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Examples and quantum sections of schematic algebras

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Abstract

In a previous paper, we defined schematic algebras as algebras having "enough" Ore-sets and developed a noncommutative scheme theory for them. The present paper is a sequel in which we prove a lifting properety for Ore-sets. This allows us to exhibit many examples of schematic algebras like homogenizations of almost commutative algebras. We use various techniques to show that other algebras like the three-dimensional Sklyanin algebras are schematic. We also calculate explicitly the quantum-sections of enveloping algebras and use local information (their one-dimensional representations) to determine the point modules. © 1997 Elsevier Science B.V.

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1. Introduction

Many of the "quantised-algebras" studied nowadays are viewed as rings of functions of a so-called quantum-space that represents a kind of noncommutative geometry but without really introducing a geometric point set or an underlying topology. In a remark in his book [13], Manin announces the failure of attempts to obtain a noncommutative scheme theory à la Grothendieck for the quantised case. However, for a certain class of graded algebras called schematic algebras, we introduced in our paper [18] a generalised Grothendieck topology. At the basis of these ideas was the definition of a noncommutative *Proj* in category theoretical terms as in [2]. We defined a noncommutative site and a quantum-site by means of words in Ore-sets and we proved that the main properties of the commutative theory are still valid. Moreover, we defined coherent sheaves and proved that they give another way to describe *Proj*, i.e. Serre's

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theorem still holds and *Proj* may indeed be viewed as a noncommutative scheme. To support the idea that this noncommutative scheme theory provides a quantum-space, we have to show that the class of schematic algebras contains the interesting algebras of quantised type. This is the aim of the present paper which is therefore complementary to [18]. We show that the geometric approach developed in [18] applies to: quantum Weyl algebras, enveloping algebras of colour Lie super algebras, Sklyanin algebras of dimension three, quantised matrix rings, gauge algebras as in [10], etc. We pay some attention to the calculation of the section-algebras and the relations (in terms of algebra generators) defining them. We apply these ideas to the determination of the so-called point varieties studied a.o. in [8, 9], and others.

We assume the reader is acquainted with the theory in [18] and refer him to loc. cit. for unexplained notations.

Finally, we would like to thank M. Van den Bergh and L. Le Bruyn for many stimulating discussions about schematic algebras.

2. Constructions of schematic algebras

Before turning to examples, we briefly recall the definition of schematic algebras. Let R be a connected k-algebra, i.e. R is positively graded and its part of degree zero is isomorphic to the base field k. Moreover, we suppose that R is Noetherian and generated by a finite number of degree 1 elements. Then R is said to be (left) *schematic* if there exists a finite number of two-sided Ore-sets S_i in R with the following property: given a graded left ideal I which has a non-trivial intersection with each Ore-set S_i, then I contains a power of $R_+ = \bigoplus_{n\geq 1} R_n$.

The aim of this paper is to show that the class of schematic algebras is sufficiently large. First we introduce some general results that allow us to generate classes of interesting examples of schematic algebras. The following "lifting property" turns out to be quite useful in that respect. Let R be a positively graded k-algebra where k is a field.

Proposition 1. *Assume that R has a central regular homogeneous element X of degree d. Denote the canonical epimorphism $R \to R/XR$ by π and suppose there exist homogeneous elements $t_j \in R$ (j in some index set J) and a natural number n such that*

$$\pi(R)_{\geq n} \subseteq \sum_{j\in J} \pi(R)\pi(t_j).$$

Then, for all $m \in \mathbb{N}$, there exists a natural number k such that

$$R_{\geq k} \subseteq RX^m + \sum_{j\in J} Rt_j$$

Proof. Put $k = n + md$ and consider a homogeneous element μ of degree $l \geq k$ in R. The claim is trivial when $m = 0$. If $m \geq 1$ then $l \geq n$, thus by hypothesis

$\exists (a_j)_{j \in J} \in h(R)$ (almost all zero), $\exists s_1 \in R_{l-d}$ such that $\mu = \sum_{j \in J} a_j t_j + s_1 X$. If $m = 1$, then the proof is finished yet. Otherwise $l - d \geq n$, so we may apply the foregoing argument to s_1 and we obtain that s_1 can be written as $\sum_{j \in J} b_j t_j + s_2 X$ for some $b_j \in h(R), s_2 \in R_{l-2d}$ (almost all b_j zero). Thus $\mu = \sum_{j \in J}(a_j + b_j X)t_j + s_2 X^2$. After at most m steps, we find that μ is in $RX^m + \sum_{j \in J} Rt_j$. $\square$

We make the following convention for the rest of this paper: a multiplicatively closed set is supposed to contain 1 and is not allowed to contain 0.

Lemma 1. *Let R be as above and suppose that S is a multiplicatively closed set in R such that $\pi(S)$ is a left Ore-set in $R/(X)$. If $\pi_n : R \to R/(X^n)$ denotes the canonical epimorphism then for every $n \in \mathbb{N}$, $\pi_n(S)$ is an Ore-set (consisting of homogeneous elements) of $R/(X^n)$.*

Proof. We proceed by induction on n, the case $n = 1$ being given. Suppose by induction that $\pi_{n-1}(S)$ is left Ore in $R/(X^{n-1})$. To prove the first Ore-condition, let $\pi_n(r)\pi_n(s) = 0$ for some $s \in S$. Then also $\pi_{n-1}(r)\pi_{n-1}(s) = 0$. Thus $\exists s' \in S$ such that $\pi_{n-1}(s')\pi_{n-1}(r) = 0$, say $s'r = uX^{n-1}$. Now certainly $\pi_n(s'rs) = 0$, which means that $usX^{n-1} = aX^n$ for some $a \in R$ or $\pi(u)\pi(s) = 0$. By assumption, $\exists t \in S$ such that $\pi(t)\pi(u) = 0$ i.e. $tu = bX$ for some $b \in R$. Now $\pi_n(ts') \in \pi_n(S)$ and $\pi_n(ts')\pi_n(r) = \pi_n(tuX^{n-1}) = \pi_n(bX^n) = 0$. To prove the second Ore-condition, given $r \in R$ and $s \in S$ then by induction $\exists t \in S, u \in R$ such that $tr - us \in X^{n-1}R$, say $tr - us = aX^{n-1}$. Since $\pi(S)$ is Ore in $R/(X)$, $\exists s' \in S, b \in R$ such that $\pi(s')\pi(a) = \pi(b)\pi(s)$, say $s'a - bs = cX$. Then $s'tr - s'us = s'aX^{n-1} = bsX^{n-1} + cX^n$, i.e. $\pi_n(s't)\pi_n(r) = \pi_n(s'u + bX^{n-1})\pi_n(s)$. $\square$

Proposition 2. *With assumptions as before, if R is Noetherian and S is saturated, meaning that $\pi^{-1}(\pi(S)) = S$, then S is a left Ore-set in R.*

