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Reference:

Boen Lynn, Guillaume Florence.- Towards a Δ -Gamma Sato multivariate model
Review of derivatives research - ISSN 1380-6645 - (2019), p. 1-39
Full text (Publisher's DOI): <https://doi.org/10.1007/S11147-019-09155-Y>
To cite this reference: <https://hdl.handle.net/10067/1574410151162165141>

Towards a Δ -Gamma Sato multivariate model

the date of receipt and acceptance should be inserted later

Abstract The increased trading in multi-name financial products has paved the way for the use of multivariate models that are at once computationally tractable and flexible enough to mimic the stylized facts of asset log-returns and of their dependence structure. In this paper we propose a new multivariate Lévy model, the so-called Δ -Gamma model, where the log-price gains and losses are modeled by separate multivariate Gamma processes, each containing a common and an idiosyncratic component. Furthermore, we extend this multivariate model to the Sato setting, allowing for a moment term structure that is more in line with empirical evidence. We calibrate the two models on single-name option price surfaces and market implied correlations and we show how the Δ -Gamma Sato model outperforms its Lévy counterpart, especially during periods of market turmoil. The numerical study also reveals the advantages of these new types of multivariate models, compared to a multivariate VG model.

Keywords multi-name option pricing · multivariate Lévy models · multivariate models with Sato marginals · Difference of Gamma processes · self-similar processes · calibration

1 Introduction

Ever since the stock market crash of October 1987, which highlighted the shortcomings of the Black-Scholes model, the search for and the development of more realistic stock price models have become important research topics in the financial world. Over the years, many univariate financial stock price models have been developed, such as the class of exponential Lévy models, which provides a variety of asset pricing models with more realistic features, like jumps, skewness and excess of kurtosis.

More recently, with the increasing amount of multi-asset derivatives, the field of interest has shifted to the modeling of the joint dynamics of a basket of assets. As an extension of the multivariate Black-Scholes model, Madan & Seneta (1990) proposed to subordinate a multivariate Brownian motion by a univariate Gamma-distributed time-change. As such, the symmetric multivariate VG model was born, which was later extended by Luciano & Schoutens (2006) to

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incorporate skewness. Since then, a variety of multivariate asset pricing models based on subordinated Brownian motions has been proposed, see f.i. Semeraro (2008), Luciano & Semeraro (2010) and Guillaume (2013). Besides via the time-change, the dependence between single-name assets can also arise by considering linear combinations of independent Lévy processes, among which at least one is common to all assets. This factor approach was first introduced by Vasicek (1987) for Brownian motions and was later used to build multivariate Lévy models as well, for example in Ballotta & Bonfiglioli (2014)¹ and Luciano et al. (2016).

While Lévy models are suited to replicate option prices for one single maturity, they perform poorly when replicating option prices across the whole set of quoted maturities, as was first illustrated by Carr et al. (2007) in the univariate case. Furthermore, as pointed out by Konikov & Madan (2001), there is a significant discrepancy between the term structure of the market implied moments and the Lévy moments. Guillaume (2011) found similar results for the original and the generalized α VG model(s) and extended them by replacing the Lévy time-changed Brownian motions by Sato time-changed Brownian motions (Guillaume (2012)), resulting in the reduced (or original) and generalized two-factor Sato models, which have moments that are more consistent with the market. These two-factor Sato models outperform the α VG models in terms of the univariate option price surface goodness of fit and have a similar goodness of fit of the correlation structure.

Besides the representation of the VG model as a subordinated Brownian motion, one can also construct the VG process as a difference of two independent Gamma processes with the same shape parameter. The idea behind this representation is that one Gamma process models the log-price gains, while the other models the log-price losses (Madan & Seneta (1990)). Finlay & Seneta (2008) extended the univariate VG model by relaxing the constraint on the shape parameter, allowing for more flexibility. The resulting process remains Lévy, but no longer necessarily VG distributed and was named the Difference of Gammas (DG) process. An empirical study showed that the DG process provides a better goodness of fit of the option price surface than the original three-parameter VG model and that it also performs better than other four-parameter models such as the CGMY model introduced in Carr et al. (2002).

In this paper we first propose a multivariate extension of the DG model of Finlay & Seneta (2008) by assuming that the multivariate asset log-returns are distributed as the difference of two multivariate Gamma processes with different shape and rate parameters. The multivariate Gamma processes are constructed as the two-parameter variant of the multivariate Gamma process proposed by Mathai & Moschopoulos (1991). The two multivariate Gamma processes each contain a common and an idiosyncratic component. This Δ -Gamma model, as we will call it, has one extra degree of freedom per asset for the option price surface calibration as well as one extra degree of freedom for calibrating the correlation structure compared to the corresponding multivariate VG model (obtained by imposing the constraint that the gain and loss Gamma processes must have the same shape parameter). The multivariate Δ -Gamma model is thus more flexible than the corresponding VG model, while still retaining the appealing feature of having simple joint and marginal characteristic functions and thus remaining tractable. Analogously to the work of Guillaume (2012), we then extend the Δ -Gamma model to the Sato setting by replacing the Lévy marginals by Sato marginals. This results in the Δ -Gamma Sato model, which will prove to outperform the Δ -Gamma Lévy model in terms of the univariate option price surface goodness of fit.

¹ The quoted paper by Ballotta and Bonfiglioli is mainly theoretical. An estimation implementation can be found in Loregian et al. (2015). Applications of this model to credit risk are presented in Ballotta & Fusai (2015) and Ballotta et al. (2018).

The construction of both the Lévy and Sato variants of the Δ -Gamma model allows for a decoupled calibration as proposed by Leoni & Schoutens (2008). We will further employ a joint calibration procedure as well to take into account both the correlation structure and the univariate option price surfaces at the same time, because the decoupled calibration might lead to a severe underestimation of the pairwise correlations, as pointed out by Guillaume (2013). For the calibration of the correlation structure, we will not rely on any historical correlation estimate. Instead, we will use a market implied correlation estimate based on simple variance swaps (see Martin (2013)).

The paper is organized as follows. Section 2 recalls the basic properties of Lévy and Sato models. In Section 3, the Δ -Gamma and Δ -Gamma Sato models are constructed and the properties and characteristics of these new models are investigated. Section 4 discusses the different calibration procedures, as well as the calibration instruments used to calibrate the univariate option price surfaces and the correlation structure. In Section 5 we assess the model performance under the risk-neutral measure in a numerical study where the different models are calibrated to a basket of four major stocks included in the Dow Jones Industrial Average. The model performance under the real-world measure is studied in Section 6. Section 7 concludes.

2 Some Lévy and Sato processes

2.1 Lévy processes

Definition 1 A stochastic process $X = \{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lévy process if X has càdlàg paths that start at zero $\mathbb{P}$ -a.s. and if X has independent and stationary increments.

Just as Brownian motions are the dynamic counterpart of the normal distribution it holds that Lévy processes are the dynamic counterpart of infinitely divisible distributions:

Definition 2 The distribution of a random variable X is infinitely divisible if, for every integer n , the characteristic function $\phi(u) = \mathbb{E}[\exp(iuX)]$ of the distribution is also the n th power of a characteristic function: $\phi(u) = (\phi_n(u))^n$.

Infinitely divisible distributions, and thus Lévy processes, are often characterized in terms of their characteristic exponent and the Lévy-Khintchine representation:

Theorem 1 *The characteristic exponent $\Psi_X(u) = \ln(\phi_X(u))$ of any infinitely divisible distribution can be written in the Lévy-Khintchine representation:*

$$\Psi_X(u) = i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (\exp(iux) - 1 - iux\mathbf{1}_{|x|<1})\nu(dx), \quad u \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ is the drift, $\sigma > 0$ the diffusion coefficient and ν a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$, named the Lévy measure. An infinitely divisible distribution with Lévy triplet (γ, σ, ν) is of infinite activity if and only if $\nu(\mathbb{R}) = \infty$.

The Lévy triplet (γ, σ, ν) uniquely defines the distribution of the Lévy process. In this paper, we will work with one particular Lévy process, namely the Gamma process.

2.1.1 The Gamma process

The Gamma process is a non-decreasing Lévy process built on the Gamma law. The characteristic function of the Gamma law with parameters $a > 0$ (shape) and $b > 0$ (rate) is given by:

$$\phi_{\text{Gamma}}(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a}, \quad u \in \mathbb{R}.$$

The Lévy triplet of a $\text{Gamma}(a, b)$ process is given by $(\gamma, \sigma, \nu) = \left(\int_0^1 x\nu(dx), 0, ax^{-1}\exp(-bx)dx\right)$, where $\nu(-\infty, 0) = 0$. The Gamma process $X = \{X_t, t \geq 0\}$ is a Lévy process such that X_t follows a $\text{Gamma}(at, b)$ distribution.

Starting from the Gamma process, one can build various stock price models. For example, the VG model, as introduced by Madan & Seneta (1990), assumes that the stock log-return follows a VG process, which can be build as a Brownian motion subordinated by a Gamma process independent of the Brownian motion, denoted by $\text{VG}(\sigma, \nu, \theta)$ and with characteristic function

$$\phi_{\text{VG}}(u; \sigma, \nu, \theta) = \left(1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu\right)^{\frac{-1}{\nu}}, \quad u \in \mathbb{R},$$

or as the difference of two independent Gamma processes with the same shape parameter, which is denoted by $\text{VG}(c, g, m)$ and has characteristic function

$$\phi_{\text{VG}}(u; c, g, m) = \left(\frac{gm}{gm + (m - g)iu + u^2}\right)^c, \quad u \in \mathbb{R}.$$

Finlay & Seneta (2008) extended the VG model by dropping the condition that the two Gamma processes must have the same shape parameter and called this new type of model the *Difference of Gammas* model. In Section 3, we will propose an extension of this model to the multivariate setting, allowing for extra flexibility.

Note that the term structure of the moments of Lévy processes implies that the skewness scales like $1/\sqrt{t}$ and the excess of kurtosis like $1/t$. However, Konikov & Madan (2001) empirically determined the term structure of the skewness and excess of kurtosis from S&P500 option prices and found that these standardized risk-neutral moments might rise slightly or be constant, but that they are not decreasing over the term.

2.2 Sato processes

Sato processes are closely related to the class of self-decomposable laws.

Definition 3 The distribution of a random variable X is self-decomposable if, for any constant c , $0 < c < 1$, $\exists$ a random variable X_c independent of X such that

$$X \stackrel{d}{=} cX + X_c. \tag{1}$$

Self-decomposable laws are a sub-class of infinitely divisible laws with a Lévy-Khintchine representation of the form

$$\ln(\phi_X(u)) = i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (\exp(iux) - 1 - iux\mathbf{1}_{|x|<1}) \frac{h(x)}{|x|} dx, \quad (2)$$

where $h(x) \geq 0$ is decreasing for positive x and increasing for negative x . Hence, self-decomposable laws are necessarily of infinite activity.

