lambda - d))B^d x_1 + (f^{(r)}(\lambda) + f^{(r)}(\lambda - 1) + \cdots + f^{(r)}(\lambda - d)) B^d x_r = (U(\lambda + 1) - U(\lambda - d))B^d y + (U'(\lambda + 1) - U'(\lambda - d))B^d x_1 + \cdots (U^{(r)}(\lambda + 1) - U^{(r)}(\lambda - d))B^d x_r = 0$, so we must have $U(\lambda + 1) - U(\lambda - d) = 0$, $U'(\lambda + 1) - U'(\lambda - d) = 0, \ldots, U^{(r)}(\lambda + 1) - U^{(r)}(\lambda - d) = 0$; i.e., $x = \lambda$ is a root of $U(x + 1) - U(x - d)$ of multiplicity $\geq r$. This implies (i) and (ii). ∎

3.3. PROPOSITION. *Let* $\dim_{\mathbb{C}} L(\lambda) = d + 1$. *Every indecomposable* R-*module in* $\mathcal{M}_{L(\lambda)}$ *is uniserial and its length is less or equal to* $r(\lambda)$.

Proof. Let M be an indecomposable R-module in $\mathcal{M}_{L(\lambda)}$. If the length of $M \leq 1$ or 2, the statements hold according to Theorem 3.2; hence we may assume that there exists a non-split sequence

$$0 \to \bigoplus_{i=1}^{m} M_i \to M \to L(\lambda) \to 0,$$

where every M_i is isomorphic to some N_r as in Lemma 3.2. Since $\operatorname{Ext}_R(L(\lambda) \cdot \oplus_{i=1}^{m} M_i) = \oplus_{i=1}^{m} \operatorname{Ext}_R(L(\lambda), M_i)$, we have the commutative diagram

$$\begin{array}{ccccccc}
0 \to & M_i & \to N_i & \to L(\lambda) & \to 0 \\
& \downarrow q & \downarrow & \| & \\
0 \to & \oplus_{i=1}^{m} M_i & \to M & \to L(\lambda) & \to 0 \\
& \downarrow p & \downarrow & \| & \\
0 \to & M_i & \to N_i & \to L(\lambda) & \to 0
\end{array}$$

where q is the canonical injection and p is the canonical projection. By the indecomposability of M, it is clear that $M \simeq N_i$ and $m = 1$. Therefore the structure of M is described as in Lemma 3.2. ∎

Actually we have proved that $\mathcal{M}_{L(\lambda)}$ is equivalent to the category of finite dimensional modules over $\mathbb{C}[x]/(x^{r(\lambda)})$.

4. EXTENSION BETWEEN HIGHEST WEIGHT MODULES

The category mod-R of finite dimensional R-modules may be decomposed as a direct sum of connected components and every connected component of mod-R is generated by some suitable connected component $\mathscr{O}_1^\lambda$ of $\mathscr{O}_1$; we denote this connect component corresponding to $\mathscr{O}_1^\lambda$ by mod$^\lambda$-R. Moreover, the simple R-modules $L(\lambda_2), \ldots, L(\lambda_n)$ in $\mathscr{O}_1^\lambda$ also constitute the class of all simple objects of mod$^\lambda$-R. Our philosophy is to study the structure of Mod$^\lambda$-R by specializing to mod$^\lambda$-R for the different dominant weights λ.

Since any highest weight module is necessarily a quotient of a Verma module, it follows from Section 1 that a highest weight module is necessarily uniserial; hence we denote it by $H_j(\lambda_i)$ and represent it pictorially as $L(\lambda_i) - L(\lambda_{i-1}) - \cdots - L(\lambda_{i-j})$. Obviously $H_j(\lambda_i)$ is finite dimensional exactly when $i - j \geq 2$. If we denote $\dim_{\mathbb{C}} L(\lambda_i)$ by d_i as before, then $\lambda_i - \lambda_{i-1} = d_i$.

In this section we aim to compute $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i))$ and to obtain the precise construction for the non-split extensions of $H_l(\lambda_k)$ by $H_j(\lambda_i)$. Our first task is to characterize when $\dim_{\mathbb{C}} \operatorname{Ext}_{\mathscr{O}}(H_l(\lambda_k), H_j(\lambda_i))$ is 1, or 0.

