Impact of probability distributions on real options valuation

Reference:
Peters Linda.- Impact of probability distributions on real options valuation
Journal of infrastructure systems / American Society of Civil Engineers - ISSN 1076-0342 - (2016), p. -
Full text (Publishers DOI): http://dx.doi.org/doi:10.1061/(ASCE)IS.1943-555X.0000289

The Impact of Probability Distributions On Real Options Valuation

Linda Peters, MSc[1]

Abstract

This paper shows that the choice for the type of probability distribution is crucial in Real Option Analysis, because it could lead to different outcomes. This is illustrated by using the beta distribution and its special cases, such as the PERT and uniform distribution to model parking garage demand uncertainty. These distributions are commonly used when there is no data available about the stochastic variable, i.e. demand uncertainty. Beta distributions are more flexible than the PERT and uniform distribution. One of the major challenges is the practicality of the beta distribution. A good solution to this challenge is provided by the PERT distribution, because of its ease-of-use and it is more flexible than the uniform distribution. In this research, the Real Options Approach of de Neufville et al. (2006) is used as the base case for modelling of the parking garage demand and is refined to allow for more generic, flexible and practical applications of the model. The impact of different probability distributions is studied on the basis of an expansion option.

CE Database Subject Headings: Real Options, Probability distributions, Comparative Studies, Decision making, Parking facilities, Spreadsheets, Flexibility

[1] Linda Peters, MSc, lindapetersmsc@gmail.com, the Netherlands

1. Introduction

Real Options provide an appropriate methodology for decision-makers to deal with long-term, highly uncertain projects involving huge investments and some freedom of choice on the timing of investment (Dixit and Pindyck, 1994). Managerial flexibility to adapt and revise future decisions in order to maximize potential gains and reduce downside losses is a necessity in uncertain and changing environments. However, there are two significant critiques of Real Options that both academics developing models and practitioners implementing these models should consider seriously. The first is that Real Options models tend to reflect "perfection" rather than economic reality (Triantis, 2005). This means that Real Option models are criticized for unrealistic assumptions regarding the volatility of the stochastic variable, a fundamental parameter for option valuation (Godinho, 2006). The second is that many practitioners view the existing models as too complicated to use and even more so to explain (Triantis, 2005).

Practitioner-oriented methods such as the Binomial Option Pricing Model (Cox et al. 1979), the simulation based method of Datar-Mathews (2007) and spreadsheet approach of the Neufville et al. (2006) have been developed to address the challenge to reduce the gap between theory and practice. Another method to support the decision-making for practitioners was developed by Arnold and Shockley (2001) and Shockley (2007). This method estimates the volatility on the basis of expert judgment by assigning subjective probabilities to different scenarios on the basis of operating cash flows of the project. Even though, the above-mentioned methodologies are preferred by practitioners, they are based on the most basic distributions, such as uniform, lognormal, triangular and normal distributions. In fact, these basic distributions may be limited in the number of the stochastic variable it can effectively represent. Since the volatility of the stochastic variable is a fundamental parameter for option valuation, this phenomenon should be studied in detail with a more generic and flexible distribution than those basic distributions. In this paper this issue will be addressed by substituting the basic uniform distribution by a more generic and flexible distribution, the beta distribution. Furthermore, the impact of the underlying probability function regarding the volatility of the stochastic variable on the outcome of the Real Options model is studied.

Flexible distributions, such as beta distributions are more capable of specifying appropriate distributions for stochastic variables and therefore apply more to real-life situations. Beta distributions are frequently used when there is no data available. Since there is usually no (historical) data for the volatility of the stochastic variable (Godinho, 2006) available, the beta distribution and its special cases are chosen for the comparative probability study in this paper. Beta distribution are more complex and less-known among the practitioners than the basic probability distributions. This lead us to the key challenges of selecting the appropriate distribution, taking into account the trade-off between flexibility and usability.

In this comparative study, the Real Options Approach of de Neufville et al. (2006) is used as the base case for modeling the parking garage demand. This spread-sheet approach evaluates the design of a parking garage, i.e. the decision for a flexible versus a fixed design. In this case-study de Neufville et al. (2006) uses a simple representation, the uniform distribution, to model the uncertainty of the demand. De Neufville et al. (2006) discusses little about the choice of the probability distribution and its impact on the design of the parking garage. The case is probably based on situations with no data available, thus motivating the choice of a distribution with equally-probable outcomes.

This paper will show the importance of a comprehensive study of different probability distributions and that the type of probability distribution used for parking garage demand does matter on the outcome to build a flexible or fixed design. The model of de Neufville et al. (2006) is extended to allow for more generic, flexible and practical applications of the model, while preserving the main Real Options model of de Neufville et al. (2006). The impact of different probability distributions is studied on the basis of an expansion option. We revisit the case in de Neufville et al. (2006) and substitute the uniform distribution by a more flexible, generic distribution, the beta distribution. This choice corresponds with the objective of this paper: to present a stochastic model that has the uniform distribution as a special case, which therefore allows us to study the effects on outcomes for departures from the uniform distribution. For instance, one of the questions we could ask ourselves is: Could varying the shape of a distribution affect demand and investment decisions?

One of the major challenges is to model an appropriate probability distribution, which is practical in its use from the one hand, but still offers enough flexibility on the other hand. In summary, to present a model that is understandable for practitioners, without having too much implications for the theoretical relevance of the model. Effort has therefore to be put in developing a method that makes use of information from experts. Such a method should be easy to use and understand by users who are not trained statisticians. A possibility to address this challenge is first to consider a simpler and more user-friendly 'three-point' distribution, also known as the PERT distribution. This distribution is commonly used by decision makers for managing projects. The greater simplicity of the PERT distribution stems from the fact that it requires one parameter less compared to the traditional Beta distribution: an "optimistic" (maximum) value, a most likely value (mode) and a "pessimistic" (minimum) value for demand. This three-parameter distribution is compared with the uniform distribution (that has two parameters, for minimum and maximum values of demand) and the four-parameter Beta distribution, both regarding their use and their effects on the outcomes of the present study.

The remainder of this paper is structured as follows. Section 2 discusses the theoretical framework of the beta distribution and its special cases, which serves as a preparation for section 3. This section provides an in-depth analysis of the demand model of de Neufville et al. (2006) and presents the refinement to the model of de Neufville et al. (2006), by substituting the uniform by the beta and PERT distributions. Section 4 discusses the results of a comparative study regarding the impact of different probability distributions on the acquisition of the expansion option. Section 5 contains the conclusion of this paper and includes a summery table that compares the different approaches according to certain criteria.