A Sato process can be constructed from any self-decomposable distribution as follows: the probability law of the Sato process at time t is obtained by scaling the self-decomposable law of X at unit time (see Carr et al. (2007)):

$$X_t \stackrel{d}{=} t^\gamma X, \quad (3)$$

where γ is the self-similar exponent. Sato processes are thus processes with independent but time inhomogeneous increments. One easily verifies that:

$$\text{Var}[X_t] = t^{2\gamma} \text{Var}[X], \quad \text{Skew}[X_t] = \text{Skew}[X], \quad \text{Kurt}[X_t] = \text{Kurt}[X],$$

indicating that the tail behavior and asymmetry of the asset log-returns is constant over the term and that the self-similar exponent γ allows for a concavity in the term structure of the variance, which is more in line with market observations, as indicated by f.i. Konikov & Madan (2001) and Guillaume (2017). Moreover, taking the law at unit time to be self-decomposable is a financially valid choice, since this means that the returns can be seen as the sum of an infinite number of independent shocks with different magnitudes (see Alexander (2008) or Guillaume (2017)).

Proposition 1 *Any linear combination of self-decomposable random variables is again a self-decomposable random variable. In particular let $X^{(1)}$ and $X^{(2)}$ be two self-decomposable random variables. Then $X^{(1)} - X^{(2)}$ is self-decomposable.*

Proof The proof follows directly from Definition 3.

3 Δ -Gamma multivariate models

Under the risk-neutral measure $\mathbb{Q}$, the n -dimensional asset price is modeled by

$$\mathbf{S}_t = \mathbf{S}_0 \exp(\mathbf{m}t + \mathbf{Y}_t) = \frac{\mathbf{S}_0 \exp((r - \mathbf{q})t + \mathbf{Y}_t)}{\mathbb{E}[\exp(\mathbf{Y}_t)]},$$

where $\mathbf{m} = r - \mathbf{q} - \ln(\phi_{\mathbf{Y}_t}(-\mathbf{i}))$, with $-\ln(\phi_{\mathbf{Y}_t}(-\mathbf{i}))$ the vector of mean-correcting terms making the model risk-neutral, r the risk-free interest rate, $\mathbf{q} = (q_1, \dots, q_n)'$ the vector of dividend yields and $\mathbf{Y} = \{\mathbf{Y}_t, t \geq 0\}$ is a n -dimensional stochastic process.

Following the same idea as Finlay & Seneta (2008), we build a *multivariate Δ -Gamma model* by considering the difference of two independent multivariate Gamma processes to model the asset log-returns:

$$\mathbf{Y}_t = \mathbf{G}_t^{(1)} - \mathbf{G}_t^{(2)}. \quad (4)$$

The processes $\mathbf{G}_t^{(k)}, k = 1, 2$ are built as the multivariate (two-parameter) Gamma process proposed by Mathai & Moschopoulos (1991):

Definition 4 Let $X^{(i)} \sim \text{Gamma}(\alpha_i, \beta_i)$, $i = 0, 1, \dots, n$ be $n + 1$ mutually independent Gamma random variables. Then $\mathbf{G} = (G^{(1)}, G^{(2)}, \dots, G^{(n)})$ where

$$G^{(j)} = \frac{\beta_0}{\beta_j} X^{(0)} + X^{(j)}, j = 1, \dots, n, \quad (5)$$

is a multivariate Gamma random variable.

From the basic properties of Gamma random variables, it is clear that $G^{(j)} \sim \text{Gamma}(\alpha_0 + \alpha_j, \beta_j)$ and that the covariance matrix is given by

$$\text{Cov}(G^{(j)}, G^{(k)}) = \frac{\beta_0^2}{\beta_j \beta_k} \text{Var}[X^{(0)}] = \frac{\alpha_0}{\beta_j \beta_k}, \quad \forall j \neq k.$$

Note that the covariance (or more precisely, the correlation) proves to be an appropriate measure of dependence, since the dependence is introduced linearly.

Consequently, under the Δ -Gamma model we have that

$$G_t^{(1,j)} = \frac{\beta_0}{\beta_j} X_t^{(0)} + X_t^{(j)} \quad \text{and} \quad G_t^{(2,j)} = \frac{\delta_0}{\delta_j} Z_t^{(0)} + Z_t^{(j)}, \quad j = 1, \dots, n \quad (6)$$

where $X^{(i)} = \{X_t^{(i)}, t \geq 0\}$, with $X_1^{(i)} \sim \text{Gamma}(\alpha_i, \beta_i)$ and $Z^{(i)} = \{Z_t^{(i)}, t \geq 0\}$, with $Z_1^{(i)} \sim \text{Gamma}(\eta_i, \delta_i)$, $i = 0, 1, \dots, n$ are independent Gamma processes. Note that additional dependence can be added by considering more than one common Gamma random variable in both the log-price gain and loss processes. In the following, we will scale the parameters β_0 and δ_0 to 1 without loss of generality. Indeed, from the space scaling property of Gamma random variables, multiplying β_0 by a constant c_1 and δ_0 by a constant c_2 does not change the marginal distribution, nor the correlation structure.

Remark 1 The proposed Δ -Gamma model belongs to the $\mathcal{DS}^1$ class of models proposed by Marfé (2013). Marfé introduced the general class of multivariate Lévy processes defined as the difference of two multivariate subordinators with a multi-factorial dependence structure and denoted it $\mathcal{DS}^m$, with m the number of common factors in each of the subordinators. As a particular case, he investigated a multivariate two-factor VG process, the $\rho\mu$ -VG(1) process, defined as the difference of two multivariate Gamma subordinators $(Y_t^+)_{t \geq 0}$ and $(Y_t^-)_{t \geq 0}$, where

$$Y_{k,t}^+ = X_{k,t}^+ + \delta_k^+ Z_t^+ \quad \text{and} \quad Y_{k,t}^- = X_{k,t}^- + \delta_k^- Z_t^-,$$

with $X_{k,t}^+ \sim \text{Gamma}((\eta_k - \eta^+)t, \mu_k)$, $X_{k,t}^- \sim \text{Gamma}((\eta_k - \eta^-)t, \rho_k)$, $Z_t^+ \sim \text{Gamma}(\eta^+t, 1)$, $Z_t^- \sim \text{Gamma}(\eta^-t, 1)$ independent processes and $\delta_k^+ = 1/\mu_k$ and $\delta_k^- = 1/\rho_k$. Compared to the $\rho\mu$ -VG(1) model investigated by Marfé, the Δ -Gamma model (4) allows for extra flexibility, and in particular for different levels of skewness and kurtosis in both the idiosyncratic and systemic upward and downward jumps.

From the Lévy triplet of the univariate Gamma distribution it follows that it is self-decomposable². Together with Proposition 1, it is clear that the multivariate Δ -Gamma model (3.1) is extendable to the class of multivariate models with Sato marginals. Following Guillaume (2012), the

² Note that the joint distribution of $\mathbf{G}$ in Definition 4 is not necessarily self-decomposable in the multivariate sense, but Q -selfdecomposable (see Barndorff-Nielsen et al. (2001)).

multivariate Δ -Gamma Sato model is then built by considering that the n -dimensional asset log-return is modeled by

$$\mathbf{Y}_t = t^\gamma \left(\mathbf{G}^{(1)} - \mathbf{G}^{(2)} \right), \quad (7)$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ and $\mathbf{G}^{(k)} = \mathbf{G}_1^{(k)}, k = 1, 2$.

From the independence of the Gamma processes $X^{(i)}, i = 0, \dots, n$ and $Z^{(i)}, i = 0, \dots, n$, it follows that the characteristic function of the n -dimensional asset log-return is

$$\begin{aligned} \phi_{\mathbf{Y}}(\mathbf{u}, t) &:= \mathbb{E} \left[\exp \left(i \sum_{j=1}^n u_j Y_t^{(j)} \right) \right] \\ &= \phi_{X^{(0)}} \left(\beta_0 \sum_{j=1}^n \frac{u_j}{\beta_j}, t \right) \phi_{Z^{(0)}} \left(-\delta_0 \sum_{j=1}^n \frac{u_j}{\delta_j}, t \right) \prod_{j=1}^n \phi_{X^{(j)}}(u_j, t) \phi_{Z^{(j)}}(-u_j, t) \end{aligned}$$

under the Δ -Gamma model (4) and

$$\phi_{\mathbf{Y}}(\mathbf{u}, t) = \phi_{X^{(0)}} \left(\beta_0 \sum_{j=1}^n \frac{u_j t^{\gamma_j}}{\beta_j}, 1 \right) \phi_{Z^{(0)}} \left(-\delta_0 \sum_{j=1}^n \frac{u_j t^{\gamma_j}}{\delta_j}, 1 \right) \prod_{j=1}^n \phi_{X^{(j)}}(u_j t^{\gamma_j}, 1) \phi_{Z^{(j)}}(-u_j t^{\gamma_j}, 1)$$

under the Δ -Gamma Sato model (7). It then follows that the univariate asset log-returns are DG distributed and DG Sato distributed³. The marginal characteristic functions are

$$\phi_{Y^{(j)}}(u, t) = \left(1 - \frac{iu}{\beta_j} \right)^{-(\alpha_0 + \alpha_j)t} \left(1 + \frac{iu}{\delta_j} \right)^{-(\eta_0 + \eta_j)t} \quad (8)$$

under the Δ -Gamma model (4) and

$$\phi_{Y^{(j)}}(u, t) = \left(1 - \frac{it^{\gamma_j} u}{\beta_j} \right)^{-(\alpha_0 + \alpha_j)} \left(1 + \frac{it^{\gamma_j} u}{\delta_j} \right)^{-(\eta_0 + \eta_j)} \quad (9)$$

under the Δ -Gamma Sato model (7).

Equations (8) and (9) indicate that setting $\alpha_j = \alpha_j^* - \alpha_0$ and $\eta_j = \eta_j^* - \eta_0$ leads to marginal characteristic functions, and hence to asset log-return moments that are independent of the parameters α_0 and η_0 of the “global” Gamma processes:

$$\phi_{Y^{(j)}}(u, t) = \left(1 - \frac{iu}{\beta_j} \right)^{-\alpha_j^* t} \left(1 + \frac{iu}{\delta_j} \right)^{-\eta_j^* t} \quad (10)$$

or

$$\phi_{Y^{(j)}}(u, t) = \left(1 - \frac{it^{\gamma_j} u}{\beta_j} \right)^{-\alpha_j^*} \left(1 + \frac{it^{\gamma_j} u}{\delta_j} \right)^{-\eta_j^*}. \quad (11)$$

Under this setting (called the decoupled setting hereafter), the models can be calibrated by making use of the decoupling calibration procedure proposed by Leoni & Schoutens (2008) which consists in first calibrating the marginals to replicate the univariate implied volatility surfaces (via $\{\alpha_j^*, \beta_j, \eta_j^*, \delta_j, (\gamma_j), j = 1, \dots, n\}$) and then calibrating the two global parameters α_0 and η_0 to replicate the correlation structure. We note that in this case, the value range of the common

³ X_t follows a DG Sato distribution if (3) holds with X_1 DG distributed.

parameters is narrowed to $\alpha_0 < \min(\alpha_j^*, j = 1, \dots, n)$ and $\eta_0 < \min(\eta_j^*, j = 1, \dots, n)$, which ensures the positivity of the shape parameters α_j and η_j of the Gamma processes. Such re-parameterization should be used whenever possible since it allows for a very fast calibration of the different Δ -Gamma models.

