JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER

Journal of Computational and Applied Mathematics 115 (2000) 309–329

www.elsevier.nl/locate/cam

Chebyshev spectral collocation methods for nonlinear isothermal magnetostatic atmospheres

A.H. Khater[a,*], A.B. Shamardan[b], D.K. Callebaut[c], R.S. Ibrahim[a]

[a] Mathematics Department, Faculty of Science, Cairo University, Beni-Suef, Egypt

[b] Mathematics Department, Faculty of Science, El-Minia University, El-Minia, Egypt

[c] Physics Department, U.I.A., University of Antwerp, B-2610 Antwerp, Belgium

Received 31 July 1998; received in revised form 20 June 1999

Abstract

The equations of magnetohydrodynamic (MHD) equilibria for a plasma in gravitational field are investigated. For equilibria with one ignorable spatial coordinate, the MHD equations are reduced to a single nonlinear elliptic equation (NLPDE) for the magnetic flux u, known as the Grad-Shafranov equation. The Chebyshev spectral collocation methods (CSCMs) are described and applied to obtain numerical solutions for nonlinear boundary value problems modelling two classes of isothermal magnetostatic atmospheres, in which the current density J is proportional to the exponential of the magnetic flux and moreover falls off exponentially with distance vertical to the base, with an "e-folding" distance equal to the gravitational scale height, for the first class and proportional to the $\sinh(u)$ for the second class. The accuracy and efficiency of this Chebyshev approach are compared favorably with those of the standard finite-difference methods.

Keywords: Nonlinear elliptic boundary value problems; Isothermal magnetostatic atmospheres; Chebyshev spectral collocation methods

1. Introduction

The equations of magnetohydrodynamic (MHD) equilibria have been used extensively to model solar magnetic structure [1,2,7–15,17–19]. The force balance in these models consists of a balance between the pressure gradient force, the Lorentz $J \times B$ force (J electric current density; B magnetic induction), and the gravitational force. The temperature distribution in the atmosphere is, in general, determined from the energy transport equation. However, in many models, the temperature distribution is specified a priori and direct reference to the energy equation is eliminated. The remaining

* Corresponding author.

PII: S0377-0427(99)00306-4

equations for the system are an equation of state for the gas (for example, the dependence of the gas pressure on density and temperature) and the steady-state Maxwell's equations.

Many models of magnetostatic equilibria assume that one of the spatial coordinates is negligible [1,2,7,9–15], leading to simple analytic models in terms of an elliptic equation for the magnetic potential u.

In solar physics the static MHD equations have been used to model diverse phenomena, such as the slow evolution stage of solar flares, or the magnetostatic support of prominences [10,12,15,19]. Moreover, the nonlinear problem has been solved in several cases [2,9–14,17,18].

In this paper, we present a set of numerical solutions by using the Chebyshev spectral collocation methods (CSCMs), for the two-dimensional boundary value problems (BVPs) in the solar surface, $z \geqslant 0$, but only for bounded domain, [7,15,1]. We consider an isothermal atmosphere with one negligible coordinate z of a rectangular Cartesian coordinate system xyz in which the gravitational force is directed in the negative y-direction. We obtained two classes of numerical solutions when $J_z \sim (\mathrm{e}^{-2\tilde{u}}\mathrm{e}^{-y/h}$ and $\sinh \tilde{u})$, respectively ($\tilde{u}$ is a dimensionless form of the magnetic potential u and h is the gravitational scale height) for which the force balance equation perpendicular to both B and e_z (e_z is the unit vector along the z-axis) reduces to the nonlinear Grad-Shafranov type elliptic equation.

The paper is organized as follows: In Section 2, the basic equations and the formulation of the problem are given. In Section 3, the CSCMs are illustrated, its accuracy is discussed and used to obtain numerical solutions of the problem. The numerical results are presented in Section 4 correspond to an isothermal magnetostatic atmosphere in which the current density is proportional to $\exp(-2\tilde{u})\exp(-y/h)$ and $\sinh(\tilde{u})$, respectively, and Section 5 gives the conclusion.

