



On principally generated quantaloid-modules in general, and skew local homeomorphisms in particular

Hans Heymans, Isar Stubbe^{*,1}

Department of Mathematics and Computer Science, University of Antwerp, Middelheimlaan 1, 2020 Antwerpen, Belgium

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ABSTRACT

Ordered sheaves on a small quantaloid \mathcal{Q} have been defined in terms of \mathcal{Q} -enriched categorical structures; they form a locally ordered category $\text{Ord}(\mathcal{Q})$. The free-cocompletion KZ-doctrine on $\text{Ord}(\mathcal{Q})$ has $\text{Mod}(\mathcal{Q})$, the quantaloid of \mathcal{Q} -modules, as its category of Eilenberg–Moore algebras. In this paper we give an intrinsic description of the Kleisli algebras: we call them the *locally principally generated \mathcal{Q} -modules*. We deduce that $\text{Ord}(\mathcal{Q})$ is biequivalent to the 2-category of locally principally generated \mathcal{Q} -modules and left adjoint module morphisms. The example of locally principally generated modules on a locale X is worked out in full detail: relating X -modules to objects of the slice category Loc/X , we show that ordered sheaves on X correspond with *skew local homeomorphisms into X* (like sheaves on X correspond with local homeomorphisms into X).

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1. Introduction

1.1. Locales and quantales, sheaves and logic

A locale X is a complete lattice in which finite infima distribute over arbitrary suprema. A particular class of examples of locales comes from topology: the open subsets of any topological space form a locale. But not every locale arises in this way, whence the slogan that locales are “pointfree topologies” [23]. There is a “pointfree” way to do sheaf theory: a sheaf F on a locale X is a functor $F: X^{\text{op}} \rightarrow \text{Set}$ satisfying gluing conditions. The collection of all such functors, together with natural transformations between them, forms the topos $\text{Sh}(X)$ of *sheaves on X* . One of the many close ties between logic and sheaf theory, which is of particular interest to us, is that the *internal logic* of $\text{Sh}(X)$ is an intuitionistic higher-order predicate logic with X as object of truth values [29,8,24]. To borrow a phrase from [38] and others, sheaf theory thus serves as *algebraic logic*.

The definition of locale can be restated: X is a complete lattice and (X, \wedge, \top) is a monoid such that the multiplication distributes on both sides over arbitrary suprema. It is natural to generalise this: a *quantale* $Q = (Q, \circ, 1)$ is, by definition, a monoid structure on a complete lattice such that the multiplication distributes on both sides over arbitrary suprema [31,

* Corresponding author.

E-mail addresses: hans.heyman@ua.ac.be (H. Heymans), isar.stubbe@ua.ac.be (I. Stubbe).

¹ Postdoctoral Fellow of the Research Foundation Flanders (FWO).

39]. Because the monoid structure of a locale is obviously commutative, but the one for a quantale need not be, one can think of quantales as “pointfree non-commutative topologies”. Examples of quantales, other than locales, can be found in algebra and geometry [33,36], in logic [52], in computer science [1,42]. In the spirit of (enriched) category theory [26], it is not hard to see that a quantale is precisely a monoid in the symmetric monoidal closed category Sup of complete lattices and suprema-preserving functions. And a *quantaloid* \mathcal{Q} is then defined as a category enriched in Sup (so a quantale is the same thing as a quantaloid with one object, precisely as a group is a groupoid with one object) [41].

The success of sheaf theory to study logic from locales, and the useful generalisation from locales to quantales (and even quantaloids), makes one wonder about the “logic of sheaves on quantales”. However, it is not at all straightforward to define “sheaves on a quantale”! Many different definitions have been proposed by many different authors, e.g. [10,32,19,15], but often only for particular classes of quantales. In previous work we have taken the following stance on the matter: whereas sheaves on a locale X can be described in terms of sets equipped with an X -valued equality relation [28,13,8], the non-commutativity of the quantale multiplication forces sheaves on a quantale Q to be sets equipped with a Q -valued *inequality* relation [9,46].² In other words, our attention goes to the category of *ordered sheaves on a quantale* (or even quantaloid), which we see as “algebraic non-commutative logic”.

More precisely, [46] studied ordered sheaves on a quantaloid \mathcal{Q} in terms of \mathcal{Q} -enriched categories, thus generalising to the non-commutative case the work of Walters [50] and Borceux and Cruciani [9] on locales. In this paper we shall show that they can equivalently be described as particular \mathcal{Q} -modules. (If \mathcal{Q} is a quantaloid, then a \mathcal{Q} -module is by definition a Sup -enriched functor $F: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$. For a quantale Q this reduces to a complete lattice on which Q acts.) This in turn can be applied to a locale X , and we find a characterisation of the relevant X -modules as particular locale morphisms with codomain X . We speak of *principally generated \mathcal{Q} -modules* in general, and *skew local homeomorphisms into X* in particular, as we shall introduce next.

1.2. Principally generated \mathcal{Q} -modules

To introduce this novel notion in \mathcal{Q} -module theory, around which this article is centred, we first recall a simple fact from order theory: The well-known adjunction between the category Ord of ordered sets and order-preserving functions on the one hand, and the category Sup of complete lattices and supremum-preserving functions on the other,

$$\text{Ord} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \text{Sup}, \tag{1}$$

has the remarkable feature that both functors involved are embeddings. This allows us to view Sup as a part of Ord , but also Ord as a part of Sup . The first viewpoint corresponds to the common understanding that a complete lattice is an ordered set in which all suprema exist and that a sup-morphism is an order-preserving function that preserves suprema. More technically: Sup is the category of Eilenberg–Moore algebras for the ‘free-cocompletion KZ-doctrine’ on Ord , which sends an ordered set to the set of its downclosed subsets ordered by inclusion. The second point of view is what the notion of a totally algebraic complete lattice is all about. Recall that an element a of a complete lattice L is *totally compact* (a.k.a. *supercompact*) when, for any downclosed $A \subseteq L$, $a \leq \bigvee A$ implies $a \in A$; and a complete lattice L is *totally algebraic* (a.k.a. *superalgebraic*) when each element is the supremum of totally compact ones [16]. It turns out that the replete image of the left adjoint in the above adjunction is precisely the subcategory of Sup of totally algebraic objects and left adjoint morphisms; thus Ord is described intrinsically (and up to equivalence) as a part of Sup .

We want to broaden the situation depicted above: instead of studying ordered sets, i.e. ordered sheaves on the two-element Boolean algebra $\mathbf{2}$, we want to consider ordered sheaves on any small quantaloid \mathcal{Q} . The latter form a 2-category $\text{Ord}(\mathcal{Q})$ that we define³ as $\text{Cat}_{\text{cc}}(\mathcal{Q}_{\text{si}})$: its objects are the Cauchy-complete categories enriched in the split-idempotent completion of the quantaloid \mathcal{Q} , and its morphisms are the \mathcal{Q} -enriched functors. It was proved by Stubbe [49] that the category of “internal sup-lattices and sup-morphisms” in $\text{Ord}(\mathcal{Q})$ is (biequivalent to) $\text{Mod}(\mathcal{Q}) := \text{QUANT}(\mathcal{Q}^{\text{op}}, \text{Sup})$, the quantaloid of \mathcal{Q} -modules. That is to say, in analogy with the situation in (1) above, there is a biadjunction

$$\text{Ord}(\mathcal{Q}) \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \text{Mod}(\mathcal{Q}) \tag{2}$$

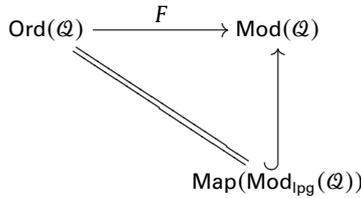
that splits the free-cocompletion KZ-doctrine on $\text{Ord}(\mathcal{Q})$ and such that moreover $\text{Mod}(\mathcal{Q})$ is (biequivalent to) the category of Eilenberg–Moore algebras for that doctrine. This describes $\text{Mod}(\mathcal{Q})$ as part of $\text{Ord}(\mathcal{Q})$, and for $\mathcal{Q} = \mathbf{2}$ we thus recover exactly half of the situation described in the first paragraph above. But what about the other half: Can we also intrinsically

² We make a remark about \mathcal{Q} -valued *equalities* at the end of this Introduction.

³ See [46] for a more “elementary” definition of ordered sheaves on a quantaloid.

characterise $\text{Ord}(\mathcal{Q})$ as part of $\text{Mod}(\mathcal{Q})$? Can we give a module-theoretic condition on an object of $\text{Mod}(\mathcal{Q})$ that makes it equivalent to the free cocompletion of an object of $\text{Ord}(\mathcal{Q})$? And how do morphisms then relate?

With our [Definition 5.3](#) and our [Theorem 5.9](#) we answer these questions in the affirmative: we prove that $\text{Map}(\text{Mod}_{\text{lp}}(\mathcal{Q}))$, defined as the subcategory of $\text{Mod}(\mathcal{Q})$ of *locally principally generated \mathcal{Q} -modules and left adjoint \mathcal{Q} -module morphisms*, is precisely the replete image of the left biadjoint functor in (2) above:



The technology that we use to solve this problem is \mathcal{Q} -enriched categorical algebra, as pioneered (in greater generality) by Benabou [4], Walters [50] and Street [43] and more recently surveyed by Stubbe [45]. More particularly, in this paper we build further on results from [48], which treats totally algebraic cocomplete \mathcal{Q} -categories, and [47], where an explicit comparison is given between cocomplete \mathcal{Q} -categories and \mathcal{Q} -modules.

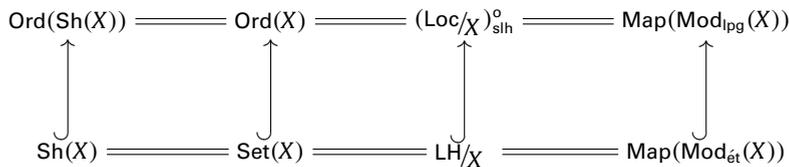
1.3. Skew local homeomorphisms

The notion of ordered sheaf on a quantaloid \mathcal{Q} is so devised that, when taking \mathcal{Q} to be the one-object suspension of a locale X (i.e. \mathcal{Q} has one object, the elements of X are viewed as arrows of \mathcal{Q} , composition of which corresponds to finite meets in X , the identity arrow thus being the top element \top of X), $\text{Ord}(X)$ is equivalent to the category of ordered objects and order-preserving morphisms in the topos $\text{Sh}(X)$ of sheaves on X [50,9,46]: $\text{Ord}(X) \simeq \text{Ord}(\text{Sh}(X))$. Our general results on \mathcal{Q} -modules from the first part of this paper surely specialise to the localic case: ordered sheaves on X can be described equivalently as locally principally generated X -modules, order-preserving morphisms then correspond to left adjoint X -module morphisms.

In analogy with ring theory it is very natural to regard a locale morphism $f: Y \rightarrow X$ as a left X -module (Y, \circ_f) with action $y \circ_f x := y \wedge f^*(x)$ [25]. This construction extends to a (contravariant) embedding of the slice category Loc/X in $\text{Mod}(X)$. Thus it is natural to try to characterise the subcategory of Loc/X which corresponds, under this embedding, to the locally principally generated X -modules and the left adjoint X -module morphisms between them, or in other words, to $\text{Ord}(X)$. In [Definitions 7.1](#) and [7.7](#) we introduce the locale theoretic notions of *skew open morphism* and *skew local homeomorphism*, and in [Theorem 7.10](#) we then prove that $(\text{Loc}/X)_{\text{slh}}^{\circ}$, by definition the (non-full) subcategory of Loc/X of skew local homeomorphisms as objects and skew open morphisms between them, is the sought-after equivalent of $\text{Ord}(X)$.

A local homeomorphism is necessarily a skew local homeomorphism; and an open locale morphism is always skew open too. Thus the category LH/X of local homeomorphisms over X is naturally a full subcategory of $(\text{Loc}/X)_{\text{slh}}^{\circ}$. This situation too can be stated in terms of X -modules: in [Definition 7.11](#) we introduce *étale X -modules* as a particular kind of locally principally generated X -modules, such that in [Theorem 7.12](#) we can prove that the full subcategory $\text{Map}(\text{Mod}_{\text{ét}}(X))$ of $\text{Map}(\text{Mod}_{\text{lp}}(X))$ defined by the étale X -modules is indeed equivalent to LH/X .

The category LH/X is a well-known equivalent of the topos $\text{Sh}(X)$, see e.g. [29, p. 524]; thus the inclusion of local homeomorphisms over X into skew local homeomorphisms over X , or equivalently the inclusion of étale X -modules into locally principally generated X -modules, is precisely the same thing as the inclusion of sheaves on X into ordered sheaves on X . Moreover, in the realm of enriched categorical structures it is a matter of fact that the “ X -sets” of Fourman and Scott [13] (see also [8], Vol. 3; [9,24], p. 502–513) are included in X -orders. All this establishes the following unifying diagram of equivalent embeddings of categories of symmetric (or discrete) objects into 2-categories of asymmetric (or ordered) objects:



This shows the relation between (ordered) sheaves on a locale X as (i) functors on X satisfying gluing conditions, (ii) X -enriched categorical structures, (iii) locale morphisms into X and (iv) X -modules.

1.4. Overview of contents

Section 2 through 6 of this paper are concerned with the translation of the notion of *ordered sheaf on a small quantaloid \mathcal{Q}* from its original definition in terms of \mathcal{Q} -enriched categorical structures [50,46] to the language of \mathcal{Q} -modules. To make this paper self-contained, we therefore start with an overview of the \mathcal{Q} -enriched categorical algebra that we need: in Section 2

we recall the definition of \mathcal{Q} -categories, functors and distributors; we speak of weighted colimits in a \mathcal{Q} -category and of Cauchy-completeness of a \mathcal{Q} -category; and we end with the definition of \mathcal{Q} -order. We have tried to include the relevant “historical” references, but in practice we refer mostly to the more recent [45–47] whose notations we follow. In Section 3 we recall some material on totally algebraic cocomplete \mathcal{Q} -categories from [48]; in fact, we recast the definition of a totally compact object in a way that suits our needs further on. (In Section 8 we explain a biadjunction involving totally algebraic cocomplete \mathcal{Q} -categories: strictly speaking it is of no technical importance for the rest of this paper, but since it may be of independent interest we have chosen to add it as an Addendum.) Section 4 then contains the crucial translation from \mathcal{Q} -enrichment to \mathcal{Q} -variation – to borrow a term from Betti *et al.* [6], later picked up by Gordon and Power [18] and Stubbe [47] – where we introduce the notion of *principally generated \mathcal{Q} -module*. Our first main theorem is that Cauchy-complete \mathcal{Q} -categories and functors between them form a category which is equivalent to that of principally generated \mathcal{Q} -modules and left adjoint module morphisms. Finally, in Section 5 we explain how a so-called *locally principally generated \mathcal{Q} -module* is the same thing as a principally generated module on the split-idempotent completion of \mathcal{Q} , thus paving the way for our second main result: \mathcal{Q} -orders (meaning Cauchy-complete categories enriched in the split-idempotent completion of \mathcal{Q}) and their morphisms are essentially the same thing as locally principally generated \mathcal{Q} -modules and left adjoint module morphisms. We discuss some examples in Section 6.