Since the joint and marginal characteristic functions are known analytically under both the Δ -Gamma and Δ -Gamma Sato models, we are able to compute any moment and any co-moment of the asset log-returns. In particular, the first five cumulants of the individual asset log-returns are given by

$$\kappa_l [Y_t^{(j)}] = (l-1)! \left(\frac{\alpha_0 + \alpha_j}{\beta_j^l} + (-1)^l \frac{\eta_0 + \eta_j}{\delta_j^l} \right) t, \quad l = 1, \dots, 5,$$

under the Δ -Gamma model (4) and

$$\kappa_l [Y_t^{(j)}] = (l-1)! \left(\frac{\alpha_0 + \alpha_j}{\beta_j^l} + (-1)^l \frac{\eta_0 + \eta_j}{\delta_j^l} \right) t^{l\gamma_j}, \quad l = 1, \dots, 5,$$

under the Δ -Gamma Sato model (7). The skewness, kurtosis and hypskewness (or fifth order standardized moment) are consequently easily recovered by

$$\text{skewness} = \frac{\kappa_3}{\kappa_2^{3/2}}, \quad \text{kurtosis} = 3 + \frac{\kappa_4}{\kappa_2^2} \quad \text{and} \quad \text{hypskewness} = 10 \text{ skewness} + \frac{\kappa_5}{\kappa_2^{5/2}}.$$

The covariance between the asset log-returns is

$$\text{Cov} (Y_t^{(j)}, Y_t^{(k)}) = \left(\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k} \right) t, \quad j \neq k$$

under the Δ -Gamma model (4) and

$$\text{Cov} (Y_t^{(j)}, Y_t^{(k)}) = \left(\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k} \right) t^{\gamma_j + \gamma_k}, \quad j \neq k$$

under the Δ -Gamma Sato model (7). Hence, the correlation between the asset log-returns is

$$\hat{\rho}_{jk} = \frac{\text{Cov} (Y_t^{(j)}, Y_t^{(k)})}{\sqrt{\text{Var} [Y_t^{(j)}] \text{Var} [Y_t^{(k)}]}} = \frac{\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k}}{\sqrt{\left(\frac{\alpha_0 + \alpha_j}{\beta_j^2} + \frac{\eta_0 + \eta_j}{\delta_j^2} \right) \left(\frac{\alpha_0 + \alpha_k}{\beta_k^2} + \frac{\eta_0 + \eta_k}{\delta_k^2} \right)}}, \quad \forall j \neq k, \quad (12)$$

under both the Δ -Gamma model (4) and the Δ -Gamma Sato model (7).

3.1 A particular case: the VG and VG Sato multivariate models

Setting $\eta_i = \alpha_i, i = 0, 1, \dots, n$, i.e. assuming that the positive and negative jumps have the same shape parameter such that they are equally skewed and have the same excess of kurtosis, leads

to reduced Δ -Gamma models whose marginals are VG and VG Sato distributed⁴. Indeed, the marginal characteristic functions then become

$$\phi_{Y_t^{(j)}} = \left(1 - iu \left(\frac{1}{\beta_j} - \frac{1}{\delta_j}\right) + \frac{u^2}{\beta_j \delta_j}\right)^{-(\alpha_0 + \alpha_j)t} \quad \text{such that} \quad Y_1^{(j)} \sim \text{VG}(\sigma_j, \nu_j, \theta_j) \quad (13)$$

where $\nu_j = \frac{1}{\alpha_0 + \alpha_j}$, $\theta_j = (\alpha_0 + \alpha_j) \left(\frac{1}{\beta_j} - \frac{1}{\delta_j}\right)$ and $\sigma_j = \sqrt{\frac{2(\alpha_0 + \alpha_j)}{\beta_j \delta_j}}$.

Moreover, imposing $\eta_i = \alpha_i, i = 0, 1, \dots, n$ under the Δ -Gamma Sato model leads to marginal characteristic functions of the form

$$\phi_{Y_t^{(j)}} = \left(1 - iut^{\gamma_j} \left(\frac{1}{\beta_j} - \frac{1}{\delta_j}\right) + \frac{u^2 t^{2\gamma_j}}{\beta_j \delta_j}\right)^{-(\alpha_0 + \alpha_j)}$$

and the asset log-return then follows a VG Sato process with parameters $\{\sigma_j, \nu_j, \theta_j, \gamma_j\}$, where σ_j, ν_j and θ_j are defined as under the Lévy VG model.

The correlation between the asset log-returns reduces to:

$$\hat{\rho}_{jk} = \frac{\alpha_0(\beta_j \beta_k + \delta_j \delta_k)}{\sqrt{(\alpha_0 + \alpha_j)(\beta_j^2 + \delta_j^2)(\alpha_0 + \alpha_k)(\beta_k^2 + \delta_k^2)}}, \quad \forall j \neq k,$$

under both the VG Lévy and Sato models.

Remark 2 The VG model introduced above belongs to the $\mathcal{DS}^1$ class of models proposed by Marfé (2013) as well. Compared to the $\rho\mu$ -VG(1) model of Marfé, the VG model (13) enhances parsimony: it is characterized by one single common parameter instead of two, while preserving the maximal attainable pairwise correlation under the decoupled setting (see Guillaume (2017)).

Remark 3 Note that the Δ -Gamma models can not account for negative correlations between different assets. This could be disadvantageous in practice, since a well-diversified portfolio should contain both positive and negative correlations⁵. The $\rho\mu$ -VG(1) model also suffers from this drawback, but Marfé (2013) suggested a simple modification of the model to incorporate negative correlations, i.e. by adding a common factor between the positive jumps of one asset and the negative jumps of another one. This modification can be applied to the Δ -Gamma models as well. Let $\mathbf{X}_t$ and $\mathbf{Z}_t$ be defined as in the beginning of this section. Next, define for $j = 1, \dots, n$ and $k = 1, \dots, n, k \neq j$ the new, mutually independent stochastic processes $X_t^{(j,k)}$ and $Z_t^{(j,k)}$, independent of $X_t^{(l)}$ and $Z_t^{(l)} \forall l = 0, \dots, n$, with $X_1^{(j,k)} \sim \text{Gamma}(\alpha_{jk}, 1)$ and $Z_1^{(j,k)} \sim \text{Gamma}(\eta_{jk}, 1)$ where $X_1^{(j,k)} = Z_1^{(k,j)}$. If $j \neq m$ and $k \neq l$, we take $X_t^{(j,k)}$ independent of $Z_t^{(l,m)} \forall j, k, l, m = 1, \dots, n, k \neq j, m \neq l$. Here, $X_t^{(j,k)}$ represents the common factor between the positive jumps of asset j and the negative jumps of asset k and $Z_t^{(j,k)}$ the common factor between the negative jumps of asset j and the positive jumps of asset k . We can then extend the Δ -Gamma models to account for negative correlations by taking $\forall j = 1, \dots, n$:

$$G_t^{(1,j)} = \frac{1}{\beta_j} X_t^{(0)} + X_t^{(j)} + \frac{1}{\beta_j} \sum_{\substack{k=1 \\ k \neq j}}^n X_t^{(j,k)} \quad \text{and} \quad G_t^{(2,j)} = \frac{1}{\delta_j} Z_t^{(0)} + Z_t^{(j)} + \frac{1}{\delta_j} \sum_{\substack{k=1 \\ k \neq j}}^n Z_t^{(j,k)}.$$

⁴ X follows a VG Sato distribution if (3) holds with X_1 VG distributed.

⁵ Empirical data shows that many stocks included in the major stock indices are in fact positively correlated, see f.i. the correlation indices issued by the Chicago Board Options Exchange (2009).

The correlation between the asset log-returns is then given by:

$$\text{Cov}\left(Y_t^{(j)}, Y_t^{(k)}\right) = \left(\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k} - \frac{\alpha_{jk}}{\beta_j \delta_k} - \frac{\eta_{jk}}{\beta_k \delta_j}\right) t, \quad j \neq k,$$

under the Δ -Gamma model and

$$\text{Cov}\left(Y_t^{(j)}, Y_t^{(k)}\right) = \left(\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k} - \frac{\alpha_{jk}}{\beta_j \delta_k} - \frac{\eta_{jk}}{\beta_k \delta_j}\right) t^{\gamma_j + \gamma_k}, \quad j \neq k,$$

under the Δ -Gamma Sato model. Although theoretically speaking this is a simple modification, it could become computationally hard to calibrate the modified Δ -Gamma models when the number of assets n increases, since the number of extra parameters is $n(n-1)$. Another way to accommodate for negative correlations is to allow the scaling factor $\lambda_j = \frac{\beta_0}{\beta_j}$ in (5) to be negative. More precisely, let

$$G^{(j)} = X^{(j)} + \lambda_j X^{(0)}, \quad j = 1, \dots, n, \quad (14)$$

where $X_1^{(i)} \sim \text{Gamma}(\alpha_i, \beta_i)$, $i = 0, 1, \dots, n$ and $\lambda_j \in \mathbb{R}$, $j = 1, \dots, n$. Then, from Proposition 1, $G^{(j)}$ is still self-decomposable such that multivariate models with Sato marginals can be built on the self-decomposable law (14). Besides, the covariances $\text{Cov}(G^{(j)}, G^{(k)}) = \lambda_j \lambda_k \frac{\alpha_0}{\beta_0^2}$, $j \neq k$ are no longer restricted to be positive. Compared to the extension of Marfé, the proposed generalization (14) allows for both positive and negative linear dependence, while enhancing the model parsimony (n extra parameters instead of $n(n-1)$ extra parameters). Note that these extensions should only be used in case of negative correlations, because they reduce the model parsimony. Since the market implied correlations considered in the numerical study in Section 5 turn out to be all positive, we will not further consider the two modified Δ -Gamma models described above.

4 Calibration

For the Δ -Gamma Lévy and Sato models, we perform the following two calibrations:

- A decoupled calibration on the decoupled setting (i.e. for the parameter set $\{\alpha_0, \eta_0, \alpha_j^*, \beta_j, \eta_j^*, \delta_j, (\gamma_j), j = 1, \dots, n\}$).
- A joint calibration on the original setting (i.e. for the parameter set $\{\alpha_0, \eta_0, \alpha_j, \beta_j, \eta_j, \delta_j, (\gamma_j), j = 1, \dots, n\}$).

For the numerical study, the performance of the Δ -Gamma and the Δ -Gamma Sato models will be compared to the performance of the VG and VG Sato models described in Section 3.1 for both calibration procedures. The results will be discussed in Section 5.