2. Basic equations and problem formulation

The relevant MHD equations consist of the equilibrium equation:

$$\rho \nabla \phi + \nabla P = J \wedge B, \tag{1}$$

coupled with Maxwell's equations:

$$J = \nabla \wedge B/(4\pi), \tag{2}$$

$$\nabla \cdot B = 0. \tag{3}$$

In Eqs. (1)–(3) P denotes the gas pressure, and ϕ denotes the gravitational potential (for a uniform gravitational field in the y-direction one has $\phi = gy$, where g is the acceleration constant due to gravity).

For equilibria that have one ignorable Cartesian coordinate z say, the equilibrium problem governed by Eqs. (1)–(3) can be reduced to the solution of a single nonlinear partial differential equation (NLPDE) for the magnetic potential $u(x, y)$ [14,10]

$$\nabla^2 u + \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{B_z^2(u)}{2}\right] + \frac{\partial}{\partial u}[4\pi P(u, \phi(x, y))] = 0. \tag{4}$$

In Eq. (4) the magnetic induction B is given by

$$B = \nabla u(x, y) \wedge e_z + B_z(u)e_z = \left[\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}, B_z\right] \tag{5}$$

and for an ideal gas pressure $P(u, \phi)$ may be written as

$$P(u, \phi) = P_0(u) \exp\left[-\int_{\phi_0}^{\phi} \frac{\mathrm{d}\phi'}{\theta(u, \phi')}\right], \tag{6}$$

where $\theta = RT$, with R the ideal gas constant and T the temperature.

A frequently studied version of (4) is the case of constant temperature T (or ϕ) in a uniform gravitational field ($\phi = gy$, $g = \text{constant}$), for which

$$P(x, y) = \left[P_0 + \frac{\lambda^2 u_0^2}{8\pi} F(\tilde{u}, x, y)\right] \mathrm{e}^{-y/h}, \tag{7}$$

where

$$h = \frac{\theta}{g} \tag{8}$$

is the gravitational scale height

$$\tilde{u} = \frac{u}{u_0} \tag{9}$$

is a dimensionless form of u while λ^2, P_0 and u_0 are constants.

In order to solve Eq. (4) the two functions $B_z^2(u)$ and $P(u, \phi(x, y))$ have to be prescribed and the appropriate boundary conditions must be specified.

The general form of Eq. (4) can be written as

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = F(\tilde{u}, x, y), \tag{10}$$

where

$$F(\tilde{u}, x, y) = -\frac{1}{u_0^2} \frac{\partial}{\partial \tilde{u}} \left(\frac{B_z^2(\tilde{u})}{2} + 4\pi P(\tilde{u}, \phi)\right). \tag{11}$$

In this paper, we investigate the numerical solutions of Eq. (10) for specific forms of $F(\tilde{u}, x, y)$.

2.1. Liouville's equation model

Let: $B_z(\tilde{u}) = 0$,

$$F(\tilde{u}, x, y) = \lambda^2 \mathrm{e}^{-2\tilde{u}} \mathrm{e}^{-y/h}, \tag{12}$$

then Eq. (10) becomes

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = \lambda^2 \mathrm{e}^{-2\tilde{u}} \mathrm{e}^{-y/h}, \tag{13}$$

Eq. (13) is a NLPDE [2,14,15].

For the problem with a finite h [15] set

$$\tilde{u} = -\frac{y}{2h} + S(x), \tag{14}$$

reducing Eq. (13) to the ordinary differential equation (ODE)

$$\frac{d^2S}{dx^2} - \lambda^2 e^{-2s} = 0, \tag{15}$$

which integrates readily to give

$$S = \log\cosh\lambda x. \tag{16}$$

This generates the magnetic field and pressure,

$$B = u_0(-1/2h, -\lambda\tanh\lambda x, 0), \tag{17}$$

$$P = P_0 e^{-y/h} + \frac{\lambda^2 u_0^2}{8\pi}\operatorname{sech}^2\lambda x. \tag{18}$$

If we set $P_0 = 0$, removing the plane-parallel component of the atmosphere, this is the well-known model for an infinite vertical sheet of diffuse plasma suspended by bowed magnetic field lines. A second particular solution was obtained by [11] with

$$\tilde{u} = -\tfrac{1}{2}[y/h + v(x, y)], \tag{19}$$

then, Eq. (13) transforms into

$$\nabla^2 v = -2\lambda^2 e^{v}. \tag{20}$$

Eq. (20) is a Liouville's equation with known closed-form solution in terms of harmonic functions. It is worth noting that the analytical solution of Liouville's equation (20) possesses Bäcklund transformations (BTs) [11,12,14] allowing one to obtain solutions.