The second part of this paper, contained in Section 7, is devoted to the application of the above to the specific case where \mathcal{Q} is the one-object suspension of a locale X (viewed as monoid (X, \wedge, \top)). Locally principally generated X -modules are then equivalent to ordered sheaves on X , which this time can really be understood as ordered objects in the topos $\text{Sh}(X)$ [50,9,46]. (If one takes for granted that locally principally generated X -modules are ordered sheaves on X , then one can start reading Section 7 right away; this second part of the paper is technically speaking rather independent from the first part.) But locally principally generated X -modules can also be expressed in terms of certain locale morphisms into X , and it is their study that we deal with here. Thus, we begin by briefly explaining, taking hints from Joyal and Tierney [25], how any locale morphism into X can be regarded as an X -module; we define *skew open morphisms* in the slice category Loc/X to correspond to left adjoint X -module morphisms. A detailed study of locally principally generated X -modules is carried out thereafter; we show in particular that any such X -module is necessarily induced by a locale morphism into X . This work being done, we come to our third main result of the paper: we define *skew local homeomorphisms* in terms of coverings by *skew open sections*, and prove that the subcategory of Loc/X with skew local homeomorphisms as objects and skew open morphisms between them, is equivalent to the category of locally principally generated X -modules and left adjoint module morphisms, *viz.* ordered sheaves on X . Remarking that local homeomorphisms are necessarily skew local homeomorphisms, we end with the identification of *étale X -modules* as those locally principally generated X -modules which correspond to local homeomorphisms, *viz.* sheaves on X .

1.5. Further work

In this paper we do not speak of the “internal logic” of the category of ordered sheaves on a quantaloid – it is still an open problem – but we hope that our contribution here will be helpful to investigate this. In this respect it should be interesting to investigate links with examples of noncommutative logics developed from a more logical (rather than algebraic, *i.e.* sheaf theoretic) point of view: for example, Goldblatt’s [17] encoding of predicates in some non-commutative logic as quantale-valued functions on a set (which can be seen as elements of a principally generated module!); or Baltag *et al.* [3], who use quantale-modules in their treatment of epistemic logic; or Rosenthal [40] model for the “bang” operator in linear logic via modules on a quantale; the construction in [12] of a logic from a very particular notion of sheaf on a restricted class of quantales; or the quantale based semantics for propositional normal modal logic in [30]; and many others.

In [37] local homeomorphisms into X are shown to correspond to Hilbert X -modules with a Hilbert basis (a special case of Hilbert Q -modules, for Q an involutive quantale [35]). Their results and our results in Section 7 (particularly **Definition 7.11**) are similar in that we both provide a description of local homeomorphisms into a locale X in terms of particular X -modules. In [21] we explain, in the generality of modules on an involutive quantale Q , the precise relationship between Hilbert Q -modules admitting a Hilbert basis on one hand, and locally principally generated Q -module satisfying a suitable *symmetry condition* on its locally principal elements on the other; we argue that the latter are precisely sets with a Q -valued *equality*, *i.e.* a “ Q -sets” rather than a “ Q -orders”. Applied to a locale X this gives (ordinary) sheaves on X ; better still, applied to suitably constructed involutive quantales we can describe all Grothendieck toposes, in a manner closely related to [51].

Our current work continues along this line and focuses on \mathcal{Q} -valued equalities in the generality of an involutive quantaloid \mathcal{Q} , more specifically on the interplay between symmetric and non-symmetric \mathcal{Q} -categories and their Cauchy completions. This is directly related to [51,7,14] but also has ties with [19,20].

2. Preliminaries

Quantaloids

Let Sup denote the category of complete lattices and maps that preserve arbitrary suprema (*suplattices* and *supmorphisms*): it is symmetric monoidal closed for the usual tensor product. A *quantaloid* is a Sup -enriched category; a

one-object quantaloid is most often thought of as a monoid in Sup: it is a *quantale*. A Sup-functor between quantaloids is sometimes called a *homomorphism*; QUANT denotes the (“illegitimate”) category of quantaloids and their homomorphisms. The standard reference on categories enriched in a symmetric monoidal category in general is [26]; for quantales and quantaloids in particular there is [39,41].

Composition with a morphism $f: X \rightarrow Y$ in a quantaloid \mathcal{Q} gives rise to adjunctions, one for each $A \in \mathcal{Q}$,

$$\mathcal{Q}(A, X) \begin{array}{c} \xrightarrow{f \circ -} \\ \perp \\ \xleftarrow{[f, -]} \end{array} \mathcal{Q}(A, Y) \quad \text{and} \quad \mathcal{Q}(Y, A) \begin{array}{c} \xrightarrow{- \circ f} \\ \perp \\ \xleftarrow{\{f, -\}} \end{array} \mathcal{Q}(X, A).$$

These right adjoints are respectively called *lifting* and *extension* (through f). We shall keep the notations “[$-$, $-$]” and “[$-$, $-$]” for liftings and extensions in any quantaloid that follows; no confusion shall arise.⁴

Given a quantaloid \mathcal{Q} we write \mathcal{Q}_{si} (“si” stands for “split the idempotents”) for the new quantaloid whose objects are the idempotent arrows in \mathcal{Q} , and in which an arrow from an idempotent $e: A \rightarrow A$ to an idempotent $f: B \rightarrow B$ is a \mathcal{Q} -arrow $g: A \rightarrow B$ satisfying $g \circ e = g = f \circ g$. Composition in \mathcal{Q}_{si} is done as in \mathcal{Q} , the identity in \mathcal{Q}_{si} on some idempotent $e: A \rightarrow A$ is e itself, and the local order in \mathcal{Q}_{si} is that of \mathcal{Q} . There is an obvious inclusion $j: \mathcal{Q} \rightarrow \mathcal{Q}_{\text{si}}$, mapping $f: A \rightarrow B$ to $f: 1_A \rightarrow 1_B$, which expresses \mathcal{Q}_{si} as the universal split-idempotent completion of \mathcal{Q} in QUANT.

When \mathcal{Q} is a small quantaloid, $\text{Mod}(\mathcal{Q})$ shall be shorthand for $\text{QUANT}(\mathcal{Q}^{\text{op}}, \text{Sup})$: the objects of this (large) quantaloid are called the *modules on \mathcal{Q}* , or briefly \mathcal{Q} -modules. Since idempotents split in Sup, and noting that \mathcal{Q}_{si} is small whenever \mathcal{Q} is, it follows that $\text{Mod}(\mathcal{Q})$ is equivalent to $\text{Mod}(\mathcal{Q}_{\text{si}})$. We shall come back to modules on \mathcal{Q} and on \mathcal{Q}_{si} in Sections 4 and 5.

A quantaloid is in particular a locally ordered category, and therefore we can straightforwardly define *adjoint pairs* in any quantaloid: $f: X \rightarrow Y$ is left adjoint to $g: Y \rightarrow X$ (and g is right adjoint to f , written $f \dashv g$) when $1_X \leq g \circ f$ and $f \circ g \leq 1_Y$. If a morphism $f: X \rightarrow Y$ has a right adjoint, then the latter is unique, and we shall often use f^* as its notation. Because left adjoints are sometimes called “maps”, $\text{Map}(\mathcal{Q})$ is our notation for the (locally ordered) category of left adjoints in a quantaloid \mathcal{Q} .

2.1. Quantaloid-enriched categories

A quantaloid is a bicategory and therefore it may serve itself as a base for enrichment [4,50,43]. The theory of quantaloid-enriched categories, functors and distributors is surveyed in [45] where many more appropriate references are given; here we can only provide a brief summary, but we follow the notations of *op. cit.* for easy cross reference. To avoid size issues we work from now on with a *small quantaloid* \mathcal{Q} .

A \mathcal{Q} -category \mathbb{A} consists of a set \mathbb{A}_0 of ‘objects’, a ‘type’ function $t: \mathbb{A}_0 \rightarrow \mathcal{Q}_0$, and for any $a, a' \in \mathbb{A}_0$ a ‘hom-arrow’ $\mathbb{A}(a', a): ta \rightarrow ta'$ in \mathcal{Q} ; these data are required to satisfy

$$\mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \quad \text{and} \quad 1_{ta} \leq \mathbb{A}(a, a)$$

for all $a, a', a'' \in \mathbb{A}_0$. A functor $F: \mathbb{A} \rightarrow \mathbb{B}$ is a map $\mathbb{A}_0 \rightarrow \mathbb{B}_0: a \mapsto Fa$ that satisfies

$$ta = t(Fa) \quad \text{and} \quad \mathbb{A}(a', a) \leq \mathbb{B}(Fa', Fa)$$

for all $a, a' \in \mathbb{A}_0$.

A \mathcal{Q} -category \mathbb{A} has an *underlying order*⁵ (\mathbb{A}_0, \leq) : for $a, a' \in \mathbb{A}_0$ define $a \leq a'$ to mean $ta = ta' =: A$ and $1_A \leq \mathbb{A}(a, a')$. If $a \leq a'$ and $a' \leq a$ we write $a \cong a'$ and say that these are *isomorphic objects* in \mathbb{A} . For parallel functors $F, G: \mathbb{A} \rightarrow \mathbb{B}$ we now put $F \leq G$ when $Fa \leq Ga$ for every $a \in \mathbb{A}_0$. With the obvious composition and identities we thus obtain a locally ordered category $\text{Cat}(\mathcal{Q})$ of \mathcal{Q} -categories and functors. Precisely because $\text{Cat}(\mathcal{Q})$ is a 2-category, we can from now on unambiguously use 2-categorical notions such as adjoint functors, Kan extensions, and so on.

To give a *distributor* (or *module* or *profunctor*) $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}$ between \mathcal{Q} -categories is to specify, for all $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$, arrows $\Phi(b, a): ta \rightarrow tb$ in \mathcal{Q} such that

$$\mathbb{B}(b, b') \circ \Phi(b', a) \leq \Phi(b, a) \quad \text{and} \quad \Phi(b, a') \circ \mathbb{A}(a', a) \leq \Phi(b, a)$$

for every $a, a' \in \mathbb{A}_0, b, b' \in \mathbb{B}_0$. Two distributors $\Phi: \mathbb{A} \dashv\vdash \mathbb{B}, \Psi: \mathbb{B} \dashv\vdash \mathbb{C}$ compose: we write $\Psi \otimes \Phi: \mathbb{A} \dashv\vdash \mathbb{C}$ for the distributor with elements

$$(\Psi \otimes \Phi)(c, a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c, b) \circ \Phi(b, a).$$

The identity distributor on a \mathcal{Q} -category \mathbb{A} is $\mathbb{A}: \mathbb{A} \dashv\vdash \mathbb{A}$ itself, i.e. the distributor with elements $\mathbb{A}(a', a): ta \rightarrow ta'$. For parallel distributors $\Phi, \Phi': \mathbb{A} \dashv\vdash \mathbb{B}$ we define $\Phi \leq \Phi'$ to mean that $\Phi(b, a) \leq \Phi'(b, a)$ for every $a \in \mathbb{A}_0, b \in \mathbb{B}_0$. It is easily seen that \mathcal{Q} -categories and distributors form a quantaloid $\text{Dist}(\mathcal{Q})$.

⁴ These right adjoints also go by the name of *residuals* when \mathcal{Q} is a quantale, i.e. a one-object quantaloid. Whereas our notations are the usual ones in category theory (for closed monoidal categories or bicategories), other notations instead of $[f, g]$ and $\{f, g\}$ that can be found in the literature include $f \rightarrow g$ and $g \leftarrow f$, or $f \multimap g$ and $f \multimap_l g$, or $f \setminus g$ and $f \backslash g$, or f^{\setminus} and f^{\setminus} .

⁵ By an *order* we mean a reflexive and transitive relation, i.e. a (small) category with at most one arrow between any two objects; some call this a preorder. We shall speak of a *partial order* or an *antisymmetric order* if we require moreover antisymmetry.

Every functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ between \mathcal{Q} -categories represents an adjoint pair of distributors:

- the left adjoint $\mathbb{B}(-, F-): \mathbb{A} \multimap \mathbb{B}$ has elements $\mathbb{B}(b, Fa): ta \longrightarrow tb$,
- the right adjoint $\mathbb{B}(F-, -): \mathbb{B} \multimap \mathbb{A}$ has elements $\mathbb{B}(Fa, b): tb \longrightarrow ta$.

The assignment $F \mapsto \mathbb{B}(-, F-)$ is a (bijective and) faithful 2-functor from $\text{Cat}(\mathcal{Q})$ to $\text{Dist}(\mathcal{Q})$; thus, whenever a distributor $\Phi: \mathbb{A} \multimap \mathbb{B}$ is represented by a functor $F: \mathbb{A} \longrightarrow \mathbb{B}$, this F is essentially unique.

2.2. Weighted colimits

In a \mathcal{Q} -enriched category \mathbb{C} we can speak of weighted limits and colimits, as introduced by Street [43] for general bicategory-enrichment. For our short account here we use [45,47] as references, but we should certainly also mention the work of Gordon and Power [18] on conical (co)limits and (co)tensors (to be explained below).

Given a distributor $\Phi: \mathbb{A} \multimap \mathbb{B}$ and a functor $F: \mathbb{B} \longrightarrow \mathbb{C}$, a functor $K: \mathbb{A} \longrightarrow \mathbb{C}$ is the Φ -weighted colimit of F when it satisfies

$$\mathbb{C}(K-, -) = [\Phi, \mathbb{C}(F-, -)]$$

in $\text{Dist}(\mathcal{Q})$ (and in that case it is essentially unique, so we write it as $\text{colim}(\Phi, F)$). The \mathcal{Q} -category \mathbb{C} is *cocomplete* when it admits all such weighted colimits. A functor $H: \mathbb{C} \longrightarrow \mathbb{C}'$ is *cocontinuous* when it preserves all colimits that happen to exist in \mathbb{C} : $H \circ \text{colim}(\Phi, F) \cong \text{colim}(\Phi, H \circ F)$. A left adjoint functor is always cocontinuous; conversely, if the domain of a cocontinuous functor is cocomplete, then that functor is left adjoint. Cocomplete \mathcal{Q} -categories and cocontinuous functors form a sub-2-category $\text{Cocont}(\mathcal{Q})$ of $\text{Cat}(\mathcal{Q})$. (Dually one can speak of *weighted limits*, *complete \mathcal{Q} -categories* and *continuous functors*; we shall not explicitly need these further on, however, it is a matter of fact that a \mathcal{Q} -category is complete if and only if it is cocomplete [45, 5.10].)

Every object X of a quantaloid \mathcal{Q} determines a one-object \mathcal{Q} -category $*_X$ whose single hom-arrow is 1_X . A *contravariant presheaf of type X* on a \mathcal{Q} -category \mathbb{A} is a distributor $\phi: *_X \multimap \mathbb{A}$; these are the objects of a cocomplete \mathcal{Q} -category $\mathcal{P}\mathbb{A}$ whose hom-arrows are given by lifting in $\text{Dist}(\mathcal{Q})$. Every object $a \in \mathbb{A}_0$ determines, and is determined by, a functor $\Delta a: *_a \longrightarrow \mathbb{A}$; thus $a \in \mathbb{A}_0$ also represents a (left adjoint) presheaf $\mathbb{A}(-, a): *_a \multimap \mathbb{A}$. The *Yoneda embedding* $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}: a \mapsto \mathbb{A}(-, a)$ is a fully faithful⁶ continuous functor. The presheaf construction $\mathbb{A} \mapsto \mathcal{P}\mathbb{A}$ extends to a 2-functor $\mathcal{P}: \text{Cat}(\mathcal{Q}) \longrightarrow \text{Cocont}(\mathcal{Q})$ which is left biadjoint to the inclusion 2-functor, with the Yoneda embeddings as its unit; thus presheaf categories are the freely cocomplete ones. In fact, a \mathcal{Q} -category \mathbb{C} is cocomplete if and only if the Yoneda embedding admits a left adjoint in $\text{Cat}(\mathcal{Q})$; if this is the case we write $\text{sup}_{\mathbb{C}}: \mathcal{P}\mathbb{C} \longrightarrow \mathbb{C}$ for that left adjoint: it maps a presheaf ϕ on \mathbb{C} to the weighted colimit $\text{sup}_{\mathbb{C}}(\phi) := \text{colim}(\phi, 1_{\mathbb{C}})$. Note by the way that $\text{sup}_{\mathbb{C}} \circ Y_{\mathbb{C}} \cong 1_{\mathbb{C}}$; actually, $Y_{\mathbb{C}}$ admits a left adjoint if and only if it admits a left inverse.