4.1. PROPOSITION. *With notation as above,* $\dim_{\mathbb{C}} \operatorname{Ext}_{\mathscr{O}}(H_l(\lambda_k), H_j(\lambda_i)) =$ 1 *whenever* $\lambda_i > \lambda_k$ *and* $\lambda_{i-j} > \lambda_{k-l}$, *and* 0 *otherwise.*

Proof. Assume that $M \in \mathscr{O}$ appears as a non-split extension given by the exact sequence

$$(h): 0 \to H_j(\lambda_i) \xrightarrow{\varphi} M \xrightarrow{\psi} H_l(\lambda_k) \to 0.$$

Let v, w be highest weight vectors for $H_j(\lambda_i)$ and $H_l(\lambda_k)$, respectively. Then there exist $x, y \in M$ such that $M = \mathbb{C}[B]x + \mathbb{C}[B]y$ (direct sum as $\mathbb{C}$-spaces), $\varphi(v) = x$ and $\psi(y) = w$. Since M is $\mathbb{C}[H]$-semisimple, the fact that (h) is non-split must be forced by the nature of the action of A on M. Since $Ax = \varphi(Av) = \varphi(0) = 0$, the only choice is to put $0 \neq Ay \in \varphi H_j(\lambda_i)$. Since the exactness of (h) is compatible with the decomposition into primary $\mathbb{C}[H]$-modules, it follows: $Ay = cB^{\lambda_i - \lambda_k - 1}x$, $c \in \mathbb{C}^*$, $\lambda_i - \lambda_{k-1} \geq 0$, whence $\lambda_i > \lambda_k$. Without loss of generality, we may take $c = 1$ here. On the other hand, $B^{d_k + d_{k-1} + \cdots + d_{k-l}}y = 0$, $B^{d_i + d_{i-1} + \cdots + d_{i-j}}x = 0$, and

$$\begin{aligned}
AB^{d_k + d_{k-1} + \cdots + d_{k-l}}y
&= \big(U(\lambda_k + 1) - U(\lambda_k - (d_k + d_{k-1} + \cdots + d_{k-l} - 1))\big) \\
&\quad \times B^{d_k + d_{k-1} + \cdots + d_{k-l} - 1}y + B^{d_k + \cdots + d_{k-l} + \lambda_i - \lambda_k - 1}x \\
&= \big(U(\lambda_k + 1) - U(\lambda_k - (\lambda_k - \lambda_{k-l-1} - 1))\big) \\
&\quad \times B^{\lambda_k - \lambda_{k-l-1} - 1}y + B^{\lambda_i - \lambda_{k-l-1} - 1}x \\
&= \big(U(\lambda_k + 1) - U(\lambda_{k-l-1} + 1)\big)B^{\lambda_k - \lambda_{k-l-1} - 1}y + B^{\lambda_i - \lambda_{k-l-1} - 1}x.
\end{aligned}$$

So $B^{\lambda_i-\lambda_{k-l-1}-1}x = 0$; hence $\lambda_i - \lambda_{k-l-1} - 1 \geq \lambda_i - \lambda_{i-j-1}$. Then $\lambda_{i-j-1} > \lambda_{k-l-1}$; hence $\lambda_{i-j} > \lambda_{k-l}$. We have proved that if $\mathrm{Ext}_{\mathscr{O}}(H_l(\lambda_k), H_j(\lambda_i)) \neq 0$, then $\dim_{\mathbb{C}} \mathrm{Ext}_{\mathscr{O}}(H_l(\lambda_k), H_j(\lambda_i)) = 1$ and $\lambda_i > \lambda_k$, $\lambda_{i-j} > \lambda_{k-l}$. If $\lambda_i > \lambda_k$, $\lambda_{i-j} > \lambda_{k-l}$, then the exact sequence (h) given above is well defined. ∎

4.2. Proposition. *Let $H_j(\lambda_i)$ and $H_l(\lambda_k)$ be two highest weight modules in* mod^{λ}-R, *and $\lambda_i > \lambda_k$, $\lambda_{i-j} > \lambda_{k-l}$; then*

$$\dim_{\mathbb{C}} \mathrm{Ext}_R\big(H_l(\lambda_k), H_j(\lambda_i)\big) = 2.$$

Proof. Suppose that v and w are the highest weight vectors of $H_j(\lambda_i)$ and $H_l(\lambda_k)$, respectively. In view of Proposition 4.1, we have the non-splitting extension

$$(h_1): 0 \to H_j(\lambda_i) \overset{\varphi_1}{\to} M_1 \overset{\psi_1}{\to} H_l(\lambda_k) \to 0$$

defined by $M_1 = \mathbb{C}[B]x_1 + \mathbb{C}[B]y_1$, direct sum as $\mathbb{C}$-spaces). $\varphi_1(v) = x_1$, $\psi_l(y_1) = w$; the action of H on M is semisimple; i.e., $Hx_1 = \lambda_i x_1$, $Hy_1 = \lambda_k y_1$. The action of A is induced by $Ax_1 = 0$, $Ay_1 = c_1 B^{\lambda_i - \lambda_k - 1}x_1$, $c_1 \in C^*$; this is well defined. On the other hand, we may define the non-split extension as

$$(h_2): 0 \to H_j(\lambda_i) \overset{\varphi_2}{\to} M_2 \overset{\psi_2}{\to} H_l(\lambda_k) \to 0,$$