Proof. Suppose $rs = 0$ for some $r \in h(R)$, $s \in S$. Since X is a central regular element, we may suppose that $\pi(r) \neq 0$. By the lemma, $\forall n \in \mathbb{N}$, $\exists a_n \in R$, $\exists s_n \in S$ such that $s_n r = a_n X^n$. Here we may take the a_n to be homogeneous. Consider the ascending chain $Ra_1 \subseteq Ra_1 + Ra_2 \subseteq \ldots$, then we conclude (using the Noetherian hypothesis) that $\exists n \in \mathbb{N}$ such that $Ra_1 + \cdots + Ra_n = Ra_1 + \cdots + Ra_{n+1}$. In particular, $\exists b_i \in h(R)$ with $a_{n+1} = \sum_{i=1}^n b_i a_i$. Now $c = s_{n+1} - \sum_{i=1}^n b_i s_i X^{n+1-i} \in S$ and $cr = 0$, so the first Ore-condition is lifted. Since X is regular in the Noetherian ring R, the two-sided ideal (X) generated by X is an Artin–Rees ideal. Thus given $r \in h(R)$, $s \in S$, taking $L = Rr + Rs$, $\exists n \in \mathbb{N}$ with $L \cap RX^n \subseteq LX$. Applying the lemma for this n yields $\exists a \in h(R)$, $t \in S$ such that $\pi_n(t)\pi_n(r) = \pi_n(a)\pi_n(s)$, hence $tr - as \in RX^n \cap L \subseteq LX$, thus $\exists b, c \in h(R)$ with $tr - as = X(br + cs)$. Finally, $(t - bX)r = (a + cX)s$ where $t + bX \in S$. $\square$

The above situation arises naturally in the theory of filtered rings. Let A be a *positively* filtered k-algebra (i.e. $F_0 R = k$), $\sigma : A \to G(A)$ the principal symbol map and $\tilde{A}$ its Rees-ring. Both $G(A)$ and $\tilde{A}$ are positively graded and there is a canonical central element X in $\tilde{A}$ of degree 1 such that $\tilde{A}/(X) \cong G(A)$. Suppose also that $\tilde{A}$ is Noetherian. This is equivalent to $G(A)$ being Noetherian or the filtration FA being Zariskian. (For a survey of the theory of Zariskian filtrations we refer to [12].) If S is a multiplicative set in A such that $\sigma(S)$ is a multiplicative set in $G(A)$, then $\tilde{S} = \{sX^{\deg \sigma(S)} \mid s \in S\}$ is a multiplicative set (consisting of homogeneous elements) in $\tilde{A}$.

Theorem 1. *Let A be positively filtered by FA such that $F_0A = k$ is a field. If $G(A)$ is schematic, then $\tilde{A}$ is schematic.*

Proof. Let $(S_i)_{i=1,\dots,n}$ be a finite number of Ore-sets in $G(A)$ defining the schematic structure. If $\pi : \tilde{A} \to G(A)$ is the canonical map, then each $T_i \overset{\text{def}}{=} \pi^{-1}(S_i)$ is a saturated multiplicatively closed set, thus Ore by Proposition 2. Select $x_i \in T_i$ for every $i \in I$. Since $G(A)$ is schematic we have that, for some $n \in \mathbb{N} : G(A)_{\geq n} \subset \sum_{i \in I} G(A)\pi(x_i)$. Since X is a central regular element in $\tilde{A}_1$, we may apply Proposition 1 to $\tilde{A}$ and so for a chosen power X^m there exists a $k \in \mathbb{N}$ such that $\tilde{A}_{\geq k} \subseteq \tilde{A}X^m + \sum_{i \in I} \tilde{A}x_i$. It follows that the T_i together with $\{X^n; n \in \mathbb{N}\}$ establish a schematic structure on $\tilde{A}$. $\square$

Example 1. The Rees-ring of an almost commutative ring is schematic. (*A* ring R is called almost commutative if there exists a filtration on R such that the associated graded ring is commutative.)

For instance, the algebra A generated by three elements X, Y and Z of degree 1 with relations:

$$XY - YX = Z^2, \qquad XZ - ZX = 0, \qquad YZ - ZY = 0,$$

is schematic because it is the Rees-ring of the first Weyl-algebra with respect to the Bernstein-filtration. This algebra is sometimes called the homogenised Weyl-algebra. It illustrates the main disadvantage of the previous theorem: the Ore-set $\{X^n \mid n \in \mathbb{N}\}$ in $\mathbb{C}[X, Y]$ is lifted to $\{X^n + Za \mid a \in A_{n-1}, n \in \mathbb{N}\}$ while it is obvious that $\{X^n \mid n \in \mathbb{N}\}$ is already an Ore-set in A. Thus the obtained Ore-sets are in some sense not the best possible. This is only relevant when we want to calculate the section-algebras explicitly, as in Section 3.

Theorem 2. *Let A be as in the previous theorem. If $G(A)$ is a schematic domain and a maximal order, then there exist Ore-sets S_i such that*

$$A = \bigcap A_{S_i}$$

Proof. Set R=$\tilde{A}$. The Ore-sets $\bar{S_i}$ that make $G(A)$ schematic may be lifted to Ore-sets $\tilde{S_i}$ that make R schematic, thus $Q_{\kappa_+}(R) = \bigcap R_{\tilde{S_i}}$. We claim that $Q_{\kappa_+}(R) = R$, i.e. that R is $\mathscr{L}(\kappa_+)$-closed. So we have to prove that $\forall I \in \mathscr{L}(\kappa_+)$:

$$R \longrightarrow \operatorname{Hom}_R(I, R), \qquad x \;\mapsto\; (r \mapsto rx)$$

is an isomorphism. The map is injective since R is a domain. Since R is a maximal order in its quotientring Q (see [17]), we may identify $\mathrm{Hom}_R(I,R)$ with $B = \{q \in Q \mid Iq \subseteq R\}$, so $R \subseteq B \subseteq Q$. If we choose a non-zero element a of I, then $aB \subseteq R$. Thus R and B are equivalent orders and $R = B$ follows. We now have obtained that $\tilde{A} = \bigcap \tilde{A}_{\tilde{S}_i}$. The sets $S_i = \{s \in A \mid \sigma(s) \in \bar{S_i}\}$ are Ore and it is now easy to see that $A = \bigcap A_{S_i}$. □

Remark 1. The theorem holds already if $G(A)$ is κ_+-closed, i.e. $Q_{\kappa_+}(G(A)) \cong G(A)$, where κ_+ is now the radical associated to the positive part of $G(A)$.

It is probably not true that the class of schematic algebras is closed under iterated Ore-extensions since Ore-sets in a ring R need not be Ore in an Ore-extension $R[x,\sigma,\delta]$. However, if a graded ring R of the type we consider is schematic by means of Ore-sets $(S_i)_i$ which are still Ore in a certain extension $R[x,\sigma,\delta]$ and x generates an Ore-set in $R[x,\sigma,\delta]$, then the following theorem proves that this extension is also schematic.

Theorem 3. *Given a positively graded ring R which is generated by R_1 and which is schematic by means of Ore-sets S_i, given σ a graded automorphism of R and δ a σ-derivation of degree* 1, *then $\forall (s_i) \in \prod S_i, \forall m \in \mathbb{N}, \exists p \in \mathbb{N}$ such that*

$$(R[x,\sigma,\delta]_+)^p \subseteq M \stackrel{\text{def}}{=} \sum_i R[x,\sigma,\delta]s_i + R[x,\sigma,\delta]x^m$$

where $R[x,\sigma,\delta]$ denotes the Ore-extension considered with gradation $(R[x,\sigma,\delta])_n = \bigoplus_{k=0}^n R_k x^{n-k}$.