Given any $X \in \mathcal{Q}$ we shall write (\mathbb{C}_X, \leq) for the (possibly empty) sub-order of (\mathbb{C}_0, \leq) containing all $c \in \mathbb{C}_0$ for which $tc = X$. If for each $X \in \mathcal{Q}$ the order (\mathbb{C}_X, \leq) is a complete lattice, we say that \mathbb{C} is *order-cocomplete*. On the other hand, for each morphism $f: A \longrightarrow B$ in \mathcal{Q} we can consider the one-element distributor $(f): *_A \multimap *_B$. Suppose that $c \in \mathbb{C}$ is of type $tc = B$, and that $\text{colim}(f, \Delta c)$ exists: it is itself a functor from $*_A$ to \mathbb{C} , and can thus be identified with an element of \mathbb{C} of type A . We write that element as $c \otimes f$ and call it the *tensor of c with f* ; \mathbb{C} has *all tensors* when all such colimits with one-element weights exist. The dual notion is *cotensor*. It has been proved in [47, 2.13] that a \mathcal{Q} -category \mathbb{C} is cocomplete if and only if it is order-cocomplete and has all tensors and cotensors; moreover, for any $\Phi: \mathbb{A} \multimap \mathbb{B}$ and $F: \mathbb{B} \longrightarrow \mathbb{C}$, the weighted colimit $\text{colim}(\Phi, F): \mathbb{A} \longrightarrow \mathbb{C}$ is the functor defined by $a \mapsto \bigvee_{b \in \mathbb{B}_0} Fb \otimes \Phi(b, a)$.

2.3. Ordered sheaves on a quantaloid

The importance of the following notion has first been recognised by Lawvere [28] in the context of categories enriched in a monoidal category.

A \mathcal{Q} -category \mathbb{C} is *Cauchy complete* if for any other \mathcal{Q} -category \mathbb{A} the map

$$\text{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{C}) \longrightarrow \text{Map}(\text{Dist}(\mathcal{Q}))(\mathbb{A}, \mathbb{C}): F \mapsto \mathbb{C}(-, F-)$$

is surjective, i.e. when any left adjoint distributor (also called *Cauchy distributor*) into \mathbb{C} is represented by a functor. This is equivalent to the requirement that \mathbb{C} admits any colimit weighted by a Cauchy distributor; and moreover such weighted colimits are *absolute* in the sense that they are preserved by any functor [44]. We write $\text{Cat}_{\text{cc}}(\mathcal{Q})$ for the full subcategory of $\text{Cat}(\mathcal{Q})$ whose objects are the Cauchy complete \mathcal{Q} -categories.

Now we have everything ready to define the central notion of this paper [46].

Definition 2.1. For a small quantaloid \mathcal{Q} , we write $\text{Ord}(\mathcal{Q})$ for the locally ordered category $\text{Cat}_{\text{cc}}(\mathcal{Q}_{\text{si}})$, and call its objects *ordered sheaves on \mathcal{Q}* , or simply *\mathcal{Q} -orders*.

⁶ A functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ is *fully faithful* when $\mathbb{A}(a', a) = \mathbb{B}(Fa', Fa)$ for every $a, a' \in \mathbb{A}_0$.

When taking \mathcal{Q} to be the (one-object suspension of) a locale X , $\text{Ord}(X)$ is the category of ordered objects and order-preserving morphisms in the topos $\text{Sh}(X)$, as first (implicitly) observed by Walters [50] (but see also [9] for the locale-specific notion, and [46] for the generalisation to quantaloids and the comparison between [50] and [9]). Obviously, this example inspired our terminology.

3. Total algebraicity revisited

We shall review and expand the material that we need from [48].

Definition 3.1 ([48]). Suppose that \mathbb{A} is a cocomplete \mathcal{Q} -category. The *totally below distributor* on \mathbb{A} is the right extension of $\mathbb{A}(-, \text{sup}_{\mathbb{A}}-)$ through $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-, -)$ in $\text{Dist}(\mathcal{Q})$:

$$\begin{array}{ccc}
 \mathcal{P}\mathbb{A} & \xrightarrow{\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-, -)} & \mathbb{A} \\
 \downarrow & \searrow \leq & \uparrow \\
 \mathbb{A}(-, \text{sup}_{\mathbb{A}}-) & & \Theta_{\mathbb{A}} := \left\{ \mathbb{A}(-, \text{sup}_{\mathbb{A}}-), \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-, -) \right\} \\
 \downarrow & & \\
 \mathbb{A} & &
 \end{array}$$

An object $a \in \mathbb{A}$ is *totally compact* when $1_{ta} \leq \Theta_{\mathbb{A}}(a, a)$. Writing $i_{\mathbb{A}}: \mathbb{A}_c \rightarrow \mathbb{A}$ for the full embedding of the totally compacts, \mathbb{A} is *totally algebraic* when the left Kan extension of $i_{\mathbb{A}}$ along itself is isomorphic to $1_{\mathbb{A}}$.

In the simplest possible case, when \mathcal{Q} is the (one-object suspension of) the two-element Boolean algebra $(\mathbf{2}, \wedge, \top)$, a \mathcal{Q} -category \mathbb{A} is an ordered set (A, \leq) , it is cocomplete precisely when (A, \leq) is a sup-lattice, and the distributor $\Theta_{\mathbb{A}}$ is the following “totally below” relation: $a' \ll a$ when, for every down-closed subset $D \subseteq A$, $a \leq \bigvee D$ implies $a' \in D$. A totally compact element is one which is totally below itself, and (A, \leq) is totally algebraic if and only if every element is the supremum of the totally compacts below it. These notions are related to, but stronger than, the “way below” relation and the “algebraic” sup-lattices [16].

Theorem 3.2 ([48]). *The 2-functor*

$$\mathcal{P}: \text{Dist}(\mathcal{Q}) \longrightarrow \text{Cocont}(\mathcal{Q}): \left(\Phi: \mathbb{A} \dashv \!\! \dashv \mathbb{B} \right) \mapsto \left(\Phi \otimes -: \mathcal{P}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{B}) \right)$$

is locally an equivalence, and its corestriction to the full sub-2-category of totally algebraic cocomplete \mathcal{Q} -categories is a biequivalence: $\text{Dist}(\mathcal{Q}) \simeq \text{Cocont}_{ta}(\mathcal{Q})$.

We may restrict the biequivalence of which this theorem speaks, to left adjoints: we then obtain the biequivalence (which we write with the same letter)

$$\mathcal{P}: \text{Map}(\text{Dist}(\mathcal{Q})) \longrightarrow \text{Map}(\text{Cocont}_{ta}(\mathcal{Q})).$$

But the definition of Cauchy completeness for \mathcal{Q} -categories implies that also

$$\text{Cat}_{cc}(\mathcal{Q}) \longrightarrow \text{Map}(\text{Dist}(\mathcal{Q})): \left(F: \mathbb{A} \longrightarrow \mathbb{B} \right) \mapsto \left(\mathbb{B}(-, F-): \mathbb{A} \dashv \!\! \dashv \mathbb{B} \right)$$

is a biequivalence, hence composing these two we get a third biequivalence (which we still write with the same letter):

Corollary 3.3. *The 2-functor*

$$\mathcal{P}: \text{Cat}_{cc}(\mathcal{Q}) \longrightarrow \text{Map}(\text{Cocont}_{ta}(\mathcal{Q})): \left(F: \mathbb{A} \longrightarrow \mathbb{B} \right) \mapsto \left(\mathbb{B}(-, F-) \otimes -: \mathbb{A} \dashv \!\! \dashv \mathbb{B} \right)$$

is a biequivalence.

The inverse biequivalence is given by “taking totally compact objects”. More precisely, if $F: \mathbb{A} \rightarrow \mathbb{B}$ is in $\text{Map}(\text{Cocont}(\mathcal{Q}))$ then it maps totally compact objects of \mathbb{A} to totally compact objects of \mathbb{B} , hence we get a functor $F_c: \mathbb{A}_c \rightarrow \mathbb{B}_c$ out of it. If \mathbb{A} and \mathbb{B} are moreover totally algebraic, then \mathbb{A}_c and \mathbb{B}_c are Cauchy complete. This in fact describes a 2-functor $(-)_c: \text{Map}(\text{Cocont}_{ta}(\mathcal{Q})) \rightarrow \text{Cat}_{cc}(\mathcal{Q})$, which turns out to be the sought-after inverse.

In fact, the biequivalence in Corollary 3.3 can also be seen as resulting from (co)restricting the following biadjunction to the objects for which the (co)unit is an equivalence:

$$\begin{array}{ccc}
 \text{Cat}(\mathcal{Q}) & \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{(-)_c} \end{array} & \text{Map}(\text{Cocont}(\mathcal{Q})).
 \end{array}$$

This is without importance for what follows, so we shall not include the details here; but since this may be of independent interest, we have written the details in a technical Addendum at the end of this paper.

Several equivalent expressions for the definition of totally compact object are given in [48] but for the purposes of this paper the following are particularly useful⁷:

Proposition 3.4. *Let $a \in \mathbb{A}$ be an object, of type $A \in \mathcal{Q}$ say, of a cocomplete \mathcal{Q} -category. The following conditions are equivalent:*

- i. a is totally compact,
- ii. for all $\phi \in \mathcal{P}\mathbb{A}$: $\phi(a) = \mathbb{A}(a, \sup_{\mathbb{A}}(\phi))$,
- iii. the functor $H_a: \mathbb{A} \rightarrow \mathcal{P}(*_A): x \mapsto \mathbb{A}(a, x)$ (“homming with a ”) is cocontinuous,
- iv. the functor $T_a: \mathcal{P}(*_A) \rightarrow \mathbb{A}: f \mapsto a \otimes f$ (“tensoring with a ”) is cocontinuous and admits a cocontinuous right adjoint.

Proof. (i \iff ii) It is easily seen (and spelled out in [48], 5.2) that, for any $x, y \in \mathbb{A}$,

$$\Theta_{\mathbb{A}}(x, y) = \bigwedge_{\phi \in \mathcal{P}\mathbb{A}} \{\mathbb{A}(y, \sup_{\mathbb{A}}(\phi)), \phi(x)\}$$

and hence, straightforwardly,

$$\begin{aligned} 1_A \leq \Theta_{\mathbb{A}}(a, a) &\iff \forall \phi \in \mathcal{P}\mathbb{A} : 1_A \leq \{\mathbb{A}(a, \sup_{\mathbb{A}}(\phi)), \phi(a)\} \\ &\iff \forall \phi \in \mathcal{P}\mathbb{A} : \mathbb{A}(a, \sup_{\mathbb{A}}(\phi)) \leq \phi(a). \end{aligned}$$

But because $\phi \leq Y_{\mathbb{A}}(\sup_{\mathbb{A}}(\phi))$ is automatic, and thus $\phi(a) \leq \mathbb{A}(a, \sup_{\mathbb{A}}(\phi))$ always holds, a being totally compact is indeed equivalent to the clause in statement (ii).

(ii \iff iii) By a straightforward calculation (e.g. using [47, Corollary 2.15]) it is easily seen that, for any $\phi \in \mathcal{P}(\mathbb{A})$, $\phi(a)$ is the ϕ -weighted colimit of H_a :

$$\text{colim}(\phi, H_a) = \bigvee_{x \in \mathbb{A}} H_a(x) \otimes \phi(x) = \mathbb{A}(a, -) \otimes \phi = \phi(a).$$

On the other hand it is clear by definition of H_a that $\mathbb{A}(a, \sup_{\mathbb{A}}(\phi)) = H_a(\sup_{\mathbb{A}}(\phi))$. Thus the formula in the second statement of the lemma can be rewritten as

$$\text{for all } \phi \in \mathcal{P}\mathbb{A}: \text{colim}(\phi, H_a) = H_a(\sup_{\mathbb{A}}(\phi)).$$

As follows straightforwardly from [45, 5.4], this in turn is equivalent to H_a preserving *all* weighted colimits.

(iii \iff iv) Due to \mathbb{A} 's cocompleteness, we surely have that all tensors with the object a exist in \mathbb{A} ; thus (the dual of) Proposition 3.2 in [47] says that H_a is the right adjoint to T_a in $\text{Cat}(\mathcal{Q})$. (A direct verification is very easy too.) Because the cocontinuous functors between cocomplete \mathcal{Q} -categories are precisely the left adjoint ones in $\text{Cat}(\mathcal{Q})$ [45, Proposition 6.8], the result follows directly. \square

4. Principally generated modules

A \mathcal{Q} -module $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ determines a cocomplete \mathcal{Q} -category $\mathbb{A}_{\mathcal{F}}$ whose set of objects is $\bigsqcup_X \mathcal{F}X$, with types given by $tx = X$ if and only if $x \in \mathcal{F}X$, and hom-arrows

$$\mathbb{A}_{\mathcal{F}}(y, x) = \bigvee \{f: tx \rightarrow ty \mid \mathcal{F}(f)(y) \leq x\}. \quad (3)$$

Similarly, a module morphism $\alpha: \mathcal{F} \Rightarrow \mathcal{G}$ determines a cocontinuous functor

$$F_{\alpha}: \mathbb{A}_{\mathcal{F}} \rightarrow \mathbb{A}_{\mathcal{G}}: x \mapsto \alpha_{tx}(x).$$

This sets up the biequivalence of $\text{Mod}(\mathcal{Q})$ with $\text{Cocont}(\mathcal{Q})$, as studied in great detail in [47, Section 4] based on work by Gordon and Power [18]. We wish to characterise, purely in terms of \mathcal{Q} -modules, those $\mathcal{F} \in \text{Mod}(\mathcal{Q})$ for which the corresponding $\mathbb{A}_{\mathcal{F}} \in \text{Cocont}(\mathcal{Q})$ is totally algebraic. In order to do so, we must introduce some notations.

Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module, and suppose that $a \in \mathcal{F}(A)$ for some $A \in \mathcal{Q}$. We shall write

$$\mathcal{Q}(-, A) \xrightarrow{\tau_a} \mathcal{F}$$

for the Sup-natural transformation that $a \in \mathcal{F}(A)$ corresponds with by the Sup-enriched Yoneda Lemma. Often we shall loosely speak of “an element a of \mathcal{F} ”, and even write “ $a \in \mathcal{F}$ ”, where actually we should be more precise and stipulate that $a \in \mathcal{F}(A)$ for some $A \in \mathcal{Q}$.

⁷ Especially condition (iii) in Proposition 3.4 is reminiscent of the notion of *atom* defined by Bunge [11] and that of *small-projective object* defined by Kelly [26].

Because $\text{Mod}(\mathcal{Q})$ is a (large) quantaloid, we can compute extensions and liftings. It is straightforward to verify, with the aid of the Sup-enriched Yoneda Lemma, that for a \mathcal{Q} -module $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ and $x \in \mathcal{F}(X)$ and $y \in \mathcal{F}(Y)$, the right lifting of τ_x through τ_y ,

$$\begin{array}{ccc}
 \mathcal{Q}(-, X) & \xrightarrow{\tau_x} & \mathcal{F} \\
 \downarrow [\tau_y, \tau_x] & \leq & \uparrow \\
 \mathcal{Q}(-, Y) & \xrightarrow{\tau_y} & \mathcal{F}
 \end{array} \tag{4}$$

is precisely represented by the \mathcal{Q} -morphism $\mathbb{A}_{\mathcal{F}}(y, x)$ in (3).