3. The beta distribution

3.1. The beta distribution

A random continuous variable is said to have a beta distribution if the function *f* below is its probability density function (pdf), with shape parameters *a*, *b* > 0:

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, 0 < x < 1 \text{ and zero otherwise.}$$

$$\text{with } B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \qquad [1]$$

The parameters *a* and *b* are called shape parameters, because the shape of the pdf is determined by these parameters. When *a* equals *b* the pdf is symmetric about its mean, which is equal to the mode (the 'most likely value'), when *a* and *b* are larger than 1. When *a* < *b* the probability mass tends to shift towards smaller values, whereas if *b* < *a* the reverse occurs. The pdf is 'bell-shaped' in case *a* > 1 and *b* > 1 and U-shaped in case 0 < *a* < 1 and 0 < *b* < 1. When *a* = *b* = 1 the beta distribution equals the uniform distribution. This shows that the uniform distribution emerges as a special case of the beta distribution, which creates a suitable context for studying the effects of departures from the uniform distribution of demand on possible investment decisions.

The mean, or expected, value of the distribution is equal to $a/(a+b)$. When *a* equals *b*, the mean equals ½ for a beta-distributed variable on the interval [0,1]. When $a < b$, the mean will be less than ½, whereas it will be greater than ½ in case $b < a$. Finally, increasing *a* or *b* or both decreases the variance.

The plots below illustrate how the parameters *a* and *b* shape the probability density function of the beta distribution. The bell-shape (with *a* = 10 and *b* = 10) and the U-shape (with *a* = 0.2 and *b* = 0.2) are easily recognized in the two upper plots. A large value of the mode, combined with a longer left tail (*a* = 10 and *b* = 2) and a small value of the mode, combined with a longer right tail (*a* = 2 and *b* = 10) are shown in de two bottom plots.

Figure 1. Probability density function of the beta distribution with *a* = 10 and *b* = 10

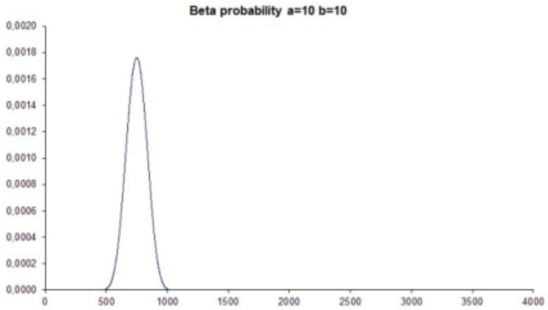


Figure 2. Probability density function of the beta distribution with *a* = 0.2 and *b*=0.2

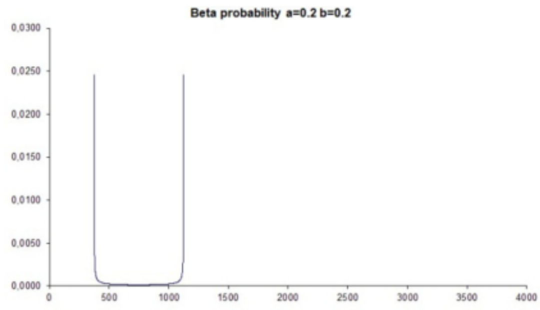


Figure 3. Probability density function of the beta distribution with a = 10 and b = 2

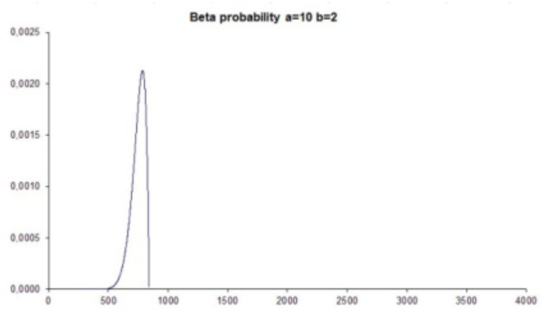
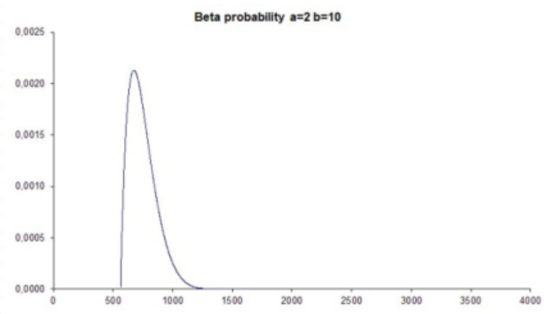


Figure 4. Probability density function of the beta distribution with a = 2 and b = 10



3.2 Motivation behind the Beta distribution

The beta distribution is a family of probability distributions of continuous random variables taking values in the interval (0,1) (Mood et al. 1974). This distribution is parameterized by two positive shape parameters, denoted by a and b in this paper, that control location parameters, such as mean and mode, and the shape of the distribution. Because the beta distribution is defined on the interval (0,1), it is often used in practical applications as a probability distribution of probabilities of events with uncertain outcomes. In many disciplines the beta distribution has been used to model random behavior and uncertainty, such as in project management (Six Sigma), business (risk modeling, actuaries, econometrics), physics and medicine.

Beta distributions are commonly used in Bayesian statistics. Their appeal can partly be ascribed to their elegance in combining subjective, a priori information (e.g., expert guesses) with actual data. These types of information are combined in a classical framework with an a priori beta distribution and a binomial Bernoulli or geometric distribution for the data, which results in an updated, a posteriori beta distribution.-This classical set-up is often used in uncertainty analyses.

It is not necessary to follow the classical Bayesian statistical framework of combining an a priori beta with a binomial distribution for the data. The generalized demand model, which will be discussed in Section 4, offers the possibility to integrate actual demand data into the model by conditioning the beta distribution on realized demand figures of past years. This will update the uncertainty of demand in present and future years in a natural way. Even though this feature is part of the generalized demand model, it does not belong to the scope of this research and therefore will not be discussed in this paper.

According to de Neufville et al. (2006), the fact that future demand is unknown provides sufficient evidence to support the use of the uniform distribution. However, without studying the impact of alternative distributions on Real Options, this conclusion may be premature. Therefore, this paper will analyze alternative distributions to show that the type of distribution does have an impact on the design of the parking garage. In this paper, the demand model of de Neufville et al. (2006) is extended, by substituting the uniform distribution by the beta distribution. The reason for selecting the beta distribution is that the beta distribution is more general (i.e., the uniform distribution is a special case of the beta distribution) and more flexible than the uniform distribution. In the context of the parking garage this means that the beta distribution allows for more flexibility in describing distributions of uncertainty in demand.

Beta distributions allow practitioners to study different forms of uncertainty and the impact on the final decision. Even though the uniform distribution is defined between lower and upper bounds it is still recommendable to use a more flexible distribution. It is highly unlikely that the probability mass is the same for all possible values, including the extreme values, of the parking demand. For instance, in case of a brand new shopping mall demand could be expected to develop much better or worse than the projection, due to well-known factors as population growth or price of gas. Or in case a parking garage is built near an existing shopping mall, demand could be pretty certain in the beginning, but less certain when time evolves in case competition from another cheaper parking garage comes in. Because of its (richer) parameterization, the beta distribution is able to handle uncertainty in demand in the aforementioned examples, where the uniform distribution would turn out to be cumbersome.