Let us assume that from Eq. (7), the relationship between P and $\tilde{u}$ is given by

$$P(x, y) = \left[P_0 + \frac{\lambda^2 u_0^2}{8\pi} e^{-2\tilde{u}}\right] e^{-y/h}. \tag{21}$$

2.2. Sinh-Poisson equation model

Let: $B_z = 0$,

$$F(\tilde{u}, x, y) = \lambda^2 \sinh(\tilde{u})\, e^{-y/h}, \tag{22}$$

then Eq. (10) becomes

$$\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial y^2} = \lambda^2 \sinh(\tilde{u})\, e^{-y/h}. \tag{23}$$

For the sinh-Poisson equation (23), the relationship between P and $\tilde{u}$ is given by

$$P(x,y)=\left[P_0-\frac{\lambda^2u_0^2}{4\pi}\cosh(\tilde{u})\right]\mathrm{e}^{-y/h}. \tag{24}$$

If $y/h \ll 1$ and $B_z=0$, Eq. (23) can be reduced to the sinh-Poisson equation

$$\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial y^2}=\lambda^2\sinh(\tilde{u}). \tag{25}$$

This equation has been solved analytically by applying BTs [11,14].

For the ordinary form of sinh-Poisson equation, set

$$\tilde{u}=S(x), \tag{26}$$

reducing Eq. (25) to the ODE

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2}-\lambda^2\sinh S(x)=0, \tag{27}$$

which integrates readily to give

$$\tilde{u}=S(x)=-4\coth^{-1}(\mathrm{e}^{2\lambda x}). \tag{28}$$

This generates the magnetic field and pressure:

$$B=u_0(0,4\lambda\,\mathrm{cosech}\,2\lambda x,0), \tag{29}$$

$$P=P_0-\frac{\lambda^2u_0^2}{4\pi}\cosh(-4\coth^{-1}\mathrm{e}^{2\lambda x}). \tag{30}$$

In the next section we will adopt a finite Chebyshev expansion to approximate $\int_{x_i}^{x}\int_{x_i}^{x}F(s,y)\,\mathrm{d}s\,\mathrm{d}s$ and $\int_{y_j}^{y}\int_{y_j}^{y}F(x,s)\,\mathrm{d}s\,\mathrm{d}s$ on the set of collocation points $\Omega=[x_i,x_{i+1}]\wedge[y_j,y_{j+1}]$, $i=\mathrm{O}(1)N_x-1$, $j=\mathrm{O}(1)N_y-1$. Also, a method based on the above approximations will de developed.

3. A spectral collocation method in two dimensions

The applicability of CSCMs for solving PDEs is not only for simple model problems, but also for large scale applications in MHD [8]. The use of Chebyshev polynomial approximations for the solution of PDEs has been advocated for example by [8,5,6,16]. The approximations which include the Galerkin and tau-spectral methods and pseudo-spectral collocation methods, are discussed in detail by [3].

3.1. The Liouville's equation

The BVP (Liouville's equation (20) and sinh-Poisson equation (25)), in the bounded domain $\Omega=[x_0,x_f]\wedge[y_0,y_f]$ is

$$v_{xx}+v_{yy}=F(v(x,y)),\qquad (x,y)\in[x_0,x_f]\wedge[y_0,y_f] \tag{31}$$

with the boundary conditions

$$\begin{aligned} &v(x, y_0) = f_1(x), \qquad v(x, y_f) = f_2(x), \quad x \in [x_0, x_f], \\ &v(x_0, y) = g_1(y), \qquad v(x_f, y) = g_2(y), \quad y \in [y_0, y_f]. \end{aligned} \tag{32}$$

To perform numerical computations of the solutions of the BVP (31) and (32), it is necessary to work in a bounded domain so as to apply CSCMs or finite difference method FDM. Moreover, one needs information about the solutions of the problems on the boundary of Ω. We can extend this solution to any desired larger but still finite domain [7,15,1].