4.1 Decoupled setting

Under the decoupled setting $\{\alpha_0, \eta_0, \alpha_j^*, \beta_j, \eta_j^*, \delta_j, (\gamma_j), j = 1, \dots, n\}$, the parameters α_j^* and η_j^* , $j = 1, \dots, n$ are chosen such that the marginal characteristic functions are independent of the global parameters α_0 and η_0 . This allows for a dissociation of the option price surface calibration and of the calibration of the correlation structure. We can thus apply the decoupled calibration

procedure proposed by Leoni & Schoutens (2008), which consists of two subsequent steps: First we calibrate the n univariate option price surfaces by minimizing an objective function⁶ of the following form:

$$\text{MRMSE} = \sum_{j=1}^n \frac{\text{RMSE}^{(j)}}{n} = \sum_{j=1}^n \left(\frac{1}{n} \sqrt{\frac{1}{N^{(j)}} \sum_{k=1}^{N^{(j)}} \left(P_k^{(j)} - \hat{P}_k^{(j)} \right)^2} \right), \quad (15)$$

where $N^{(j)}$ is the number of quoted option prices for stock j and $P_k^{(j)}$ and $\hat{P}_k^{(j)}$ denote the k th quoted option price and model option price of stock j , respectively. The model option prices are computed using the Carr-Madan formula (see Carr & Madan (1998)). Since the marginal characteristic functions do not have any parameters in common, the marginal parameters $\{\alpha_j^*, \beta_j, \eta_j^*, \delta_j, (\gamma_j)\}$ are calibrated independently for each stock $j = 1, \dots, n$ by minimizing $\text{RMSE}^{(j)}$. We then fix the marginal parameters $\{\alpha_j^*, \beta_j, \eta_j^*, \delta_j, (\gamma_j), j = 1, \dots, n\}$ to their optimal value obtained from the option price surface calibration and perform a correlation calibration to obtain the optimal value for α_0 and η_0 .

The correlation (12) can be rewritten in terms of the decoupled parameterization as

$$\hat{\rho}_{jk} = \frac{\frac{\alpha_0}{\beta_j \beta_k} + \frac{\eta_0}{\delta_j \delta_k}}{\sqrt{\left(\frac{\alpha_j^*}{\beta_j^2} + \frac{\eta_j^*}{\delta_j^2} \right) \left(\frac{\alpha_k^*}{\beta_k^2} + \frac{\eta_k^*}{\delta_k^2} \right)}}, \quad \forall j \neq k. \quad (16)$$

An objective function of the form

$$\text{RMSE}^\rho = \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^n \sum_{k>j} (\rho_{jk} - \hat{\rho}_{jk})^2}, \quad (17)$$

where ρ_{jk} and $\hat{\rho}_{jk}$ denote the market implied and the model correlation between the j th and k th log-returns, respectively, will be minimized in order to obtain the most favorable values for α_0 and η_0 . Note that high values of correlation can only be reached if the α_j^* and η_j^* are not too constraining since the admissible values of the global parameters are bounded from above:

$$\alpha_0 \in (0, \min(\alpha_j^*, j = 1, \dots, n)) \quad \text{and} \quad \eta_0 \in (0, \min(\eta_j^*, j = 1, \dots, n)). \quad (18)$$

Despite this limitation, the decoupled calibration procedure should be used whenever leading to an accurate fit of the dependence structure, since it allows for a very fast calibration of the different Δ -Gamma models.

⁶ To check the robustness of the calibration results to a change of the objective function, we also performed the calibration of the different models using the multivariate Average Relative Percentage Error

$$\text{MARPE} = \sum_{j=1}^n \frac{\text{ARPE}^{(j)}}{n} = \sum_{j=1}^n \left(\frac{1}{n} \sqrt{\frac{1}{N^{(j)}} \sum_{k=1}^{N^{(j)}} \frac{|P_k^{(j)} - \hat{P}_k^{(j)}|}{P_k^{(j)}}} \right)$$

instead of the objective function (15) to fit the single-name option prices. The use of the functional MARPE instead of MRMSE leads to similar results for both the decoupled and the joint calibration as the ones in Section 5, which are available to the interested reader upon request.

4.2 Original setting

Under the original setting $\{\alpha_0, \eta_0, \alpha_j, \beta_j, \eta_j, \delta_j, (\gamma_j), j = 1, \dots, n\}$, the marginal characteristic functions depend on the whole set of parameters, such that we can not decouple the marginal calibrations from the correlation structure calibration. Hence, we minimize an objective function of the form:

$$\text{MRMSEJ} = \sum_{j=1}^n \frac{\text{RMSE}^{(j)}}{n} + \alpha^\rho \text{MRMSE}^* \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^n \sum_{k>j} (\rho_{jk} - \hat{\rho}_{jk})^2}, \quad (19)$$

as proposed by Guillaume (2013), where MRMSE^* is the optimal value of the multivariate root mean square error obtained by fitting the option price surfaces only, i.e. when taking $\alpha^\rho = 0$ in (19). The scaling of the correlation goodness of fit by this factor ensures that both terms of Equation (19) are of the same magnitude order. The parameter $\alpha^\rho \geq 0$ allows the user to specify the relative importance of the correlation matching; a parameter α^ρ equal to 0 indicating that the correlation calibration is not a desired feature and that the model is calibrated on the univariate option price surfaces only.

4.3 Calibration instruments

4.3.1 Underlying stocks

We consider a basket containing four major stocks included in the Dow Jones Industrial Average (DJX) during the period extending between the fourth of January 2007 and the 20th of October 2009 where we take quotes once every two weeks. The recent financial crisis of 2008 is included in the period under consideration, so that we can compare the model performance for different market regimes, namely periods with low, medium and high market volatility. The stocks under consideration are Microsoft Corp. (MSFT), General Electric Co. (GE), Pfizer Inc. (PFE) and Intel Corp. (INTC), which are the stocks included in the DJX with the highest open interest during the considered period.

Since options on the stocks included in the DJX are of the American type, we first extract European option prices from the quoted American option prices in order to use the Carr-Madan formula, which only holds for European type options. In this way, we can avoid resorting to computationally intensive numerical methods to compute model option prices. Extracting European option prices from the quoted American option prices is done by applying the iterative Implied Binomial (iIB) tree approach introduced by Tian (2011). The first step of this iIB tree approach consists in calibrating a standard implied binomial tree to a rough guess for the European option prices (e.g. the quoted American option prices). The calibrated standard binomial tree is then used to calculate the early exercise premium on the American options, which will subsequently be removed from the corresponding American option quote. The obtained prices are a refined guess for the European option prices. Next, we again calibrate a standard binomial tree on this refined guess for the European option prices and start the whole process from the beginning. This is done iteratively until the early exercise premiums converge. For the calibration of the Δ -Gamma models, we only use put options, since some call option prices obtained by the iIB tree approach can turn out to be negative due to small rounding errors.

Note that the stocks under consideration pay quarterly dividends, while the Carr-Madan formula assumes a continuous dividend yield. We therefore transform the discrete dividend yield into a continuous dividend yield.

4.3.2 Correlation instruments

Since the univariate option price surfaces are calibrated under the risk-neutral measure, we would like to do the same with the correlation structure, omitting any use of historical time series. Therefore, we will use market correlation estimates, which can be inferred from market quotes for any actively traded index, for the calibration of the correlation structure, as proposed by Guillaume (2012). Examples of such market correlation estimates are the S&P500 correlation indices issued by the CBOE (see, for example, Chicago Board Options Exchange (2009) for more information on these correlation indices). These indices however implicitly assume a market where stock indices are expressed as geometric averages of the underlying assets, whereas in practice indices are constructed as arithmetic averages. We will therefore resort to a different, model-free correlation index, the average linear return correlation:

$$\rho^{\text{ALRC}}(t) = \frac{\text{Var}\left(\frac{S_t - S_0}{S_0}\right) - \sum_{j=1}^n (w_j^*)^2 \text{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right)}{2 \sum_{j < k} w_j^* w_k^* \sqrt{\text{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right)} \sqrt{\text{Var}\left(\frac{S_t^{(k)} - S_0^{(k)}}{S_0^{(k)}}\right)}},$$

where the variance of the linear return of the underlyings and of the index is extracted from the single-name option price surfaces and of the index option price surface, respectively, using the option payoff spanning formula of Breeden & Litzenberger (1978) and where $w_j^* = w_j S_0^{(j)} / S_0$. Here, w_j denotes the weight corresponding to the j th stock in the market, $S_t = \sum_{j=1}^n w_j S_t^{(j)}$, where S_t and $S_t^{(j)}$, $j = 1, \dots, n$ denote the price of the index and of the j th underlying asset at time t , respectively. For a price-weighted index like the Dow Jones we consider here, $w_j = \frac{1}{\delta}$, where δ is the index divisor. We then approximate the correlation between any pair of asset linear returns by the ALRC⁷:

$$\rho\left(\frac{S_{T_\rho}^{(j)}}{S_{t_0}^{(j)}}, \frac{S_{T_\rho}^{(k)}}{S_{t_0}^{(k)}}\right) \approx \rho^{\text{ALRC}}(T_\rho),$$

where T_ρ is the time horizon of the correlation, which we will take equal to one year. The correlation between the asset log-returns, $\rho(Y_{T_\rho}^{(j)}, Y_{T_\rho}^{(k)})$ can subsequently be approximated by using Taylor series expansions.

⁷ Whenever possible, the calibration of the correlation structure should be done on pair-wise market implied correlations, which requires a deep and active option trading market for any basket composed of any pair of the n underlyings. For each pair (j, k) of assets, we then have

$$\rho^{(jk)}(t) = \frac{\text{Var}\left(\frac{S_t^{(jk)} - S_0^{(jk)}}{S_0^{(jk)}}\right) - \omega_i^{(jk)} \text{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right) - \omega_j^{(jk)} \text{Var}\left(\frac{S_t^{(k)} - S_0^{(k)}}{S_0^{(k)}}\right)}{2 (\omega_j^{(jk)})^* (\omega_k^{(jk)})^* \sqrt{\text{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right)} \sqrt{\text{Var}\left(\frac{S_t^{(k)} - S_0^{(k)}}{S_0^{(k)}}\right)}},$$

where $S_t^{(jk)} = \omega_j^{(jk)} S_t^{(j)} + \omega_k^{(jk)} S_t^{(k)}$ and $(\omega_j^{(jk)})^* = \omega_j^{(jk)} S_0^{(jk)} / S_0^{(jk)}$.

5 Results

In this section we will compare the performance of the Δ -Gamma and the corresponding VG models, in the Lévy as well as in the Sato setting. This is done by first looking at the option price surface goodness of fit, and then at the correlation goodness of fit. The model performance of the multivariate Black-Scholes model serves as a benchmark.