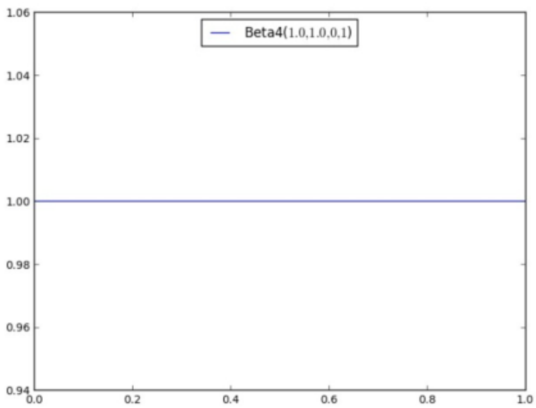
The family of beta distributions has the additional advantage that commonly used distributions in practice, such as the PERT, can be derived from the beta distribution as special cases. These distributions require less information than the 'full form' beta distribution (a minimum and maximum value and a mode). Because of the restricted parameterization and the type of required input, these special distributions are easier to understand and specify by practitioners. These type of distributions are often used in project management (six sigma). PERT is an often-used distribution to model expert opinion. The PERT will be discussed in section 3.3.

3.3 PERT and Special cases of the beta distribution

3.3.1. The uniform distribution

The uniform or rectangular distribution is a special case of the beta distribution when the shape parameters are defined as a = b = 1. It's particularly useful in theoretical statistics, because it's convenient to deal with it mathematically (Mood et al. 1974). The shape of the probability density function of the uniform distribution is rectangular as is shown below in figure 5.

Figure 5. Probability density function of uniform distribution on the interval [0, 1]



De Neufville et al. (2006) uses a simple representation, the uniform distribution, to model the uncertainty of demand. The uniform distribution is the most basic distribution applied to situations with limited or no available data. This distribution assigns equal probabilities to all possible outcomes of a random variable (i.e., demand in this case). De Neufville et al. (2006) argue that the choice for the uniform distribution is obvious and give little justification for the demand model. According to de Neufville et al. (2006) there is no reason to assume that certain values of demand are more likely than other values in situations with sparse or no data. This applies, for instance, to situations with a totally new venture, unique in location, with no prospects of the availability of reliable information. Therefore, building on the above-mentioned arguments, the use of the uniform distribution would be enough to demonstrate that a flexible design can have great advantages over the fixed design.

Uniform distributions have no parameters, apart from a minimum and maximum value, so that the impact of distributional assumptions regarding demand uncertainty on ROV can be studied in a limited way. However, in case of uncertainty in demand and other practical situations, a more flexible distribution is required to study the impact of uncertainty in demand and its shape over time. For instance, uncertainty may be large in the first years when knowledge about demand is limited, but may gradually decrease with time when more information becomes available about the development of demand over time. The reverse situation could take place as well, thus starting with a small uncertainty in the first year, which increases with time. Moreover, additional information about the uncertainty could be gathered from experts and based on this information a minimum, maximum and most likely value could be incorporated into the demand model.

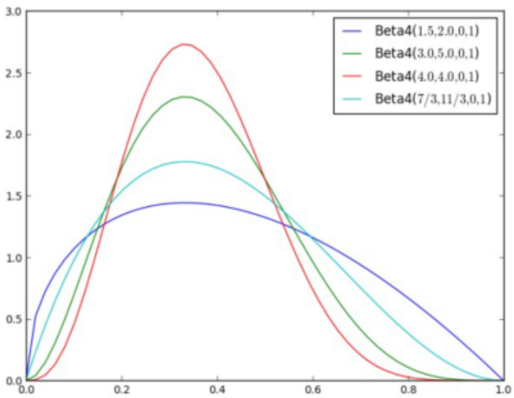
In summary, uniform distribution is a useful model for a few random phenomena. In case of demand uncertainty, where the outcome has not the same probability and more information is available, the beta distribution or PERT distribution provide better alternatives. The PERT distribution is explained in the next section.

3.3.2. The PERT distribution

The beta distribution discussed in Section 3.1 is based on two parameters a and b and is defined on the interval [0,1]. For convenience, we refer to this standard, two-parameter distribution simply as the Beta-2 distribution. The Beta-2 can be easily extended to a four-parameter distribution (Beta-4) by replacing the end points 0 and 1 of the Beta-2 by a minimum (*Min*) and a maximum value (*Max*). However, this Beta-4 distribution has its practical use hampered, because it is not intuitive to directly estimate the shape parameters a and b (Buchsbaum, 2012).

A class of beta distributions that is intuitively more appealing for estimation purposes is the so-called PERT distribution, in short the PERT-distribution. The PERT-distribution is a special case of the Beta-4 distribution, as it has in fact three parameters. The PERT-distribution has one parameter less than the Beta-4, as it supposes a fixed relation between the values of parameters a and b.[2] A second property of the PERT-distribution is that the existence of a (unique) mode is assumed, so that $a, b > 1$. All that is needed to specify a PERT-distribution is a "most likely value", where the highest concentration of probability mass is supposed to be found. Examples of PERT-distributions are given in the figure below.

Figure 6. Examples of PERT distribution on the interval [0, 1]



[2] A relation between the parameters a and b could be established in different ways, for instance, by eliciting a measure of variation in the uncertainty of demand from an expert. We will not go into detail on this aspect, as it falls beyond the scope of the paper.

The parameters *Min*, *Mode*, and *Max* of the PERT-distribution are easily translated to practice as the "most pessimistic", "most likely" and "most optimistic" scenarios. In addition, these scenarios can be easily recognized from the PERT-curve. In summary, the PERT-distribution is very practical in its use, because the three most important scenarios have a one-to-one relation with its shape and therefore stimulate a more intuitive understanding of the relation between the shape parameters and its graphical visualization.

The practical advantage of the PERT-distribution in comparison to the Beta-4 distribution has implications for the flexibility of the model. The PERT-distribution cannot produce all the possible shapes of the Beta-4 distribution. The PERT-distribution does not cover cases where a and/or b are smaller than 1 (e.g., U-shaped density functions, for which a, b < 1). This is called the trade-off between usability and flexibility, meaning the impact of an increased flexibility on the compromises regarding usability and complexity of the model. An interesting question therefore is to what extent the uncertainty in the results (in particular distributions for ROV) will differ for beta distributions that fall outside the class of PERT-distributions. Do these 'exceptional distributions' lead to different distributions of ROV or not, in which case it would be sufficient to focus on PERT-distributions? Some light can be shed on this question by setting up a simulation study.