Of course it makes sense to speak of adjoints in $\text{Mod}(\mathcal{Q})$; if a \mathcal{Q} -module morphism $\alpha: \mathcal{F} \Rightarrow \mathcal{G}$ has a right adjoint,⁸ we shall usually denote it as $\alpha^*: \mathcal{G} \Rightarrow \mathcal{F}$.

Proposition 4.1. *Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module, and $\mathbb{A}_{\mathcal{F}}$ the associated cocomplete \mathcal{Q} -category. We have the following:*

- i. $a \in \mathbb{A}_{\mathcal{F}}$ is totally compact if and only if $a \in \mathcal{F}$ and τ_a is a left adjoint.
- ii. $\mathbb{A}_{\mathcal{F}}$ is totally algebraic if and only if

$$\text{for each } x \in \mathcal{F}, \tau_x = \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \mathcal{F} \text{ and } \tau_a \text{ is a left adjoint} \right\}. \tag{5}$$

Proof. (i) A representable module $\mathcal{Q}(-, A): \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ corresponds under the biequivalence $\text{Mod}(\mathcal{Q}) \simeq \text{Cocont}(\mathcal{Q})$ with $\mathcal{P}(*_A)$, and a Sup-natural transformation $\tau_a: \mathcal{Q}(-, A) \Rightarrow \mathcal{F}$ corresponds to the “tensoring with a ” cocontinuous functor $T_a: \mathcal{P}(*_A) \rightarrow \mathbb{A}_{\mathcal{F}}: f \mapsto a \otimes f = F(f)(a)$. Thus the fourth statement in Proposition 3.4 proves the claim made here.

(ii) With suitable application of the Sup-enriched Yoneda Lemma, it is easily deduced from (i) that the formula in (5), which is stated in terms of the \mathcal{Q} -module \mathcal{F} , says precisely the same thing as

$$\text{for all } x \in \mathbb{A}_{\mathcal{F}}, x = \bigvee \{ a \otimes \mathbb{A}_{\mathcal{F}}(a, x) \mid a \in (\mathbb{A}_{\mathcal{F}})_c \},$$

which is stated in terms of the cocomplete \mathcal{Q} -category $\mathbb{A}_{\mathcal{F}}$. But the right-hand side in this latter formula is the explicit way of writing the value in x of the (pointwise) left Kan extension of $i_{\mathbb{A}_{\mathcal{F}}}: (\mathbb{A}_{\mathcal{F}})_c \rightarrow \mathbb{A}_{\mathcal{F}}$ along itself (see e.g. [45, p. 26] combined with [47, Corollary 2.15]): hence it says that this left Kan extension is the identity functor on $\mathbb{A}_{\mathcal{F}}$. \square

The preceding result prompts the following definition.

Definition 4.2. Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module. An $a \in \mathcal{F}$ is a *principal element* if τ_a is a left adjoint in $\text{Mod}(\mathcal{Q})$. Writing the set of principal elements of \mathcal{F} as \mathcal{F}_{pr} , \mathcal{F} is *principally generated* if

$$\text{for each } x \in \mathcal{F}, \tau_x = \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \mathcal{F}_{\text{pr}} \right\}. \tag{6}$$

We can add the following succinct characterisation.

Proposition 4.3. *Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module. The set of principal elements of \mathcal{F} is*

$$\mathcal{F}_{\text{pr}} = \left\{ \tau(1_X) \mid X \in \mathcal{Q} \text{ and } \tau: \mathcal{Q}(-, X) \Rightarrow \mathcal{F} \text{ is a left adjoint} \right\},$$

and \mathcal{F} is *principally generated* if and only if

$$\text{id}_{\mathcal{F}} = \bigvee \left\{ \tau \circ \tau^* \mid X \in \mathcal{Q} \text{ and } \tau: \mathcal{Q}(-, X) \Rightarrow \mathcal{F} \text{ is a left adjoint} \right\}. \tag{7}$$

Proof. The first part of the Proposition is an application of the Sup-enriched Yoneda Lemma.

As for the second part, for any $x \in \mathcal{F}$ we can compute, by general calculation rules for liftings in a quantaloid,⁹ that

$$\left(\bigvee \left\{ \tau_a \circ \tau_a^* \mid a \in \mathcal{F}_{\text{pr}} \right\} \right) \circ \tau_x = \bigvee \left\{ \tau_a \circ \tau_a^* \circ \tau_x \mid a \in \mathcal{F}_{\text{pr}} \right\} = \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \mathcal{F}_{\text{pr}} \right\}.$$

If we assume (7), i.e. if the bracketed expression on the far left equals $\text{id}_{\mathcal{F}}$, then clearly (6) follows. Conversely, assuming (6), i.e. assuming that the far right expression equals $\tau_x = \text{id}_{\mathcal{F}} \circ \tau_x$, and this for every $x \in \mathcal{F}$, implies – because the representables are generators in $\text{Mod}(\mathcal{Q})$ – that the bracketed expression on the far left must be equal to $\text{id}_{\mathcal{F}}$. \square

⁸ Whether α has a right adjoint in $\text{Mod}(\mathcal{Q})$ or not, each of its components certainly has a right adjoint in Ord , say $\alpha'_x: \mathcal{G}(X) \rightarrow \mathcal{F}(X)$, and these always form a lax natural Ord -transformation $\alpha': \mathcal{G} \Rightarrow \mathcal{F}$. If α has a right adjoint in $\text{Mod}(\mathcal{Q})$, then – for reasons of unicity of adjoints in Ord – it must be α' . In other words, α has a right adjoint in $\text{Mod}(\mathcal{Q})$ if and only if the lax natural Ord -transformation α' is strictly natural and its components preserve suprema.

⁹ In any quantaloid, if $f: A \rightarrow B$ is left adjoint to $f^*: B \rightarrow A$, then for any $g: C \rightarrow B$ we have $[f, g] = f^* \circ g$.

From this we can now deduce an elegant characterisation of principally generated \mathcal{Q} -modules, entirely in terms of universal constructions¹⁰ in the cocomplete quantaloid $\text{Mod}(\mathcal{Q})$ (we thank S. Lack for a stimulating discussion on this topic).

Corollary 4.4. *An $\mathcal{F} \in \text{Mod}(\mathcal{Q})$ is principally generated if and only if it is the adjoint retract of a direct sum of representable \mathcal{Q} -modules.*

Proof. This holds by application of Lemma 4.5 below to the family of all left adjoint \mathcal{Q} -module morphisms from representable \mathcal{Q} -modules to \mathcal{F} ; compare with (6). \square

Lemma 4.5. *Let F be an object in a quantaloid with direct sums and consider a family of left adjoints with codomain F :*

$$\left\{ F \begin{array}{c} \xleftarrow{f_i} \\ \perp \\ \xrightarrow{f_i^*} \end{array} R_i \right\}_{i \in I}.$$

Writing $f: F \rightarrow \bigoplus_i R_i$ and $f^*: \bigoplus_i R_i \rightarrow F$ for the unique factorisations of the f_i 's and the f_i^* 's we have that $f \circ f^* = \bigvee_i (f_i \circ f_i^*)$ and $f^* \circ f \geq 1_{\bigoplus_i R_i}$. Hence $1_F = \bigvee_i (f_i \circ f_i^*)$ if and only if f and f^* express F as an adjoint retract of $\bigoplus_i R_i$.

Proof. Recall that, in any quantaloid, an object $\bigoplus_i R_i$ is the direct sum of a family of objects $(R_i)_{i \in I}$ if and only if there are morphisms $s_j: R_j \rightarrow \bigoplus_i R_i$ and $p_j: \bigoplus_i R_i \rightarrow R_j$, for all $j \in I$, such that $\bigvee_i (s_i \circ p_i) = 1_{\bigoplus_i R_i}$ and $p_j \circ s_i = \delta_{i,j}$ (where $\delta_{i,j}: X_i \rightarrow X_j$ is zero when $i \neq j$ and the identity otherwise). It follows that also $p_j \circ f^* = f_j^*$ and $f \circ s_j = f_j$ hold. To prove the equality, we compute that

$$f \circ f^* = f \circ \left(\bigvee_i (s_i \circ p_i) \right) \circ f^* = \bigvee_i ((f \circ s_i) \circ (p_i \circ f^*)) = \bigvee_i (f_i \circ f_i^*).$$

The inequality is (because we are dealing with a 2-categorical (co)product) equivalent to requiring $p_j \circ 1_{\bigoplus_i R_i} \circ s_k \leq p_j \circ f^* \circ f \circ s_k$ for all $j, k \in I$. But the left-hand side equals $\delta_{j,k}$ and the right-hand side equals $f_j^* \circ f_k$, and since it is true that $1_{R_j} \leq f_j^* \circ f_j$, we are done. \square

We shall write $\text{Mod}_{\text{pg}}(\mathcal{Q})$ for the full subquantaloid of $\text{Mod}(\mathcal{Q})$ determined by the principally generated \mathcal{Q} -modules, and thus obtain:

Theorem 4.6. *For any small quantaloid \mathcal{Q} , the biequivalence $\text{Mod}(\mathcal{Q}) \simeq \text{Cocont}(\mathcal{Q})$ restricts to a biequivalence $\text{Mod}_{\text{pg}}(\mathcal{Q}) \simeq \text{Cocont}_{\text{ta}}(\mathcal{Q})$.*

Combined with the earlier observation in Corollary 3.3 we get:

Corollary 4.7. *For any small quantaloid \mathcal{Q} , the locally ordered categories $\text{Cat}_{\text{cc}}(\mathcal{Q})$ and $\text{Map}(\text{Mod}_{\text{pg}}(\mathcal{Q}))$ are biequivalent.*

Although slightly off-topic, we find it important to remark that Corollary 4.4 implies that the principally generated modules on \mathcal{Q} form a closed class of colimit weights in the sense of Albert and Kelly [2] and Kelly and Schmitt [27]; in fact, this class is the closure of the class of (weights for) direct sums and adjoint retracts. The general theory explained for \mathcal{V} -enriched categories in the cited references implies that, for any small quantaloid \mathcal{Q} , $\text{Mod}_{\text{pg}}(\mathcal{Q})$ is precisely the free cocompletion of \mathcal{Q} for direct sums and adjoint retracts, or equivalently, the free cocompletion of \mathcal{Q} for all colimits weighed by a principally generated module. Since we know that $\text{Dist}(\mathcal{Q}) \simeq \text{Cocont}_{\text{ta}}(\mathcal{Q}) \simeq \text{Mod}_{\text{pg}}(\mathcal{Q})$, this at once describes the universal property of the distributor quantaloid too. In [45] it is shown that $\text{Dist}(\mathcal{Q})$ is the universal “direct sum and split monad” completion of \mathcal{Q} ; but it is trivial that, in a quantaloid, splitting monads are the same thing as adjoint retracts. In the latter reference it is moreover shown that direct sums and splitting monads suffice to admit all lax limits and all lax colimits. Combining all this, it thus follows that the principally generated modules, as a class of weights, describe precisely the lax (co)completion of \mathcal{Q} .

5. Locally principally generated modules

It is well-known that idempotents in Sup split: for an idempotent $e: L \rightarrow L$ we let

$$L_e \begin{array}{c} \xrightarrow{s_e} \\ \xrightarrow{p_e} \end{array} L$$

denote the obvious splitting with $L_e := \{x \in L \mid e(x) = x\}$; of course any other splitting of e is isomorphic to this one.

¹⁰ In any quantaloid, products coincide with coproducts, and are often called *direct sums*. An object Y is an *adjoint retract* of an object X when there exists a left adjoint $p: X \rightarrow Y$ whose right adjoint $s: Y \rightarrow X$ is its splitting ($p \circ s = 1_Y$).

Given a quantaloid \mathcal{Q} , let \mathcal{Q}_{si} denote its split-idempotent completion; note that \mathcal{Q}_{si} is small because \mathcal{Q} is. Writing $j: \mathcal{Q} \rightarrow \mathcal{Q}_{\text{si}}$ for the obvious inclusion,

$$- \circ j: \text{Mod}(\mathcal{Q}_{\text{si}}) \rightarrow \text{Mod}(\mathcal{Q})$$

is an equivalence of quantaloids. We wish to describe the full subcategory of $\text{Mod}(\mathcal{Q})$ that is equivalent to $\text{Mod}_{\text{pg}}(\mathcal{Q}_{\text{si}})$ under the action of $- \circ j$. Thereto we shall first fix some notations.

We shall write

$$(-)_{\text{si}}: \text{Mod}(\mathcal{Q}) \rightarrow \text{Mod}(\mathcal{Q}_{\text{si}})$$

for the inverse equivalence to $- \circ j$: it sends a \mathcal{Q} -module \mathcal{F} to the \mathcal{Q}_{si} -module \mathcal{F}_{si} defined (up to isomorphism) by:

- for an object $e: A \rightarrow A$ of \mathcal{Q}_{si} , $\mathcal{F}_{\text{si}}(e) := \mathcal{F}(A)_{\mathcal{F}(e)}$,
- for a morphism $f: e \rightarrow e'$ in \mathcal{Q}_{si} , $\mathcal{F}_{\text{si}}(f) := \mathcal{F}(f)$.

If $e: A \rightarrow A$ is an idempotent in \mathcal{Q} , then the representable Sup-natural transformation $\mathcal{Q}(-, e): \mathcal{Q}(-, A) \Rightarrow \mathcal{Q}(-, A)$ is idempotent too. All idempotents in $\text{Mod}(\mathcal{Q})$ split, so this one does too: we shall write

$$\mathcal{F}_e \begin{array}{c} \xrightarrow{\sigma_e} \\ \xleftarrow{\pi_e} \end{array} \mathcal{Q}(-, A)$$

for the obvious splitting over $\mathcal{F}_e := \mathcal{Q}_{\text{si}}(-, e) \circ j = \mathcal{Q}_{\text{si}}(j-, e)$, and we refer to such a \mathcal{Q} -module \mathcal{F}_e as the *fixpoint \mathcal{Q} -module* for $e: A \rightarrow A$.