Let $\Delta = \{y_0 < y_1 < \cdots < y_{N_y}\}$ be a partition of the interval $[y_0, y_f]$, with the step size $\Delta y_j = y_{j+1} - y_j$. Each subinterval $[y_j, y_{j+1}]$ is divided by the Chebyshev collocation points

$$y_{jk} = y_j + \Delta y_j d_k, \quad d_k = \frac{1}{2}\left[1 - \cos\left(\frac{k\pi}{v_y}\right)\right], \quad k = \mathrm{O}(1)v_y,\ j = \mathrm{O}(1)N_y. \tag{33}$$

Similarly, each subinterval $[x_i, x_{i+1}]$ is divided by the Chebyshev collocation points

$$x_{il} = x_i + \Delta x_i c_l, \quad c_l = \frac{1}{2}\left[1 - \cos\left(\frac{l\pi}{v_x}\right)\right], \quad l = \mathrm{O}(1)v_x,\ i = \mathrm{O}(1)N_x. \tag{34}$$

For $y \in [y_j, y_{j+1}]$ and $j = \mathrm{O}(1)N_y - 1$, the approximation of $F(x, y)$ in the y-direction is

$$\tilde{F}(x, y) = {\sum_{r=0}^{v_y}}'' a_r^{(j+1)}(x) T_r\left(\frac{2(y - y_j)}{\Delta y_j} - 1\right) \tag{35}$$

where

$$a_r^{(j+1)}(x) = \frac{2}{v_y} {\sum_{k=0}^{v_y}}'' \tilde{F}\left(x, y_j + \frac{\Delta y_j}{2}(1 + \cos(k\pi/v_y)\right) T_r(\cos(k\pi/v_y)), \quad r = \mathrm{O}(1)v_y. \tag{36}$$

Here $T_r(x)$ is the rth Chebyshev polynomial. A summation symbol with double primes denotes a sum with the first and last terms halved.

Now, we can easily show that the following relations are true:

$$\begin{aligned} \mathrm{CH}(r, \theta) &= \int_0^\theta T_r(2s - 1)\,\mathrm{d}s \\ &= \begin{cases} \dfrac{T_{r+1}(2\theta - 1)}{4(r+1)} - \dfrac{T_{r-1}(2\theta - 1)}{4(r-1)} + \dfrac{(-1)^{r+1}}{2(r^2 - 1)} & \text{if } r \geqslant 2, \\ \frac{1}{8}[T_2(2\theta - 1) - 1] & \text{if } r = 1, \\ \frac{1}{2}[T_1(2\theta - 1) + 1] & \text{if } r = 0. \end{cases} \end{aligned} \tag{37}$$

Using (35)–(37), the indefinite integral $\int_{y_j}^{y_j+\Delta y_j\theta} \tilde{F}(x,\xi)\,d\xi$ takes the form

$$\int_{y_j}^{y_j+\Delta y_j\phi} \tilde{F}(x,\xi)\,d\xi = \Delta y_j \sum_{k=0}^{v_y} b_k(\phi)\tilde{F}(x, y_{jk}), \tag{38}$$

where

$$b_k(\phi) = \frac{1}{2v_y(1+\delta_{v_yk}+\delta_{0k})}\left\{\sum_{j=0}^{v_y}\frac{T_j(y_k)}{(1+\delta_{v_yj})(j+1)}(T_{j+1}(2\phi-1)+(-1)^j)\right.$$
$$\left. -\sum_{j=2}^{v_y}\frac{T_j(y_k)}{(1+\delta_{v_yj})(j-1)}(T_{j-1}(2\phi-1)+(-1)^j)\right\}, \quad k=\mathrm{O}(1)v_y, \tag{39}$$

δ_{lk} is the Kronecker delta, $y_k = -\cos(k\pi/v_y)$.