$M_2 = \mathbb{C}[B]x_2 + \mathbb{C}[B]y_2$ (direct sum as $\mathbb{C}$-space), $\varphi_2(v) = x_2$, $\psi_2(y_2) = w$. The action of A is induced by $Ax_2 = 0$, $Ay_2 = 0$; however, the action of H is defined by $Hx_2 = \lambda_i x_2$, $Hy_2 = \lambda_k y_2 + c_2 B^{\lambda_i - \lambda_k} x_2$, $c_2 \in \mathbb{C}^*$. Because

$$\begin{aligned}
&AB^{\lambda_k - \lambda_{k-l-1}} y_2 \\
&\quad = AB^{d_k + d_{k-1} + \cdots + d_{k-l}} y_2 \\
&\quad = \big(U(\lambda_k + 1) - U(\lambda_k - (d_k + d_{k-1} + \cdots + d_{k-l} - 1))\big) \\
&\qquad \times B^{d_k + d_{k-1} + \cdots + d_{k-l} - 1} y_2 \\
&\qquad + c_2\big(U'(\lambda_k + 1) - U'(\lambda_k - (d_k + d_{k-1} + \cdots + d_{k-l} - 1))\big) \\
&\qquad \times B^{d_k + d_{k-1} + \cdots + d_{k-l} + \lambda_i - \lambda_k - 1} x_2 \\
&\quad = \big(U(\lambda_k + 1) - U(\lambda_{k-l-1} + 1)\big) B^{\lambda_k - \lambda_{k-l-1} - 1} y_2 \\
&\qquad + c_2\big(U'(\lambda_k + 1) - U'(\lambda_{k-l-1} + 1\big) B^{\lambda_i - \lambda_{k-l-1} - 1} x_2
\end{aligned}$$

and, for $\lambda_{i-j} > \lambda_{k-l}$, then $\lambda_{i-j-1} > \lambda_{k-l-1}$, then $\lambda_i - \lambda_{k-l-1} - 1 \geq \lambda_i - \lambda_{i-j-1}$; hence $B^{\lambda_i - \lambda_{k-l-1} - 1} x_2 = 0$. It follows that $U(\lambda_k - 1) - U(\lambda_{k-l-1} + 1) = 0$. So we obtain $HB^{\lambda_k - \lambda_{k-r-1}} y_2 = 0$. From this we see that (h_2) is well defined. Clearly (h_1) and (h_2) are $\mathbb{C}$-independent in $\mathrm{Ext}_R(H_l(\lambda_k), H_j(\lambda_i))$. We want next to know what is $(h_1) + (h_2)$ in $\mathrm{Ext}_R(H_l(\lambda_k), H_j(\lambda_i))$. Let $\Delta : H_l(\lambda_k) \to H_l(\lambda_k) \oplus H_l(\lambda_k)$ be the R-map defined by $\Delta(w) = (w, w)$ and $\nabla : H_j(\lambda_i) \oplus H_j(\lambda_i) \to H_j(\lambda_i)$ defined by $\nabla(\alpha_1 \nu, \alpha_2 \nu) = (\alpha_1 + \alpha_2)\nu$, $\alpha_i \in \mathbb{C}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & H_j(\lambda_i) \oplus H_j(\lambda_i) & \xrightarrow{\binom{\varphi_1}{\varphi_2}} & M_1 \oplus M_2 & \xrightarrow{\binom{\psi_1}{\psi_2}} & H_l(\lambda_k) \oplus H_l(\lambda_k) & \to 0 \\
 & \downarrow \nabla & & \downarrow & & \| & \\
0 \to & H_j(\lambda_i) & \to & N & \to & H_l(\lambda_k) \oplus H_l(\lambda_k) & \to 0 \\
 & \| & & \uparrow & & \uparrow \Delta & \\
0 \to & H_j(\lambda_i) & \overset{\varphi}{\to} & M & \downarrow \psi & H_l(\lambda_k) & \to 0
\end{array}$$