4. A generalized demand model

De Neufville et al. (2006) propose a two-step simulation approach of demand over time. The first step consists of simulating three values for demand: a realisation of demand in the first year, a realisation of demand in year 10, and an additional demand after year 10 ("final demand"). Each of the three samples are drawn from uniform distributions, with bounds at ±50 percent of "projected demand" (i.e., the expectations of the uniform distributions). The three demand realisations are then used to calculate realisations for other years according to the following function:

$$d_t = d_\infty - (d_\infty - d_1)\exp\big(-\beta(t-1)\big), \qquad [2]$$

where d_t denotes realised demand in year t, d_∞ stands for "final demand" (as t increases) and β is a parameter that is set at a non-negative value in de Neufville et al. (2006).

The second step consists of introducing a second random component, which is used to 'disturb' the realised demand profile of the first simulation step (calculated according to [2]). This is done by drawing growth rates of realised demand in subsequent years, also from uniform distributions. The resulting realisation of demand over time is used to calculate NPV's, which, after repeating the two simulation steps a predefined number of times, generates a probability distribution of NPV in situations with and without expansion option. The decision whether or not to expand is based on the difference between real option value (ROV) and real option cost (ROC), where ROV is calculated as the difference between the average NPV's with and without expansion option.

The model of de Neufville et al. (2006) has a number of shortcomings:

- Uniform distributions have no parameters, apart from a minimum and maximum value, so that the impact of distributional assumptions regarding demand uncertainty on ROV can be studied in a limited way (in fact only through the minimum and maximum values of the intervals on which the uniform distributions are defined);
- The model therefore leads to only one theoretical outcome for ROV, as the uniform distributions only yield one expected NPV with expansion option and one expected NPV without expansion option;
- Repeated simulations with de Neufville's model result in empirical probability distributions of NPV without expansion option with a mode at the highest NPV, that is, the probability mass functions ('histograms') peak at the highest NPV. The question is whether this is plausible.

In order to have more flexibility to study the impact of uncertainty in demand and its shape over time, a random process of demand has been developed that extends the ideas of de Neufville et al. (2006). It is well known that uniform distributions are special cases of beta distributions (Mood et al., 1974). Beta distributions can take a variety of forms and are therefore very useful for the purpose of this study. Beside the uniform distribution, beta density functions can be symmetric around the mean, skewed with mode smaller or larger than the mean, and can also take U-shaped forms. The random process of demand is constructed as follows.

Distribution of demand uncertainty in first year

Let D_1 denote a random variable for demand in the first year and let U_1 have a beta distribution on the interval (0, 1) with parameters a, $b > 0$. The beta probability density function is proportional to $x^{a-1}(1 - x)^{b-1}$, with $x \in (0, 1)$. The random variable D_1 can now be defined as follows:

$$D_1 = (d_1^+ - d_1^-)U_1 + d_1^-, \quad [3]$$

where d_1^- and d_1^+ denote lower and upper bounds for demand in year 1, respectively. We follow de Neufville et al. (2006) and set $d_1^+ = 3d_1^-$ (according to their ±50% of demand projection).
Expected demand in year 1 is equal to

$$D_1 = (d_1^+ - d_1^-)\frac{a}{a+b} + d_1^-, \quad [4]$$

which in the simulations is set equal to the demand projection in de Neufville et al.'s case (750 spaces), for each choice of the values of a and b in our simulations. Note that for $a = b = 1$, we have the uniform distribution chosen by de Neufville et al. (2006). Variance, or uncertainty in demand, decreases when a or b increases.

Distribution of demand from second year onwards

Contrary to de Neufville et al.'s two-step procedure, a random process is proposed here that is simulated by sampling a demand in year t, conditionally upon the sample in the preceding year $t - 1$. This means that demand random variables in different years are dependent. The dependence is incorporated as follows into the random process.

Again, de Neufville et al.'s model is taken as a point of departure. Their "dynamic demand model" [2] is rewritten here as:

$$ED_t = cED_{t-1} + (1 - c)ED_\infty \quad [5]$$

where $c = exp(-\beta)$, and ED_t and ED_∞ denote expected demand in year t and final demand, respectively. Expression [5] can be viewed as the expectation of the conditionally expected demand in year t, given demand D_{t-1} in year $t - 1$. The conditional expectation of demand in year t is constructed as follows:

$$E(D_t|D_{t-1}) = cD_{t-1} + (1 - c)ED_\infty \quad [6]$$

We take this as the expectation of a conditional beta distribution in year t, which is specified in a similar way as the distribution of demand in the first year.

The shape parameters a and b in year 1 are chosen in such a way that it represents the beta distribution that most closely represents our prior knowledge. As additional feature two parameters c_a and c_b are added. Parameter c_a indicates the change of parameter a. The value of a decreases in time when c_a is greater than 0 and increases in time when c_a is less than 0. When c_a equals zero, the value of a is constant over time. Likewise, parameter c_b indicates the change of parameter b. In this way one could simulate a situation with much uncertainty in the first years but little uncertainty in later years by using negative c_a and c_b. On the other hand, if increasing uncertainty is required c_a and c_b can be set to a positive value. The implementation of c_a and c_b result in a change of a and b over time, so when $a(t)$ is defined as parameter a at time t and $b(t)$ is defined as parameter b at time t, the parameters $a(t)$ and $b(t)$ caused by parameters c_a and c_b are defined as follows:

$$a(t) = a(1)\exp\big(-c_a(t-1)\big) \quad [7]$$

$$b(t) = b(1)\exp\big(-c_b(t-1)\big) \quad [8]$$

4.1 The generalized demand model using the PERT distribution

Once the dynamic demand model of de Neufville et al. (2006) is extended with the beta distribution, the PERT distribution can easily be incorporated too, because the parameters of the Beta distribution are obtained directly from the parameters of the PERT distribution.

Next to replacing the uniform distribution by the beta distribution, the parameters c_a and c_b were added as an additional feature to indicate the change of the parameters a and b of the beta distribution over time. Again, three parameters c_{Min}, c_{Mode} and c_{Max} are added to adjust the change of the shape parameters regarding the PERT-distribution. The parameters c_{Min}, c_{Mode} and c_{Max} are defined in a similar way as with the Beta distribution and are as follows:

$$Min(t) = Min(1)\exp\left(-c_{Min}(t-1)\right) \tag{9}$$

$$Mode(t) = Mode(1)\exp\left(-c_{Mo}(t-1)\right) \tag{10}$$

$$Max(t) = Max(1)\exp\left(-c_{Max}(t-1)\right) \tag{11}$$

A note should be made that the change parameters can cause the parameters Min, $Mode$ and Max to violate the condition $Min < Mode < Max$ for instance by increasing Min and decreasing $Mode$. These side effects are easily avoided at the implementation of the model.