5.1 Option price surface goodness of fit

The option price surface goodness of fit is measured by the multivariate RMSE (15), which is shown in Figure 1 together with the VIX volatility index, which measures the future expected market volatility over the next thirty calendar days. The global financial crisis of 2008, which occurred in the aftermaths of the bankruptcy of Lehman Brothers, is characterized by a high value of the VIX, indicating a high level of market uncertainty. We observe that, as expected, the Lévy models lead to a better fit of the univariate option price surfaces than the multivariate Black-Scholes model, for both the decoupled and the joint calibration procedure. However, during the heart of the credit crisis spanning from October 2008 to December 2008, the Lévy MRMSE is significantly higher than during periods of low or medium market volatility. The Sato models on the other hand systematically outperform the Black-Scholes and the Lévy models, with no significant difference between periods of low, medium or high market volatility. These observations are in line with the fact illustrated by Carr et al. (2007) in the univariate case, that Sato models, unlike Lévy models, are able to replicate option prices across the whole set of quoted maturities and strikes. Besides, they corroborate the empirical evidence highlighted by Guillaume (2012) that Sato models highly outperform Lévy models during distress periods. Table 1 displays the average gain⁸ in the option price surface goodness of fit obtained by moving from the Lévy to the Sato setting. As one can see, using Sato processes instead of Lévy processes can lead to an option price surface replication which is around 47% more accurate on average. This improvement is slightly bigger for the VG models than for the Δ -Gamma models as well as in the decoupled setting than in the original setting. Besides, one can observe from Figure 1 that the Δ -Gamma models outperform the corresponding VG models, in both the Lévy and the Sato setting. This was to be expected, since the Δ -Gamma models have one extra parameter per asset for the option price surface calibration and are thus more flexible. The average gain⁹ in accuracy obtained by moving from the VG model to the Δ -Gamma model is given in Table 2. The improvement in the goodness of fit obtained by using the Δ -Gamma model instead of the VG model is more pronounced for the Lévy models than for the Sato models and is more noticeable in the decoupled setting than in the original setting. However, the improvement in the goodness of fit when moving from the VG models to the Δ -Gamma models is far less significant than the improvement when moving from the Lévy to the Sato setting. Finally, from Table 3, which summarizes the average accuracy gain¹⁰ obtained when performing a decoupled calibration instead of a joint calibration, one observes that the joint calibration leads to a slightly higher multivariate RMSE. This is easily explained by the fact that the correlation goodness of fit is taken into account in the objective function of the joint calibration.

⁸ $\text{mean} \left[(\text{MRMSE}^{Lévy} - \text{MRMSE}^{Sato}) / (\text{MRMSE}^{Lévy}) \right]$
⁹ $\text{mean} \left[(\text{MRMSE}^{VG} - \text{MRMSE}^{\Delta\text{Gamma}}) / (\text{MRMSE}^{VG}) \right]$
¹⁰ $\text{mean} \left[(\text{MRMSE}^{Joint} - \text{MRMSE}^{Decoup}) / (\text{MRMSE}^{Joint}) \right].$

Fig. 1: Comparison of the univariate option price surface calibration goodness of fit under Lévy and Sato models (a), a closer look on the Sato models (b) and evolution of the volatility index VIX through time (c).

Table 1: Average gain in option price surface GoF: moving from the Lévy to the Sato setting.

	Av. gain Δ -Gamma	Av. gain VG
Decoup	47.83%	48.41%
Joint	45.94%	47.46%

Table 2: Average gain in option price surface GoF: moving from the VG to the Δ -Gamma model.

	Av. gain Lévy	Av. gain Sato
Decoup	8.50 %	4.52%
Joint	6.71%	1.95%

Table 3: Average gain in option price surface GoF: moving from the joint to the decoupled calibration method.

	Av. gain Lévy	Av. gain Sato
VG	2.01%	3.39%
Δ -Gamma	3.88 %	5.90%

5.2 Correlation goodness of fit

5.2.1 Correlation fit under the decoupled calibration

Figure 2 shows the correlation RMSE (17) under the Δ -Gamma and corresponding VG Lévy and Sato models. From Figure 2 it is clear that, although the Δ -Gamma models have one extra degree of freedom to match the correlation, they do not outperform the corresponding VG models in terms of the correlation goodness of fit under the decoupled setting. Moreover, the Sato-type models significantly outperform the Lévy-type models in terms of the correlation fitting. However, the correlation fit is of relatively poor quality for all the models when working in the decoupled setting as indicated by the average correlation RMSE reported in Table 4.

Looking at Figure 3, which shows the market and model correlations, we observe that the Δ -Gamma model mostly severely underestimates the linear correlations and that the other models do not perform well either under the decoupled setting. There might be several explanations for this poor quality fit. For one, there are only one (for the VG models) or two (for the Δ -Gamma models) parameters to fit the $\frac{1}{2}(n^2 - n)$ linear correlations between the n underlyings. Secondly, by first calibrating the idiosyncratic parameters as one does in the decoupled calibration, one imposes an upper bound (18) on the set of admissible values for the common parameters α_0 and η_0 , which might be too restrictive. To have more insight on the reason of the poor fit of the market implied correlations under the decoupled setting, one can compare the upper bound on the common parameters with the calibrated value.

Fig. 2: Comparison of the correlation structure goodness of fit under the Lévy models ((a) and (b)) and under the Sato models (c) with $\alpha^{\rho} = 1$ for the joint calibration.

Table 4: Average values of the correlation RMSE under the decoupled calibration.

	VG	Δ -Gamma	VG Sato	Δ -Gamma Sato
mean RMSE $^\rho$	0.1389	0.2738	0.1105	0.1528

a) VG models

It turns out that the optimal value almost never coincides with the upper bound for the VG Lévy and Sato models (see Figure 4). This gives evidence for the fact that the poor fit of the correlation structure of the VG-type models is mainly due to the small number of common parameters (one in this case). This is further illustrated on Figure 4, which also shows the ratio $\rho_{jk}/\bar{\rho}_{jk}$ for the VG Lévy and Sato models, where $\bar{\rho}_{jk}$ is the *idiosyncratic part of the correlation*:

$$\bar{\rho}_{jk} = \frac{\beta_j \beta_k + \delta_j \delta_k}{\sqrt{\alpha_j^* (\beta_j^2 + \delta_j^2) \alpha_k^* (\beta_k^2 + \delta_k^2)}}, \quad \forall j \neq k.$$

This ratio is basically the market implied common parameter per pair of assets (denoted by α_{jk}^{opt}), i.e. the value of α_0 leading to a perfect match of the market correlation ρ_{jk} . There are now three possible situations:

1. All the α_{jk}^{opt} are below the upper bound:

$$\max_{j,k} (\alpha_{j,k}^{opt}) \leq \min_j (\alpha_j^*).$$

In this case, the upper bound will not be reached by α_0 in the calibration. A poor correlation fit is due to the α_{jk}^{opt} 's that differ significantly from each other and is explained by the too low number of common parameters only.

2. At least one α_{jk}^{opt} lies above the upper bound and the upper bound is reached:

$$\max_{j,k} (\alpha_{j,k}^{opt}) > \min_j (\alpha_j^*) \quad \text{and} \quad \alpha_0 = \min_j (\alpha_j^*).$$

The constraint (18) on α_0 then explains the typically high value of RMSE $^\rho$.

3. At least one α_{jk}^{opt} lies above the upper bound and the upper bound is not reached:

$$\max_{j,k} (\alpha_{j,k}^{opt}) > \min_j (\alpha_j^*) \quad \text{and} \quad \alpha_0 < \min_j (\alpha_j^*).$$

The constraint (18) on α_0 might also explain the typically high value of RMSE $^\rho$, but the too low number of parameters prevails over the existence of an upper bound on α_0 to explain the poor correlation fit.

Table 5 displays the percentage of quoting days for which a particular situation occurs and the average correlation RMSE on these quoting days. For the VG-type models, the poor correlation fit is explained by the too low number of common parameters on 85.30% (i.e. 16.18% + 69.12%) and 88.23% (i.e. 33.82% + 54.41%) of the quoting days under consideration for the Lévy and Sato setting, respectively. For these quoting days, a natural extension of the VG models is to add additional common parameters (to match the correlation structure) and the Δ -Gamma models are one example of such an extension. For the remaining 14.70% of the quoting days for the VG Lévy and 11.77% of the quoting days for the VG Sato model, the RMSE $^\rho$ is predominated by the restriction on α_0 , i.e. by the upper bound imposed by the calibration procedure. Looking at the average correlation RMSE displayed in Table 5, one can observe that for the VG-type models,

the correlation fit for the quoting days where $\max_{(j,k)} (\alpha_{j,k}^{opt}) \leq \min_j (\alpha_j^*)$ is, on average, three to four (respectively two) times better than when $\max_{(j,k)} (\alpha_{j,k}^{opt}) > \min_j (\alpha_j^*)$ under the Lévy (respectively Sato) setting, but remains significant. Furthermore, the correlation RMSE is on average the highest when the common parameter α_0 reaches its upper bound.

b) Δ -Gamma models

For the Δ -Gamma Lévy, respectively Sato, model at least one of the two common parameters coincides with the upper bound for 92.65%, respectively 72.06%, of the quoting days under consideration (see Figures 5, 6¹¹ and Table 6), indicating that the poor fit of the correlation structure of the Δ -Gamma models under the decoupled setting is essentially due to the calibration procedure which restricts too severely the range of acceptable values of the common parameters and not to the too small number of common parameters as for the VG models.

Note that one can observe in general a lower maximal attainable correlation for the Δ -Gamma models than for the VG models. This is in line with the fact that, under the decoupled setting, the fit of the correlation structure under the Δ -Gamma models mainly fails due to the calibration setting while the poor fit of the correlation structure under the VG models is due to both the reduced number of common parameters and the choice of the calibration procedure. These observations provide evidence against the use of the decoupling calibration procedure for the Δ -Gamma models and, to a lower extent nonetheless, for the corresponding VG models, especially for highly correlated markets. Indeed, the restrictions on the common parameters severely limit the range of attainable correlations from above, as indicated by (16) and as shown in Figure 7.

5.2.2 Correlation fit under the joint calibration

Performing a joint calibration, where we calibrate the idiosyncratic and the common parameters at the same time, the market correlations are more effectively matched than under the decoupled calibration procedure, as indicated by Figures 2 and 8. Moreover, the Δ -Gamma models usually outperform the VG models in terms of the correlation fitting under the joint calibration. However, for the Sato models we observe a few quoting days for which the correlation RMSE remains non-negligible. This might be explained by the lower MRMSE* under the Sato models (compared to the Lévy models), which determines the weight assigned to RMSE ^{ρ} in the objective function. Table 7 summarizes the average values of the correlation RMSE for the different models in the original parameter setting over the period under consideration. Compared to the corresponding VG model, the correlation RMSE is three and 1.5 times smaller under the Δ -Gamma Lévy and Δ -Gamma Sato model, respectively. This can be explained by the additional parameter of the Δ -Gamma models (η_0) that can be calibrated to improve the fit of the dependence structure.

¹¹ Note that the parameters of the positive jump process are less stable than those of the negative jump process for both the Δ -Gamma Lévy and Sato models, hence the variability in magnitude order for the upper bound on α_0 .

Fig. 3: Model and market implied correlations between the stocks under consideration obtained with the decoupled calibration procedure: (a) GE-MSFT, (b) GE-PFE, (c) GE-INTC, (d) MSFT-PFE, (e) MSFT-INTC, (f) PFE-INTC

Fig. 4: Evolution of the common component parameter through time for the VG Lévy (a) and the VG Sato model (b).

Table 5: Possible cases for $\alpha_{j,k}^{opt}$, percentage of quoting days and the correlation RMSE per case.