The third row (h) just corresponds to $(h_1) + (h_2)$ in $\mathrm{Ext}_R(H_l(\lambda_k), H_j(\lambda))$. It is routine to check that (h) is defined by the following rules $M = \mathbb{C}[B]x + \mathbb{C}[B]y$ (direct sum as $\mathbb{C}$-spaces). $\varphi(\nu) = x \cdot \psi(y) = w$. $Ax = 0$, $Ay = c_1 B^{\lambda_i - \lambda_k - 1} x$, and $Hx = \lambda_i x$, $Hy = \lambda_k y + c_2 B^{\lambda_i - \lambda_k} x$. Clearly if $M = \mathbb{C}[B]x + \mathbb{C}[B]y$, $\varphi(\nu) = x$, $\psi(y) = w$ is an extension of $H_l(\lambda_k)$ by $H_j(\lambda_i)$, the actions of A and H on M must have this form. This finishes the proof of the proposition. ∎

4.3. Proposition. *Let $H_j(\lambda_i)$ and $H_l(\lambda_k)$ be two highest weight modules in* mod^{λ}-R:

1. *If $\lambda_i : > \lambda_k$ and $\lambda_{i-j} = \lambda_{k-l}$ then* $\dim_{\mathbb{C}} \mathrm{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 1$.
2. *If $\lambda_i > \lambda_k$, $\lambda_{i-j} < \lambda_{k-l}$, then* $\dim_{\mathbb{C}} \mathrm{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 0$.

Proof. 1. Assume that

$$(h): 0 \to H_j(\lambda_i) \overset{\varphi}{\to} M \overset{\psi}{\to} H_l(\lambda_k) \to 0$$

is an exact sequence. Take v, w to be the highest weight vectors of $H_j(\lambda_i)$ and $H_l(\lambda_k)$, respectively; then there exist $x, y \in M$ such that $M = \mathbb{C}[B]x + \mathbb{C}[B]y$ (direct sum as $\mathbb{C}$-spaces), $\varphi(\nu) = x$, $\psi(y) = w$, and $Ax = 0$, $Hx =$

$\lambda_i x$. Assume that $Hy = \lambda_k y + t_1 B^{\lambda_i - \lambda_k} x$ and $Ay = t_2 B^{\lambda_i - \lambda_k - 1} x$. Because $B^{\lambda_k - \lambda_{k-l-1}} y = 0$ we also have

$$AB^{\lambda_k - \lambda_{k-l-1}} y = (U(\lambda_k + 1) - U(\lambda_{k-l-1} + 1))B^{\lambda_k - \lambda_{k-l-1} - 1} y$$
$$+ t_1(U'(\lambda_k + 1) - U'(\lambda_{k-l-1} + 1))$$
$$\times B^{\lambda_i - \lambda_{k-l-1} - 1} x + t_2 B^{\lambda_i - \lambda_{k-l-1} - 1} x$$

Since $\lambda_{k-l} \geq \lambda_{i-j}$ and $\lambda_{k-l-1} \geq \lambda_{i-j-1}$, we have $\lambda_i - \lambda_{k-l-1} - 1 \leq \lambda_i - \lambda_{i-j-1} - 1$, $B^{\lambda_i - \lambda_{k-l-1} - 1} x \neq 0$. So we must have

$$t_1(U'(\lambda_k + 1) - U'(\lambda_{k-l-1} + 1)) + t_2 = 0. \qquad (*)$$

Thus, $HB^{\lambda_k - \lambda_{k-l-1}} y = \lambda_{k-l-1} B^{\lambda_k - \lambda_{k-l-1}} y + t, \ B^{\lambda_i - \lambda_k + \lambda_k - \lambda_{k-l-1}} x = t_1 B^{\lambda_i - \lambda_{k-l-1}} x$. If $\lambda_{i-j} = \lambda_{k-l}$, then $B^{\lambda_i - \lambda_{k-l-1}} x = B^{\lambda_i - \lambda_{i-j-1}} x = 0$, the action of H is well defined and the solution space of (*) is one-dimensional, so we have $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k) \cdot H_j(\lambda_i)) = 1$.

2. If $\lambda_{i-j} < \lambda_{k-l}$ then $\lambda_i - \lambda_{k-l-1} \leq \lambda_i - \lambda_{i-j-1} - 1$ and $B^{\lambda_i - \lambda_{k-l-1}} x \neq 0$. We must have $t_1 = 0$ and from Eq. (*) in the proof of 1; it follows that (h) splits. ∎

4.4. Proposition. *Let* $H_j(\lambda_i)$ *and* $H_l(\lambda_k)$ *be the two highest weight modules in* $\operatorname{mod}^{\lambda}$-$R$. *If* $\lambda_i < \lambda_k$ *then* $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 0$.

Proof. As in the proof of Proposition 4.3, it is now very easy to see that $Ax = 0$, $Ay = 0$, $Hx = \lambda_i x$, $Hy = \lambda_k y$; this means that the sequence (h) splits. ∎

It remains to consider the case $\lambda_i = \lambda_k$.