Proof. Because R is schematic, we know there is an $n \in \mathbb{N}$ such that $(R_+)^n \subseteq \sum Rs_i$. Put $p = n + m$ and choose $l \geq p$. We proceed by induction on t, the number of x occurring in a monomial of $R[x,\sigma,\delta]$ of length l. The case $t = 0$ follows from $l \geq m$. Suppose all monomials of length l with occurrences of x smaller than t belong to M. If $t \geq m$, then, modulo elements of M, we may permute all x to the last place and we end with a term in $R[x,\sigma,\delta]x^m$. If $t < m$, then again modulo M we may rewrite the monomial in the form $x^t a$ with $a \in R_{l-t}$ and this one is in M because $l - t \geq n$. □

Using this theorem, we can provide new examples of schematic algebras.

Example 2. The coordinate ring of quantum 2×2-matrices $\mathcal{O}_q(M_2(\mathbb{C}))$ ($q \in \mathbb{C}$, see [7]) is schematic. This algebra is by definition generated by elements a,b,c,d subject to the relations:

$$\begin{array}{lll} ba = q^{-2}ab, & ca = q^{-2}ac, & bc = cb, \\ db = q^{-2}bd, & dc = q^{-2}cd, & ad - da = (q^2 - q^{-2})bc. \end{array}$$

Using the Diamond lemma (cf. [5]) and Proposition 2, or checking repeatedly the conditions on an Ore-extension of a schematic algebra in order to be schematic (see the comments before Lemma 1), one proves that $\mathcal{O}_q(M_2(\mathbb{C}))$ is schematic by means of the Ore-sets formed by the powers of the generators a,b,c and d.

Example 3. Quantum Weyl algebras as defined by J. Alev and F. Dumas in [1] are schematic. We briefly recall the definition. Given an $n \times n$ matrix $\Lambda = (\lambda_{ij})$ $(n \geq 2)$ with $\lambda_{ij} \in k^*$ and a row vector $\overline{q} = (q_1, \dots, q_n)$, all $q_i \neq 0$, one defines the Quantum Weyl algebra $A_n = A_n^{\overline{q},\Lambda}$ as the algebra generated by $x_1, \dots, x_n, y_1, \dots, y_n$ and subjected to relations $(i < j)$:

$$x_i x_j = \mu_{ij} x_j x_i, \qquad x_i y_j = \lambda_{ji} y_j x_i, \qquad y_j y_i = \lambda_{ji} y_i y_j, \qquad x_j y_i = \mu_{ij} y_i x_j,$$
$$x_j y_j = 1 + q_j y_j x_j + \sum_{i<j} (q_i - 1) y_i x_i,$$

where $\mu_{ij} = \lambda_{ij} q_i$.

Proof. We may view A_n as an iterated Ore-extension where the variables are added in the order $x_1, y_1, x_2, y_2, \dots$. With respect to the standard filtration, the successive associated graded rings have one of the following forms: $G(A_{k-1})[x_k, \sigma]$ or $G(A_{k-1})[x_k, \sigma][y_k, \tau, \delta] = G(A_k)$. It is easy to see that $\{x_k^n \mid n \in \mathbb{N}\}$ is an Ore-set in $G(A_{k-1})[x_k, \sigma]$ so the previous theorem entails that $G(A_{k-1})[x_k, \sigma]$ is schematic. This implies that $G(A_{k-1})[x_k, \sigma]$ satisfies the conditions of the previous theorem. So we are left to prove that $\{y_k^n \mid n \in \mathbb{N}\}$ is an Ore-set in $G(A_{k-1})[x_k, \sigma][y_k, \tau, \delta]$. Since this is a commuting set, it suffices to establish the exchange condition for all pairs (x_i, y_k), (y_i, y_k) $(1 \leq i \leq k-1)$ and for (x_k, y_k). This follows immediately from the relations, except for the last one, so let us consider that one explicitly. In $G(A_k)$, we have

$$x_k y_k = q_k y_k x_k + \sum_{i<k} (q_i - 1) y_i x_i$$

or

$$y_k x_k = q_k^{-1} x_k y_k - q_k^{-1} \sum_{i<k} (q_i - 1) y_i x_i.$$

Multiplying on the left by y_k and using the relation $y_k y_i x_i = \lambda_{ki} y_i y_k x_i = y_i x_i y_k$, we find

$$y_k^2 x_k = q_k^{-1} \left(y_k x_k - \sum_{i<k} (q_i - 1) y_i x_i \right) y_k.$$

It follows from the foregoing theorem that $G(A_k)$ is schematic, thus A_k is schematic. $\square$

Example 4 (*Sklyanin-algebras*). Consider a three-dimensional Sklyanin-algebra over a field K. According to [4], this is a graded K-algebra A_K generated by three homogeneous elements X, Y and Z of degree 1 with relations of the following form:

$$aXY + bYX + cZ^2 = 0,$$
$$aYZ + bZY + cX^2 = 0,$$
$$aZX + bXZ + cY^2 = 0,$$

where a, b, c are in K.

We need some easy lemmas:

Lemma 2. *If R is a graded k-algebra such that its center $Z(R)$ is Noetherian and such that R is a finitely generated $Z(R)$-module, then R is schematic.*

Lemma 3. *If $K \subseteq L$ is a field extension, A is a schematic K-algebra and if $B = A \otimes_K L$ is a domain, then B is a schematic L-algebra.*

Theorem 4. *Three-dimensional Sklyanin-algebras over $\mathbb{C}$ are schematic.*

Proof. Let S denote the subring of $\mathbb{C}$ generated by a, b and c. If we choose $\mathfrak{m}$ a maximal ideal of S, then $S_\mathfrak{m}$ is a local ring of K, the quotient field of S, with finite residue field. Consequently, there exists a discrete valuation ring R of K which dominates $S_\mathfrak{m}$ and whose residue field F is finite. It follows from [4] that A_F satifies the conditions of the first lemma, yielding that A_F is schematic. If h is a uniformiser of R, then

$$A_R/(h) \cong A_R \otimes_R F \cong A_F$$

Since h is a central element of degree 0 in A_R, we are able to lift the Ore-sets of A_F to homogeneous Ore-sets S_i in A_R by Proposition 2.

Moreover, given $s_i \in S_i$, then there exists a natural number n such that $\forall x \in (A_R)_{\geq n}, \exists r_{\neq 0} \in R$ such that $rx \in \sum_i A_R x_i$: we know there is an $n \in \mathbb{N}$ such that $\forall m \geq n, \forall x \in (A_R)_m, \exists y \in (A_R)_m$ with $x - hy \in I = \sum_i A_R x_i$. Fix an element x in A_R of degree bigger then n, then iterating this process yields elements $x_1 = x, x_2, \ldots$, all of which have the same degree and which satisfy $x_j - hx_{j+1} \in I$. Since A_R is Noetherian (cf. [4, Lemma 8.9]) and h is a central element, we have that (h) is an Artin–Rees ideal of A_R. Applied to the left ideal $L = I + A_R x$, this gives a $j \in \mathbb{N}$ such that $L \cap (h^j) \subseteq hL$. Now $x - h^j x_{j+1}$ is in I and this implies that $h^j x_{j+1} \in L \cap (h^j) \subseteq hL$. Thus we may write $h^j x_{j+1}$ in the form $h(a+dx)$ where a is in I and d is in $(A_R)_0 = R$. Consequently, $(hd + 1)x \in I$ and it is trivial that $r = hd + 1$ is non-zero.