Let

$$\Phi(x,y) = \frac{\partial^2 v}{\partial x^2}, \qquad \Psi(x,y) = \frac{\partial^2 v}{\partial y^2},$$

using partial integration, one obtains two equations for v as

$$v(x,y) = \int_{x_i}^{x}(x-s)\Phi(s,y)\,ds$$
$$+\frac{x-x_i}{\Delta x_i}\left(v(x_{i+1},y)-v(x_i,y)-\int_{x_i}^{x_{i+1}}(x_{i+1}-s)\Phi(s,y)\,ds\right)+v(x_i,y), \tag{40}$$

$$v(x,y) = \int_{y_i}^{y}(y-s)\Psi(x,s)\,ds$$
$$+\frac{y-y_j}{\Delta y_i}\left(v(x,y_{j+1})-v(x,y_j)-\int_{y_j}^{y_{j+1}}(y_{j+1}-s)\Psi(x,s)\,ds\right)+v(x,y_j) \tag{41}$$

and Eq. (31) takes the form

$$\Phi(x,y)+\Psi(x,y) = F(v(x,y)). \tag{42}$$

Using CSCMs to approximate the integrals in Eqs. (40)–(42), we obtain

$$v(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)$$
$$= \Delta x_i\sum_{k=0}^{v_x} b_k(\theta)(\theta-c_k)\Phi(x_{ik}, y_j+\phi\Delta y_j)$$
$$+\theta\left(v(x_{i+1}, y_j+\phi\Delta y_j)-v(x_i, y_j+\phi\Delta y_j)-\Delta x_i\sum_{k=0}^{v_x} b_k(1)(1-c_k)\Phi(x_{ik}, y_j+\phi\Delta y_j)\right)$$
$$+v(x_i, y_j+\phi\Delta y_j), \tag{43}$$

$$\begin{aligned} &v(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)\\ &\quad =\Delta y_j\sum_{k=0}^{v_y} b_k(\phi)(\phi-d_k)\Psi(x_i+\theta\Delta x_i, y_{jk})\\ &\qquad +\phi\left(v(x_i+\theta\Delta x_i, y_{j+1})-\ v(x_i+\theta\Delta x_i, y_j)-\Delta y_j\sum_{k=0}^{v_y} b_k(1)(1-d_k)\Psi(x_i+\theta\Delta x_i, y_{jk})\right)\\ &\qquad + v(x_i+\theta\Delta x_i, y_j), \end{aligned} \tag{44}$$

$$\Phi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)+\Psi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)=F(v(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)). \tag{45}$$

Solving the above system of nonlinear equations, we obtain the approximate solutions $v(x,y)$ on the graded mesh.

We notice that matrix B of El-gendi's method [4], is the same matrix $[b_{jl}]$.

Use the above method to solve the BVP (20), (32) in Section 3.1, taking into account Eqs. (43), (44) and replacing Eq. (45) by the following equation:

$$\Phi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)+\Psi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)=-2\lambda^2 e^{v(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)} \tag{46}$$

where

$$\begin{aligned} x_{il}&=x_i+\frac{\Delta x_i}{2}\left[1-\cos\frac{l\pi}{v_x}\right], \qquad l=O(1)v_x, \quad i=O(1)N_x,\\ y_{jk}&=y_j+\frac{\Delta y_j}{2}\left[1-\cos\frac{k\pi}{v_y}\right], \qquad k=O(1)v_y, \quad j=O(1)N_y. \end{aligned} \tag{47}$$

3.2. The sinh-Poisson equation

Similarly, we can give the Chebyshev approximate solution to the BVP (25), (32) via the solution of (32), (43), (44) and replacing (45) by the following equation:

$$\Phi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)+\Psi(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)=\lambda^2\sinh(v(x_i+\theta\Delta x_i, y_j+\phi\Delta y_j)). \tag{48}$$

3.3. The accuracy order of CSCMs

In order to study the convergence of Chebyshev spectral collocation methods CSCMs for solving the nonlinear boundary value problem (BVP) (31) and (32), we need the following lemma:

Lemma 1.