		$\max_{j,k}(\alpha_{j,k}^{opt}) \leq \min_j(\alpha_j^*)$	$\max_{j,k}(\alpha_{j,k}^{opt}) > \min_j(\alpha_j^*)$	
			$\alpha_0 = \min_j(\alpha_j^*)$	$\alpha_0 < \min_j(\alpha_j^*)$
VG	% quoting days	16.18%	14.71%	69.12%
	mean (RMSE p)	0.0579	0.2038	0.1441
VG Sato	% quoting days	33.82%	11.77%	54.41%
	mean (RMSE p)	0.0637	0.1582	0.1292

Fig. 5: Evolution of the common component parameters through time for the Δ -Gamma Lévy model.

Fig. 6: Evolution of the common component parameters through time for the Δ -Gamma Sato model.

Table 6: Percentage of quoting days for which the upper bound on (one of) the common parameter(s) is reached (Tolerance = 10^{-6}).

	VG	Δ -Gamma (α)	Δ -Gamma (η)	VG Sato	Δ -Gamma Sato (α)	Δ -Gamma Sato (η)
Per par.	14.71%	69.12%	86.76%	11.77%	55.88%	54.41 %
Total	14.71%	92.65 %		11.77%	72.06%

Fig. 7: Maximal attainable pairwise correlations under the Lévy and Sato models in the decoupled setting.

Fig. 8: Model and market implied correlations between the stocks under consideration obtained with the joint calibration procedure: (a) GE-MSFT, (b) GE-PFE, (c) GE-INTC, (d) MSFT-PFE, (e) MSFT-INTC, (f) PFE-INTC

Table 7: Average values of correlation RMSE joint calibration.

	VG	Δ -Gamma	VG Sato	Δ -Gamma Sato
mean RMSE ^{ρ}	0.0035	0.0012	0.0165	0.0106

Remark 4 The previous results show that the extra common parameter of the Δ -Gamma models results in a better correlation fit than could be obtained with the VG models under the joint calibration procedure. To investigate whether adding extra common parameters can further improve the correlation goodness of fit, we calibrate a two-factor version of the Δ -Gamma Lévy and Sato models to the same basket of four stocks. These two-factor models are constructed as in Remark 3.1 by adding one extra common risk factor to both the log-price gain and the log-price loss processes. The calibration is performed on a representative selection of 12 quoting days over the period extending between the fourth of January 2007 and the 20th of October 2009 using the joint calibration procedure¹². The average values of the MRMSE and RMSE ^{ρ} can be found in Table 8, together with the average values for the original Δ -Gamma Lévy and Sato models on these 12 quoting days and the average relative reduction¹³ in the objective function when moving from a one-factor to a two-factor model. We observe a reduction in both the average MRMSE and the average RMSE ^{ρ} for the two-factor Δ -Gamma Lévy and Sato models. This reduction is more significant for the correlation RMSE than for the MRMSE and more significant in the Lévy setting than in the Sato setting, especially for the correlation RMSE (64.55% versus 4.2%). Hence, such extension is in this case only recommended in the Lévy setting, since the model parsimony decreases while not significantly improving the overall fit in the Sato setting.

Table 8: Average value of MRMSE and RMSE ^{ρ} on the selected 12 quoting days over the period 04/01/2007 - 20/10/2009 under the different model settings and the average relative reduction in objective function value when moving from a one-factor to a two-factor model.

	<u>Δ-Gamma Lévy</u>			<u>Δ-Gamma Sato</u>		
	one-factor	two-factor	reduction	one-factor	two-factor	reduction
mean(MRMSE)	0.1296	0.1287	6.88e-03	0.0722	0.0721	2.08e-03
mean(RMSE ^{ρ})	1.45e-03	5.15e-04	0.6455	0.0306	0.0294	0.0420

5.3 Comparison with other VG-type models

The marginal distribution of the VG model proposed in this paper, shortly referred to as the Δ -VG model, coincides with the one of the α -VG model of Semeraro (2008) and of the $\rho\alpha$ -VG model

¹² A decoupled calibration can also be performed by taking $\alpha_j^* = \alpha_j + \alpha_{0,1} + \alpha_{0,2}$ and $\eta_j^* = \eta_j + \eta_{0,1} + \eta_{0,2}$. The marginal structure under the decoupled setting will then be the same as under the decoupled one-factor Δ -Gamma models. Besides, the constraints $\alpha_{0,1} + \alpha_{0,2} \in (0, \min(\alpha_j^*, j = 1, \dots, n))$ and $\eta_{0,1} + \eta_{0,2} \in (0, \min(\eta_j^*, j = 1, \dots, n))$ would have to hold such that the maximal attainable pairwise correlation would be the same as under the one-factor model. Hence, such extension is not recommended under the decoupled setting since the model parsimony will decrease while not improving the model flexibility.

¹³ $\text{mean} \left(\frac{\text{RMSE}^{1factor} - \text{RMSE}^{2factor}}{\text{RMSE}^{1factor}} \right)$

of Luciano et al. (2016), i.e. the marginals are all VG-distributed. The dependence structure, however, is different under each of these models, which directly impacts the correlation GoF under the decoupled setting. Guillaume (2017) compared the range of correlation under the α -VG model and the Δ -VG model and concluded that the Δ -VG model always has a higher maximal attainable correlation than the α -VG model (see Guillaume (2017) for more information). Hence, the upper bound on the common parameter is expected to be reached more often under the α -VG model leading to a worse correlation fit. A similar comparison can be made between the Δ -VG model and the $\rho\alpha$ -VG model when we consider positive correlations (see Appendix A). Again, the maximal attainable (positive) correlation under the Δ -VG model is higher than under the $\rho\alpha$ -VG model, so the upper bound will be reached more often under the $\rho\alpha$ -model whenever positive market correlations are to be matched¹⁴. However, this does not imply that we expect a worse correlation fit under the $\rho\alpha$ -VG model. Indeed, the $\rho\alpha$ -VG model has extra parameters to fit the pairwise correlations and is hence more flexible, such that a better fit can be obtained for positive market correlations whenever the upper bound on the common parameter α is not too restrictive. The linear VG model proposed by Ballotta & Bonfiglioli (2014) has VG distributed marginals as well, but an extra condition on the marginal parameters is imposed to ensure that the marginals are VG distributed. Hence, the option price surface GoF will be slightly worse under the linear VG model than under the other VG-type models considered here. Moreover, an analogous analytical comparison between the correlation structure of the linear VG model and the Δ -VG model is not possible, since the marginal structure is not the same.

As a numerical study, the four different VG-type models are calibrated on the same basket of four stocks, on a selection of five different quoting days, which represent different levels of market turmoil. The results are summarized in Table 9. As the marginal distribution is the same under the α -VG, $\rho\alpha$ -VG and the Δ -VG model, they lead to the same option price surface GoF. The linear VG model leads to a slightly higher MRMSE value, due to the extra restrictions on the marginal parameters. When comparing the correlation GoF, it is observed that the Δ -VG and the $\rho\alpha$ -VG models always outperform the α -VG model and that the $\rho\alpha$ -model outperforms the Δ -VG model except on the third quoting day. Comparing the correlation GoF of the linear VG to the Δ -VG model, the Δ -VG model outperforms the linear VG model on the quoting days with a low to medium VIX level but when the VIX level is high, the linear VG model leads to better fit. We conclude that the Δ -VG models are a valuable alternative to existing multivariate VG models. The results for the Sato specification of the models are similar, since the correlation structure is the same as under the Lévy specification¹⁵.

Table 9: Value of the objective functions MRMSE and RMSE^ρ for the different VG-type Lévy models under the decoupled calibration.

VIX	08/01/2008		20/05/2008		04/06/2008		28/10/2008		11/12/2008	
	25.43		17.58		20.80		66.96		55.78	
	MRMSE	RMSE^ρ	MRMSE	RMSE^ρ	MRMSE	RMSE^ρ	MRMSE	RMSE^ρ	MRMSE	RMSE^ρ
linVG	0.1799	0.1783	0.1204	0.2424	0.1047	0.2977	0.2840	0.1554	0.2805	0.1406
α VG	0.1756	0.2012	0.1133	0.2177	0.0927	0.3887	0.2813	0.2938	0.2802	0.1478
$\rho\alpha$ -VG	0.1756	0.1360	0.1133	0.0302	0.0927	0.2481	0.2813	0.1826	0.2802	0.1230
Δ -VG	0.1756	0.1696	0.1133	0.1123	0.0927	0.1965	0.2813	0.2225	0.2802	0.1475

¹⁴ When a negative market correlation is observed, the $\rho\alpha$ -VG model will always outperform the Δ -VG model, since the latter can only model positive correlations. One of the extensions proposed in Remark 3 should then be applied to the Δ -VG model in order to incorporate negative correlations.

¹⁵ The results are available to the interested reader.

6 Historical calibration

In this section we compare the performance of the VG and the Δ -Gamma model when performing a historical moment matching calibration¹⁶. We thus estimate the model parameters under the real world measure $\mathbb{P}$ instead of under the risk-neutral measure $\mathbb{Q}$ we employed in the previous sections¹⁷. Under the real world measure, the stock price process is given by:

$$\mathbf{S}_t = \mathbf{S}_0 \exp(\boldsymbol{\mu}t + \mathbf{Y}_t),$$

where $\boldsymbol{\mu}$ is the drift and $\mathbf{Y}_t$ is defined as in Section 3. Analogously to the market implied calibration, we can make a distinction between the original parametrization and the decoupled parametrization. In the decoupled parametrization, we set $\alpha_j^* = \alpha_0 + \alpha_j$ and $\eta_j^* = \eta_0 + \eta_j$ for $j = 1, \dots, n$ and we estimate the idiosyncratic parameters first by matching the model moments with the historical moments. Subsequently we calibrate the common parameters by matching the model and the historical correlations. In the original parametrization, we estimate all the parameters at the same time by minimizing the distance between the model and historical moments, including the correlation.

6.1 Decoupled setting

In the decoupled parametrization, we can find a closed-form solution to the univariate moment matching system for both the VG and the Δ -Gamma model. For the Δ -Gamma model, we can find the optimal parameter set $\{\mu_j, \alpha_j^*, \beta_j, \eta_j^*, \delta_j\}$ for $j = 1, \dots, n$ by equating the standardized moments under the Δ -Gamma model with the historical moments:

$$\begin{cases} m &= \mu_j \Delta t + \left(\frac{\alpha_j^*}{\beta_j} - \frac{\eta_j^*}{\delta_j} \right) \Delta t, \\ v &= \left(\frac{\alpha_j^*}{\beta_j^2} + \frac{\eta_j^*}{\delta_j^2} \right) \Delta t, \\ s &= \frac{2}{v^{3/2}} \left(\frac{\alpha_j^*}{\beta_j^3} - \frac{\eta_j^*}{\delta_j^3} \right) \Delta t, \\ k &= 3 + \frac{6}{v^2} \left(\frac{\alpha_j^*}{\beta_j^4} + \frac{\eta_j^*}{\delta_j^4} \right) \Delta t, \\ h &= 10s + \frac{24}{v^{5/2}} \left(\frac{\alpha_j^*}{\beta_j^5} - \frac{\eta_j^*}{\delta_j^5} \right) \Delta t, \end{cases} \quad (20)$$

where m is the mean, v the variance, s the skewness, k the kurtosis and h the hyperskewness of the daily log-returns $\ln(S_j(t_k)/S_j(t_{k-1}))$. We take $\Delta t = 1/252$, denoting a one-trading day period. The system (20) can be rewritten in terms of the Δ -Gamma model parameters, leading

¹⁶ Note that we do not use the MLE to calibrate the different models under the $\mathbb{P}$ -measure because the marginal densities are expressed in terms of special functions (Whittaker W functions for the Δ -Gamma model and modified Bessel functions of the third kind for the VG model). To evaluate such functions, we would have to resort to numerical approximations that often lead to significant numerical errors and hence, to unreliable parameter estimates for the models under consideration. On the other hand, the method of moments leads to a straightforward calibration, since the parameter estimates are given analytically as a function of the empirical moments.