5. Case study and Simulation results

5.1 Case Study de Neufville et al. (2006)

De Neufville et al. (2006) incorporates Real Options theory into physical system design. The paper of de Neufville et al. (2006) shows how designers and decision-makers of infrastructure systems can evaluate flexibility in engineering systems in fairly simple ways. In the paper of de Neufville et al. (2006) an easy-to-use spreadsheet approach is presented to evaluate whether to choose for a flexible or fixed design in the context of a parking garage. This spreadsheet approach is designed for valuing Real Options. The results of the spreadsheet will show what is the better design.

The case-example of de Neufville et al. (2006) deals with a multi-level car park for a commercial center in region that is growing as population expands. Economic analysis recognizes that actual demand is uncertain, given the long time horizon (de Neufville et al. (2006)). Designing a big parking garage, the possibility exist that demand will be smaller and the cost of the big parking garage cannot be recovered. At the same time, designing a small parking garage reduces the opportunity to take advantage of the upside potential if the demand grows rapidly. In order to deal with this dilemma, the concept of staging the parking garage is introduced. Staging avoids the development of unnecessary capacity (de Neufville et al. (2006)). There is a possibility to create a real option into the parking garage design by strengthening the footings and columns of the original building so additional levels can be added to the parking garage easily. However, creating such a possibility incorporates costs and is called a premium. This premium is the price to acquire the real option for future expansion, a right but not the obligation to exercise this option.

The purpose of the spreadsheet model for real options of de Neufville et al. (2006) is to present an easy to use model and provides insight into the way that flexibility minimizes exposure to risk and maximizes the potential for gain under favorable circumstances. This method consists of two components; the deterministic component, the Net Present Value base case and a stochastic component, the base case with uncertainty in demand and the expansion option.

Firstly, decision-makers will have to decide on acquiring the option prior to building the parking garage. In other words, when the outcome of the model results in acquiring the option, designers will add stronger footings and columns to the original building of the parking garage. Secondly, the acquisition of the option, the decision on exercising the option is expected to be taken, and can only take place when the option is acquired. It is very important to distinguish between acquiring and exercising the option. The moment of exercising the option takes place when realized demand exceeds the capacity in two consecutive years.

5.2 NPV base case

It is important to verify whether it is relevant to incorporate uncertainty in the decision-making process. This is done by a projection of the future costs and revenues of the project, which results in a design with the maximum NPV. In case of the parking garage, one can increase the number of levels until the maximum NPV is obtained. This is called the base case, against which flexible solutions are compared.

5.3 Randomized NPV

In contrast to the base case NPV, which includes a deterministic demand model, a stochastic demand is introduced with calculation of the Randomized NPV. This means that uncertainty is incorporated into the demand and therefore the main difference between the deterministic and the stochastic demand model is the volatility.

By running 2000 different scenarios, 2000 different NPV's were generated. From these NPV's, expectation (average), standard deviation, 95% confidence interval and a distribution of possible NPV's for the project can be determined. From the distribution of possible NPV's a cumulative distribution is derived, that shows the probabilities that the worst case scenarios could occur. In other words, it shows the probability that an NPV might be less or equal to a threshold. On the basis of the randomized NPV it can be decided whether it makes sense to incorporate uncertainty in the investment decision.

5.4 Expansion option

An expansion option is considered if market demand turns out to be more favorable than expected, therefore management may increase capacity or accelerate resource utilization (Schwartz and Trigeorgis, 2001). In the example of de Neufville et al. (2006), the capacity of the parking garage compared to the market demand has an impact on the final decision whether or not to acquire and perhaps later on exercise the expansion option. The Real Option Value is determined by comparing average NPV between the value with and without flexibility. Based on the difference between the value of option and the costs of flexibility (costs to design stronger footings and columns) the decision is made whether or not to acquire the option.

5.5 Simulation results

In this section the impact of different probability distributions on the acquisition of the expansion option is studied. The main purpose of this simulation study is to demonstrate that more flexible and general distributions than the uniform distribution result in a different outcome. A more accurate and realistic specification regarding the uncertainty of parking garage demand, even without data, is given by the PERT and Beta distribution. Therefore, the use of the uniform distribution is not enough to demonstrate that a flexible design can have great advantages over the fixed design.

The design of this simulation study is two-fold: firstly, an individual simulation study incorporating the beta distribution. Secondly, a comparative study where the uniform, beta and PERT distribution are compared to one another. The individual simulation study, which compares simulation results of only the beta distribution, show that different simulation results arise when using the same generic beta distribution. The base case values of the simulations regarding the comparative study are chosen as such, that the compared distributions in question show opposite outcomes regarding the acquisition of the expansion option. This means a “Yes” or “No” for the expansion option. Since only two outcomes of the simulation are possible and the main purpose of this simulation study is to demonstrate a difference in outcome between the distributions, only two distributions are compared at the time.

In this simulation study, the individual simulation with the beta distribution will be discussed in the following section. Hereafter, a discussion will take place regarding the comparative study about the different type of distributions.

5.6 Individual distribution simulation results.

We used the random process of the previous section to simulate a number of cases. The first case considers 16 different combinations of parameter values for a and b of the beta distribution of demand in the first year, which covers cases with $a > b$, $b > a$ and $a = b$. In this way, the shape of the beta distribution is varied in order to quantify its effect on the decision of whether or not to buy the expansion option. For the second year onward, the parameters a_t and b_t are increased, so that uncertainty in the demand decreases with time. We used the values $c_a = c_b = -0.5$ for each pair (a, b).

For each pair (a, b), we increased the number of samples from the random process from 2,000 to 10,000 in order to further decrease the sample variance in the average NPV with and without expansion. Table 1 shows the differences between ROV and ROC for each of the 16 pairs (a, b). The results show that both types of decision emerge (buying or not buying the option). Buying the option seems to take place more frequently when $b > a$ rather than $a > b$. Cases where $b > a$ correspond to demand distributions with longer tails towards larger values of demand. The results also suggest that buying the option occurs more frequently when a increases as well, along with b.

The simulation results should be treated merely as indications for decision making at this premature stage of application of the random process. More pairs (a, b) should be considered in order to obtain a better picture of the effect of distributional assumptions on the decision of whether to expand or not. Nevertheless, the first results clearly show that decision makers should be aware of the impact of the shape of probability distributions of uncertainty in demand on decision making. This is one of the main messages that this paper should transmit to the field of real options, which is an important additional insight with respect to the study of De Neufville et al. (2006).

Table 1. Difference between ROV and ROC for 16 different pairs (a, b) of the beta distribution of demand in year 1, with 10,000 samples per pair.

	b = 1	3	5	10
a = 1	-£86,190	-£114,755	-£41,528	£13,508
3	-£12,249	-£2,044	-£40,530	£12,019
5	-£12,629	-£6,847	£3,788	£3,854
10	-£25,416	-£14,965	£31,711	£41,802

5.6 Comparative study.

In this comparative study, the uniform, beta and PERT distribution are compared to one another. In the following summary table the results of the four comparative simulations are presented.