If we put A_K equal to $A_R \otimes_R K$, then it is easy to see that the Ore-sets S_i remain Ore in A_K by means of which A_K is schematic. Thus $A_\mathscr{C}$ is also schematic because

$$A_\mathscr{C} \cong A_K \otimes_K \mathbb{C}. \qquad \square$$

Remark 2. This method might be useful for showing that other nice algebras given by generators and relations are schematic.

The next examples of schematic algebras are not necessarily domains.

Example 5 (*Color Lie Super Algebras*). It follows from Example 1 that the Rees-ring of an enveloping algebra of a finite dimensional Lie algebra is schematic, but this is even true for color Lie super algebras: let L be a finite dimensional Γ-graded ε-Lie algebra where Γ denotes a finite abelian group and where ε is a mapping from $\Gamma \times \Gamma$

to $\mathbb{C}^*$ satisfying the following relations:

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)=1, \qquad \varepsilon(\alpha,\beta+\gamma)=\varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma), \qquad \varepsilon(\alpha+\beta,\gamma)=\varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma),$$

for all $\alpha,\beta,\gamma\in\Gamma$. Thus, L is a Γ-graded vectorspace $L=\bigoplus_{\gamma\in\Gamma}L_\gamma$ together with a graded bilinear mapping $\langle,\rangle$ satisfying the following identities:

$$\langle a,b\rangle = -\varepsilon(\alpha,\beta)\langle b,a\rangle$$

$$\varepsilon(\gamma,\alpha)\langle a,\langle b,c\rangle\rangle+\varepsilon(\alpha,\beta)\langle b,\langle c,a\rangle\rangle+\varepsilon(\beta,\gamma)\langle c,\langle a,b\rangle\rangle=0$$

for all $a\in L_\alpha,\ b\in L_\beta,\ c\in L_\gamma,\ \alpha,\beta,\gamma\in\Gamma$. The next properties of ε follow immediately from its definition and will be needed in the sequel:

$$\varepsilon(0,\alpha)=\varepsilon(\alpha,0)=1, \qquad \varepsilon(\alpha,\alpha)=\pm1 \quad \text{for all } \alpha\in\Gamma.$$

Define

$$\Gamma_0=\{\alpha\in\Gamma \mid \varepsilon(\alpha,\alpha)=1\}, \qquad L^0=\bigoplus_{\alpha\in\Gamma_0}L_\alpha.$$

Then $\Gamma_0=\Gamma$ or Γ_0 is a subgroup of index 2 in Γ. If $T(L)$ denotes the tensoralgebra on L and $J(L)$ is the ideal of $T(L)$ generated by all $a\otimes b-\varepsilon(\alpha,\beta)b\otimes a-\langle a,b\rangle$ $(a\in L_\alpha, b\in L_\beta)$ then $U(L)=T(L)/J(L)$ is the universal enveloping algebra of L. $U(L)$ is Γ-graded and has a positive filtration by putting $F_nU(L)$ equal to the canonical image of $T^nU(L)$. The generalized Poincaré–Birkhoff–Witt theorem states that the associated graded algebra $G(L)$ is isomorphic as a $\mathbb{Z}\times\Gamma$-algebra to the quotient of $T(L)$ by the ideal generated by all $a\otimes b-\varepsilon(\alpha,\beta)b\otimes a$ with $a\in L_\alpha, b\in L_\beta$. Consequently, if $\{e_i\}_{i=1\dots n}$ is a homogeneous basis for L, say $\deg(e_i)=\delta(i)\in\Gamma$, then all elements $e_1^{i_1}\cdots e_n^{i_n}$ with $0\le i_j\le 1$ if $\delta(j)\notin\Gamma_0$ form a basis for the vectorspace $U(L)$. As usual, we denote the principal map from $U(L)$ to $G(L)$ by σ, thus $\forall a\in F_nU(L)\setminus F_{n-1}U(L)$, $\sigma(a)=a \bmod F_{n-1}U(L)$. Write $\langle e_i,e_j\rangle=\sum_{k=1}^n\beta_{ij,k}e_k$. The algebra R generated over $\mathbb{C}$ by $E_1,\dots,E_n,X_0$ with relations

$$E_iX_0=X_0E_i, \qquad E_iE_j-\varepsilon(\delta(i),\delta(j))E_jE_i=\sum_{k=1}^n\beta_{ij,k}E_kX_0,$$

may be viewed as the Rees-ring of $U(L)$, so $R/X_0R\cong G(L)$.

Theorem 5. *$G(L)$ is schematic and consequently R is schematic.*

Proof. Say $\varepsilon(\delta(i),\delta(i))=1$ for $1\le i\le r$ and -1 for $r<i\le n$. We claim that for all $i\le r$ the set $S_i=\{1,\sigma(e_i),\sigma(e_i)^2,\dots\}$ is an Ore-set in $G(L)$. We first prove that $\sigma(e_i)$ is a regular element of $G(L)$. If $L=L^0$ then $G(L)$ is a domain and we are done. So assume that Γ_0 is a subgroup of index 2 in Γ. Let A be the algebra with generators $e_1,\dots,e_r$ and relations $e_ie_j=\varepsilon(\delta(i),\delta(j))e_je_i$ and let B be the algebra with generators $e_{r+1},\dots,e_n$ and relations $e_ie_j=\varepsilon(\delta(i),\delta(j))e_je_i$. Then one knows that $G(L)\cong A\otimes_\varepsilon B$ where $A\otimes_\varepsilon B$ is the usual vector space tensor product but the multiplication is determined

by $(e_i \otimes e_j)(e_k \otimes e_l) = \varepsilon(\delta(j), \delta(k))e_ie_k \otimes e_je_l$. Now $\sigma(e_i)$ corresponds under this isomorphism to the homogeneous element $e_i \otimes 1$ which is regular in $A \otimes_\varepsilon B$.

The second Ore-condition is easy to verify since S_i is a commuting set and $\forall j = 1,\dots,n : \sigma(e_j)\sigma(e_i) = \varepsilon(\delta(j), \delta(i))\sigma(e_i)\sigma(e_j)$.

