The n-fold compound option

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Abstract

This paper revisits the compound options as introduced by R. Geske [2]. Geske presented a theory for pricing an option on an option which he defined as a compound option. He developed a closed form expression for this kind of options.
In this paper we will extend the notion of compound option to the n-fold compound option or compound option of order n.
Moreover an interesting relationship between a k-variate normal distribution function and a (k+1)-variate normal distribution function is proved for this intention.

keywords: financial, n-fold compound options, multivariate normal CDF.

1 Introduction

As was mentioned by Geske [2], any opportunity with a choice whose value depends on an underlying asset can be viewed as an option.
The specific opportunity for an option are its boundary conditions. Many opportunities have a sequential nature, where latter opportunities are available only if earlier opportunities are undertaken. Such is the nature of the compound option (by Geske [2]) or option on an option.
We reintroduce the concept of a compound (call) option.
A 2-fold compound call option (or compound option of order 2) is a call option on a call option i.e. a call option with the underlying being a call option itself. So such a contract entitles one to the following payoff at t_1

$$\max\{C(t_1, S(t_1), t_2, K_2), K_1\},$$

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that is, at t_1, the investor (holder) receives the maximum of the amount K_1 and the value of a European call on the asset S with exercise date and price given by t_2 and K_2 respectively. In other words the investor holds a call, exercisable at t_1, on the underlying call which is exercisable at t_2.
In this paper this idea is generalized to a compound call of order n (with exercise date and price given by t_1 and K_1) with as underlying asset a compound call of order n-1 (with exercise date and price given by t_2 and K_2) which itself is a call on a call of order n-2 ... until the final underlying asset, a European call, to be a call of order 1 (with exercise date and price given by t_n and K_n).
The price at time t_0 of such a call of order n is denoted by

$$C^{(n)}(t_0, S(t_0; (t_j, K_j)_{j=1}^n)),$$

with (t_j, K_j) the exercise date and price of the call of order (n-j+1), underlying the call of order n.
In section 2 the valuation equation for this n-fold compound option (compound of order n) is presented and proved by induction. The proof is based on the PDE representation form and involves a result on a relationship between the k-variate and (k+1)-variate normal distribution function which is treated in appendix A.
The reason for the consideration of these n-fold compound options is the possible application of such derived financial products in the field of

- growing business
- insurance business where the relevance of such products was shown by Simon and Van Wouwe [6] to leave the insured the opportunity to get out of a life insurance contract on certain surrender dates and to be in the possibility to put a price on such an opportunity.

Notations:

V	:	current market value of the firm,
S	:	current market value of the stock, viewed as a call option on the value of the firm V,
C	:	current value of the compound call-option,
t	:	current time,
t^*	:	maturity date of investment for the compound call option C,
T	:	maturity date of investment for the call option S,
r	:	risk-free rate of interest,
σ	:	instantaneous variance of the return on the assets of the firm,

$$
\begin{array}{rcl}
K & : & \text{exercise price for the compound call option } C, \\
M & : & \text{exercise price for the call option } S, \\
N_1(.) & : & \text{univariate cumulative normal distribution function}, \\
N_2(h, k : \rho) & : & \text{bivariate cumulative normal distribution function} \\
& & \text{with } h \text{ and } k \text{ as upper limits and } \rho \text{ as the correlation coefficient}, \\
\overline{V} & : & \text{solution of the equation } S(V, t^*) = K.
\end{array}
$$

The economic assumptions, which are to be considered for the valuation equation of a compound option in a continuous time and using a hedging argument, are:

- there is no credit risk, only market risk,
- the market is maximally efficient, i.e. it is infinitely liquid and does not exhibit any friction,
- continuous trading is possible,
- the time evolution of the asset price is stochastic and exhibits geometric Brownian motion,
- the risk-free interest rate r and the volatility σ are constant,
- the underlying pays no dividends,
- the underlying is arbitrarely divisible,
- the market is arbitrage-free.