$$\sum_{l=0}^{v}(c_l)^m b_l(\theta)=\frac{\theta^{m+1}}{m+1}, \quad m=O(1)v. \tag{49}$$

Proof. We can easily prove

$$\sum_{l=0}^{v}{}'' \sin^{2m}\frac{l\pi}{2v}T_r(-\cos(l\pi/v))=\begin{cases}(v/4^m)C_{m-r}^{2m}(1+\delta_{rv}), & m\geqslant r,\\ 0 & \text{otherwise,}\end{cases}$$

then, for $m=0,1,\dots,v$ we have

$$\sum_{l=0}^{v}(c_l)^m b_l(\theta)=\frac{2}{v}\sum_{r=0}^{v}{}''\left(\sum_{l=0}^{v}{}''\right)(c_l)^m T_r(-\cos(l\pi/v))\int_0^\theta T_r(2s-1)\,\mathrm{d}s$$

$$=2\int_0^\theta\sum_{r=0}^{m}\frac{C_{m-r}^{2m}}{4^m}\frac{1+\delta_{rv}}{1+\delta_{r0}+\delta_{rv}}T_r(2s-1)\,\mathrm{d}s$$

$$=\int_0^\theta\left[\cos\left(\frac{1}{2}\cos^{-1}(2s-1)\right)\right]^{2m}\,\mathrm{d}s$$

$$=\int_0^\theta s^m\,\mathrm{d}s=\frac{\theta^{\,m+1}}{m+1}.$$

This completes the proof. □

Using the above CSCMs to solve Eq. (31), we can give the local truncation error $\eta_i(\theta,y)$ and $\eta_j(x,\phi)$ of Eqs. (43) and (44).

Using Taylor expansion for Eq. (43), we get

$$\mathrm{LHS}=v(x_i,y)+\sum_{\delta=0}^{\infty}\frac{(\theta\Delta x)^\delta}{\delta!}\frac{\partial^\delta v}{\partial x^\delta}\bigg|_{(x_i,y)},$$

$$\mathrm{RHS}=(\Delta x)^2\sum_{\delta=0}^{v_x}\left(b_\delta(\theta)(\theta-c_\delta)\sum_{l=0}^{\infty}\frac{(\Delta x c_\delta)^l}{l!}\frac{\partial^{l+2}v}{\partial x^{l+2}}\bigg|_{(x_i,y)}\right)$$

$$+\theta\left(\sum_{l=0}^{\infty}\frac{(\Delta x c_{v_x})^l}{l!}\frac{\partial^l v}{\partial x^l}\bigg|_{(x_i,y)}-v(x_i,y)\right.$$

$$\left.-(\Delta x)^2\sum_{\delta=0}^{v_x}b_\delta(1)(1-c_\delta)\sum_{l=0}^{\infty}\frac{(\Delta x c_\delta)^l}{l!}\frac{\partial^{l+2}v}{\partial x^{l+2}}\bigg|_{(x_i,y)}-v(x_i,y)\right)$$

$$+v(x_i,y)+\eta_i(\theta,y).$$

Then, after some calculations, the local truncation error $\eta_i(\theta,y)$ takes the form

$$\eta_i(\theta,y)=\sum_{l=2}^{\infty}\frac{(\Delta x)^l}{(l-2)!}\frac{\partial^l v}{\partial x^l}\bigg|_{(x_i,y)}\left(\frac{\theta^{\,l}-\theta}{l(l-1)}-\sum_{n=0}^{v_x}[b_n(\theta)(\theta-c_n)-\theta b_n(1)(1-c_n)]c_n^{l-2}\right).$$