If $\mathscr{L}(\kappa_i)$ is the filter corresponding to the Ore-set S_i ($\delta(i) \in \Gamma_0$) and a left ideal I is in the intersection of these filters, then we know that for all i there exists an integer n_i such that $\sigma(e_i)^{n_i} \in I$. Put $m = n - r + \sum_{i=1}^r n_i$, then it follows from the generalized Poincaré–Birkhoff–Witt theorem that $(G(L)_+)^m \subseteq I$. Consequently, $\mathscr{L}(\kappa_+) \stackrel{\text{def}}{=} \{I \lhd_l G(L) \mid \exists\, n \in \mathbb{N} : (G(L)_+)^n \subseteq I\} = \cap \mathscr{L}(\kappa_i)$. $\square$

It is an interesting question whether there are algebras (besides the free algebras) which do not have enough Ore-sets in the sense of the definition of schematic algebras. It is hard to prove this directly, therefore we first make the following observation:

Proposition 3. *If R is (left) schematic, then $\mathrm{Ext}^n_R(k_R, R)$ is torsion for all $n \in \mathbb{N}$.*

Proof. It is well-known that for a finitely generated right R-module M and a two-sided Ore-set S, we have

$$Q_S(\mathrm{Ext}^n_R(M,R)) \cong \mathrm{Ext}^n_{Q_S(R)}(Q_S(M), Q_S(R)).$$

It follows that if R is schematic by means of the two-sided Ore-sets S_i, then the localisation of $\mathrm{Ext}^n_R(k_R, R)$ at each S_i is zero, hence it is torsion. $\square$

Example of a non-schematic algebra. The next algebra is taken from [16] where it is used to show that not all Noetherian connected algebras satisfy the condition χ defined in [3].

Let U be the graded algebra $k\langle x, y\rangle/(yx - xy - x^2)$ and suppose $char(k) = 0$. If R denotes the subalgebra of U, generated by y and xy, then $\mathrm{Ext}^1_R(k_R, R)) \cong U/R$ as graded left R-modules (see [16]). An easy induction shows that for all $n \in \mathbb{N}$: $y^n x \equiv n! x^{n+1} \bmod R$, hence U/R is not torsion as a left R-module. Using the previous proposition, we conclude that R is not left schematic.

This counterexample is in a sense unsatisfactory since R is not generated in degree 1. As in the commutative theory, we should define a weighted *Proj* for algebras not generated in degree 1.

3. Quantum sections of enveloping algebras

3.1. Introduction

Start with a finite dimensional Lie algebra $\mathfrak{g} = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$ and denote the structure constants by $\alpha_{ij,k}$, i.e.

$$[x_i, x_j] = \sum_{k=1}^n \alpha_{ij,k} x_k.$$

The enveloping algebra $U(\mathfrak{g})$ is equipped with a natural filtration such that its associated graded ring is the polynomial algebra in n variables. The corresponding Rees-ring $H(\mathfrak{g})$ is then generated by $n+1$ degree 1 elements $X_1,\dots,X_n,Z$ and the defining relations are:

$$X_iX_j - X_jX_i = \sum_{k=1}^{n} \alpha_{ij,k}X_kZ, \quad \forall i,j \in \{1,\dots,n\},$$
$$X_iZ - ZX_i = 0, \quad \forall i \in \{1,\dots,n\}.$$

Therefore it is sometimes called the homogenised enveloping algebra. There are canonical isomorphisms

$$H(\mathfrak{g})/(Z-1) \simeq U(\mathfrak{g}) \quad \text{and} \quad H(\mathfrak{g})/(Z) \simeq \mathbb{C}[x_1,\dots,x_n].$$

They imply that $H(\mathfrak{g})$ has excellent homological properties: it is Auslander-regular of Gelfand–Kirillov dimension $n+1$ and it satisfies the Cohen–Macaulay property, cf. [12].

As in [8, 9], we define the *quantum space* $\mathbb{P}_q(\mathfrak{g})$ of $\mathfrak{g}$ to be *Proj*$(H(\mathfrak{g}))$. (We explain in [18] the way we look at this quotient category.) We conclude that $H(\mathfrak{g})$ is schematic using Theorem 1 and the second isomorphism mentioned above. Thus according to [18], we can associate a geometric picture to $\mathbb{P}_q(\mathfrak{g})$. As this picture heavily uses localisations and the Ore-sets furnished by Theorem 1 are rather big, looking for smaller ones pays off. In [6], the authors have determined the smallest Ore-set of $U(\mathfrak{g})$ containing a given element x of $\mathfrak{g}$: it is generated by the set $\{x - e \mid e \in \mathbb{Z}E\}$ where E is the set of all eigenvalues of the adjoint representation of x. If we make the convention that corresponding upper- and lower-case letters will denote corresponding elements of $H(\mathfrak{g})_1$ and $\mathfrak{g}$, then it is trivial that the smallest Ore-set $S(X)$ of $H(\mathfrak{g})$ we are looking for is generated by the set $\{X - eZ : e \in \mathbb{Z}E\}$. This Ore-set consists of homogeneous elements, thus $H(\mathfrak{g})_{S(X)}$ is a graded algebra and we may define:

Definition 1. The *quantum sections* $\Gamma(X)$ of the quantum space $\mathbb{P}_q(\mathfrak{g})$ on "the open set corresponding to x" is the degree zero part of the graded algebra $H(\mathfrak{g})_{S(X)}$.

For example, from the general theory of filtered rings, it follows that $\Gamma(Z) = (H(\mathfrak{g})_{\{1,Z,Z^2,\dots\}})_0 \cong U(\mathfrak{g})$. The quantum sections are again Auslander-regular, because $H(\mathfrak{g})_{S(X)}$ is strongly graded, meaning that $(H(\mathfrak{g})_{S(X)})_i(H(\mathfrak{g})_{S(X)})_j = (H(\mathfrak{g})_{S(X)})_{i+j}$ and in particular that the category of graded $H(\mathfrak{g})_{S(X)}$-modules is equivalent with the category of $\Gamma(X)$-modules (cf. [14]). However, there are also noticeable differences between the quantum sections $\Gamma(X)$ and $U(\mathfrak{g})$. For instance, they do not have to be finitely generated and with the filtration induced by the canonical generators (as in the next subsection) they are virtually never almost commutative. Further, they usually have larger Gelfand–Kirillov dimension by [11, Theorem 8].

Choose a basis $(y_i)_{i=1,\dots,n}$ of $\mathfrak{g}$, then it is easily seen that $H(\mathfrak{g})$ is schematic by means of the Ore-sets $S(Y_i)$ together with $\{Z^n \mid n \in \mathbb{N}\}$. It follows from Theorem 2 that $U(\mathfrak{g})$ can be written as an intersection of certain localisations at Ore-sets generated by degree one elements. We now prove a stronger version of this theorem, namely that $U(\mathfrak{g})$ can be written as an intersection of two localisations.

Proposition 4. *If $i \neq j$ then $U(\mathfrak{g}) = S_{x_i}^{-1}U(\mathfrak{g}) \cap S_{x_j}^{-1}U(\mathfrak{g})$ where S_{x_k} is the multiplicatively closed set generated by all $x_k - \alpha$ with α in the abelian group generated by all eigenvalues of* $\mathrm{ad}x_k$ $(k = i,j)$.