According to the article of Geske [2] we have the following partial differential equations:

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= rC - rV\frac{\partial C}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} && (1) \\
\frac{\partial S}{\partial t} &= rS - rV\frac{\partial S}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 S}{\partial V^2}, && (2)
\end{aligned}
$$

with boundary conditions:

$$
\begin{aligned}
C_{t^*} &= \max(0, S_{t^*} - K) && (3) \\
S_T &= \max(0, V_T - M). && (4)
\end{aligned}
$$

Solving equation (2) leads to the well-known Black-Scholes-Merton equation:

$$
S(V, t) = V.N_1(d_1) - M.e^{-r(T-t)}.N_1(d_2),
$$

with

$$
\begin{aligned}
d_2 &= \frac{ln\frac{V}{M} + (r - \frac{\sigma^2}{2}).(T-t)}{\sigma\sqrt{T-t}} \\
d_1 &= d_2 + \sigma\sqrt{T-t}.
\end{aligned}
$$

However, there is a slight difference for the value of the compounded call option, which was showed by Geske [2]:

$$
C(V,t) \quad = \quad V.N_2\left(h_1, d_1; \rho\right) - M.e^{-r(T-t)}.N_2\left(h_2, d_2; \rho\right) - K.e^{-r(t^*-t)}.N_1\left(h_2\right),
$$

with

$$
\begin{aligned}
h_2 &= \frac{ln\frac{V}{\overline{V}} + \left(r - \frac{1}{2}\sigma^2\right)(t^* - t)}{\sigma\sqrt{t^* - t}} \\
h_1 &= h_2 + \sigma\sqrt{t^* - t}.
\end{aligned}
$$

2 Valuation of the n-fold compound option

Definition 2.1 By induction:
A compound call option (of order 2) is a call option on a call option. This can be generalized to a compound call of order $k+1$ (with exercise date and price given by t_1 and K_1) with an underlying call of order k.

Now consider an expansion of the symbols from paragraph 1:

t_i	:	maturity date of investment for the compound call option C_i,
K_i	:	exercise price for the compound call option C_i,
C_i	:	current value of the compound call option on the option C_{i+1},
$N_k(a_1, a_2, \ldots, a_k; A)$	:	k-variate cumulative normal distribution function with a_i as upper limits and A as the correlation matrix.

Because each C_i is function of the value of the firm V and the time t, these calls all have the same PDE:

$$\frac{\partial C_i}{\partial t} = r.C_i - r.V.\frac{\partial C_i}{\partial V} - \frac{1}{2}\sigma^2.V^2.\frac{\partial^2 C_i}{\partial V^2},$$

but with a different boundary condition:

$$C_i(V, t_i) = \max(0, C_{i+1}(V, t_i) - K_i)\,.$$

The most outer call is simply derived according to Black, Scholes and Merton, while the next one can be defined following the method of Geske [2].
Further we repeatedly add a time step and solve the corresponding PDE.
This results in the following theorem:

Theorem 2.1 Suppose for $s = k+1, k, ..., 2$ the calls C_s are known and given by:

$$\begin{aligned} C_s &= V.N_{k+2-s}\left(a_s, a_{s+1}, ..., a_{k+1}; A_s^{k+2-s}\right) \\ &\quad - \sum_{m=s}^{k+1} K_m.e^{-r(t_m-t)}.N_{m+1-s}\left(b_s, b_{s+1}, ..., b_m; A_s^{m+1-s}\right), \end{aligned}$$

where we use the notations

$$a_\ell = b_\ell + \sigma\sqrt{t_\ell - t} \qquad \ell = 2, 3, ..., k+1$$

$$b_\ell = \frac{ln\frac{V}{\overline{V}_\ell} + (r - \frac{\sigma^2}{2})(t_\ell - t)}{\sigma\sqrt{t_\ell - t}} \qquad \ell = 2, 3, ..., k+1$$

$\overline{V}_\ell$ solution of the equation $C_{\ell+1}(V, t_\ell) = K_\ell$ $\qquad \ell = 2, 3, ..., k$

$$
\begin{aligned}
\overline{V}_{k+1} &= M \\
\rho_{ij} &= \sqrt{\frac{t_i - t}{t_j - t}} \qquad\qquad i < j \\
A_s^\ell &= \left(a_{ij}^\ell\right)_{i,j=1,2,\ldots,\ell} \qquad \text{where } \begin{cases} a_{ii} &= 1 \\ a_{ij} &= \rho_{i+s-1,j+s-1} \quad i < j \end{cases} .
\end{aligned}
$$