Proposition 5.1. *Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module, and \mathcal{F}_{si} the associated \mathcal{Q}_{si} -module.*

- i. *Given an idempotent $e: A \rightarrow A$ in \mathcal{Q} , $a \in \mathcal{F}_{\text{si}}(e)$ is a principal element of the \mathcal{Q}_{si} -module \mathcal{F}_{si} if and only if $a \in \mathcal{F}(A)$ satisfies $\mathcal{F}(e)(a) = a$ and the \mathcal{Q} -module morphism $\tau_a \circ \sigma_e: \mathcal{F}_e \Rightarrow \mathcal{F}$ is a left adjoint.*
- ii. *\mathcal{F}_{si} is a principally generated \mathcal{Q}_{si} -module if and only if*

$$\text{for each } x \in \mathcal{F}, \tau_x = \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid \begin{array}{l} \text{there exists an idempotent } e \\ \text{in } \mathcal{Q} \text{ such that } \mathcal{F}(e)(a) = a \text{ and} \\ \tau_a \circ \sigma_e: \mathcal{F}_e \Rightarrow \mathcal{F} \text{ is a left adjoint} \end{array} \right\}. \tag{8}$$

Proof. (i) Because we put $\mathcal{F}_{\text{si}}(e) = \mathcal{F}(A)_{\mathcal{F}(e)}$, it is trivial that $a \in \mathcal{F}_{\text{si}}(e)$ if and only if $a \in \mathcal{F}(A)$ satisfying $\mathcal{F}(e)(a) = a$. By the Sup-enriched Yoneda Lemma, we know that $a \in \mathcal{F}(A)$ corresponds uniquely to a \mathcal{Q} -module morphism, which we called $\tau_a: \mathcal{Q}(-, A) \Rightarrow \mathcal{F}$; but also $a \in \mathcal{F}_{\text{si}}(e)$ corresponds uniquely to a \mathcal{Q}_{si} -module morphism: let us call it $\rho_a: \mathcal{Q}_{\text{si}}(-, e) \Rightarrow \mathcal{F}_{\text{si}}$. By Definition 4.2, $a \in \mathcal{F}_{\text{si}}(e)$ is a principal element if ρ_a is a left adjoint (in $\text{Mod}(\mathcal{Q}_{\text{si}})$, that is). Because of the equivalence of $\text{Mod}(\mathcal{Q})$ and $\text{Mod}(\mathcal{Q}_{\text{si}})$, expressed by $(-)_{\text{si}}$ and $- \circ j$, and because $\mathcal{F}_e = \mathcal{Q}_{\text{si}}(-, e) \circ j$, ρ_a corresponds uniquely to a left adjoint \mathcal{Q} -module morphism $\zeta_a: \mathcal{F}_e \Rightarrow \mathcal{F}$. But because \mathcal{F}_e is the \mathcal{Q} -module over which the idempotent $\mathcal{Q}(-, e): \mathcal{Q}(-, A) \Rightarrow \mathcal{Q}(-, A)$ splits (and recall that we write the splitting with inclusion σ_e and projection π_e) we necessarily have that $\zeta_a = \tau_a \circ \sigma_e$ (and $\tau_a = \zeta_a \circ \pi_e$). Thus it is indeed sufficient and necessary that $\tau_a \circ \sigma_e$ be a left adjoint \mathcal{Q} -module morphism.

(ii) Again following Definition 4.2, and with the notations that we introduced in the first part of this proof, \mathcal{F}_{si} is a principally generated \mathcal{Q}_{si} -module if

$$\text{for each } x \in \mathcal{F}_{\text{si}}, \rho_x = \bigvee \left\{ \rho_a \circ [\rho_a, \rho_x] \mid a \in (\mathcal{F}_{\text{si}})_{\text{pr}} \right\}.$$

This supremum of \mathcal{Q}_{si} -module morphisms can be written in terms of \mathcal{Q} -module morphisms, for similar reasons as in the first part of the proof:

$$\text{for each } x \in \mathcal{F}_{\text{si}}, \zeta_x = \bigvee \left\{ \zeta_a \circ [\zeta_a, \zeta_x] \mid a \in (\mathcal{F}_{\text{si}})_{\text{pr}} \right\}.$$

Using the notation

$$\alpha := \left\{ (a, e) \mid \begin{array}{l} e: A \rightarrow A \text{ an idempotent in } \mathcal{Q} \text{ and } a \in \mathcal{F} \text{ such} \\ \text{that } \mathcal{F}(e)(a) = a \text{ and } \tau_a \circ \sigma_e \text{ is left adjoint} \end{array} \right\}$$

we can spell this out as:

$$\begin{array}{l} \text{for each idempotent } d: X \rightarrow X \text{ in } \mathcal{Q} \\ \text{and each } x \in \mathcal{F} \text{ such that } \mathcal{F}(d)(x) = x, \end{array} \tag{9}$$

$$\tau_x \circ \sigma_d = \bigvee \left\{ (\tau_a \circ \sigma_e) \circ [(\tau_a \circ \sigma_e), (\tau_x \circ \sigma_d)] \mid (a, e) \in \alpha \right\}.$$

Assume now that (9) holds. Taking in particular $d = 1_A$ (in which case $\sigma_d = \sigma_{1_A}$ is the identity transformation) it implies that for all $x \in \mathcal{F}$

$$\begin{aligned} \tau_x &= \bigvee \left\{ (\tau_a \circ \sigma_e) \circ [(\tau_a \circ \sigma_e), \tau_x] \mid (a, e) \in \alpha \right\} \\ &= \bigvee \left\{ (\tau_a \circ \sigma_e) \circ \pi_e \circ [\tau_a, \tau_x] \mid (a, e) \in \alpha \right\} \\ &= \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid (a, e) \in \alpha \right\} \\ &= \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \beta \right\} \end{aligned}$$

by a suitable application of Lemma 5.2 stated below (and the proof of which is straightforward) to pass from the first to the second line, by the fact that $\tau_a \circ \sigma_e \circ \pi_e = \tau_a \circ \mathcal{Q}(-, e) = \tau_a$ (which is the equivalent of $\mathcal{F}(e)(a) = a$) to pass from the second to the third line, and where we introduced another auxiliary notation in the last line:

$$\beta := \left\{ a \mid \text{there exists an idempotent } e: A \longrightarrow A \text{ in } \mathcal{Q} \text{ such that } (a, e) \in \alpha \right\}.$$

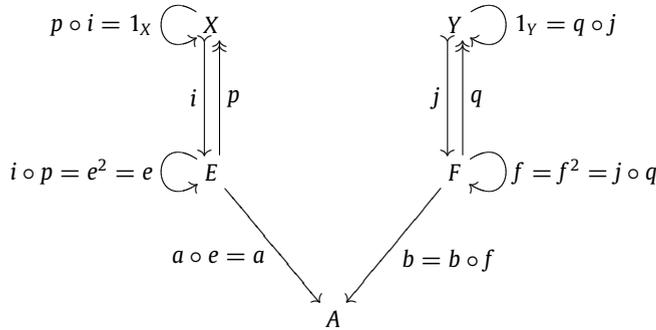
A priori an $a \in \mathcal{F}$ may be locally principal at several different idempotents, in which case α contains several “copies” of a (one for each idempotent it is locally principal at) but β contains a only once. But because an expression like $\tau_a \circ [\tau_a, \tau_x]$ does not contain any reference to the idempotents at which a is locally principal, we can make the last step in the above series of equalities. Hence we derived the condition expressed in (8).

Conversely, assume the validity of (8). Then, for every idempotent $d: X \longrightarrow X$ in \mathcal{Q} and $x \in \mathcal{F}$ such that $\mathcal{F}(d)(x) = x$, we can compute in a similar fashion that

$$\begin{aligned} \tau_x \circ \sigma_d &= \left(\bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \beta \right\} \right) \circ \sigma_d \\ &= \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \circ \sigma_d \mid a \in \beta \right\} \\ &= \bigvee \left\{ \tau_a \circ (\sigma_e \circ \pi_e) \circ [\tau_a, \tau_x] \circ \sigma_d \mid (a, e) \in \alpha \right\} \\ &= \bigvee \left\{ (\tau_a \circ \sigma_e) \circ [(\tau_a \circ \sigma_e), (\tau_x \circ \sigma_d)] \mid (a, e) \in \alpha \right\}. \end{aligned}$$

Thus we obtain the condition expressed in (9). \square

Lemma 5.2. *In any quantaloid, for a diagram like*



we have $q \circ [b, a] \circ i = [b \circ j, a \circ i]$.

The result in Proposition 5.1 suggests a new definition, to be compared with Definition 4.2:

Definition 5.3. Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \longrightarrow \text{Sup}$ be a \mathcal{Q} -module. An $a \in \mathcal{F}$ is a *locally principal element* (at an idempotent $e: A \longrightarrow A$ in \mathcal{Q}) if (there is an idempotent $e: A \longrightarrow A$ in \mathcal{Q} such that) $\mathcal{F}(e)(a) = a$ and $\tau_a \circ \sigma_e$ is a left adjoint in $\text{Mod}(\mathcal{Q})$. Writing \mathcal{F}_{pr} for the set of locally principal elements of \mathcal{F} , we say that \mathcal{F} is *locally principally generated* if

$$\text{for each } x \in \mathcal{F}, \tau_x = \bigvee \left\{ \tau_a \circ [\tau_a, \tau_x] \mid a \in \mathcal{F}_{\text{pr}} \right\}.$$

Thus, a locally principal element of $\mathcal{F}: \mathcal{Q}^{\text{op}} \longrightarrow \text{Sup}$ at an identity of \mathcal{Q} is the same thing as, simply, a principal element of \mathcal{F} : idempotents in \mathcal{Q} are viewed as *localities* (or “*opens*”). It follows from the comparison of Definitions 4.2 and 5.3 that a principally generated \mathcal{Q} -module is necessarily also locally principally generated; but the converse is not true in general, as the next example shows.

Example 5.4. A fixpoint \mathcal{Q} -module $\mathcal{F}_e: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ for an idempotent $e: A \rightarrow A$ in \mathcal{Q} trivially has $e \in \mathcal{F}_e(A)$ as locally principal element (at e , as a matter of fact); it follows straightforwardly that \mathcal{F}_e is locally principally generated. However, \mathcal{F}_e need not have any principal element and thus need not be principally generated; for a concrete example, let \mathcal{Q} be the one-object suspension of the three-element chain $\{0 \leq e \leq 1\}$.

We have the following useful characterisation of locally principally generated \mathcal{Q} -modules.

Proposition 5.5. *Let $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ be a \mathcal{Q} -module. The set of locally principal elements is*

$$\mathcal{F}_{\text{lpr}} = \left\{ \zeta(e) \mid \begin{array}{l} e \text{ is an idempotent in } \mathcal{Q} \text{ and} \\ \zeta: \mathcal{F}_e \Rightarrow \mathcal{F} \text{ is a left adjoint} \end{array} \right\},$$

and \mathcal{F} is locally principally generated if and only if

$$\text{id}_{\mathcal{F}} = \bigvee \left\{ \zeta \circ \zeta^* \mid \begin{array}{l} e \text{ is an idempotent in } \mathcal{Q} \text{ and} \\ \zeta: \mathcal{F}_e \Rightarrow \mathcal{F} \text{ is a left adjoint} \end{array} \right\}. \tag{10}$$

Proof. This follows straightforwardly from Proposition 4.1 and (the proof of) Proposition 5.1. \square

Much like Corollary 4.4 does for principally generated \mathcal{Q} -modules, we can now give a characterisation in terms of universal constructions in $\text{Mod}(\mathcal{Q})$ of locally principally generated \mathcal{Q} -modules.

Corollary 5.6. *An $\mathcal{F} \in \text{Mod}(\mathcal{Q})$ is locally principally generated if and only if it is an adjoint retract of a direct sum of fixpoint \mathcal{Q} -modules.*

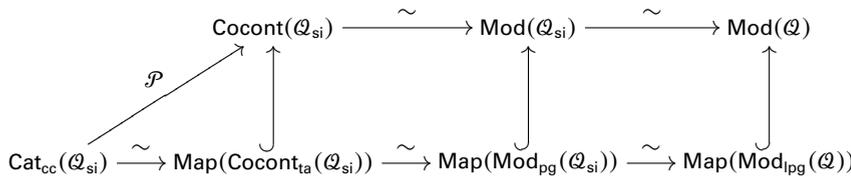
Proof. Apply Lemma 4.5 to the family of left adjoint \mathcal{Q} -module morphisms from all fixpoint \mathcal{Q} -modules to \mathcal{F} ; compare with (10). \square

Remark 5.7. A \mathcal{Q} -module $\mathcal{F}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ is *projective* when the representable $\text{Mod}(\mathcal{Q})(\mathcal{F}, -): \text{Mod}(\mathcal{Q}) \rightarrow \text{Sup}$ preserves epimorphisms. It is known (see e.g. [48, Proposition 9.5]) that this is equivalent to \mathcal{F} being *small-projective* (in the sense of Kelly [26]: $\text{Mod}(\mathcal{Q})(\mathcal{F}, -)$ preserves all small weighted colimits), and equivalent to \mathcal{F} being a retract of a direct sum of representable \mathcal{Q} -modules. It thus follows from Corollaries 4.4 and 5.6 that any (locally) principally generated \mathcal{Q} -module is necessarily projective in $\text{Mod}(\mathcal{Q})$. But the difference between “projective” and “(locally) principally generated” lies precisely in the strictly stronger requirement that, for the latter to hold, \mathcal{F} needs to be an *adjoint* retract of a direct sum of representable \mathcal{Q} -modules (fixpoint \mathcal{Q} -modules).

Let $\text{Mod}_{\text{lpg}}(\mathcal{Q})$ denote the full subquantaloid of $\text{Mod}(\mathcal{Q})$ whose objects are the locally-principally generated \mathcal{Q} -modules.

Theorem 5.8. *For a small quantaloid \mathcal{Q} , the biequivalence $\text{Mod}(\mathcal{Q}_{\text{si}}) \simeq \text{Mod}(\mathcal{Q})$ restricts to a biequivalence $\text{Mod}_{\text{pg}}(\mathcal{Q}_{\text{si}}) \simeq \text{Mod}_{\text{lpg}}(\mathcal{Q})$.*

The biequivalences in Corollary 3.3 and Theorem 4.6 apply to any small quantaloid, hence in particular to the split-idempotent completion \mathcal{Q}_{si} of a small quantaloid \mathcal{Q} . The combination with the biequivalence in Theorem 5.8 then shows that the following diagram commutes:



Up to the identification of $\text{Ord}(\mathcal{Q})$, the locally ordered category of ordered sheaves on a small quantaloid \mathcal{Q} , with the locally ordered category $\text{Cat}_{\text{cc}}(\mathcal{Q}_{\text{si}})$, the composition of the 2-functors

$$\text{Ord}(\mathcal{Q}) = \text{Cat}_{\text{cc}}(\mathcal{Q}_{\text{si}}) \xrightarrow{\mathcal{P}} \text{Cocont}(\mathcal{Q}_{\text{si}}) \xrightarrow{\sim} \text{Mod}(\mathcal{Q}_{\text{si}}) \xrightarrow{\sim} \text{Mod}(\mathcal{Q})$$

is precisely the left biadjoint 2-functor $F: \text{Ord}(\mathcal{Q}) \rightarrow \text{Mod}(\mathcal{Q})$ in (1) (see also [49, 3.3]). Hence we proved that this 2-functor factors as a (composition of) biequivalence(s) followed by an inclusion.

Theorem 5.9. *The locally ordered category $\text{Ord}(\mathcal{Q})$ of ordered sheaves on a small quantaloid \mathcal{Q} is biequivalent to $\text{Map}(\text{Mod}_{\text{lpg}}(\mathcal{Q}))$, the locally ordered category of locally principally generated \mathcal{Q} -modules and left adjoint \mathcal{Q} -module morphisms between them.*

6. Examples

In the three examples that follow we shall consider a one-object quantaloid \mathcal{Q} . In this situation we prefer to view \mathcal{Q} as monoid $Q := (\mathcal{Q}(*, *), \circ, 1)$ in Sup , and a \mathcal{Q} -module $\mathcal{M}: \mathcal{Q}^{\text{op}} \rightarrow \text{Sup}$ as the object $M := \mathcal{M}(*)$ of Sup together with an action¹¹

$$M \times Q \longrightarrow M: (x, q) \mapsto x \circ q := \mathcal{M}(q)(x).$$

With slight abuse of notation we shall write $[x, y]$ for the element of Q that represents the lifting $[\tau_x, \tau_y]$ in $\text{Mod}(Q)$; that is to say, $[x, y] = \bigvee \{q \in Q \mid x \circ q \leq y\}$ (compare with (3) and (4)). An element $a \in M$ is principal if and only if $[a, -]: M \rightarrow Q$ is a Sup -morphism that preserves the action of Q .