Proof. Suppose $z \in T \stackrel{\text{def}}{=} S_{x_i}U(\mathfrak{g}) \cap S_{x_j}U(\mathfrak{g})$. Then $z = s^{-1}a = t^{-1}b$ for some $a,b \in U(\mathfrak{g})$ and $s \in S_{x_i}, t \in S_{x_j}$, hence $\sigma(s) = x_i^n, \sigma(t) = x_j^m$ by definition of S_{x_i} and S_{x_j}. $G(U(\mathfrak{g}))$ is a domain, so σ is multiplicative and $\sigma(z) = x_i^{-n}\sigma(a) = x_j^{-m}\sigma(b)$, or $x_j^m\sigma(a) = x_i^n\sigma(b) \in (x_i^n)$. Since $j \neq i$, we must have that $\sigma(a) \in (x_i^n)$ and thus $\deg\sigma(a) \geq n$. Therefore $\deg\sigma(z) \geq 0$ holds for all z in T. Suppose now that z is in T but not in $U(\mathfrak{g})$. We can write $\sigma(a) = x_i^n\sigma(c_1)$ for some c_1 in $U(\mathfrak{g})$. Then $\sigma(z) = \sigma(c_1)$, $0 \neq z - c_1 \in T$ and $\deg\sigma(z - c_1) < \deg\sigma(z)$. After applying this argument a finite number of times, we find $z - c_1 - \cdots - c_r \in T$ but $\deg\sigma(z - c_1 - \cdots - c_r) < 0$, a contradiction. □

3.2. *Defining equations of quantum sections*

The purpose of the theory in [18] is to reduce questions about the graded algebra R to questions about the ungraded algebras $(Q_S(R))_0$, making a projective theory locally affine as desired. It is therefore important to know these section-algebras very well. In this subsection, we describe the section-algebra $\Gamma(X)$ of an enveloping algebra explicitly by generators and relations. It is clear that its relations are simplified by starting with an appropriate basis for $\mathfrak{g}$, for instance one could choose a basis containing x such that the matrix of $\mathrm{ad}x$ is in Jordan-normal form. However, we do not want a basis depending on x since we will compare different section-algebras. Thus fix a basis $x_1,\ldots,x_n$ of $\mathfrak{g}$ and say that

$$[x_i,x_j] = \sum_{k=1}^{n} \alpha_{ij,k}x_k, \quad \forall i,j \in \{1,\ldots,n\}.$$

Fix $x \in \mathfrak{g}$, say $x = \sum_{i=1}^{n} \gamma_i x_i$. We obtain the relations for $\Gamma(X)$ as follows: suppose that

$$\mathrm{ad}x(x_i) = \sum_{k=1}^{n} \beta_{ik}x_k, \quad \forall i = 1,\ldots,n$$

then we know how X commutes with the X_i in $H(\mathfrak{g})$:

$$XX_i - X_iX = \sum_{k=1}^{n} \beta_{ik}X_kZ, \quad \forall i = 1,\ldots,n.$$

If we multiply these equations on both sides by X^{-1} and substitute the resulting equations in $(X^{-1}X_i)(X^{-1}X_j) - (X^{-1}X_j)(X^{-1}X_i)$ and in $(X^{-1}X_i)(X^{-1}Z) - (X^{-1}Z)(X^{-1}X_i)$ then we obtain a subset of the relations of $\Gamma(X)$.

The only problem is to show that one gets all relations of $\Gamma(X)$ in this way. We proceed as follows: if B is the $\mathbb{C}$-algebra generated by $X_1,\ldots,X_n,Z$ and T_e (one for each $e \in \mathbb{Z}E$) with relations:

$$X_iX_j - X_jX_i - \sum_{k=1}^{n} \alpha_{ij,k} ZX_k \quad (i = 1,\dots,n),$$

$$X_iZ - ZX_i \quad (i = 1,\dots,n),$$

$$(X - eZ)T_e - 1 \quad (e \in \mathbb{Z}E), \qquad T_e(X - eZ) - 1 \quad (e \in \mathbb{Z}E)$$

(where $X = \sum_{i=1}^{n} \gamma_i X_i$), then it is clear that $B \cong H(\mathfrak{g})_S$. Because the degree 1-element X is invertible in B, this algebra is of the form $B_0[X, T_0, \varphi]$ where φ is the inner automorphism of B_0 induced by X. We want to rewrite the relations above as much as possible in terms of degree zero elements of B. Define $Y_i = T_0X_i$ $(i = 1,\dots,n)$, $Z_0 = T_0Z$ and $X_e = T_eX$ $(e \in \mathbb{Z}E \setminus \{0\})$. Define V as the $\mathbb{C}$-vectorspace generated by Y_i, X_e, Z_0, X and T_0 and let TV denote the tensoralgebra on V where all generators have degree 0, except X and T_0 which have degree 1 and -1, respectively. Let J be the two-sided graded ideal of TV generated by

$$\sum_{i=1}^{n} \gamma_i Y_i - 1$$

$$Y_iY_j - Y_jY_i - \sum_{k=1}^{n} \alpha_{ij,k} Z_0 Y_k - \sum_{k=1}^{n} \beta_{ik} Z_0 Y_k Y_j + \sum_{k=1}^{n} \beta_{jk} Z_0 Y_k Y_i \quad (i,j = 1,\dots,n)$$

$$Y_iZ_0 - Z_0Y_i - \sum_{k=1}^{n} \beta_{ik} Z_0 Y_k Z_0 \quad (i = 1,\dots,n)$$

$$Y_iX - XY_i + \sum_{k=1}^{n} \beta_{ik} X Z_0 Y_k \quad (i = 1,\dots,n)$$

$$XZ_0 - Z_0X$$

$$XT_0 - 1 \text{ and } T_0X - 1$$

$$X_e(1 - eZ_0) - 1 \quad \text{and} \quad (1 - eZ_0)X_e - 1 \quad (e \in \mathbb{Z}E \setminus \{0\}).$$

Then $B \cong TV/J$. If W denotes the subspace of V generated by the Y_i, X_e and by Z_0, then there is a canonical map $TW/(TW \cap J_0) \to (TV/J)_0$ which is easily seen to be injective. It is surjective because X and T_0 are normalizing and the subalgebra of B_0 generated by the elements of W is invariant under φ. Thus

$$\Gamma(X) \cong B_0 \cong TW/(TW \cap J_0).$$

We conclude:

Theorem 6. *The section-algebra $\Gamma(X)$ is the algebra generated by X_e, (one for each $e \in \mathbb{Z}E$), $Y_i (i = 1,\dots,n)$ and Z_0 with relations*:

$$\sum_{i=1}^{n} \gamma_i Y_i = 1, \qquad Y_iY_j - Y_jY_i = \sum_{k=1}^{n} \alpha_{ij,k} Z_0 Y_k + \sum_{k=1}^{n} \beta_{ik} Z_0 Y_k Y_j - \sum_{k=1}^{n} \beta_{jk} Z_0 Y_k Y_i,$$

$$Y_iZ_0 - Z_0Y_i = \sum_{k=1}^{n} \beta_{ik} Z_0 Y_k Z_0, \qquad X_e(1 - eZ_0) = 1, \qquad (1 - eZ_0)X_e = 1.$$

Consequently, $\Gamma(X)$ is affine if and only if x is nilpotent.

Example 6. We calculate the quantum sections of sl_2. Fix the usual basis $\{e, f, h\}$ with

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

The sections $\Gamma(E)$ are generated by Y_2, Y_3 and Z_0 which satisfy the relations:

$$Y_2Y_3 - Y_3Y_2 = 4Z_0Y_2 + Z_0Y_3^2, \qquad Y_2Z_0 - Z_0Y_2 = Z_0Y_3Z_0,$$

$$Y_3Z_0 - Z_0Y_3 = -2Z_0^2.$$

The sections $\Gamma(F)$ are obtained by means of an isomorphism from $\Gamma(E)$ which maps $Z_0 = E^{-1}Z$ to $-F^{-1}Z$, $Y_2 = E^{-1}F$ to $F^{-1}E$ and $Y_3 = E^{-1}H$ to $F^{-1}H$.