Then the $(k+1)$ fold compound option can be found to be:

$$
\begin{aligned}
C_1 &= V.N_{k+1}\left(a_1, a_2, \ldots, a_{k+1}; A_1^{k+1}\right) \\
&\quad - \sum_{m=1}^{k+1} K_m.e^{-r(t_m - t)}.N_m\left(b_1, b_2, \ldots, b_m; A_1^m\right).
\end{aligned}
$$

Proof
Since C_1 is a call option, the following PDE holds for C_1:

$$
\frac{\partial C_1}{\partial t} = r.C_1 - r.V.\frac{\partial C_1}{\partial V} - \frac{1}{2}\sigma^2.V^2.\frac{\partial^2 C_1}{\partial V^2},
$$

with $C_1(V, t_1) = \max(0, C_2(V, t_1) - K_1)$ as boundary condition.
Making use of the results in appendix B, the PDE for C_1 can be transformed into a diffusion equation:

$$
\frac{\partial \tilde{x}}{\partial s} = \frac{\partial^2 \tilde{x}}{\partial p^2},
$$

with adjusted boundary conditions for the variables p and s

$$
\tilde{x}(p, 0) = \begin{cases} C_2(V, t_1) - K_1 & \text{if } V \geq \overline{V}_1 \\ 0 & \text{if } V < \overline{V}_1 \end{cases}
$$

($\overline{V}_1$ being the solution of $C_2 - K_1 = 0$.)
(see [7] for a proof of the monotonicity of C with respect to V),

or

$$
\tilde{x}(p, 0) = \begin{cases} V.N_k(d_2, d_3, \ldots, d_{k+1}; A_2^k) & \\ - \sum_{m=2}^{k+1} K_m.e^{-r(t_m - t_1)}.N_{m-1}(f_2, f_3, \ldots, f_m; A_2^{m-1}) - K_1 & \text{if } p \geq 0 \\ 0 & \text{if } p < 0. \end{cases}
$$

By introducing the notations

$$d_\ell = f_\ell + \sigma\sqrt{t_\ell - t_1} \qquad \ell = 2,3,...,k+1$$

$$f_\ell = \frac{ln\frac{V}{\overline{V}_\ell} + (r - \frac{\sigma^2}{2})(t_\ell - t_1)}{\sigma\sqrt{t_\ell - t_1}} \qquad \ell = 2,3,...,k+1$$

and for $\ell = 2,...,k$ the matrices

$$A_2^\ell = \left(a_{ij}^\ell\right)_{i,j=1,2,...,\ell} \qquad \text{where} \left\{ \begin{array}{lll} a_{ii} & = & 1 \\ a_{ij} & = & a_{ji} = \rho_{i+1,j+1}\big|_{t_1} \qquad i<j \\ & = & \sqrt{\dfrac{t_{i+1} - t_1}{t_{j+1} - t_1}} \end{array} \right.$$

we find the following expression for V at $t = t_1$:

$$V = \overline{V}_1 \, exp\left(\frac{\frac{1}{2}\sigma^2}{r - \frac{1}{2}\sigma^2}\,p\right),$$

which will be used later on.
Using a Green's function as delta-function, the expression for $\tilde{x}(p,s)$ can be written as:

$$\tilde{x}(p,s) = \int_{-\infty}^{+\infty} \tilde{x}(p',0).G(p-p',s)\,dp'$$

$$\text{with}\, G(p-p',s) = \frac{1}{\sqrt{4\pi s}}.e^{-\frac{(p-p')^2}{4s}},$$

or

$$\begin{aligned}\tilde{x}(p,s) = & \int_0^{+\infty} \overline{V}_1.exp\left(\frac{\frac{1}{2}\sigma^2}{r-\frac{1}{2}\sigma^2}\,p'\right).N_k(d_2,...,d_{k+1};A_2^k).\frac{1}{\sqrt{4\pi s}}.exp\left(-\frac{(p-p')^2}{4s}\right)dp' \\ & -\int_0^{+\infty}\sum_{m=2}^{k+1} K_m.e^{-r(t_m-t_1)}.N_{m-1}(f_2,...,f_m;A_2^{m-1}).\frac{1}{\sqrt{4\pi s}}.exp\left(-\frac{(p-p')^2}{4s}\right)dp' \\ & -\int_0^{+\infty} K_1.\frac{1}{\sqrt{4\pi s}}.exp\left(-\frac{(p-p')^2}{4s}\right)dp'.\end{aligned}$$