Table 2. Summary table comparative study

	Beta vs Uniform 1		Beta vs Uniform 2		Uniform vs PERT		PERT vs Beta	
	Beta	Uniform	Beta	Uniform	Uniform	PERT	PERT	Beta
ROV - ROC	£777,560	-£241,989	£1,633,139	-£241,989	-£176,477	£1,082,057	£1,082,057	-£568,408
Acquire?	Yes	No	Yes	No	No	Yes	Yes	No

The simulation values are the result of the difference between Real Options Value (ROV) and Real Option Cost (ROC). A positive value/profit is equivalent to a “yes” regarding the acquisition of the expansion option, whereas a negative value/loss is equivalent to a “no” regarding the acquisition of the expansion option.

Table 2. shows the difference between ROV and ROC for the different probability distributions. Applying different distributions with the same base case values per test case result in different outcomes, a “yes” or “no” regarding the acquisition of the expansion option. The most generic and therefore flexible distribution, the beta distribution, shows a “yes” when compared to the uniform distribution, provided the same parameters are selected as in this simulation study. However, when compared to the most practical distribution, the PERT, the beta distribution shows a “no” for the acquisition of the expansion option. This does not necessarily mean that the PERT is the most suitable distribution for parking garage demand, since the Beta distribution can take on shapes that cannot be produced by the PERT. In both comparisons that involve the uniform distribution, the simulation result in a “no” for the uniform distribution regarding the acquisition of the expansion option. These results clearly show again that decision makers should be aware of the impact of the shape of probability distributions of uncertainty in demand and that the uniform distribution is too basic to describe uncertainty of parking garage demand. In the following sections the simulation results will be further explained.

5.7 Simulation results – Beta:Yes vs Uniform:No

In this first simulation example, the beta distribution is compared to the uniform distribution. Here, the uncertainty of parking garage demand fluctuates in different ways over time. This is sketched in two situations. In the first situation, uncertainty may be large in the first years when knowledge about demand is limited, but increases gradually over time when more information about this demand becomes available. The second situation contains the reverse of situation one, where demand uncertainty is small in the first year and increases over time. The beta distribution is capable to vary the extent of uncertainty in demand in different ways over time, whereas the uniform distribution is not suitable to handle these kind of situations since it assumes equal probability for each event.

Additional flexibility with the beta distribution is created through assigning different probabilities to specific events, performed through varying parameters a and b. There is more chance that the option will be exercised when $b > a$. If $b > a$ the mass of the probability distribution occurs more in the right tail. This means a high probability that the projected realized demand exceeds the projected demand from the demand model and probably leads to exercising the expansion option. This results in a higher real options value, since the capacity of the garage is optimally used. And therefore results in acquiring the expansion option. Because of the flexibility of the beta distribution, where in the two examples the probability mass is situated at the extremes, the investment decision could result in a “yes” regarding the acquisition of the expansion option, which is not the case for the uniform distribution.

The base case values for this simulation study example are further explained in the appendix. The results are presented in the summary table 2.

Situation 1. New shopping mall in a brand new area

A parking garage is built near a complete new shopping mall in a brand new area. This means the uncertainty of the demand in the first years is high and later on this uncertainty decreases due to experience. In figure 7 the U-shape of the beta distribution in the first year indicates that the probability mass is situated at the extremes. The worst and best case scenario of demand could be due to well-known factors as population growth, price of gas or competition from another mall or parking garage. From the simulation results of the spreadsheets in table 2 it turns out that the option should be acquired.

Figure 7. Probability density function of situation 1 at year 1

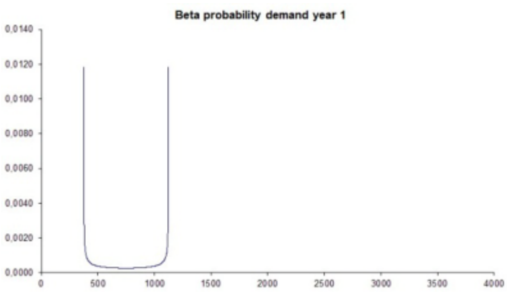
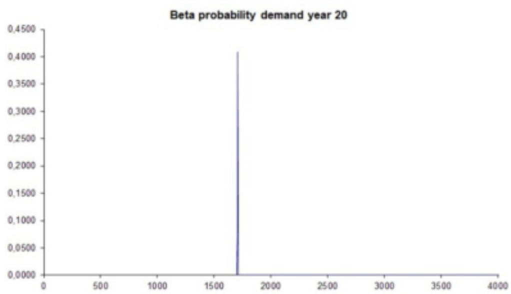


Figure 8. Probability density function of situation 1 at year 20



In the first year the uncertainty of demand is high, with a standard deviation of 280. The probability mass is not situated around the projected demand of 750, but at the upper bound (375) and lower bound (1125) of the interval on which the pdf is defined. Later on, the demand becomes less uncertain. For instance, a run of 2000 simulations resulted in a projected demand of 1661 with standard deviation of only 1 in year 20. This decrease of uncertainty in demand is due to the negative values of c_a and c_b.

Situation 2. Parking garage in an existing area

In this case the demand is not uncertain in the first years, but uncertainty will increase later on. The parking garage is built in an existing area where there is a lot of knowledge and experience about the development in demand. Later on a competing shopping mall is built, and due to this development there is less knowledge and experience available on the demand. This results in more uncertainty in demand. From the results of the spreadsheet it turns out that the option should be acquired.

Figure 9. Probability density function of situation 2 at year 1

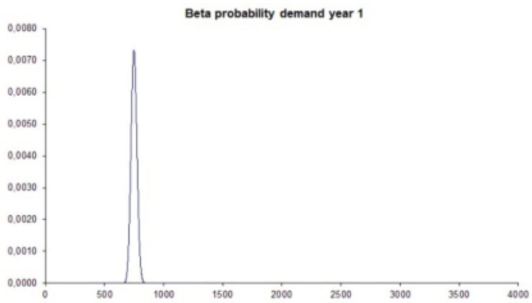
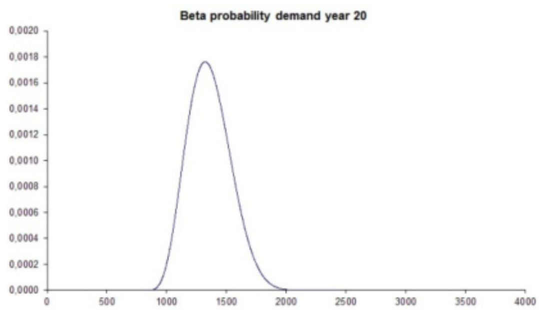


Figure 10. Probability density function of situation 2 at year 20



In the first year there is not much of uncertainty in demand with a standard deviation of 24.The probability mass is situated around the projected demand of 750.Later on, the uncertainty increases. For instance, a run of 2000 simulations resulted in a projected demand of 1745 with standard deviation of 230 in year 20. Yet the probability mass remains situated around the projected demand. This increase of uncertainty in demand is due to the positive values of c_a and c_b.