The sections $\Gamma(H)$ are generated by Y_1 and Y_2 over the localisation of $\mathbb{C}[Z_0]$ at the multiplicatively closed set generated by $\{1 + 2nZ_0 \mid n \in \mathbb{Z}\}$. The ideal of relations has the following generators:

$$(1 - 2Z_0)Y_1Y_2 = Z_0 + (1 + 2Z_0)Y_2Y_1,$$

$$Y_1Z_0 - Z_0Y_1 = 2Z_0Y_1Z_0, \qquad Y_2Z_0 - Z_0Y_2 = -2Z_0Y_2Z_0.$$

3.3. Point modules

Let R be a positively graded k-algebra $R = k \oplus R_1 \oplus \cdots$, generated by a finite number of degree 1 elements. Recall from [4] that a point module of R is a cyclic graded R-module M with Hilbert series $(1 - t)^{-1}$, i.e. $\dim_k M_i = 1$ $(i \geq 0)$ and $\dim_k M_i = 0$ $(i < 0)$. Suppose that S is a homogeneous Ore-set in R such that $S \cap R_1 \neq \emptyset$. Then it is easy to prove that each point module M is either S-torsion or S-torsionfree. In the latter case, we have that $\dim_k(Q_S(M))_0 = 1$, i.e. M induces a one-dimensional representation of $(Q_S(M))_0$. Moreover, this correspondence is one-one, because if k is a $(Q_S(R))_0$-module, then $(Q_S(R) \otimes_{(Q_S(R))_0} k)_{\geq 0}$ is an S-torsionfree point module.

Suppose now that R is schematic by means of Ore-sets S_i such that $S_i \cap R_1 \neq \emptyset$ for each i. Then we get all point modules of R in this way, but point modules which are S_i-torsionfree for m indices i will appear m times. The solution to get a one–one correspondence is the introduction of sheaves, as in [18]. In the terminology introduced there, we have: there is a one-one correspondence between point modules and coherent sheaves $\mathscr{F}$ whose sections satisfy: $\dim_k(\mathscr{F}(S_i))_0 \leq 1$ (equality for at least one i). Note that the degree zero part of the coherent sheaf associated to a point module M is a sheaf in the classical sense because one can easily check that $(Q_S(Q_T(M)))_0 \cong (Q_{S \vee T}(M))_0$ for all homogeneous Ore-sets S and T of R. (Recall from [18] that in general a module M is determined by its loclisations only if the “mixed” localisations like $Q_S(Q_T(M))$ are included.)

We may apply the foregoing to the homogenised enveloping algebra $R = H(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ because it is schematic by means of Ore-sets generated in degree 1. In particular, we may easily determine the point variety of $H(\mathfrak{g})$ thereby recovering the result of [9]. Recall from [8, 9] that the point variety $\mathscr{V}$ is the variety in $\mathbb{P}(R_1^*)$

determining the point modules of R in the following way: let $0{\neq}f \in R_1^*$ belong to $\mathscr{V}$, choose n linearly independant elements a_i of R_1 such that $f(a_i) = 0$ then $R/\sum_i Ra_i$ is a point module of R. We retain the notation of section 1 and let S_i be the smallest Ore-set of $R = H(\mathfrak{g})$ containing X_i. The one-dimensional representations f of $\Gamma(X_l)$ $(l = 1,\dots,n)$ are determined by $n+1$ scalars $f(Y_1),\dots,f(Y_n),f(Z_0)$ satisfying the rules:

$$f(Y_l) = 1,$$

$$f(Z_0)\left(\sum_{k=1}^{n} \alpha_{ij,k} f(Y_k) + \sum_{k=1}^{n} \beta_{ik} f(Y_k)f(Y_j) - \sum_{k=1}^{n} \beta_{jk} f(Y_k)f(Y_i)\right) = 0,$$

$$f(Z_0)^2\left(\sum_{k=1}^{n} \beta_{ik} f(Y_k)\right) = 0, \qquad \forall e \in \mathbb{Z}E\setminus\{0\} \quad f(Z_0) \neq \frac{1}{e},$$

where i,j run through the set $\{1,\dots,n\}$. The corresponding point module (which is S_l-torisonfree) is R/I where I is the graded left ideal

$$\sum_{i\neq l} R(X_i - f(Y_i)X_l) + R(Z - f(Z_0)X_l).$$

The corresponding element of the point variety is

$$(f(Y_1) : \dots : 1 : \dots : f(Y_n) : f(Z_0)) \in \mathbb{P}^n(\mathbb{C}) = \mathbb{P}(R_1^*)$$

(the 1 is on the l-th place).

The one-dimensional representations of $\Gamma(Z) = U(\mathfrak{g})$ satisfy the rule $\sum_{k=1}^{n} \alpha_{ij,k} f(X_k) = 0$. Such a representation f belongs to the point module $R/\sum_{i=1}^{n} R(X_i - f(X_i)Z)$ and is represented by

$$(f(x_1) : \dots : f(x_n) : 1) \in \mathbb{P}^n(\mathbb{C}) = \mathbb{P}(R_1^*).$$

We are now able to describe the point variety $\mathscr{V}$ locally: the intersection of $\mathscr{V}$ with $\bigcap_{e\in\mathbb{Z}E} D(X_l - eZ)$ is given by those $0{\neq}f \in R_1^*$ such that

$$f([X_i,X_j]X_l + [X_l,X_i]X_j - [X_l,X_j]X_i)f(Z) = 0,$$

$$f([X_l,X_i])f(Z)^2 = 0, \qquad f(X_l - eZ) \neq 0.$$

The intersection of $\mathscr{V}$ with $D(Z)$ is

$$\left\{ 0{\neq}f \in R_1^* \,\middle|\, \sum_{k=1}^{n} \alpha_{ij,k} f(X_k) = 0 \right\}.$$

It follows that $\mathscr{V} = V((X_k[X_i,X_j] + X_j[X_k,X_i] + X_i[X_j,X_k])Z,\ [X_i,X_j]Z^2)$ and we regain a result contained in [9].

Example 7. We calculate the point variety $\mathscr{V}$ of $\mathfrak{g} = sl_2(\mathbb{C})$.

The points of $\mathscr{V}$ corresponding to a one-dimensional representation of $\Gamma(E)$ have the form: $(1 : f(Y_2) : f(Y_3) : f(Z_0))$ where

$$f(Z_0)(4f(Y_2) + f(Y_3)^2) = 0,$$
$$f(Y_3)f(Z_0)^2 = 0, \qquad -2f(Z_0)^2 = 0.$$

The $S(F)$-torsionfree point modules are given by those points $(f(Y_1) : 1 : f(Y_3) : f(Z_0))$ of $\mathbb{P}^3(\mathbb{C})$ which satisfy

$$f(Z_0)(4f(Y_1) + f(Y_3)^2) = 0, \qquad f(Y_3)f(Z_0)^2 = 0, \qquad 2f(Z_0)^2 = 0.$$

As ad$h$ possesses non-zero eigenvalues, the one-dimensional representations of $\Gamma(H)$ correspond with $(f(Y_1) : f(Y_2) : 1 : f(Z_0))$ where

$$f(Z_0)(1 + 4f(Y_1)f(Y_2)) = 0,$$
$$2f(Y_1)f(Z_0)^2 = 0, \qquad -2f(Y_2)f(Z_0)^2 = 0,$$
$$f(Z_0) \neq \frac{1}{2n} \quad (n \in \mathbb{Z} - \{0\}).$$

As $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, the only one-dimensional representation of $U(\mathfrak{g})$ is the trivial one. It corresponds to the origin $(0 : 0 : 0 : 1)$ in $\mathbb{P}^3(\mathbb{C})$

The point variety $\mathscr{V}$ is thus the plane at infinity together with the origin, in agreement with the result of [8]. The extra inequalities in the equations for $\Gamma(H)$ turn out to be superfluous, but they are important for a Lie algebra $\mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.