A substitution of $b = \dfrac{p'-p}{\sqrt{2s}}$ in all of the three integrals and a second substitution $b' = b - \sigma\sqrt{t_1 - t}$ in the first integral, lead to the following expression:

$$\tilde{x}(p,s) = \int_{-d_1}^{+\infty} V.e^{r(t_1-t)}.N_k(d_2^*,...,d_{k+1}^*;A_2^k).\frac{1}{\sqrt{2\pi}}.exp\left(-\frac{b'^2}{2}\right)db'$$

$$
\begin{aligned}
&-\int_{-f_1}^{+\infty}\sum_{m=2}^{k+1} K_m.e^{-r(t_m-t_1)}.N_{m-1}(f_2^*,...,f_m^*;A_2^{m-1}).\frac{1}{\sqrt{2\pi}}.exp\left(-\frac{b^2}{2}\right)\,db\\
&-\int_{-f_1}^{+\infty} K_1.\frac{1}{\sqrt{2\pi}}.exp\left(-\frac{b^2}{2}\right)\,db.
\end{aligned}
$$

In these calculations the new integration boundaries can be found as:

$$
\begin{aligned}
-f_1 &= -\frac{ln\frac{V}{\overline{V}_1}+\left(r-\frac{1}{2}\sigma^2\right)(t_1-t)}{\sigma\sqrt{t_1-t}}\\
-d_1 &= -(f_1+\sigma\sqrt{t_1-t}),
\end{aligned}
$$

while for $\ell = 2,3,...,k+1$ we get:

$$
\begin{aligned}
f_\ell &= \frac{ln\frac{\overline{V}_1}{\overline{V}_\ell}+\frac{\frac{1}{2}\sigma^2}{r-\frac{1}{2}\sigma^2}\,p'+\left(r-\frac{1}{2}\sigma^2\right)(t_\ell-t_1)}{\sigma\sqrt{t_\ell-t_1}}\\
f_\ell^* &= \frac{b_\ell+\rho_{1\ell}\,b}{\sqrt{1-\rho_{1\ell}^2}} \qquad \text{if } \rho_{1\ell}=\sqrt{\frac{t_1-t}{t_\ell-t}}
\end{aligned}
$$

and

$$
\begin{aligned}
d_\ell &= \frac{ln\frac{\overline{V}_1}{\overline{V}_\ell}+\frac{\frac{1}{2}\sigma^2}{r-\frac{1}{2}\sigma^2}\,p'+\left(r+\frac{1}{2}\sigma^2\right)(t_\ell-t_1)}{\sigma\sqrt{t_\ell-t_1}}\\
d_\ell^* &= \frac{a_\ell+\rho_{1\ell}\,b'}{\sqrt{1-\rho_{1\ell}^2}}.
\end{aligned}
$$

An application of theorem A.1 in the first integral leads to the final expression for $\tilde{x}(p,s)$:

$$
\begin{aligned}
\tilde{x}(p,s) &= V.e^{r(t_1-t)}.N_{k+1}(d_1,a_2,...,a_{k+1};A_1^{k+1})\\
&\quad -\sum_{m=2}^{k+1} K_m.e^{-r(t_m-t_1)}.N_m(f_1,b_2,...,b_m;A_1^m)-K_1.N_1(f_1),
\end{aligned}
$$

where the correlation matrices can be written as:

$$
A_1^{k+1} = (a_{ij})_{i,j=1,2,\cdots,k+1} \qquad \text{with} \left\{\begin{array}{lll} a_{ii} &=& 1\\ a_{ij} &=& a_{ji} \;=\; \rho_{ij} \qquad \text{if } i<j \end{array}\right.
$$

and for $m = 2, 3, \cdots, k+1$:

$$A_1^m = (a_{ij})_{i,j=1,2,\cdots,m} \qquad \text{with} \begin{cases} a_{ii} = 1 \\ a_{ij} = a_{ji} = \rho_{ij} \qquad \text{if } i < j \end{cases} .$$