Using Lemma (1), we have

$$\eta_i(\theta,y)=\mathrm{O}((\Delta x)^{v_x+1}).$$

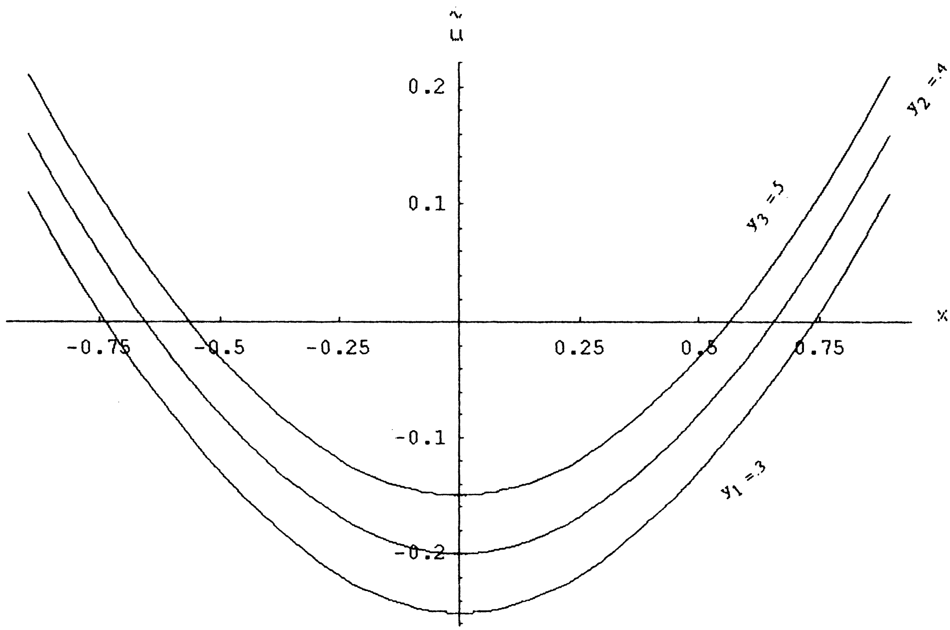

Fig. 1. The potential field lines $\tilde{u}$ for the ordinary form of Liouville's equation with a finite h for $\lambda = 1$, $h = 1$.

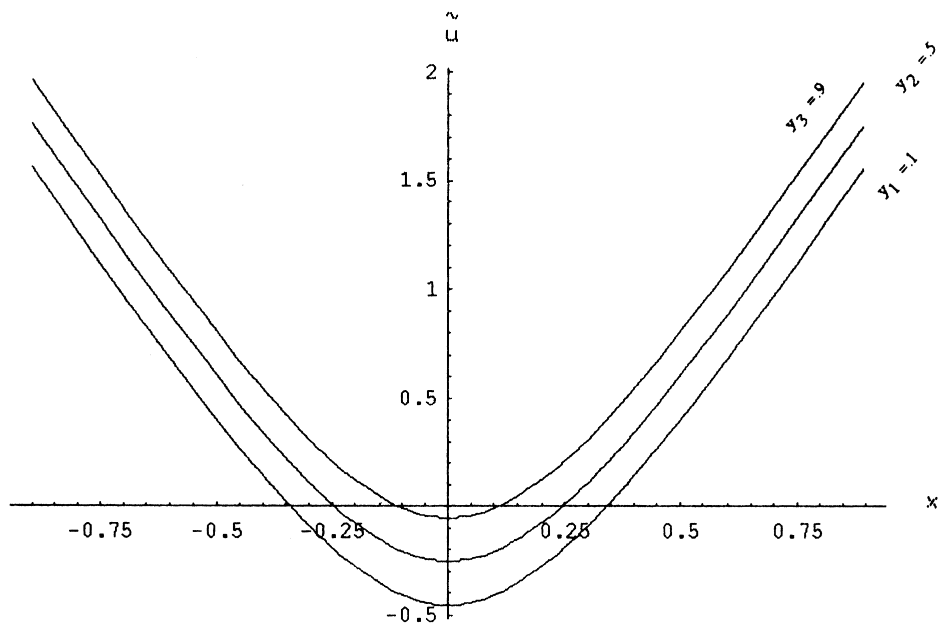

Fig. 2. The potential field lines $\tilde{u}$ for the ordinary form of Liouville's equation with a finite h for $\lambda = 3$, $h = 1$.

In the similar way for Eq. (44), we obtain

$$\eta_j(x,\phi) = O((\Delta y)^{v_y+1}).$$

Hence, the truncation error of the above method is $O((\Delta x)^{v+1}, (\Delta y)^{v+1})$, where $v = \min(v_x, v_y)$.