4.5. Proposition. *Let* $H_j(\lambda_i)$ *and* $H_l(\lambda_k)$ *be two highest weight modules, and suppose* $\lambda_i = \lambda_k$ *then*

(i) *when* $\lambda_{i-j} > \lambda_{k-l}$, $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 1$;

(ii) *when* $\lambda_{i-j} = \lambda_{k-l}$, *then* $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 1$ *if and only if* $U(\lambda_i + 1) = U(\lambda_{i-j-1} + 1)$ *and* $U'(\lambda_i + 1) = U'(\lambda_{i-j-1} + 1)$

(iii) *when* $\lambda_{i-j} < \lambda_{k-l}$, $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i) = 0$.

Proof. (i) Assume that

$$(h): 0 \to H_j(\lambda_i) \xrightarrow{\varphi} M \xrightarrow{\psi} H_l(\lambda_k) \to 0$$

is an exact sequence. Take v, w to be the highest weight vectors of $H_j(\lambda_i)$ and $H_l(\lambda_k)$, respectively. Then there exist $x, y \in M$ such that $M = \mathbb{C}[B]x + \mathbb{C}[B]y$ (direct sum as $\mathbb{C}$-spaces), $\varphi(v) = x$, $\psi(y) = w$; then

$Ax = 0$, $Hx = \lambda_i x$. Since $\lambda_i = \lambda_k$ we may assume that $Hy = \lambda_k y + cx$ and $Ay = 0$. Because $B^{\lambda_k - \lambda_{k-l-1}} y = 0$. Hence,

$$AB^{\lambda_k - \lambda_{k-l-1}} y = \left(U(\lambda_k + 1) + U(\lambda_{k-l-1} + 1)\right) B^{\lambda_k - \lambda_{k-l-1} - 1} y$$
$$+ c\left(U'(\lambda_k + 1) + U'(\lambda_{k-l-1} + 1\right) B^{\lambda_k - \lambda_{k-l-1} - 1}$$
$$HB^{\lambda_k - \lambda_{k-l-1}} y = \mathbb{C} B^{\lambda_k - \lambda_{k-l-1}} x.$$

If $\lambda_{i-j} > \lambda_{k-l}$, $\lambda_{i-j-1} > \lambda_{k-l-1}$, $\lambda_i - \lambda_{i-j-1} \leq \lambda_k - \lambda_{k-l-1} - 1$, then $B^{\lambda_k - \lambda_{k-l-1} - 1} x = 0$; of course, $B^{\lambda_k - \lambda_{k-l-1}} x = 0$, $U(\lambda_k + 1) - U(\lambda_{k-l-1} + 1) = 0$. Hence $AB^{\lambda_k - \lambda_{k-l-1}} y = 0$, $HB^{\lambda_k - \lambda_{k-l-1}} y = 0$, and the actions of A and H on M are well defined.

(ii) The proof of (ii) is completely similar to that of Theorem 3.2.

(iii) If $\lambda_{i-j} < \lambda_{k-l}$ then $\lambda_{i-j-1} < \lambda_{k-l-1}$, $\lambda_k - \lambda_{k-l-1} \leq \lambda_i - \lambda_{i-j-1} - 1$, so $B^{\lambda_k - \lambda_{k-l-1}} x \neq 0$; then $c = 0$. This implies that $\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i)) = 0$. ∎

The statements of this section may now be summed up by Table I.

5. EXISTENCE OF ENOUGH PROJECTIVES IN MOD-R

As pointed out before, finite dimensional R-modules need not be semisimple in general for the rings we are considering here. In the foregoing sections we did obtain a fairly satisfactory description of extensions of finite dimensional simple and highest weight R-modules. What is missing in such a description for the whole of mod-R is the complete knowledge concerning the projective objects in mod-R. A step in this direction should be to establish, just like Bernstein, Gelfand, and Gelfand did for the category $\mathcal{O}$, that there are enough projective modules in mod-R

TABLE I

			$\dim_{\mathbb{C}} \operatorname{Ext}_R(H_l(\lambda_k), H_j(\lambda_i))$
(a)	$\lambda_i > \lambda_k$	$\lambda_{i-j} > \lambda_{k-l}$	2
		$\lambda_{i-j} = \lambda_{k-l}$	1
		$\lambda_{i-j} < \lambda_{k-l}$	0
(b)	$\lambda_i = \lambda_k$	$\lambda_{i-j} > \lambda_{k-l}$	1
	$\lambda_{i-j} = \lambda_{k-l}$	$U'(\lambda_i + 1) = U'(\lambda_{i-j-1} + 1)$	1
		$U'(\lambda_i + 1) \neq U'(\lambda_{i-j-1} + 1)$	0
		$\lambda_{i-j} < \lambda_{k-l}$	0
(c)	$\lambda_i < \lambda_k$		0