The final form for the $(k+1)$ fold compound call option therefore equals

$$\begin{aligned} C_1(V,t) &= V.N_{k+1}(a_1, a_2, ..., a_{k+1}; A_1^{k+1}) \\ &\quad - \sum_{m=2}^{k+1} K_m.e^{-r(t_m - t)}.N_m(b_1, b_2, ..., b_m; A_1^m) - K_1.e^{-r(t_1-t)}.N_1(b_1), \end{aligned}$$

with

$$a_\ell = b_\ell + \sigma\sqrt{t_\ell - t} \qquad \ell = 1, 2, ..., k+1$$

$$b_\ell = \frac{ln\frac{V}{\overline{V}_\ell} + (r - \frac{\sigma^2}{2})(t_\ell - t)}{\sigma\sqrt{t_\ell - t}} \qquad \ell = 1, 2, ..., k+1$$

$\overline{V}_\ell$ determined by $C_{\ell+1}(V, t_\ell) = K_\ell$ $\qquad \ell = 1, 2, ..., k$

$$\overline{V}_{k+1} = M$$

$$\rho_{ij} = \sqrt{\frac{t_i - t}{t_j - t}} \qquad i < j$$

$$A_1^\ell = \left(a_{ij}^\ell\right)_{i,j=1,2,...,\ell} \qquad \text{where} \begin{cases} a_{ii} = 1 \\ a_{ij} = a_{ji} = \rho_{ij} \quad i < j \end{cases} .$$

3 Conclusions

The notion of a n-fold compound option is introduced as a generalization of the compound option by Geske [2]. The closed-form analytic expression for this n-fold compound option is proved by induction and by using some interesting results on the relationship between (k+1)-variate normal distributions and k-variate normal distributions.

A A useful relationship between the $k+1$ th multivariate distribution function and the k th multivariate distribution function

In theorem A.1 we will need the following two lemmas about a matrix and its inverse. Both of the lemmas can be proved in a straightforward way.

Lemma A.1 If

$$A = \left[\begin{array}{c|ccc} 1 & \frac{-a_{12}}{\sqrt{1-a_{12}^2}} & \cdots & \frac{-a_{1k}}{\sqrt{1-a_{1k}^2}} \\ \hline 0 & \frac{1}{\sqrt{1-a_{12}^2}} & & \mathrm{O} \\ \vdots & & \ddots & \\ 0 & \mathrm{O} & & \frac{1}{\sqrt{1-a_{1k}^2}} \end{array}\right]$$

then

$$A^{-1} = \left[\begin{array}{c|ccc} 1 & a_{12} & \cdots & a_{1k} \\ \hline 0 & \sqrt{1-a_{12}^2} & & \mathrm{O} \\ \vdots & & \ddots & \\ 0 & \mathrm{O} & & \sqrt{1-a_{1k}^2} \end{array}\right]$$

by the use of the principle of partitioning.

Lemma A.2

$$\text{If } A = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B}^{-1} & \\ 0 & & & \end{array}\right], \text{ then } A^{-1} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B} & \\ 0 & & & \end{array}\right].$$

Let N_k be the k-variate normal distribution function and N_{k-1} the (k-1)-variate normal distribution function, the following expression can be determined between N_k and N_{k-1}.

Theorem A.1

$$\int_{-m_1}^{+\infty} \frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}x_1^2}.N_{k-1}\left(\frac{m_2+\rho_{12}\,x_1}{\sqrt{1-\rho_{12}^2}},\cdots,\frac{m_k+\rho_{1k}\,x_1}{\sqrt{1-\rho_{1k}^2}};B\right)\,dx_1 = N_k(m_1,\cdots,m_k;C),$$

with $C=(c_{ij})_{i,j=1,\cdots,k}$ a symmetric matrix with

$$\begin{cases} c_{11} &= 1 \\ c_{1j} &= \rho_{1j} \\ c_{ij} &= \rho_{1i}\,\rho_{1j}+\sqrt{(1-\rho_{1i}^2)\,(1-\rho_{1j}^2)}\,b_{i-1,j-1} \end{cases}$$

and where for convenience we put

$$N_0 = 1.$$

Proof by induction
For $k=1$ the result is straightforward.
For the second part of the proof we first rewrite the integral as