Now suppose moreover that the underlying complete lattice of Q is totally algebraic (in the classical sense, as recalled in the Introduction). Principality of an element $a \in M$ is then equivalent to the following two requirements:

- i. $[a, x \circ c] \leq [a, x] \circ c$, for all $x \in M$ and all totally compact $c \in Q$,
- ii. $[a, \bigvee_{i \in I} x_i] \leq \bigvee_{i \in I} [a, x_i]$, for all $(x_i)_{i \in I} \in M$.

This is equivalent to asking, for all $x, (x_i)_{i \in I} \in M$ and totally compact $c, d \in Q$,

- i. if $a \circ d \leq x \circ c$ then there exists a totally compact $k \in Q$ such that $a \circ k \leq x$ and $d \leq k \circ c$,
- ii. if $a \circ d \leq \bigvee_{i \in I} x_i$ then there exists an $i \in I$ such that $a \circ d \leq x_i$.

In particular, for a principal element $a \in M$ and a totally compact element $d \in Q$, the element $a \circ d \in M$ is totally compact in (the underlying complete lattice of) M . Let \mathcal{Q}_c , resp. M_c , denote the partially ordered sets of totally compact elements of the underlying complete lattices of \mathcal{Q} , resp. M . For an element x of a \mathcal{Q} -module M we have

$$\begin{aligned} x &= \bigvee \left\{ a \circ [a, x] \mid a \in M_{\text{pr}} \right\} \\ \iff x &= \bigvee \left\{ a \circ q \mid a \in M_{\text{pr}}, q \in \mathcal{Q}, a \circ q \leq x \right\} \\ \iff x &= \bigvee \left\{ a \circ d \mid a \in M_{\text{pr}}, d \in \mathcal{Q}_c, a \circ d \leq x \right\} \\ \implies x &= \bigvee \left\{ b \mid b \in M_c, b \leq x \right\}. \end{aligned}$$

Hence, every principally generated module on a totally algebraic quantale has a totally algebraic underlying sup-lattice.

We shall now spell out three quite different applications.

Example 6.1 (Complete Lattices). The quantaloid Sup can be identified with $\text{Mod}(\mathbf{2})$, where $\mathbf{2}$ stands for the two-element chain with its obvious quantale structure, $(\{0 \leq 1\}, \wedge, 1)$. For a complete lattice/ $\mathbf{2}$ -module S , the conditions above say that:

- i. $a \neq 0_S$,
- ii. a is totally compact.

Hence, the principal elements of S are the non-zero totally compact elements, while 0_S is not principal but (the unique element that is) *locally* principal at 0. Thus S is a totally algebraic complete lattice if and only if it is a principally generated $\mathbf{2}$ -module, if and only if it is a locally principally generated $\mathbf{2}$ -module.

Example 6.2 (Automata). Let $(N, \cdot, 1)$ be a monoid (in Set), then the powerset of N can be equipped with the pointwise multiplication

$$A \cdot B := \{a \cdot b \mid a \in A, b \in B\} \text{ for } A, B \subseteq N$$

and thus $(2^N, \cdot, \{1\})$ is a quantale (it is the free quantale on $(N, \cdot, 1)$, see [39]). The totally compact elements of 2^N are the singletons and the empty subset. An element $a \in M$ of a 2^N -module M is principal if and only if, for all $x, (x_i)_{i \in I} \in M$ and $n, p \in N$,

- ia. $a \circ n \neq 0_M$,
- ib. if $a \circ n \leq x \circ p$ then there exists a $k \in N$ such that $a \circ k \leq x$ and $n = k \circ p$,
- ii. if $a \circ n \leq \bigvee_{i \in I} x_i$ then there exists an $i \in I$ such that $a \circ n \leq x_i$.

(We wrote $a \circ n$ instead of $a \circ \{n\}$ for notational convenience.) Note that (ia) + (ii) are equivalent to $a \circ n$ being principal in the underlying complete lattice of the module M .

This example can be interesting for the theory of automata (or labelled transition systems): by Corollary 4.7, the principally generated 2^N -modules can be identified with Cauchy complete 2^N -enriched categories. It is well-known that categories enriched in a free quantale are precisely non-deterministic automata with N as set of labels [5,39].

¹¹ As mentioned earlier, \mathcal{Q} -modules are essentially “the same thing” as cocomplete \mathcal{Q} -categories; actions of \mathcal{Q} correspond with tensors in \mathcal{Q} -categories. In previous work [47] we therefore denoted actions with “ \otimes ”, the usual symbol for tensors in enriched categories. However to avoid any confusion with pure tensors in a tensor product of sup-lattices, we here adopt a “ \circ ” as notation.

Example 6.3 ($\text{Rel}(S, S)$ -modules). Let Rel denote the quantaloid of sets and relations. For any set S , $Q_S := \text{Rel}(S, S)$ is a totally algebraic quantale: its totally compact elements are the empty set and the singletons $(s, t) \in S \times S$ (we omit the curly brackets for clarity). A Q_S -module M is “the same thing” as the skeletal (i.e. having no non-identical isomorphic objects) cocomplete Q_S -category \mathbb{A}_M (as explained in the beginning of Section 4). On the other hand, to give a Q_S -category \mathbb{A} , with object set \mathbb{A}_0 , is equivalent to giving an order relation \preceq on the set $\mathbb{A}_0 \times S$; the correspondence is given by:

$$(a, s) \preceq (b, t) \text{ if and only if } (s, t) \in \mathbb{A}(a, b).$$

Writing the equivalence relation on $\mathbb{A}_0 \times S$ induced by the order \preceq as \approx , it can be verified that \mathbb{A} is skeletal and cocomplete if and only if \preceq satisfies:

- i. for all $s \in S$, $(a, s) \approx (b, s)$ implies $a = b$,
- ii. for all $(a_i)_{i \in I} \in \mathbb{A}_0$ there exists a $\bigvee_{i \in I} a_i \in \mathbb{A}_0$ such that $(\bigvee_{i \in I} a_i, s) \preceq (b, t)$ if and only if $(a_i, s) \preceq (b, t)$ for all $i \in I$,
- iii. for all $a \in \mathbb{A}_0$ and $s, t \in S$ there exists $b \in \mathbb{A}_0$ such that $(b, t) \approx (a, s)$ and moreover, whenever $u \neq t$, (b, u) is a bottom element for the order $(\mathbb{A}_0 \times S, \preceq)$.

Conditions (ii) and (iii) are easily deduced from the equivalence of cocomplete \mathcal{Q} -categories to conically cocomplete and tensored \mathcal{Q} -categories [26,47]: $\bigvee_{i \in I} a_i$ in (ii) is the conical colimit of $(a_i)_{i \in I}$ in \mathbb{A} (and thus its order theoretical join in (\mathbb{A}_0, \leq)), while b in (iii) is the tensor product $a \circ (s, t)$. Moreover, these conditions imply

ii'. For all $(b_i)_{i \in I} \in \mathbb{A}_0$ there exists a $\bigwedge_{i \in I} b_i \in \mathbb{A}_0$ such that $(a, s) \preceq (\bigwedge_{i \in I} b_i, t)$ if and only if $(a, s) \preceq (b_i, t)$ for all $i \in I$.

Hence, a Q_S -module M can be given in terms of an order \preceq on $M \times S$, satisfying conditions (i–iii). An element $a \in M$ is then principal if and only if, for all $s, t \in S$,

- i. $(a, s) \not\approx \perp$, where \perp denotes a bottom element for $(M \times S, \preceq)$,
- ii. if $(a, s) \preceq (\bigvee_{i \in I} x_i, t)$ then there exists an $i \in I$ such that $(a, s) \preceq (x_i, t)$.

The order \preceq on $M \times S$, corresponding to a Q_S -module M has the characteristics of an entailment (especially (ii) and (ii') above): a couple $(a, s) \in M \times S$ can be thought of as an occurrence of an event $a \in M$ at a place $s \in S$. Quantales of the form $\text{Rel}(S, S)$ arise in the context of relational representations of spatial quantales \mathcal{Q} , i.e. quantale homomorphisms $\rho: \mathcal{Q} \rightarrow \text{End}(2^S) = \text{Rel}(S, S)$ [34].

In the next section we shall dwell on the case where \mathcal{Q} is the one-object suspension of a locale X . The formulation of ordered sheaves on X by means of locally principally generated X -modules allows for a neat translation to “skew local homeomorphisms” into X .

7. Skew local homeomorphisms

7.1. Induced modules on a locale X

In what follows, Loc denotes the (2-)category of locales. We follow the notational convention of Johnstone [22, p. 40] for morphisms in Loc : thus a locale morphism $f: Y \rightarrow X$ is an adjoint pair

$$\begin{array}{ccc}
 & f^* & \\
 Y & \longleftarrow & X \\
 & \perp & \\
 & f_* & \\
 & \longrightarrow &
 \end{array}$$

in the 2-category of partially ordered sets such that the left adjoint preserves finite infima. We do *not* follow the convention of Johnstone [22] and MacLane and Moerdijk [29] when it comes to defining an order on the hom-sets in Loc : for $f, g: Y \rightarrow X$ in Loc we define that $f \leq g$ if $f_* \leq g_*$. That is to say: we have that $\text{Loc} \cong \text{Frm}^{\text{coop}}$ as 2-categories (whereas the cited references have $\text{Loc} \cong \text{Frm}^{\text{op}}$).¹²

Considering a locale X as a monoid (X, \wedge, \top_X) in Sup it makes sense to write $\text{Mod}(X)$ for the quantaloid of modules on the locale. Instead of writing these modules as contravariant Sup -enriched presheaves on the one-object suspension of the locale, we rather consider them as objects of Sup on which (X, \wedge, \top_X) acts on the right: we write (M, \circ) for a Sup -object M together with the action $(m, x) \mapsto m \circ x$. In the same vein, an X -module morphism $\alpha: (M, \circ) \rightarrow (N, \circ)$ is a Sup -morphism $\alpha: M \rightarrow N$ which is equivariant for the respective actions.

¹² The reason for our preference is in the first place notational convenience, especially in the 2-functors considered further on. However, there is maybe a deeper reason why this different ordering of locale morphisms is natural here: in the cited references locale morphisms are studied as inducing geometric morphisms between toposes of sheaves; the ordering of locale morphisms is chosen to correspond with the usual notion of natural transformation between geometric morphisms. We however shall study locale morphisms (or rather, morphisms in the slice category Loc/X) as inducing order-preserving morphisms between the (ordered) sheaves themselves; and the ordering of the locale morphisms is chosen to correspond with the natural ordering of those morphisms between sheaves.

Given an $f: Y \rightarrow X$ in Loc , it is easily seen that putting

$$y \circ_f x := y \wedge f^*(x) \tag{11}$$

for $y \in Y$ and $x \in X$ results in an action of the monoid (X, \wedge, \top_X) on Y in Sup . In other words, from $f: Y \rightarrow X$ in Loc we get an object $(Y, \circ_f) \in \text{Mod}(X)$. Moreover, suppose that

$$\begin{array}{ccc}
 Y & \xrightarrow{h} & Z \\
 f \searrow & & \swarrow g \\
 & X &
 \end{array}
 \tag{12}$$

is a commutative triangle in Loc , then $h^*: Z \rightarrow Y$ is a morphism in Sup satisfying $h^*(z \circ_f x) = h^*(z) \circ_g x$, for all $x \in X, z \in Z$. That is to say, $h^*: (Z, \circ_g) \rightarrow (Y, \circ_f)$ is a morphism in $\text{Mod}(X)$. All this adds up to an injective and faithful (but not full) 2-functor

$$(\text{Loc}/X)^{\text{coop}} \rightarrow \text{Mod}(X). \tag{13}$$

We are now interested in left adjoint X -module morphisms:

Definition 7.1. A morphism $h: f \rightarrow g$ in Loc/X as in (12) is *skew open* if the corresponding order-preserving function $h^*: Z \rightarrow Y$ has a left adjoint $h_!: Y \rightarrow Z$ satisfying the “balanced Frobenius identity¹³”: for all $y \in Y$ and $x \in X$,

$$h_!(y \wedge f^*(x)) = h_!(y) \wedge g^*(x). \tag{14}$$

Example 7.2. For an $h: Y \rightarrow Z$ in Loc the following are equivalent:

- i. $h: Y \rightarrow Z$ is open in Loc (according to the “usual” definition of openness as in e.g. [29, p. 500]),
- ii. for any $f: Y \rightarrow X$ and $g: Z \rightarrow X$ in Loc such that $g \circ h = f$, the morphism $h: f \rightarrow g$ in Loc/X is skew open,
- iii. considering $h: Y \rightarrow Z$ and $1_Z: Z \rightarrow Z$ as objects in Loc/Z , the (unique) morphism $h: h \rightarrow 1_Z$ in Loc/Z is skew open.

Clearly the identity morphisms in Loc/X are skew open, and the composition of skew open morphisms is again skew open; it thus makes sense to speak of the sub-2-category $(\text{Loc}/X)^\circ$ of Loc/X with the same objects but only its skew open morphisms.

Upon inspection it is easily seen that, for any two locale morphisms $f: Y \rightarrow X$ and $g: Z \rightarrow X$, there is an isomorphism of ordered sets

$$(\text{Loc}/X)^\circ(f, g) \cong \text{Map}(\text{Mod}(X))((Y, \circ_f), (Z, \circ_g))$$

given by sending a skew open morphism h to the X -module morphism $h_!$ with right adjoint h^* . Sending skew open morphisms in Loc/X to their utmost left adjoints (i.e. $h \mapsto h_!$) thus gives rise to an injective and fully faithful 2-functor

$$(\text{Loc}/X)^\circ \rightarrow \text{Map}(\text{Mod}(X)). \tag{15}$$

In the codomain category of this functor we are now interested in the locally principally generated objects. In the next subsection we develop that notion further.

7.2. Locally principally generated X -modules

Let X be a locale. As is customary in locale theory, see e.g. [29, p. 486], for any $u \in X$ we generically write $i: \downarrow u \rightarrow X$ for the corresponding open sublocale of X , i.e. it is the open Loc -morphism defined by $i_*(v) := (u \Rightarrow v), i^*(x) := (x \wedge u)$ and $i_!(v) := v$. As noted before, it is therefore also skew open in Loc/X as (unique) morphism from $i: \downarrow u \rightarrow X$ to the terminal object $1_X: X \rightarrow X$,

$$\begin{array}{ccc}
 \downarrow u & \xrightarrow{i} & X \\
 i \searrow & & \swarrow 1_X \\
 & X &
 \end{array}
 \tag{16}$$

¹³ Putting $Z = X$ and $g = 1_X$ this reduces to what is called the “Frobenius identity” in [29, p. 500]; we call this generalisation “balanced” because we get the (“unbalanced”) Frobenius identity by plugging in a terminal object.

All elements of the Sup-monoid (X, \wedge, \top_X) are idempotent, thus each $u \in X$ gives rise to an idempotent representable X -module morphism on the (only) representable X -module (X, \wedge) . The image under the functor in (15) of the $(\text{Loc}/X)^\circ$ -morphism in (16) is precisely the splitting of this idempotent:

$$(\downarrow u, \wedge) \begin{array}{c} \xrightarrow{i_!} \\ \xleftarrow{i^*} \end{array} (X, \wedge) \begin{array}{c} \xrightarrow{- \wedge u} \\ \xrightarrow{- \wedge u} \end{array} (X, \wedge). \tag{17}$$

It is noteworthy that this is actually an *adjoint splitting*, since $i_! \dashv i^*$ in $\text{Mod}(X)$, and that $- \wedge u$ – because 1_X is terminal in Loc/X and the functor in (14) is fully faithful – this is the *only* adjunction in $\text{Mod}(X)$ between $(\downarrow u, \wedge)$ and (X, \wedge) .