in the sense that every $M \in$ mod-R has a projective cover $P \to M$. Any $M \in$ mod-R may be decomposed as $M = \bigoplus_{\sigma \in \mathbb{C}} M_\sigma = \{x \in M, (H - \sigma)^q x = 0$ for some $q \in \mathbb{N}\}$ by the classical Jordan theorem. For $x \in M_\sigma$ let us introduce the "depth" of x as $D(x) = \min\{q, (H - \sigma)^q x = 0\}$ and the depth of M as $D(M) = \max\{D(x), x \in H_\sigma, \sigma \in W(M)\}$, where $W(M) = \{\sigma \in \mathbb{C}, M_\sigma \neq 0\}$. Since M is finite dimensional we have $D(M) < \infty$.

For the rings R_f we are considering there does exist a nice upperbound for $D(M)$ in terms of $\deg_x f(x)$. The consequent theorem is the key result of this section; we will give the proof after giving a fundamental lemma of independent interest.

5.1. THEOREM. *For any* $M \in$ mod-R, $D(M) \leq (\deg f(x))^3$.

First let us point out that we may reduce the problem of proving the theorem to the consideration of $M \in \text{mod}^\lambda$-R for any dominant weight λ. A vector $x \in M_\sigma$ is then a highest weight vector if $Ax = 0$.

5.2. LEMMA. *If $x \in M_\sigma$ is a highest weight vector, then* $D(x) \leq (\deg f(x))^2$.

Proof. Let $L(\lambda_2), L(\lambda_3), \ldots, L(\lambda_n)$, $\lambda_2 < \lambda_3 < \cdots < \lambda_n = \lambda$, be the all simple objects in mod^λ-R. Since $H(H - \sigma)^{D(x)-1}x = \sigma(H - \sigma)^{D(x)-1}x$, we obtain $A(H - \sigma)^{D(x)-1}x = 0$. Since there exists an R-homomorphism $V(\lambda_i) \to M$ for some λ_i, mapping the highest weight vector of $V(\lambda_i)$ to $(H - \sigma)^{D(x)-1}x$, we must have $\sigma = \lambda_i$. Without loss of generality in the following proof, we only deal with the $\sigma = \lambda_n = \lambda$. Set: $x_0 = x$, $x_1 = (H - \lambda_n)x_0, \ldots, x_{D(x)-1} = (H - \lambda_n)^{D(x)-1}x_0$, these vectors are $\mathbb{C}$-linearly independent. We have deduced earlier that there exist λ_{i1}, $1 \leq i_1 \leq n$, such that $B^{\lambda_n - \lambda_{i_1}}x_0 = 0$, but $B^{\lambda_n - \lambda_{i_1} - 1}x_0 \neq 0$. Therefore,

$$\begin{aligned}
0 &= AB^{\lambda_n - \lambda_{i_1}}x_0 \\
&= \big(U(\lambda_n + 1) - U\big(\lambda_n - (\lambda_n - \lambda_{i_1}) + 1\big)\big)B^{\lambda_n - \lambda_{i_1} - 1}x_0 \\
&\quad + \big(U'(\lambda_n + 1) - U'\big(\lambda_n - (\lambda_{i_1}) + 1\big)\big)B^{\lambda_n - \lambda_{i_1} - 1}x_1 + \cdots \\
&\quad + \big(U^{(D(x)-1)}(\lambda_n + 1) - U^{(D(x)-1)}(\lambda_n - (\lambda_n - \lambda_1) + 1)\big) \\
&\quad \times B^{\lambda_n - \lambda_{i_1} - 1}x_{D(x)-1}.
\end{aligned}$$

If $B^{\lambda_n - \lambda_{i_1} - 1}x_{j_1 - 1} \neq 0$ but $B^{\lambda_n - \lambda_{i_1} - 1}x_{j_1} = 0$, then $j_1 > 0$ and

$$B^{\lambda - \lambda_{i_1} - 1}x_j = \begin{cases} 0 & \text{if } j \geq j_1, \\ \text{nonzero} & \text{if } j \leq j_1 - 1. \end{cases}$$

It is easily seen that $B^{\lambda_n-\lambda_{i_1}-1}x_0,\dots,B^{\lambda_n-\lambda_{i_1}-1}x_{j_1-1}$ are $\mathbb{C}$-linearly independent; hence

$$\begin{aligned} &U(\lambda_n+1)-U\big(\lambda_n-(\lambda_n-\lambda_{i_1})+1\big)\\ &\quad=U'(\lambda_n+1)-U'\big(\lambda_n-(\lambda_n-\lambda_{i_1})+1\big)\\ &\quad=\cdots=U^{(j_1-1)}(\lambda_n+1)-U^{(j_1-1)}\big(\lambda_n-(\lambda_n-\lambda_{i_1})+1\big)\\ &\quad=0. \end{aligned}$$