¹⁷ Note that a calibration under the $\mathbb{P}$ -measure only makes sense for the Lévy models since Sato processes are by construction processes with non-stationary increments, such that an historical calibration of Sato models would then lead to parameter estimates that strongly depend on the origin of the time series.

to a closed-form solution of the moment matching system¹⁸:

$$\begin{cases} \mu_j &= \frac{m}{\Delta t} - \left(\frac{\alpha_j^*}{\beta_j} - \frac{\eta_j^*}{\delta_j} \right), \\ \alpha_j^* &= \left(\frac{v}{\Delta t} - \frac{\eta_j^*}{\delta_j^2} \right) \beta_j^2, \\ \eta_j^* &= \frac{\delta_j^3}{(\beta_j + \delta_j) \Delta t} \left(v - \frac{1}{2} s v^{3/2} \beta_j \right), \\ \beta_j &= \frac{1}{\frac{1}{3}(k-3)v\delta_j + s\sqrt{v}}, \\ \delta_j &= \frac{-3(h-10s)\sqrt{v} + 6(k-3)s\sqrt{v} - D}{-4(k-3)^2 v + 3(h-10s)sv}, \end{cases} \quad (21)$$

where $D = \sqrt{3v} \sqrt{3\tilde{h}^2 + 32\tilde{k}^3 - 36\tilde{h}\tilde{k}s - 36\tilde{k}^2 s^2 + 36\tilde{h}s^3}$, with $\tilde{h} = h - 10s$ and $\tilde{k} = k - 3$. The solution (21) belongs to the parameter space $\{\mu_j \in \mathbb{R}, \alpha_j^* > 0, \beta_j > 0, \eta_j^* > 0, \delta_j > 0\}$ if and only if

$$k - 3 > \frac{3s^2}{2} \quad \text{and} \quad 3s(h - 10s) < 4(k - 3)^2. \quad (22)$$

Since it must hold that $k - 3 > \frac{3s^2}{2} > 0$, it is clear that the Δ -Gamma model is suited for modeling positive excess of kurtosis. Note that if the historical moments do not satisfy the condition (22), we can adjust the system (20) as proposed by Guillaume & Schoutens (2013), i.e. by replacing the empirical moments k and h (i.e. the moments of highest order in the moment-matching system) such that (22) is satisfied.

For the VG(σ, ν, θ) model, Guillaume & Schoutens (2013) proved that a closed-form solution to the market implied moment matching system is given by:

$$\begin{cases} \sigma^2 &= -\frac{c_2}{3c_1} + (R + \sqrt{D})^{1/3} + (R - \sqrt{D})^{1/3}, \\ \nu &= \frac{s^2 v^3}{\Delta t^2 (\sigma^2 + 2\frac{v}{\Delta t})^2 (\frac{v}{\Delta t} - \sigma^2)}, \\ \theta &= \text{sign}(s) \sqrt{\frac{v}{\Delta t} - \sigma^2}, \end{cases} \quad (23)$$

where

$$\begin{cases} c_1 &= \frac{(\frac{k}{3}-1)\Delta t}{v s^2}, \\ c_2 &= 3\frac{\frac{k}{3}-1}{s^2} - 1, \\ c_3 &= 2\frac{v^2}{\Delta t^2} \left(1 - 2\frac{\frac{k}{3}-1}{s^2} \right), \end{cases}$$

and where

$$D = Q^3 + R^2, \quad Q = -\frac{c_2^2}{9c_1^2} \quad \text{and} \quad R = -\frac{c_3}{2c_1} - \left(\frac{c_2}{3c_1} \right)^3.$$

Here, σ^2 is the solution to the cubic equation $c_1 x^3 + c_2 x^2 + c_3 = 0$ and (23) belongs to the parameter space $\{\sigma_j > 0, \nu_j > 0, \theta_j \in \mathbb{R}\}$ if and only if there exists at least one positive real root of this cubic equation, i.e. if and only if (see Guillaume & Schoutens (2013))

$$6 + 3s^2 - 2k < 0.$$

¹⁸ A similar moment matching system can also be used under the $\mathbb{Q}$ -measure to calibrate the marginal parameters under the decoupled setting, provided the calibration is performed on one single maturity. When considering the whole set of quoted maturities, the moment matching system becomes overdetermined, such that one has to resort to some optimization problem and one loses the main advantage of the method (see Guillaume & Schoutens (2013)).

We can use this closed-form solution in the historical setting as well, leading to the closed-form solution for the optimal historical parameters $\{\mu_j, \alpha_j^*, \beta_j, \delta_j\}$ for $j = 1, \dots, n$:

$$\begin{cases} \mu_j &= \frac{m}{\Delta t} - \theta_j, \\ \alpha_j^* &= \frac{1}{\nu_j}, \\ \beta_j &= \left(\sqrt{\frac{\theta_j^2 \nu_j^2}{4} - \frac{\sigma_j^2 \nu_j}{2}} + \frac{\theta_j \nu_j}{2} \right)^{-1}, \\ \delta_j &= \left(\sqrt{\frac{\theta_j^2 \nu_j^2}{4} - \frac{\sigma_j^2 \nu_j}{2}} - \frac{\theta_j \nu_j}{2} \right)^{-1}, \end{cases} \quad (24)$$

where σ_j^2 , ν_j and θ_j are defined as in (23).

Once the idiosyncratic parameters are determined using (21) or (24), we calibrate the common parameter(s) by minimizing the distance between the model correlations and the historical correlations, i.e. by minimizing

$$\text{HRMSE}^\rho = \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^n \sum_{k>j} (\tilde{\rho}_{jk} - \hat{\rho}_{jk})^2},$$

where $\tilde{\rho}_{jk}$ and $\hat{\rho}_{jk}$ denote the historical and the model correlation between the j th and the k th asset log-return, respectively. The historical correlation between any pair of daily asset log-returns is estimated by the sample correlation:

$$\tilde{\rho}_{jk} = \frac{\sum_{l=1}^d (x_l^{(j)} - \overline{x^{(j)}}) (x_l^{(k)} - \overline{x^{(k)}})}{\sqrt{\sum_{l=1}^d (x_l^{(j)} - \overline{x^{(j)}})^2 \sum_{l=1}^d (x_l^{(k)} - \overline{x^{(k)}})^2}},$$

where $x_l^{(j)}$, $l = 1, \dots, d$ are the daily asset log-returns and $\overline{x^{(j)}}$ the sample mean of the daily asset log-returns of asset j , $j = 1, \dots, n$.

The restrictions on the common parameters (18) must again be satisfied. Tables 10 and 11 summarize the results of the moment matching estimation for the Δ -Gamma and the VG model, respectively, for the period spanning from the fourth of January 2007 to the 20th of October 2009. Note that both moment matching systems (21) and (24) admit a solution for each asset for the period under consideration, such that the moments can be perfectly matched. One can see that the Δ -Gamma model outperforms the VG model in terms of correlation fitting, but that again the restriction on the common parameter η_0 is too restrictive as it was usually observed under the $\mathbb{Q}$ -measure. On the other hand, the poor fit of the historical correlations of the VG model is due to the too small number of common parameters (i.e. one) only since the upper bound on α_0 is then not reached.

6.2 Original setting

In the original parameter setting, finding a closed-form expression for the optimal parameters $\{\mu_j, \alpha_0, (\eta_0), \alpha_j, \beta_j, (\eta_j), \delta_j, j = 1, \dots, n\}$ (i.e. solving the moment matching system) is in most

Table 10: Moment matching estimation of the Δ -Gamma model: historical parameters and the corresponding correlation RMSE.

	μ	α^*	β	η^*	δ		
GE	2.7065	79.5564	28.7239	253.2281	43.4926	α_0	35.6550
MSFT	2.4518	65.2007	31.3554	254.1927	55.7760	η_0	161.1023
PFE	0.0324	155.4082	58.4633	161.1023	56.6456	RMSE $_{\rho}$	0.1106
INTC	-0.0565	270.9313	57.9019	244.2340	52.5815		

Table 11: Moment matching estimation of the VG model: historical parameters and the corresponding correlation RMSE.

	μ	α^*	β	δ		
GE	-0.4604	141.4355	34.5615	35.5539	α_0	85.1901
MSFT	-0.3739	118.5087	37.8574	42.5888	RMSE $_{\rho}$	0.1380
PFE	-0.0335	158.3978	58.8382	56.3288		
INTC	0.2650	256.1802	56.8173	53.4142		

cases impossible. Indeed, the moment matching system can not be reduced to n independent systems of dimension equal to the number of idiosyncratic parameters anymore since the global parameters affect the moments of each asset log-return. Hence, we will perform a moment matching estimation by numerically minimizing the distance between the historical and model moments. In particular, we consider the correlation matching a desired feature and we propose consequently to minimize an objective function of the following form:

$$\text{HRMSE} = \frac{\sum_{j=1}^n \sqrt{\frac{1}{N} \sum_{k=1}^N (\text{MOM}_k^{(j)} - \widehat{\text{MOM}}_k^{(j)})^2}}{n} + \alpha_{\rho} \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^n \sum_{k>j} (\rho_{jk} - \hat{\rho}_{jk})^2}, \quad (25)$$

where $\text{MOM}_k^{(j)}$ and $\widehat{\text{MOM}}_k^{(j)}$ denote the historical and the model (standardized) moment of order k of stock j , respectively. N is the order of the highest order (standardized) moment we want to match and is taken equal to 5 for both the Δ -Gamma model and the VG model, such that we can compare the two models. Again, the parameter α_{ρ} allows the user to specify the relative importance of the correlation fitting and is taken equal to one. The resulting parameters are given in Tables 12 and 13 for the Δ -Gamma and the VG model, respectively. Again, we observe that calibrating all the parameters at the same time results in a better correlation fit and that the Δ -Gamma model significantly outperforms the VG model, in terms of both the historical moment replication and in the correlation fitting. We do note that, nevertheless, the correlation RMSE values for both models are significantly higher under the $\mathbb{P}$ -measure than under the $\mathbb{Q}$ -measure. Extra parameters could be necessary to match at the same time five moments and one co-moment under the historical setting. Looking at the calibrated parameter values of the Δ -Gamma model, we observe values of η close to zero for the PFE and INTC stocks. Since a Gamma distribution with a shape parameter close to zero resembles the Dirac-Delta function with all the mass in zero, the calibration results indicate that the PFE and the INTC stocks have idiosyncratic processes that exhibit mostly positive jumps and that the negative jumps of these assets are mainly due to the common component, i.e. due to the underlying market that was characterized by high levels of market fear and herd behavior during the period under consideration. The obtained model moments are displayed in Tables 15 and ??, together with the historical moments, for the Δ -Gamma and the VG model, respectively. For the VG model, one can observe that especially

the skewness is poorly matched. For both models we can conclude that the kurtosis and the hyperskewness are matched better than the lower moments. This is mainly due to the choice of the functional (RMSE), which assesses less weight to the lower moments since they have a lower amplitude. A better fit of the lower moments could be reached by allocating more weight to the lower moments in the objective function (25) or by considering the average relative percentage error (ARPE) to assess the goodness of fit of the moments.