Applying Definition 5.3 to an X -module (M, \circ) we get the following. An element $p \in M$ is *locally principal at* $u \in X$ if and only if $p \circ u = p$ and the composite X -module morphism

$$(\downarrow u, \wedge) \xrightarrow{i_!} (X, \wedge) \xrightarrow{p \circ -} (M, \circ)$$

admits a right adjoint in $\text{Mod}(X)$. Let $(M, \circ)_{\text{lpr}}$ denote the set of elements of M which are locally principal at some $u \in X$. Then (M, \circ) is *locally principally generated* if and only if, for each $m \in M$,

$$m = \bigvee \{p \circ [p, m] \mid p \in (M, \circ)_{\text{lpr}}\},$$

where $[p, m] := \bigvee \{u \in X \mid p \circ u \leq m\}$.

We shall recast the latter definition in a more pleasant form.

Proposition 7.3. *Let (M, \circ) be an X -module.*

- i. *If $p \in M$ is locally principal at $u \in X$, then for any $m \in M$, $p \circ [p, m]$ is locally principal at $u \wedge [p, m]$.*
- ii. *For any $m \in M$, $\{p \circ [p, m] \mid p \in (M, \circ)_{\text{lpr}}\} = \downarrow m \cap (M, \circ)_{\text{lpr}}$.*
- iii. *(M, \circ) is locally principally generated if and only if*

$$\text{for all } m \in M, m = \bigvee (\downarrow m \cap (M, \circ)_{\text{lpr}}). \tag{18}$$

Proof. (i) For shorthand we introduce $q := p \circ [p, m]$ and $v := u \wedge [p, m]$. Then it is easily verified that

$$\begin{aligned} q \circ v &= (p \circ [p, m]) \circ (u \wedge [p, m]) \\ &= p \circ ([p, m] \wedge u \wedge [p, m]) \\ &= (p \circ u) \circ [p, m] \\ &= p \circ [p, m] \\ &= q. \end{aligned}$$

Moreover the diagram

$$\begin{array}{ccc} (\downarrow v, \wedge) & \xrightarrow{i_!} & (X, \wedge) \xrightarrow{q \circ -} (M, \circ) \\ \downarrow & & \parallel \\ (\downarrow u, \wedge) & \xrightarrow{i_!} & (X, \wedge) \xrightarrow{p \circ -} (M, \circ) \end{array}$$

in $\text{Mod}(X)$, where the left downward arrow is the obvious inclusion of $\downarrow v$ into $\downarrow u$, commutes: for $w \leq v$ we can compute that

$$q \circ w = (p \circ [p, m]) \circ w = p \circ ([p, m] \wedge w) = p \circ w.$$

But $(\downarrow v, \wedge) \longrightarrow (\downarrow u, \wedge): w \mapsto w$ is a left adjoint in $\text{Mod}(X)$, hence the top composite morphism is a left adjoint whenever the bottom composite morphism is.

(ii) Because $p \circ - \dashv [p, -]$ as order-preserving maps between X and M , it is trivial that $p \circ [p, m] \leq m$ and $(p \leq m \Rightarrow p \circ [p, m] = p)$, for any $p, m \in M$. We have just shown that if p is locally principal then so is $p \circ [p, m]$. Hence the equality of these sets.

(iii) Is now immediate. \square

We can also translate to an X -module (M, \circ) the condition in Proposition 5.5 that expresses that it is locally principally generated if and only if

$$\text{id}_{(M, \circ)} = \bigvee \{\zeta \circ \zeta^* \mid u \in X, \zeta \in \text{Map}(\text{Mod}(X))((\downarrow u, \wedge), (X, \wedge))\}, \tag{19}$$

where we write ζ^* for the right adjoint to ζ . This fact allows us to prove the following remarkable property:

Proposition 7.4. *Let (M, \circ) be a locally principally generated X -module.*

- i. *For $m, n \in M$, $m = n$ if and only if for every $u \in X$ and every left adjoint X -module morphism $\zeta: (\downarrow u, \wedge) \longrightarrow (M, \circ)$ we have $\zeta^*(m) = \zeta^*(n)$.*
- ii. *For every $m, n \in M$ and $x \in X$, $(m \wedge n) \circ x = m \wedge (n \circ x)$.*
- iii. *M is a locale and $f^*: X \longrightarrow M: x \mapsto \top_M \circ x$ is the inverse image of a locale morphism $f: M \longrightarrow X$ for which $(M, \circ_f) = (M, \circ)$.*

Proof. (i) One direction is trivial; for the other one expands $m = \text{id}_{(M, \circ)}(m)$ and $n = \text{id}_{(M, \circ)}(n)$ by means of the formula given above.

(ii) Let $u \in X$ and $\zeta \in \text{Map}(\text{Mod}(X))((\downarrow u, \wedge), (X, \wedge))$ with right adjoint ζ^* . Then $\zeta^*((m \wedge n) \circ x) = \zeta^*(m \wedge n) \wedge x = (\zeta^*(m) \wedge \zeta^*(n)) \wedge x$ because ζ^* is a module morphism (and thus turns the “ $- \circ x$ ” into a “ $- \wedge x$ ”) and because it is a right adjoint (and thus preserves infima). But “for the same reasons” we also have that $\zeta^*(m \wedge (n \circ x)) = \zeta^*(m) \wedge \zeta^*(n \circ x) = \zeta^*(m) \wedge (\zeta^*(n) \wedge x)$. Thus $\zeta^*((m \wedge n) \circ x) = \zeta^*(m \wedge (n \circ x))$ for all u and all ζ , and we conclude by the above that $(m \wedge n) \circ x = m \wedge (n \circ x)$.

(iii) Let $m, (m_i)_{i \in I}$ be elements of M . Let $u \in X$ and $\zeta: (\downarrow u, \wedge) \longrightarrow (X, \wedge)$ a left adjoint in $\text{Mod}(X)$ with right adjoint ζ^* . Using that ζ^* is both a left and a right adjoint in Ord one computes that

$$\zeta^* \left(m \wedge \bigvee_i m_i \right) = \zeta^*(m) \wedge \bigvee_i \zeta^*(m_i)$$

but also that

$$\zeta^* \left(\bigvee_i (m \wedge m_i) \right) = \bigvee_i (\zeta^*(m) \wedge \zeta^*(m_i)).$$

In both right-hand sides we now find elements of the locale $\downarrow u$, where \wedge distributes over \bigvee , and hence these expressions are equal. This holds for all u and all ζ , so by the the first statement we obtain $m \wedge \bigvee_i m_i = \bigvee_i (m \wedge m_i)$, which means that M is a locale. Finally, the function $f^*: X \longrightarrow M$ is certainly a Sup-morphism: because the action of (X, \wedge, \top_X) on M preserves suprema “in both variables”. But moreover, for $x, y \in X$, we may compute – using the formula in (ii) with $m = \top_M \circ x$ and $n = \top_M$ to pass from the second line to the third – that

$$\begin{aligned} f^*(x \wedge y) &= \top_M \circ (x \wedge y) \\ &= (\top_M \circ x) \circ y \\ &= (\top_M \circ x) \wedge (\top_M \circ y) \\ &= f^*(x) \wedge f^*(y). \end{aligned}$$

Thus f^* is indeed the inverse image part of a locale morphism $f: Y \longrightarrow X$. Putting $n = \top_M$ in the formula in (ii) it follows that moreover

$$m \wedge f^*(x) = m \wedge (\top_M \circ x) = m \circ x,$$

that is to say, $(M, \circ_f) = (M, \circ)$ as claimed. \square

We now go on to define the notion of “skew local homeomorphism”.

7.3. Skew local homeomorphisms

Let $f: Y \longrightarrow X$ be in Loc and $u \in X$; we keep the notation $i: \downarrow u \longrightarrow X$ for the corresponding open sublocale of X . Recall from [29, p. 524] that the elements of the set

$$S_f(u) := \text{Loc}/X (i, f)$$

are the *sections of f at u* . This defines a sheaf $S_f: X^{\text{op}} \longrightarrow \text{Set}$, and this construction extends to a functor $\text{Loc}/X \longrightarrow \text{Sh}(X)$ whose restriction to local homeomorphisms is an equivalence of categories.

A particular feature of local homeomorphisms is that, whenever $f = g \circ h$ in Loc , if f and g are local homeomorphisms then so is h ; recall also that a local homeomorphism is always open in Loc (see *loc. cit.*). Thus, if $f: Y \longrightarrow X$ is a local homeomorphism then every $s \in S_f(u)$ is an *open section* in the sense that $s: \downarrow u \longrightarrow Y$ is an open locale morphism. With this in mind the following is a natural generalisation.

Definition 7.5. For $f: Y \longrightarrow X$ in Loc and $i: u \longrightarrow X$, we put

$$S_f^\circ(u) := (\text{Loc}/X)^\circ(i, f)$$

and call its elements the *skew open sections of f at u* .

Example 7.6. Every open section $s: \downarrow u \rightarrow Y$ of a locale map $f: Y \rightarrow X$ is necessarily skew open too; but the converse need not hold. However, if $f: Y \rightarrow X$ is a local homeomorphism then $S_f(u) = S_f^\circ(u)$ for all $u \in X$.

A morphism $f: Y \rightarrow X$ in Loc is a local homeomorphism if and only if Y can be covered by its open sections [24, vol. 2, p. 503], i.e.

$$\top_Y = \bigvee \left\{ s_i(u) \mid u \in X, s \in S_f(u) \text{ and } s \text{ is open in } \text{Loc} \right\}.$$

In this case, every $y \in Y$ can be covered by open sections of f , by taking the restrictions of the open sections of f to y . This motivates our main definition in this section:

Definition 7.7. A morphism $f: Y \rightarrow X$ in Loc is a *skew local homeomorphism* if

$$1_Y = \bigvee \{s_i \circ s^* \mid u \in X, s \in S_f^\circ(u)\}.$$

For the record we immediately add:

Example 7.8. Every local homeomorphism is a skew local homeomorphism. A skew local homeomorphism is a local homeomorphism if and only if its (skew open) sections are all open.

Skew local homeomorphisms can be characterised in different ways:

Proposition 7.9. Let $f: Y \rightarrow X$ be in Loc .

- i. There is a bijection between skew open sections of f at $u \in X$ and locally principal elements of the X -module (Y, \circ_f) at the idempotent $u \in (X, \wedge, \top_X)$; if $s \in S_f^\circ(u)$ then $s_i(u) \in Y$ is the corresponding locally principal element.
- ii. The following statements are equivalent:
 - (a) (Y, \circ_f) is a locally principally generated X -module,
 - (b) for all $y \in Y, y = \bigvee (\downarrow y \cap \{s_i(u) \mid u \in X, s \in S_f^\circ(u)\})$,
 - (c) f is a skew local homeomorphism.

Proof. (i) By the fully faithful 2-functor in (15) we know, for each $i: \downarrow u \rightarrow X$, that

$$(\text{Loc}/X)^\circ(i, f) \cong \text{Map}(\text{Mod}(X))(\downarrow u, \wedge), (Y, \circ_f);$$

the left-hand side is precisely $S_f^\circ(u)$, and the bijection is given from left to right by sending an $s \in S_f^\circ(u)$ to s_i . As in Proposition 5.5, the right hand side is in bijection with the set of elements of (Y, \circ_f) which are locally principal at u , by sending s_i to $s_i(u)$.

(ii) Immediate from (iii) in Proposition 7.3, and (19). \square

Let $(\text{Loc}/X)_{\text{slh}}^\circ$ denote the full subcategory of $(\text{Loc}/X)^\circ$ whose objects are the skew local homeomorphisms. It follows from the above results that the fully faithful 2-functor in (15) (co)restricts to a fully faithful 2-functor

$$(\text{Loc}/X)_{\text{slh}}^\circ \rightarrow \text{Map}(\text{Mod}_{\text{lpg}}(X)). \tag{20}$$

This 2-functor is easily seen to be injective on objects; but due to Proposition 7.4 it is surjective too: for every locally principally generated X -module (M, \circ) the locale morphism $f: M \rightarrow X$ with inverse image $f^*(x) = \top_M \circ x$, which satisfies $(M, \circ) = (M, \circ_f)$, is a skew local homeomorphism. The consequence of our work is then the following result.

Theorem 7.10. For any locale X , the 2-functor in (20) is an isomorphism of locally ordered categories:

$$(\text{Loc}/X)_{\text{slh}}^\circ \cong \text{Map}(\text{Mod}_{\text{lpg}}(X));$$

both of these are thus equivalent to $\text{Ord}(X) \simeq \text{Ord}(\text{Sh}(X))$, the ordered sheaves on X viewed as enriched categorical structures, resp. the internal orders in the topos $\text{Sh}(X)$.

We have seen in Example 7.8 that any local homeomorphism is necessarily a skew local homeomorphism; and we have seen in Example 7.2 that any open locale morphism is necessarily skew open too. It follows that LH/X is a full subcategory of $(\text{Loc}/X)_{\text{slh}}^\circ$. We just proved the latter to be isomorphic to $\text{Map}(\text{Mod}_{\text{lpg}}(X))$, thus it makes sense to determine those locally principally generated X -modules which, under this isomorphism, correspond to local homeomorphisms.

Definition 7.11. A locally principally generated X -module (M, \circ) is an *étale X -module* when every left adjoint X -module morphism $\zeta: (\downarrow u, \wedge) \rightarrow (M, \circ)$ satisfies, for all $v \in \downarrow u$ and $m \in M$,

$$\zeta(v \wedge \zeta^*(m)) = \zeta(v) \wedge m.$$

It is straightforward from Example 7.8 and Proposition 7.9 that a skew local homeomorphism $f: Y \rightarrow X$ is a local homeomorphism if and only if (Y, \circ_f) is an étale X -module. Letting $\text{Mod}_{\text{ét}}(X)$ stand for the full sub-2-category of $\text{Mod}_{\text{lpg}}(X)$ consisting of étale X -modules, we can conclude with the following summary.

Theorem 7.12. For any locale X there is a commuting square

$$\begin{array}{ccc} (\text{Loc}/X)_{\text{sh}}^{\circ} & \xlongequal{\quad} & \text{Map}(\text{Mod}_{\text{lpq}}(X)) \\ \uparrow & & \uparrow \\ \text{LH}/X & \xlongequal{\quad} & \text{Map}(\text{Mod}_{\text{ét}}(X)) \end{array}$$

in which the equalities denote isomorphisms of (locally ordered) categories and the upward arrows are full embeddings. The categories in the bottom row are equivalent to $\text{Sh}(X)$, the locally ordered categories in the top row are equivalent to $\text{Ord}(\text{Sh}(X))$, and the inclusions view “sets as discrete (or symmetric) orders”.

8. Addendum: Biadjunction, biequivalence

It was shown in [48, Section 8] that, for any \mathcal{Q} -category \mathbb{C} , the totally compact objects of the presheaf category $\mathcal{P}\mathbb{C}$ form precisely the Cauchy-completion of \mathbb{C} : $(\mathcal{P}\mathbb{C})_c = \mathbb{C}_{cc}$. That is to say, a contravariant presheaf $\phi: *_A \rightarrow \mathbb{C}$ is totally compact in $\mathcal{P}\mathbb{C}$ if and only if it has a right adjoint in $\text{Dist}(\mathcal{Q})$ (“ ϕ is Cauchy”). Representable contravariant presheaves certainly are Cauchy, thus the Yoneda embedding $Y_{\mathbb{C}}: \mathbb{C} \rightarrow \mathcal{P}\mathbb{C}$, which sends an object $c \in \mathbb{C}$ to the representable $\mathbb{C}(-, c): *_c \rightarrow \mathbb{C}$, corestricts to $(\mathcal{P}\mathbb{C})_c$:

$$\bar{Y}_{\mathbb{C}}: \mathbb{C} \rightarrow (\mathcal{P}\mathbb{C})_c: c \mapsto \mathbb{C}(-, c). \quad (21)$$

The following is a mere triviality.