5.8 Simulation results - Uniform: No vs PERT: Yes

In the previous section the beta distribution was compared to the uniform distribution. The beta distribution could be problematic to apply in practice, because the value of its parameters do not have a one-to-one relation with its shape. On the contrary, the PERT does have an intuitive relation between the parameters and its shape. Therefore, the PERT is more practical in its use than the beta distribution. Because of the practical application of the PERT, the PERT distribution is compared to the uniform distribution in this example. The case is studied where the simulations of the uniform distribution show a "no" for the acquisition of the expansion option. The same base case is used, but replaced with the PERT distribution and parameters of the PERT distribution are chosen in such a way that it will result in a "yes" regarding the acquisition of the expansion option. The base case values that are used for this study are shown in the appendix.

5.9 Simulation results – PERT: Yes vs Beta: No

In this simulation study the practical PERT distribution is compared to the most flexible distribution, the beta distribution. The main question that is addressed in this simulation study is whether a more practical and less complex distribution would lead to the same results as the most flexible distribution. In that case it would be much easier for decision-makers to specify the distribution. Unfortunately, the simulation results of this study will show a different outcome regarding the acquisition of the option. As was mentioned before, there are shapes of the beta distribution, which cannot be produced by the PERT. Therefore for this simulation study the U-shape is selected for the beta distribution, a shape that cannot be produced by the PERT. A U-shape means that the shape parameters a, b <1. On the contrary, a prerequisite to define the PERT as a beta distribution is to define parameters a, b >1, meaning that the beta distribution has a unique mode.

For this simulation study the same base case values are used as in section 5.8. These base case values are shown in the appendix.

The results of the simulation involving the PERT distribution will lead to a different outcome, that is a "yes" regarding the acquisition of the expansion option, whereas the simulation results of the beta distribution result in a "no" regarding the acquisition of the expansion option. This implies that the beta distribution shouldn't be replaced by a more simplistic distribution, such as the PERT distribution. In other words, shapes that can only be produced, such as the U-shape, by the beta distribution should be incorporated in the analysis, since this distribution shows a different outcome than the PERT.

6. Conclusion

In this paper we have studied the impact of different probability distributions on the outcome of the Real Options Model. The aim of this paper is two-fold. First, to investigate the effects of different types of distributions of demand uncertainty on real options outcomes and second to study the usability of the model by decision-makers taking into account the trade-off between flexibility and usability.

This paper shows that different types of probability distribution could lead to different outcomes on the decision whether or not to acquire the expansion option. This is the result of a comparative study between the uniform distribution and a more generic and flexible distribution, the beta distribution. Because of this flexibility, the beta distribution is more capable of specifying a more appropriate distribution for demand uncertainty with no data available than the uniform distribution. The beta distribution further refines the model of de Neufville et al. (2006), since the uniform distribution is a special case of the beta distribution. Another advantage of the beta distribution is that it can be modeled in Excel ®. The above-mentioned refinement of the model of de Neufville et al. (2006) by the beta distribution and the modeling in Excel ® enable to maintain the advantages and objectives regarding the model of de Neufville et al. (2006).

One of the challenges of the beta distribution is its practical use, because it has proven to be difficult to assign direct estimates to the shape parameters a and b. As an alternative to overcome this intuitive hurdle is the introduction of a more user-friendly distribution, the PERT. The input parameters of the PERT only require the minimum, most likely and maximum value and this is an often-used approach in for example managing projects. The practicality of the PERT in comparison to the beta distribution has implications on the flexibility, as the PERT cannot generate all the possible shapes of the beta distribution.

The main purpose of the simulation was to research whether more generic distributions than the uniform distribution would result in a different outcome regarding the design of the parking garage. From the simulation results comparison between the uniform and beta distribution, it turns out that simulations incorporating the beta distribution result in a different outcome, that is acquiring the expansion option, compared not acquiring the expansion option in case of the uniform distribution. The U-shape of the beta distribution, i.e. probability mass is situated at the extremes, is incorporated in this simulation example. The worst and best case scenario of demand could be due to well-known factors as population growth, price of gas or competition from another mall or parking garage. That's why the shape of the beta distribution is more intuitive and related to practice than the uniform distribution. Another interesting observation was that repeated simulations with de Neufville's model result in empirical probability distributions of NPV without expansion option with a mode at the highest NPV, that is, the probability mass functions ('histograms') peak at the highest NPV. The question is whether this is plausible. The PERT, another more generic distribution than the uniform distribution, showed "expansion" according to the simulation results, whereas the uniform distribution showed the opposite result, "no expansion".

Based on this, it can be concluded that different distributions show a different impact on the outcome of the Real Options Model, ie. the decision for a flexible or fixed design of the parking garage.

Another purpose of the simulation was to study whether the more practical version of the beta distribution, the PERT, would lead to different results than the beta distribution. The U-shape of the beta distribution resulted in "no expansion option", while the PERT showed a "yes" for the expansion option. From this conclusion can be drawn that shapes that cannot be produced by the PERT distribution should be taken into account as a candidate for specifying the appropriate distribution for the uncertainty of demand. In other words, this means that the parameter values for $a < 1$ and $b < 1$ should be allowed in the simulation. A more in-depth simulation study should be addressed in a separate paper in order to draw more sophisticated conclusions about the impact of probability distribution on the design of the parking garage.

In order to have a good understanding of the impact of various probability distributions and the comparison between the distributions, a summary is provided in the following table. This table summarizes all the approaches according to the following criteria:

- Flexibility for refining the probability distribution
- Development of uncertainty in demand over time
- Implications of the probability distribution form on decision making
- Improvement of model based on realized demand

Table 3. Summary

Method/Assumptions	Flexibility for refining the probability distribution	Development of uncertainty in demand over time	Implications of the probability distribution form on decision making	Improvement of model based on realized demand
de Neufville et al.(uniform distribution)	Cumbersome and limited by sizing the intervals on which the uniform distributions are defined	Even more cumbersome and limited by sizing the intervals on which the uniform distributions are defined for each year	Cumbersome and limited by sizing the intervals on which the uniform distributions are defined	No opportunity is built in for conditioning the model on realized demand from past years
de Neufville et al. extended with beta distribution and monitoring	Can handle a variety of shapes by setting just two parameters a and b, beside a minimum and maximum value	Can be handled elegantly by allowing a and b to be time dependent (through parameters c_a and c_b, see text)	Different values for parameters a and b may result in different decisions. The role of the probability distribution can be made comprehensible easily	The model is conditioned on realized demand of the preceding years, in order to make a decision based on the latest information
De Neufville et al. extended with PERT distribution and monitoring	Is also flexible, but to a lesser extent than the full form beta distribution as it has one parameter less (a mode beside minimum and maximum)	Can be handled in a similar way as with beta distribution	Same as beta distribution with the mode as an extra parameter	Same as beta distribution

7. Further research

In order to select a suitable probability distribution to describe the uncertainty of a stochastic variable, such as parking garage demand, it is important to study both theory and practice. One of the important aspects is to research on what basis or criteria decisions are taken by practitioners. For example is the decision-making in line with Real Options theory? Does the model provide sufficient support for decision-makers in practice? Will decision-makers have enough confidence to make their decision on the basis of a spread-sheet? What decision-rules are part of the decision-making?