Table 12: Joint moment matching estimation of the Δ -Gamma model: historical parameters, objective function value and correlation RMSE.

	μ	α	β	η	δ		
GE	1.8743	78.5262	39.5146	24.3114	59.8436	α_0	0.9927
MSFT	2.2094	64.2104	34.7540	25.1784	61.8205	η_0	229.0061
PFE	1.2254	97.3219	68.0765	4.3588e-9	80.8106	HRMSE	0.1009
INTC	-0.4853	288.5866	60.9918	1.7002e-8	53.4864	RMSE $_{\rho}$	0.0789

Table 13: Joint moment matching estimation of the VG model: historical parameters, objective function value and correlation RMSE.

	μ	α	β	δ		
GE	-0.8729	56.8767	33.4097	38.0590	α_0	87.2198
MSFT	-0.8157	36.7464	34.9937	45.0297	HRMSE	0.2122
PFE	-0.0437	71.1517	59.8157	57.4335	RMSE $_{\rho}$	0.1351
INTC	0.2969	168.6437	56.8408	53.0782		

Table 14: Joint moment matching estimation of the Δ -Gamma model : historical and model moments.

		m	v	s	k	h
GE	historical	-0.0014	9.1387e-4	0.0801	8.3495	10.0116
	model	-0.0014	4.8279e-4	0.0799	8.3494	10.0116
MSFT	historical	-1.0379e-4	5.8741e-4	0.3624	9.4669	22.7607
	model	-1.0384e-4	4.7815e-4	0.3624	9.4669	22.7607
PFE	historical	-6.0899e-4	3.7967e-4	-0.1165	7.7818	-2.8163
	model	-6.5206e-4	2.2334e-4	-0.2909	7.7482	-2.8080
INTC	historical	-8.8176e-5	6.7122e-4	-0.1298	5.9623	-2.8588
	model	-7.5388e-5	6.2656e-4	-0.1115	5.9662	-2.8601

Table 15: Joint moment matching estimation of the Δ -Gamma model : historical and model moments.

		m	v	s	k	h
GE	historical	-0.0014	9.1387e-4	0.0801	8.3495	10.0116
	model	-0.0014	9.0705e-4	0.3632	8.3345	10.0015
MSFT	historical	-1.0379e-4	5.8741e-4	0.3624	9.4669	22.7607
	model	-1.0372e-4	6.4433e-4	0.7450	9.4703	22.7483
PFE	historical	-6.0899e-4	3.7967e-4	-0.1165	7.7818	-2.8163
	model	-6.0910e-4	3.6617e-4	-0.1087	7.7815	-2.8166
INTC	historical	-8.8176e-5	6.7122e-4	-0.1298	5.9623	-2.8588
	model	-8.8136e-5	6.7465e-4	-0.1439	5.9685	-2.8580

7 Conclusion

This paper proposed several multivariate extensions of the Difference of Gammas model of Finlay & Seneta (2008). The log-return process is first constructed as the difference of two multivariate Gamma processes with different shape and rate parameters, one process modeling the log-price gains and the other the log-price losses. Each multivariate Gamma process is defined following Mathai & Moschopoulos (1991) and thus contains an idiosyncratic and a common component. The resulting multivariate Δ -Gamma model is an extension of the multivariate VG model, which arises by assuming that the log-price gains and losses have the same shape parameter. Subsequently, the Δ -Gamma model was extended to the Sato setting, resulting in a multivariate model with Sato marginals with a term structure that is more in line with market observations and allowing for a better replication of the univariate option prices, in both the strike and the time-to-maturity dimension.

The numerical study revealed that the Δ -Gamma models outperform the VG models in terms of the single name option price surface replication, but that the precision gain of moving from a VG framework to a Δ -Gamma framework is small compared to the precision gain of using multivariate processes with Sato marginals instead of multivariate Lévy processes. As such, one may conclude that multivariate models with Sato marginals should be preferred over multivariate Lévy models, especially during market distress periods. We showed that the construction of both the Lévy and Sato models allowed for a decoupled calibration procedure, where the idiosyncratic parameters are first calibrated on univariate option price surfaces only and the common parameters are subsequently determined by fitting the model correlations to the market ones. This decoupling procedure however imposes an upper bound on the common parameters that turned out to be too restrictive, resulting in a poor correlation fit. The numerical study has highlighted the fact that a joint calibration, where all the parameters are calibrated at once by minimizing an objective function that includes a penalty term for the correlation fitting, results in a significant improvement of the correlation fit while retaining a univariate option price surface fit that is comparable to the one obtained with the decoupled calibration. Besides, the results have illustrated the relatively poor performance of the VG models compared to their Δ -Gamma extension, pointing out the necessity of the extra common shape parameter to fit the correlation structure.

Lastly, the performance of the VG and the Δ -Gamma model in a historical moment matching estimation was investigated. It turned out that again the Δ -Gamma model outperforms the VG model and that the joint calibration should be preferred over the decoupled calibration procedure to calibrate Δ -Gamma models.

Acknowledgements The authors thank the anonymous referee for the useful suggestions.

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A Maximal attainable correlation: Δ -VG model versus $\rho\alpha$ -VG model

Let $\mathbf{B}(s)$ be a multi-parameter Brownian motion corresponding to a Brownian motion with independent components and Lévy triplet $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{0})$ with $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and let $\bar{\mathbf{B}}_t$ be a multivariate Brownian motion with correlations ρ_{jk}^B , $j, k = 1, \dots, n$ and Lévy triplet $(\boldsymbol{\mu}^\rho, \boldsymbol{\Sigma}^\rho, \mathbf{0})$ with $\boldsymbol{\mu}^\rho = (\mu_1\nu_1, \dots, \mu_n\nu_n)$ and $\boldsymbol{\Sigma}^\rho = (\rho_{jk}^B \sigma_j \sigma_k \sqrt{\nu_j \nu_k})_{j,k=1}^n$ with $\rho_{jj}^B = 1$. Under the $\rho\alpha$ -VG model of Luciano et al. (2016) the n -dimensional asset log-returns are modeled by the factor-based subordinated Brownian motion

$$\mathbf{Y}_t = \begin{pmatrix} B^{(1)}(X_t^{(1)}) + \bar{B}^{(1)}(Z_t) \\ \vdots \\ B^{(n)}(X_t^{(n)}) + \bar{B}^{(n)}(Z_t) \end{pmatrix},$$

where $X_t^{(j)}$ and Z_t are independent subordinators, independent of $\mathbf{B}_t$ and $\bar{\mathbf{B}}_t$, with law at unit time $X_1^{(j)} \sim \text{Gamma}(\frac{1}{\nu_j} - \alpha, \frac{1}{\nu_j})$, $j = 1, \dots, n$ and $Z \sim \text{Gamma}(\alpha, 1)$ (see Luciano et al. (2016) for further details).

The linear correlation between two assets j and k , $j \neq k$, is given by

$$\hat{\rho}_{jk}^{\rho\alpha VG} = \alpha \frac{\nu_j \nu_k \theta_j \theta_k + \rho_{jk}^B \sqrt{\nu_j \nu_k} \sigma_j \sigma_k}{\sqrt{(\sigma_j^2 + \nu_j \theta_j^2)(\sigma_k^2 + \nu_k \theta_k^2)}}. \quad (26)$$

Under the VG model (13) the correlations in the (σ, ν, θ) -parametrization become

$$\hat{\rho}_{jk}^{\Delta VG} = \alpha_0 \frac{\nu_j \nu_k \theta_j \theta_k + \sqrt{\nu_j \nu_k} \sqrt{(2\sigma_j^2 + \nu_j \theta_j^2)(2\sigma_k^2 + \nu_k \theta_k^2)}}{2\sqrt{(\sigma_j^2 + \nu_j \theta_j^2)(\sigma_k^2 + \nu_k \theta_k^2)}} \quad \forall j \neq k.$$

The ratio of linear correlation is given by

$$\frac{\hat{\rho}_{jk}^{\rho\alpha VG}}{\hat{\rho}_{jk}^{\Delta VG}} = \frac{\alpha}{\alpha_0} \frac{2\theta_j \theta_k + 2\rho_{jk}^B \frac{\sigma_j \sigma_k}{\sqrt{\nu_j \nu_k}}}{\theta_j \theta_k + \sqrt{\left(\frac{2\sigma_j^2}{\nu_j} + \theta_j^2\right) \left(\frac{2\sigma_k^2}{\nu_k} + \theta_k^2\right)}}.$$

Note that, while the Δ -VG model correlation can only be positive, the correlation under the $\rho\alpha$ -VG model can also take on negative values. Hence, we will compare the maximal attainable positive correlation under both models, where we only consider positive correlations for the $\rho\alpha$ -VG model. The correlation coefficient (26) is positive whenever

$$\max \left(-1, -\text{sign}(\theta_1 \theta_k) \sqrt{\frac{\theta_j^2 \theta_k^2 \nu_j \nu_k}{\sigma_j^2 \sigma_k^2}} \right) \leq \rho_{jk}^B \leq 1. \quad (27)$$

The common parameters α and α_0 have the same upper bound $0 < \alpha < \min_j \left(\frac{1}{\nu_j} \right)$ and $0 < \alpha_0 < \min_j \left(\frac{1}{\nu_j} \right)$ whenever the marginals are the same. Hence, the ratio of maximal attainable correlation is

$$\frac{\hat{\rho}_{jk}^{\rho\alpha VG, \max}}{\hat{\rho}_{jk}^{\Delta VG, \max}} = \frac{2 + \text{sign}(\theta_j \theta_k) \rho_{jk}^B \sqrt{\frac{2\sigma_j^2 2\sigma_k^2}{\nu_j \theta_j^2 \nu_k \theta_k^2}}}{1 + \text{sign}(\theta_j \theta_k) \sqrt{\left(\frac{2\sigma_j^2}{\nu_j \theta_j^2} + 1\right) \left(\frac{2\sigma_k^2}{\nu_k \theta_k^2} + 1\right)}},$$

which is always smaller than one whenever (27) holds, implying that the Δ -VG has a wider range of positive correlation than the $\rho\alpha$ -VG model. Hence, the upper bound on α for the $\rho\alpha$ -VG model is expected to be reached more often than the upper bound on α_0 for the Δ -VG model whenever the market correlation (and hence the model correlation) is positive. Note that this does not indicate that the Δ -VG model will lead to a better correlation fit. Indeed, due the extra flexibility of the $\rho\alpha$ -VG model introduced by the ρ_{jk}^B , the $\rho\alpha$ -VG model will result in a better correlation fit whenever the upper bound on α is not too restrictive.