Lemma 8.1. For any \mathcal{Q} -category \mathbb{C} , the functor $\bar{Y}_{\mathbb{C}}: \mathbb{C} \rightarrow (\mathcal{P}\mathbb{C})_c$ is fully faithful, and it is an equivalence if and only if \mathbb{C} is Cauchy-complete.

On the other hand we can compute, for any cocomplete \mathcal{Q} -category \mathbb{A} and any $\phi \in \mathcal{P}(\mathbb{A}_c)$, the ϕ -weighted colimit of the inclusion $i_{\mathbb{A}}: \mathbb{A}_c \rightarrow \mathbb{A}$. This defines a functor

$$R_{\mathbb{A}}: \mathcal{P}(\mathbb{A}_c) \rightarrow \mathbb{A}: \phi \mapsto \text{colim}(\phi, i_{\mathbb{A}}) \quad (22)$$

about which we record some auxiliary results.

Lemma 8.2. For any cocomplete \mathcal{Q} -category \mathbb{A} , the functor $R_{\mathbb{A}}: \mathcal{P}(\mathbb{A}_c) \rightarrow \mathbb{A}$ is cocontinuous and admits a cocontinuous right adjoint. Moreover, $R_{\mathbb{A}}$ is always fully faithful, and it is an equivalence if and only if \mathbb{A} is totally algebraic.

Proof. We claim that $R_{\mathbb{A}}$ is left adjoint to $H: \mathbb{A} \rightarrow \mathcal{P}(\mathbb{A}_c): a \mapsto \mathbb{A}(i_{\mathbb{A}}-, a)$. Indeed, for $\phi \in \mathcal{P}(\mathbb{A}_c)$ and $x \in \mathbb{A}$,

$$\begin{aligned} \mathbb{A}(R_{\mathbb{A}}\phi, x) &= \mathbb{A}(\text{colim}(\phi, i_{\mathbb{A}}), x) \\ &= \left[\phi, \mathbb{A}(i_{\mathbb{A}}-, x) \right] \\ &= \mathcal{P}(\mathbb{A}_c)(\phi, Hx). \end{aligned}$$

Next we prove that H itself is cocontinuous; it suffices to show that it preserves suprema of contravariant presheaves: for any $\phi \in \mathcal{P}\mathbb{A}$, $H(\sup_{\mathbb{A}}(\phi)) = \text{colim}(\phi, H)$. Note first that, by Proposition 3.4,

$$\begin{aligned} H(\sup_{\mathbb{A}}(\phi)) &= \mathbb{A}(i_{\mathbb{A}}-, \sup_{\mathbb{A}}(\phi)) \\ &= \phi(i_{\mathbb{A}}-) \\ &= \mathbb{A}(i_{\mathbb{A}}-, -) \otimes \phi. \end{aligned}$$

But then also

$$\begin{aligned} \mathcal{P}(\mathbb{A}_c)(H(\sup_{\mathbb{A}}(\phi)), -) &= \left[\mathbb{A}(i_{\mathbb{A}}-, -) \otimes \phi, - \right] \\ &= \left[\phi, [\mathbb{A}(i_{\mathbb{A}}-, -), -] \right] \\ &= \left[\phi, \mathcal{P}(\mathbb{A}_c)(H-, -) \right] \end{aligned}$$

which is the universal property that we had to check.

To see that $R_{\mathbb{A}}$ is fully faithful it is (necessary and) sufficient to show that the unit of the adjunction $R_{\mathbb{A}} \dashv H$ is an isomorphism (cf. [48,49, 2.3] for example). Thus, for $c \in \mathbb{A}_c$ and $\phi \in \mathcal{P}(\mathbb{A}_c)$ we compute that

$$\begin{aligned} ((H \circ R_{\mathbb{A}})(\phi))(c) &= \mathbb{A}(i_{\mathbb{A}}c, \text{colim}(\phi, i_{\mathbb{A}})) \\ &= H_c(\text{colim}(\phi, i_{\mathbb{A}})) \quad (\text{with notations as in Proposition 3.4}) \\ &= \text{colim}(\phi, H_c \circ i_{\mathbb{A}}) \quad (\text{because of Proposition 3.4}) \\ &= \text{colim}(\mathbb{A}(-, i_{\mathbb{A}}-) \otimes \phi, H_c). \end{aligned}$$

By an argument in the proof of Proposition 3.4, we know that a presheaf-weighted colimit of H_c is the value of that weight in c ; here this allows us to equate

$$\text{colim}(\mathbb{A}(-, i_{\mathbb{A}}-) \otimes \phi, H_c) = \mathbb{A}(c, i_{\mathbb{A}}-) \otimes \phi = \mathbb{A}_c(c, -) \otimes \phi = \phi(c),$$

taking into account that $c \in \mathbb{A}_c$ and $i_{\mathbb{A}}$ is a full embedding. This indeed proves that $H \circ R_{\mathbb{A}} = 1_{\mathcal{P}(\mathbb{A}_c)}$.

Finally, knowing that $R_{\mathbb{A}}$ is always fully faithful, it is an equivalence if and only if also the counit of the adjunction $R_{\mathbb{A}} \dashv H$ is an isomorphism. Spelled out this means that, for every $a \in \mathbb{A}$, $a \cong \text{colim}(\mathbb{A}(i_{\mathbb{A}}-, a), i_{\mathbb{A}})$, which precisely says that $1_{\mathbb{A}}$ is the (pointwise) left Kan extension of $i_{\mathbb{A}}$ along itself. \square

Theorem 8.3. *There is a biadjunction*

$$\text{Cat}(\mathcal{Q}) \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{(-)_c} \end{array} \text{Map}(\text{Cocont}(\mathcal{Q}))$$

where the involved 2-functors are defined as:

$$\begin{aligned} \mathcal{P} \left(\mathbb{C} \xrightarrow{F} \mathbb{D} \right) &:= \mathcal{P}\mathbb{C} \xrightarrow{\mathbb{D}(-, F-) \otimes -} \mathcal{P}\mathbb{D}, \\ \left(\mathbb{A} \xrightarrow{F} \mathbb{B} \right)_c &:= \mathbb{A}_c \xrightarrow{F} \mathbb{B}_c \end{aligned}$$

and counit and unit are given by the functors in (21) and (22).

Proof. We shall prove that, for any cocomplete \mathcal{Q} -category \mathbb{A} ,

$$\left(\mathbb{A}_c, R_{\mathbb{A}}: \mathcal{P}(\mathbb{A}_c) \longrightarrow \mathbb{A} \right)$$

is a biuniversal right reflection along the 2-functor $\mathcal{P}: \text{Cat}(\mathcal{Q}) \longrightarrow \text{Map}(\text{Cocont}(\mathcal{Q}))$. The latter is indeed a 2-functor: for an $F: \mathbb{C} \longrightarrow \mathbb{D}$ in $\text{Cat}(\mathcal{Q})$ we have adjoints

$$\begin{array}{ccc} & \mathbb{D}(-, F-) \otimes - & \\ & \downarrow \perp & \\ \mathcal{P}\mathbb{C} & \xleftarrow{[\mathbb{D}(-, F-), -]} & \mathcal{P}\mathbb{D} \text{ in } \text{Cat}(\mathcal{Q}) \\ & \uparrow \perp & \\ & \mathbb{D}(F-, -), - & \end{array}$$

(with all compositions and liftings computed in $\text{Dist}(\mathcal{Q})$) so \mathcal{P} lands in the 2-category $\text{Map}(\text{Cocont}(\mathcal{Q}))$; and 2-functoriality is obvious. Moreover, Lemma 8.2 provides the information that $R_{\mathbb{A}}$ is a morphism of $\text{Map}(\text{Cocont}(\mathcal{Q}))$. So all we need to show, is that $R_{\mathbb{A}}$ has the required 2-universal property, i.e. the order-preserving function

$$\text{Cat}(\mathcal{Q})(\mathbb{C}, \mathbb{A}_c) \longrightarrow \text{Map}(\text{Cocont}(\mathcal{Q}))(\mathcal{P}\mathbb{C}, \mathbb{A}): F \mapsto R_{\mathbb{A}} \circ \mathcal{P}F \tag{23}$$

is an equivalence of ordered sets. We shall prove first that it is essentially surjective, and then that it is order-reflecting.

Essential surjectivity. Suppose given a left adjoint $G: \mathcal{P}\mathbb{C} \longrightarrow \mathbb{A}$ in $\text{Cocont}(\mathcal{Q})$, or equivalently, suppose given adjoints

$$\mathcal{P}\mathbb{C} \begin{array}{c} \xrightarrow{G} \\ \perp \\ \xleftarrow{H} \\ \perp \\ \xrightarrow{K} \end{array} \mathbb{A} \text{ in } \text{Cat}(\mathcal{Q}).$$

For $\psi \in \mathcal{P}\mathbb{A}$ we can compute, with straightforward arguments involving liftings and compositions in the quantaloid $\text{Dist}(\mathcal{Q})$, that $\psi(G \circ Y_{\mathbb{C}}-)$ is the ψ -weighted colimit of H :

$$\begin{aligned} \left[\psi, \mathcal{P}\mathbb{C}(H-, -) \right] &= \left[\psi, [\mathcal{P}\mathbb{C}(Y_{\mathbb{C}}-, H-), -] \right] \\ &= \left[\psi, [\mathbb{A}(G \circ Y_{\mathbb{C}}-, -), -] \right] \text{ (because } G \dashv H) \\ &= \left[\mathbb{A}(G \circ Y_{\mathbb{C}}-, -) \otimes \psi, - \right] \end{aligned}$$

$$\begin{aligned}
 &= \left[\mathbb{A}(-, G \circ Y_{\mathbb{C}}-), \psi, - \right] \\
 &= \left[\psi(G \circ Y_{\mathbb{C}}-), - \right] \quad (\text{by Yoneda Lemma for } \mathcal{Q}\text{-cats}) \\
 &= \mathcal{P}\mathbb{C}(\psi(G \circ Y_{\mathbb{C}}-), -).
 \end{aligned}$$

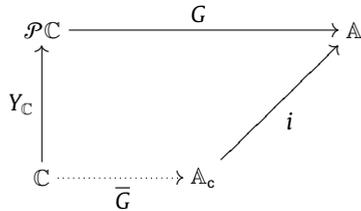
But for any $x \in \mathbb{C}$, we can also compute that

$$\begin{aligned}
 \mathbb{A}(GY_{\mathbb{C}}(x), \text{sup}_{\mathbb{A}}(\psi)) &= \mathcal{P}\mathbb{C}(Y_{\mathbb{C}}(x), H(\text{sup}_{\mathbb{A}}(\psi))) \quad (\text{by adjunction } G \dashv H) \\
 &= (H \circ \text{sup}_{\mathbb{A}}(\psi))(x) \quad (\text{by Yoneda Lemma for } \mathcal{Q}\text{-cats}) \\
 &= \text{colim}(\psi, H)(x) \quad (\text{because } H \text{ is cocontinuous}).
 \end{aligned}$$

Putting these together we have, for any $\psi \in \mathcal{P}\mathbb{A}$ and $x \in \mathbb{C}$, that

$$\psi(GY_{\mathbb{C}}(x)) = \mathbb{A}(GY_{\mathbb{C}}(x), \text{sup}_{\mathbb{A}}(\psi))$$

which, according to [Proposition 3.4](#), means that for any $x \in \mathbb{C}$ the object $GY_{\mathbb{C}}(x)$ of \mathbb{A} is totally compact. In other words, the given $G: \mathcal{P}\mathbb{C} \rightarrow \mathbb{A}$ factors as

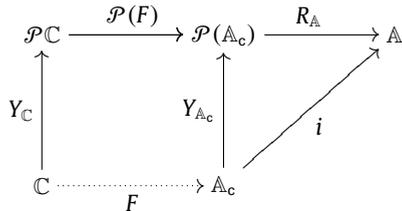


where $\bar{G}(x) := G(Y_{\mathbb{C}}(x))$. It is a matter of calculation, using cocontinuity of G amongst other things, to see that $G = R_{\mathbb{A}} \circ \mathcal{P}(\bar{G})$: for $\phi \in \mathcal{P}\mathbb{C}$,

$$\begin{aligned}
 (R_{\mathbb{A}} \circ \mathcal{P}(\bar{G}))(\phi) &= \text{colim}(\mathcal{P}(\bar{G})(\phi), i_{\mathbb{A}}) \\
 &= \text{colim}(\mathbb{A}_c(-, \bar{G}-) \otimes \phi, i_{\mathbb{A}}) \\
 &= \text{colim}(\mathbb{A}(-, i_{\mathbb{A}}-) \otimes \mathbb{A}_c(-, \bar{G}-) \otimes \phi, 1_{\mathbb{A}}) \\
 &= \text{colim}(\mathbb{A}(-, i_{\mathbb{A}} \circ \bar{G}-) \otimes \phi, 1_{\mathbb{A}}) \\
 &= \text{colim}(\mathbb{A}(-, GY_{\mathbb{C}}-) \otimes \phi, 1_{\mathbb{A}}) \\
 &= \text{colim}(\phi, G \circ Y_{\mathbb{C}}) \\
 &= G \circ \text{colim}(\phi, Y_{\mathbb{C}}) \\
 &= G(\phi).
 \end{aligned}$$

Thus we proved that the function in (23) is essentially surjective.

Order-reflection. Remark first that for any $F: \mathbb{C} \rightarrow \mathbb{A}_c$ in $\text{Cat}(\mathcal{Q})$, the outer diagram



commutes: both the left-hand square and the right-hand triangle are easily checked by computation. Now suppose that some $F, G: \mathbb{C} \rightarrow \mathbb{A}_c$ in $\text{Cat}(\mathcal{Q})$ are such that $R_{\mathbb{A}} \circ \mathcal{P}(F) \leq R_{\mathbb{A}} \circ \mathcal{P}(G)$ in $\text{Map}(\text{Cocont}(\mathcal{Q}))$. Then we can deduce from the above that

$$i_{\mathbb{A}} \circ F = R_{\mathbb{A}} \circ \mathcal{P}(F) \circ Y_{\mathbb{C}} \leq R_{\mathbb{A}} \circ \mathcal{P}(G) \circ Y_{\mathbb{C}} = i_{\mathbb{A}} \circ G.$$

But because $i_{\mathbb{A}}: \mathbb{A}_c \rightarrow \mathbb{A}$ is a full embedding, it follows that necessarily $F \leq G$ from the start. This proves that the function in (23) is also order-reflecting.

It is now a matter of routine computation to verify that the right biadjoint to the 2-functor $\mathcal{P}: \text{Cat}(\mathcal{Q}) \rightarrow \text{Map}(\text{Cocont}(\mathcal{Q}))$ is indeed given by “restricting to totally compacts”:

$$(-)_c: \text{Map}(\text{Cocont}(\mathcal{Q})) \rightarrow \text{Cat}(\mathcal{Q}): (F: \mathbb{A} \rightarrow \mathbb{B}) \mapsto (F: \mathbb{A}_c \rightarrow \mathbb{B}_c),$$

and that the unit of the biadjunction is indeed given by those corestrictions of Yoneda embeddings as in (21). \square

The (co)restriction of biadjoint 2-functors to those objects for which the (co)unit is an equivalence, is a biequivalence of 2-categories. In the case of interest above, we recover via [Lemmas 8.1](#) and [8.2](#) the biadjunction in [Corollary 3.3](#).

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