$$\int_{-m_1}^{+\infty} \frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}x_1^2}.N_{k-1}\left(\frac{m_2+\rho_{12}\,x_1}{\sqrt{1-\rho_{12}^2}},\cdots,\frac{m_k+\rho_{1k}\,x_1}{\sqrt{1-\rho_{1k}^2}};B\right)\,dx_1$$

$$=\int_{-m_1}^{+\infty}\int_{-\left(\frac{m_2+\rho_{12}\,x_1}{\sqrt{1-\rho_{12}^2}}\right)}^{+\infty}\cdots\int_{-\left(\frac{m_k+\rho_{1k}\,x_1}{\sqrt{1-\rho_{1k}^2}}\right)}^{+\infty}\frac{1}{\sqrt{(2\pi)^k\,\det B}}.e^{-\frac{1}{2}\,P(x_1,\ldots,x_k)}\,dx_1\,dx_2\cdots dx_k\,,$$

$$\text{with}\,P(x_1,\ldots,x_k)=[x_1\quad x_2\quad\cdots\quad x_k]\,.\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B}^{-1} & \\ 0 & & & \end{array}\right].\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

Making use of the substitution

$$y_j = x_j\,\sqrt{1-\rho_{1j}^2}+\rho_{1j}\,x_1 \qquad \text{for } j=2,3,\cdots,k$$

we can rewrite this expression as

$$=\int_{-m_1}^{+\infty}\cdots\int_{-m_k}^{+\infty}\frac{1}{\sqrt{(2\pi)^k\,\det B\,\prod_{j=2}^{k}(1-\rho_{1j}^2)}}.e^{-\frac{1}{2}\,P^*(x_1,y_2,\ldots,y_k)}\,dx_1\,dy_2\cdots dy_k.$$

In this formula, we introduced the matrix:

$$P^*(x_1, y_2, \ldots, y_k) = \begin{bmatrix} x_1 & y_2 & \cdots & y_k \end{bmatrix} .D. \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B}^{-1} & \\ 0 & & & \end{array}\right] .D^t. \begin{bmatrix} x_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix},$$

where

$$D = \left[\begin{array}{c|ccc} 1 & \frac{-\rho_{12}}{\sqrt{1-\rho_{12}^2}} & \cdots & \frac{-\rho_{1k}}{\sqrt{1-\rho_{1k}^2}} \\ \hline 0 & \frac{1}{\sqrt{1-\rho_{12}^2}} & & \mathrm{O} \\ \vdots & & \ddots & \\ 0 & \mathrm{O} & & \frac{1}{\sqrt{1-\rho_{1k}^2}} \end{array}\right].$$

Since we want to express the k-variate integration by means of a k-variate normal CDF, we now have to determine the correlation matrix C with:

$$C^{-1} = D. \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B}^{-1} & \\ 0 & & & \end{array}\right] .D^t.$$

An application of lemma A.1 and A.2 leads to

$$C = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \rho_{12} & \sqrt{1-\rho_{12}^2} & & \mathrm{O} \\ \vdots & & \ddots & \\ \rho_{1k} & \mathrm{O} & & \sqrt{1-\rho_{1k}^2} \end{array}\right] . \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{B} & \\ 0 & & & \end{array}\right]$$
$$. \left[\begin{array}{c|ccc} 1 & \rho_{12} & \cdots & \rho_{1k} \\ \hline 0 & \sqrt{1-\rho_{12}^2} & & \mathrm{O} \\ \vdots & & \ddots & \\ 0 & \mathrm{O} & & \sqrt{1-\rho_{1k}^2} \end{array}\right],$$

or

$$C = \left[\begin{array}{c|ccc} 1 & \rho_{12} & \cdots & \rho_{1k} \\ \hline \rho_{12} & & & \\ \vdots & & \text{F} & \\ \rho_{1k} & & & \end{array}\right],$$

with F obtained by partitioning as

$$\begin{bmatrix} \rho_{12} \\ \vdots \\ \rho_{1k} \end{bmatrix} . \begin{bmatrix} \rho_{12} & \cdots & \rho_{1k} \end{bmatrix} + \begin{bmatrix} \sqrt{1-\rho_{12}^2} & & \text{O} \\ & \ddots & \\ \text{O} & & \sqrt{1-\rho_{1k}^2} \end{bmatrix} .B. \begin{bmatrix} \sqrt{1-\rho_{12}^2} & & \text{O} \\ & \ddots & \\ \text{O} & & \sqrt{1-\rho_{1k}^2} \end{bmatrix}.$$