Hence $x=\lambda_n$ is a root of $U(x+1)=u(X-(\lambda_n-\lambda_i)+1)$ having multiplicity j_1. Because $B^{\lambda_n-\lambda_{i1}-1}x_{j1}=0$, we know that $B^{\lambda_n-\lambda_{i_1}+1}x_{j1}=0$ and there exists λ_{i_2}, $i_2\geq i_1+1$, such that $B^{\lambda_n-\lambda_{i2}}x_{j_1}=0$, but $B^{\lambda_n-\lambda_{i2}-1}x_{j2}\neq 0$. Therefore,

$$\begin{aligned} 0&=AB^{\lambda_n-\lambda_{i2}}x_{j_1}\\ &=\big(U(\lambda_n+1)-U\big(\lambda_n-(\lambda_n-\lambda_{i_2})+1\big)\big)B^{\lambda_n-\lambda_{j2}-1}x_{j_1}\\ &\quad+\big(U'(\lambda_n+1)-U'\big(\lambda_n-(\lambda_n-\lambda_{i_2})+1\big)\big)B^{\lambda_n-\lambda_{i2}-1}x_{j_1+1}+\cdots\\ &\quad+\big(U^{(D(x)-j_1-1)}(\lambda_n+1)-U^{(D(x)-j_1-1)}\big(\lambda_n-(\lambda_n-\lambda_{i_2})+1\big)\big)\\ &\quad\times B^{\lambda_n-\lambda_{i_2}-1}x_{D(x)-1}. \end{aligned}$$

If $B^{\lambda_n-\lambda_{i_2}-1}x_{j_2-1}\neq 0$, $B^{\lambda_n-\lambda_{i_2}-1}x_{j_2}=0$, then $j_2>j_1$ and

$$B^{\lambda_n-\lambda_{i_2}-1}x_j=\begin{cases}0 & \text{if } j\geq j_2,\\ \text{nonzero} & \text{if } j\leq j_2-1.\end{cases}$$

Hence $x=\lambda_n$ is a root of $U(x+1)-U(x-(\lambda_n-\lambda_{i_2})+1)$ having multiplicity j_2-j_1. Since $B^{\lambda_n-\lambda_{i_2}-1}x_{j_2}=0$, hence $B^{\lambda_n-\lambda_{i_2}+1}x_{j_2}=0$ etc. In general we obtain that $x=\lambda_n$ is a root of $U(x+1)-U(x-(\lambda_n-\lambda_{i_k})+1)$ having multiplicity j_k-j_{k-1}. Of course $j_k-j_{k-1}\leq \deg(U(x+1)-U(x-(\lambda_n-\lambda_{ik})+1)\leq \deg f(x)$. Because $j_n>j_{n-1}>\cdots>j_1>0$ it follows that $j_n\geq D(x)$ and then $D(x)\leq j_n-j_{n-1}+j_{n-1}-j_{n-2}+\cdots+j_2-j_1+j_1\leq n\deg f(x)\leq(\deg f(x))^2$ (since $n\leq\deg f(x)$). ∎

Let M' be the submodule of M generated by x and let M'' be the submodule generated by $\{(H-\sigma)y\mid y\in M'_\sigma\}$. It is easily seen that M'/M'' is indecomposable in $\mathscr{O}$. Put $M(\mathscr{O},x)=M'/M''$, then we have the following.

5.3. LEMMA. *The length of* $M(\mathscr{O}, x) \leq \deg f(x)$.

Proof. Write $lM(\mathscr{O}, x)$ for the length of $M(\mathscr{O}, x)$. No confusion arises if we write x also for the canonical image of x in $M(\mathscr{O}, x)$. Assume that $B^{m_1}x \neq 0$, $B^{m_1+1}x = 0$ and $A^{m_2}x \neq 0$, $A^{m_2+1}x = 0$ then $\{B^{m_1}x, B^{m_1-1}x, \ldots, Bx, x, Ax, \ldots, A^{m_2-1}x, A^{m_2}x\}$ is a $\mathbb{C}$-basis of $M(\mathscr{O}, x)$. Therefore, every non-zero weight space of $M(\mathscr{O}, x)$ is one-dimensional, the multiplicity of any $L(\lambda_i)$ occurring as a composition factor of $M(\mathscr{O}, x)$ in any given composition series of it is at most 1. Therefore $lM(\mathscr{O}, x) \leq n - 1 \leq \deg f(x)$.