Therefore one of the recommendations is to apply and test this model in practice. This could be done by asking experts to specify the probability distribution of parking garage demand. In order to verify the accuracy of the expert's opinion, this opinion should be calibrated with real data or facts. In other words, to test the expert judgment with real data or facts to select the best possible distribution. Another important element is to study the decision behavior of practitioners and verify whether the models are useful to support the decision-making process.

In this paper a few cases have been simulated to show that different probability distributions result in a different outcome on the decision whether or not to acquire the expansion option. In order to have a more sophisticated grasp on the impact of probability distributions on the decision regarding the design of the parking garage, it is recommended to write a separate paper about simulations. This paper could be focused on how to structure a simulation process, investigate several decision rules regarding the simulation and so on. For example in this paper a simulation decision rule is based on the average ROV with and without expansion option. In this future paper, it is interesting to study the results when this decision rule is replaced by another.

Another recommendation for further research is the replacement of the demand model of de Neufville et al. (2006) by other models or probability distributions that describe the uncertain demand evolution such as the geometric Brownian process, Poisson distribution and so on. A key element in this research is to find an appropriate method to integrate expert judgments and real data (such as realized demand) into the demand model. This means for example to determine whether the realized demand (based on real data) fits the Binomial distribution in order to be the conjugate for the Beta distribution.

8. Acknowledgements

The author appreciates the cooperation of Dr. A.G Chessa who worked with the author on the implementation of the beta distribution. I am very grateful for the review of, Dr. D. Cassimon, Dr. Mr. P.J. Engelen and Prof. Dr. Ir. A. K. Hajdasinski.

9. Appendix.

			Beta vs Uniform		Uniform vs PERT	PERT vs Beta
			1	2		
	Parameter	Unit	Value		Value	Value
Parking Garage Model De Neufville et al. (2006)	Demand in year 1	spaces	750		750	750
	Additional demand by year 10	spaces	750		750	750
	Additional demand after year 10	spaces	250		500	500
	Average annual revenu	per space used	£5.000		£5.000	£5.000
	Average operating costs	per space available	£1.000		£1.000	£1.000
	Land lease and other fixed costs	p.a.	£1.800.000		£1.800.000	£1.800.000
	Capacity cost	per space	£8.000		£8.000	£8.000
	Capacity cost	growth per level for every level above 2	10%		10%	10%
	Capacity limit	cars per level	200		200	200
	Capacity	levels	6		6	6
	Time horizon	years	15		15	15
	Discount rate		12%		12%	12%
	Realized demand in yr 1 within	of demand projection	40%		50%	50%
	Additional demand by year 10	of projection	50%		50%	50%
	Additional demand after year 10	of demand projection	50%		50%	50%
	Annual volatility of demand growth	points of growth projection	15%		15%	15%
Beta distribution	a (shape parameter)		0.4	100		0.5
	b (shape parameter)		0.4	200		0.5
	c_a (change of a)		-0.2	0.15		0
	c_b (change of b)		-0.6	0.15		0
PERT distribution	min (minimum)				550	550
	mode				750	750
	max (maximum)				900	900
	c_{min} (change of min)				0.15	0.15
	c_{mode} (change of mode)				0.15	0.15
	c_{max} (change of max)				0.15	0.15

10. References

[1] Bain, L. J. and Engelhardt, M. (1992). *Introduction to probability and mathematical statistics*, second edition, Boston, PWS-KENT Publishing Company

[2] Copeland, T. and Antikarov, V. (2001). *Real Options - A Practitioner's Guide*, New York, NY, TEXERE

[3] Neufville, R. de, Scholtes, S. and Wang ,T, (2006). "Valuing Options by Spreadsheet: Parking Garage Case Example," *ASCE Journal of Infrastructure Systems*, vol.12, no 2. 107-111

[4] Mood, A.M., Graybill, F.A., Boes, D.C. (1974). *Introduction to the Theory of Statistics,* third edition, New York,NY, McGraw-Hill, Inc.

[5] Mun, J. (2006). *Real Options Analysis*, second edition, Hoboken, New Jersey, John Wiley & Sons, Inc.

[6] Schwartz, E.S. and Trigeorgis L., (Eds.) (2001) *Real Options and Investment under Uncertainty,* Cambridge, Massachusetts, The MIT Press

[7] Iversen; G.R., (1984), *Bayesian Statistical Inference*, California, SAGE publications

[8] Neufville, R de, Scholtes, S., (2011), Flexibility in Engineering Design, Massachusetts, Massachusetts Institute of Technology Press

[9] Dixit, A and Pindyck, R. (1994) Investment under Uncertainty, Princeton University Press, Princeton, NJ.

[10] Buchsbaum, P., (2012), *Modified PERT Simulation*, Brasil, Great Solutions

[11] Triantis, A (2005), Realizing the Potential of Real Options Does Theory Meet Practice by Triantis, Journal of Applied Corporate Finance, Vol. 17 (2), Spring 2005, pp.8-16

[12] Godinho, P. (2006). Monte Carlo Estimation of Project Volatility for Real Options Analysis. Journal of Applied Finance, Vol. 16, No. 1, Spring/Summer 200

[13] Cox, J., Ross, S., Rubinstein, M., (1979), "Option Pricing: A Simplified Approach, Journal of Financial Economics 7, 229 – 263, North-Holland Publishing Company

[14] Datar, Vinay T. and Mathews, Scott H., (2004) European Real Options: An Intuitive Algorithm for the Black-Scholes Formula. Journal of Applied Finance, Vol. 14, No. 1, Spring/Summer 2004

[15] Datar Vinay T. and Mathews, Scott H., (2007) A Practical Method for Valuing Real Options: The Boeing Approach, Journal of Applied Corporate Finance, Vol. 19, No. 2, Spring 2007

[16] Shockley, R., (2007), An Applied Course in Real Option Valuation, Thomson South Western, Mason

[17] Arnold, T., Shockley, R., (2001). Value creation at Anheuser-Busch: a Real Options example. Journal of Applied Corporate Finance 14 (2), 52–61.