Hence this results in:

$$C = (c_{ij})_{i,j=1,\cdots,k} \qquad \text{with} \qquad \begin{cases} c_{11} &= 1 \\ c_{1j} &= \rho_{1j} \\ c_{ij} &= \rho_{1i}\,\rho_{1j} + \sqrt{(1-\rho_{1i}^2)\,(1-\rho_{1j}^2)}\,b_{i-1,j-1} \end{cases}$$

Now, since $P^*(x_1, y_2, \ldots, y_k)$ can be written as

$$P^*(x_1, y_2, \ldots, y_k) = \begin{bmatrix} x_1 & y_2 & \cdots & y_k \end{bmatrix} .C^{-1}. \begin{bmatrix} x_1 \\ y_2 \\ \cdots \\ y_k \end{bmatrix}$$

and

$$\det C = \prod_{j=2}^{k} (1-\rho_{1j}^2).\det B$$

for the integral we find

$$\int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \, \det B \, \prod_{j=2}^{k} (1-\rho_{1j}^2)}} . e^{-\frac{1}{2}\, P^*(x_1, y_2, \ldots, y_k)}\, dx_1\, dy_2 \ldots\, dy_k$$

$$= \int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \, \det C}} . e^{-\frac{1}{2}\, P^*(x_1, y_2, \ldots, y_k)}\, dx_1\, dy_2 \ldots\, dy_k$$

or

$$= N_k(m_1, \ldots, m_k; C),$$

which completes the proof.

B From PDE to diffusion equation

Consider the PDE:

$$\frac{\partial C_i}{\partial t} = r.C_i - r.V.\frac{\partial C_i}{\partial V} - \frac{1}{2}\sigma^2.V^2.\frac{\partial^2 C_i}{\partial V^2},$$

with boundary condition at time t_i given by

$$C_i(V,t_i) = \max(0, C_{i+1}(V,t_i) - K_i).$$

Let $\overline{V}_i$ be defined as the value for which $C_{i+1}(\overline{V}_i, t_i) = K_i$.
Making use of some substitutions, we can rewrite this PDE as a diffusion equation.
Indeed, first we choose w as

$$w = ln\frac{V}{\overline{V}_i},$$

and we define the function $x(w,t)$ as

$$\begin{aligned} x(w,t) &= e^{r(t_i-t)}.C_i(V,t) \\ &= e^{r(t_i-t)}.C_i(\overline{V}_i.e^w, t). \end{aligned}$$

Secondly, we rescale the independent variables as

$$\begin{cases} w' = \dfrac{r-\frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2}w \\ s = \dfrac{\left(r-\frac{1}{2}\sigma^2\right)^2}{\frac{1}{2}\sigma^2}(t_i - t), \end{cases}$$

we define the function $\hat{x}(w', s)$

$$\hat{x}(w',s) = x(w,t).$$

With

$$p = w' + s,$$

finally we rewrite the dependent variable as

$$\tilde{x}(p,s) = \hat{x}(w',s).$$

Then it follows in a straightforward way that this last function satisfies the diffusion equation

$$\frac{\partial \tilde{x}}{\partial s} = \frac{\partial^2 \tilde{x}}{\partial p^2}.$$

See [5] for more details.

C Green's function

Consider Green's function

$$G(z-z',t') = \frac{1}{\sqrt{4\pi t'}}.e^{-\frac{(z-z')^2}{4t'}},$$

which clearly satisfies the diffusion equation

$$\frac{\partial G}{\partial t'} = \frac{\partial^2 G}{\partial z^2}.$$

Note that G behaves like a delta-function in $t'=0$:

* If $z \neq z'$ and $t' \to 0$ then $G(z-z',t') \to 0$
* If $z = z'$ and $t' \to 0$ then $G(z-z',t') \to \infty$
*

$$\begin{aligned} \int_{-\infty}^{+\infty} G(z-z',t')\,dz' &= -\int_{+\infty}^{-\infty} \frac{1}{\sqrt{2\pi}}\,exp(-\frac{q^2}{2})\,dq \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} exp(-\frac{q^2}{2})\,dq \\ &= 1 \end{aligned}$$

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