Now we are ready to prove Theorem 5.1.

5.4. *Proof of Theorem* 5.1. We shall prove that $D(x) \leq lM(\mathscr{O}, x)$ (deg $f(x)^2$ by induction. First, if $lM(\mathscr{O}, x_1) = 1$; assume that $A^m x \neq 0$ and $A^{m+1}x = 0$, since $A^m x$ and x generate each other in $M(\mathscr{O}, x)$ we may without loss of generality take $A^m x$ instead of x, this means that we may restrict to the case where x is a highest weight vector. In view of Lemma 5.1, $D(x) \leq (\deg f(x))^2$. Second, for $x \in M_\sigma$ with $A^m x \neq 0$ and $A^{m+1}x = 0$, let N be the submodule of M generated by $A^m x$. Then $D(x) \leq D(\bar{x}) + D(N)$, where $\bar{x}$ is the canonical image of x in M/N. Now $lM/N(\mathscr{O}, \bar{x}) \leq lM(\mathscr{O}, x) - 1$, by the induction assumption. $D(\bar{x}) \leq lM/N(\mathscr{O}, \bar{x})$ $(\deg f(x))^2$; hence $D(x) \leq lM/N(\mathscr{O}, x)\,(\deg f(x))^2 + D(N)$ in view of Lemma 5.2. $D(x) \leq (lM(\mathscr{O}, x) - 1)(\deg f(x))^2 + (\deg f(x))^2 \leq lM(\mathscr{O}, x)$ $(\deg f(x))^2$, so according to Lemma 5.2. we have that $D(x) \leq (\deg f(X))^3$. ∎

It is likely that there may exist better bounds for $D(X)$!

5.5. THEOREM. *Let* $M \in \operatorname{mod}^\lambda\text{-}R$, *then there exist a projective object* $P \in \operatorname{mod}^\lambda\text{-}R$ *and a surjective map* $P \to M$.

Proof. Because all simple objects of $\operatorname{mod}^\lambda\text{-}R$ are exactly the $L(\lambda_2), \ldots, L(\lambda_n)$, there exists a $k \in \mathbb{N}$ such that $A^k x = 0$ and $B^k x = 0$ for any $x \in M$ (note that k only depends on λ, not on M). Take $I(\lambda_i) = RB^k + R(H - \lambda_i)^{(\deg f(x))^3} + RA^k$, and $Q(\lambda_i) = R/I(\lambda_i)$; put $q = \bar{1} \in Q(\lambda_i)$.Clearly $Q(\lambda_i) \in \operatorname{mod}^\lambda\text{-}R$. Since q is of weight λ_i, so is $\varphi(q)$ for any $\varphi \in \operatorname{Hom}_R(Q(\lambda_i), M)$. Because $B^k M = A^k M = 0$ and $D(M) \leq (\deg f(x))^3$, the map $\operatorname{Hom}_R(Q(\lambda_i), M) \to M_{\lambda_i}$ defined by $\varphi \mapsto \varphi(q)$ is surjective. On the other hand, if $\varphi \neq 0$ then $\varphi(q) \neq 0$ because q generates $Q(\lambda_i)$. Thus $\operatorname{Hom}_R(Q(\lambda_i), M) \to M_{\lambda_i}$ is an isomorphism; this means that the functors $M \mapsto \operatorname{Hom}(Q(\lambda_i), M)$ and $M \mapsto M_{\lambda_i}$, are isomorphic. However, $M \mapsto M_{\lambda_i}$ is exact on $\operatorname{mod}^\lambda R$; this implies that $Q(\lambda_i)$ is projective in $\operatorname{mod}^\lambda\text{-}R$. Because M is a finite dimensional R-module in $\operatorname{mod}^\lambda\text{-}R$, M is generated by a finite number of elements of weights $\lambda_2, \lambda_3, \ldots, \lambda_n$. Set $Q = \bigoplus_{i=2}^n Q(\lambda_i)$ then any $M \in \operatorname{mod}^\lambda\text{-}R$ is a homomorphic image of a direct sum of a finite number of copies of Q. ∎

A standard technique in the representation theory of Artinian rings leads to the following corollary; its proof is similar to the proof included in [S].

5.6. COROLLARY. *Any indecomposable projective object P in* mod^{λ}-R *has a unique maximal submodule* $\text{rad}(P)$. *This provides us with a one-to-one correspondence between indecomposable projective objects and simple objects* $L(\lambda_i)$, $i = 2, \ldots, n$, *in* mod^{λ}-R.

5.7. *Problem. Let $T(\lambda_i)$ be the projective cover of $L(\lambda_i)$, the existence of which is ensured by the foregoing results. What is the precise structure of $T(\lambda_i)$?*

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