The results of this section can immediately be generalised to colour Lie super algebras. We do not give the result explicitly but we point the way by solving the main obstacle: given some element x of a colour Lie super algebra L, find the minimal Ore-set of the Rees-ring R of L containing X. This is an easy generalisation of the argument of [6]: fix $\alpha \in \Gamma_0$ and an element $x \in L_\alpha$. Suppose the order of α in Γ is t and let $\{\beta_1, \ldots, \beta_m\}$ be a complete set of coset representatives of the subgroup of Γ generated by α. Let $U(L)^{\beta_i} = \bigoplus_{j=0}^{t-1} U(L)_{\beta_i + j\alpha}$. Define an endomorphism d_x of $U(L)$ by declaring its image for each basis element:

$$d_x(e_1^{i_1} \ldots e_n^{i_n}) = xe_1^{i_1} \ldots e_n^{i_n} - \varepsilon(\alpha, \gamma)e_1^{i_1} \ldots e_n^{i_n}x,$$

where $\gamma = \sum_{j=1}^{n} i_j\delta(j)$. Now it is easy to verify that $d_x(F_mU(L)) \subseteq F_mU(L)$ for all $m \in \mathbb{N}$. Thus, given $r \in U(L)$, there exists a finite dimensional d_x-stable vectorspace V containing r such that the restriction of d_x to V is triagonalizable. Let E be the set of all eigenvalues of $d_x : U(L) \to U(L)$ and F the additive subgroup of $\mathbb{C}$ generated by $\bigcup_{\gamma \in \Gamma}^{m} \varepsilon(\alpha, \gamma)E$.

Theorem 7. *The multiplicatively closed set S generated by $\{x - e \mid e \in F\}$ is a left Ore-set in $U(L)$.*

Proof. Given r in $U(L)$, there exist d_x-stable spaces $0 = V_0 \subset V_1 \subset \cdots \subset V_l$ such that $r \in V_l$ and $\forall i = 1,\dots,l$, $\exists \alpha_i \in E$ such that $d_x(v + V_{i-1}) = \alpha_i v + V_{i-1}$, $\forall v \in V_i$. If $v \in V_i$ and $d_x(v) - \alpha_i v = w \in V_{i-1}$, then

$$\begin{aligned}(x-\alpha_i)v &= d_x(v) + \left(\sum_{\gamma\in\Gamma} \varepsilon(\alpha,\gamma)v_\gamma\right)x - \alpha_i v\\ &= w + \left(\sum_{\gamma\in\Gamma} \varepsilon(\alpha,\gamma)v_\gamma\right)x\\ &\in U(L)x + V_{i-1}.\end{aligned}$$

Thus $(x-\alpha_1)\cdots(x-\alpha_l)r \in U(L)x$ with $(x-\alpha_1)\cdots(x-\alpha_l) \in S$. If $e \in F$ then also $(x-e-c_1)\cdots(x-e-c_l)r \in U(L)(x-e)$ but here the c_i are eigenvalues of d_{x-e}. Now if c is an eigenvalue of d_{x-e}, then since $d_{x-e}(U(L)^{\beta_i}) \subseteq U(L)^{\beta_i}$ there exists $i \in \{1,\dots,m\}$ and a non-zero $v \in U(L)^{\beta_i}$ such that $d_{x-e}(v) = cv$. But then

$$\begin{aligned}d_x(v) &= d_{x-e}(v) + ev - \varepsilon(\alpha,\beta_i)ev\\ &= (c + e - \varepsilon(\alpha,\beta_i)e)v.\end{aligned}$$

Consequently, $c + e - \varepsilon(\alpha,\beta_i)e \in E$, thus $c \in F$ and $(x-e-c_1)\cdots(x-e-c_l) \in S$.

S satisfies the second Ore-condition because $\sigma(x-e) = \sigma(x)$ is regular in $G(L)$ as we have already proved in the previous theorem. □

Consequently, the set $\tilde{S} = \{sX_0^{\deg\sigma(s)} \mid s \in S\}$ is right Ore in R. Choose a homogeneous basis $\{e_i\}_{i=1\dots n}$ of L such that $e_1 = x$. Let

$$\langle e_i, x\rangle = \sum_{l=1}^{n} \alpha_{il} e_l \quad \text{for all } i = 1,\dots,n,$$

$$\langle e_i, e_j\rangle = \sum_{k=1}^{n} \beta_{ij,k} e_k \quad \text{for all } i,j = 1,\dots,n.$$

In R, these relations become

$$E_i X - \varepsilon(\delta(i),\alpha) X E_i = \sum_{l=1}^{n} \alpha_{il} E_l X_0.$$

Multiplying on both sides with X^{-1} yields

$$X^{-1}E_i - \varepsilon(\delta(i),\alpha) E_i X^{-1} = \sum_{l=1}^{n} \alpha_{il} X^{-1} E_l X_0 X^{-1}.$$

Put $Z_i = X^{-1}E_i$, $T = X^{-1}X_0$, thus $Z_1 = 1$. Then

$$\begin{aligned}&\varepsilon(\delta(i),\alpha)Z_iZ_j - \varepsilon(\delta(j),\alpha)\varepsilon(\delta(i),\delta(j))Z_jZ_i\\ &\quad = \varepsilon(\delta(i),\alpha)X^{-1}(E_iX^{-1})E_j - \varepsilon(\delta(j),\alpha)\varepsilon(\delta(i),\delta(j))X^{-1}(E_jX^{-1})E_i\end{aligned}$$

$$= X^{-1}\left(X^{-1}E_i - \sum_{l=1}^{n} \alpha_{il}X^{-1}E_lX_0X^{-1}\right)E_j$$

$$-\varepsilon(\delta(i),\delta(j))X^{-1}\left(X^{-1}E_j - \sum_{l=1}^{n} \alpha_{jl}X^{-1}E_lX_0X^{-1}\right)E_i$$

$$= \sum_{k=1}^{n} \beta_{ij,k}TZ_k - \sum_{l=1}^{n} \alpha_{il}TZ_lZ_j + \varepsilon(\delta(i),\delta(j))\sum_{l=1}^{n} \alpha_{jl}TZ_lZ_i$$

and $\varepsilon(\delta(i),\alpha)Z_iT = TZ_i - \sum_{l=1}^{n} \alpha_{il}TE_lT$ are the relations of $Q_S^q(R)$ up to the T-adic completion.

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