4. Numerical results

4.1. Liouville's equation model

The application of the CSCMs developed in Section 3, for an isothermal magnetostatic atmosphere in which the current density is proportional to $e^{-2\tilde{u}-y/h}$ of the magnetic potential are illustrated in the following examples:

Example 1. For the ordinary form of Liouville's equation (BVP):

$$\frac{d^2S}{dx^2} - \lambda^2 e^{-2S} = 0, \quad -1 \leqslant x \leqslant 1 \tag{50}$$

with boundary conditions

$$\begin{aligned} S(-1) &= \log\cosh(-\lambda), \\ S(1) &= \log\cosh(\lambda). \end{aligned} \tag{51}$$

This problem has exact solution [15]

$$S(x) = \log\cosh\lambda x. \tag{52}$$

From Eq. (14), the magnetostatic potential take the form

$$\tilde{u} = -\frac{y}{2h} + S(x) = -\frac{y}{2h} + \log\cosh\lambda x, \quad h \text{ is finite.} \tag{53}$$

Applicable for some relevant values of λ. The approximations of the magnetic field $\tilde{u}$ of Eq. (53) are displayed in Figs. 1 and 2 and the approximations of the gas pressure are given in Figs. 3 and 4.

Example 2. Consider the Liouville's BVP

$$v_{xx} + v_{yy} = -2\lambda^2 e^{v}, \quad -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \tag{54}$$

which has the exact solution

$$v(x,y) = 2\log(\sqrt{2}\,\operatorname{sech}(\lambda(x+y))), \quad -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1. \tag{55}$$

From Eq. (19), we get the magnetostatic potential solution of Eq. (13) in the form

$$\tilde{u}(x,y) = -\frac{1}{2}\left\{\frac{y}{h} + 2\log(\sqrt{2}\,\operatorname{sech}(\lambda(x+y)))\right\}. \tag{56}$$

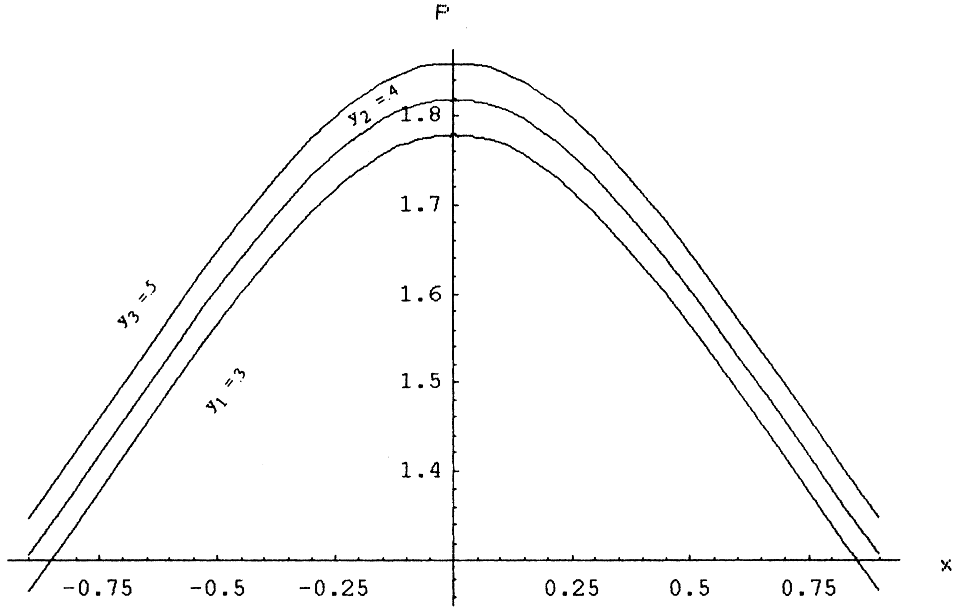

Fig. 3. The gas pressure for the ordinary form of Liouville's equation with a finite h for $\lambda = 1$, $h = 1$, $P_0 = 1$ and $u_0^2 = 8\pi$.

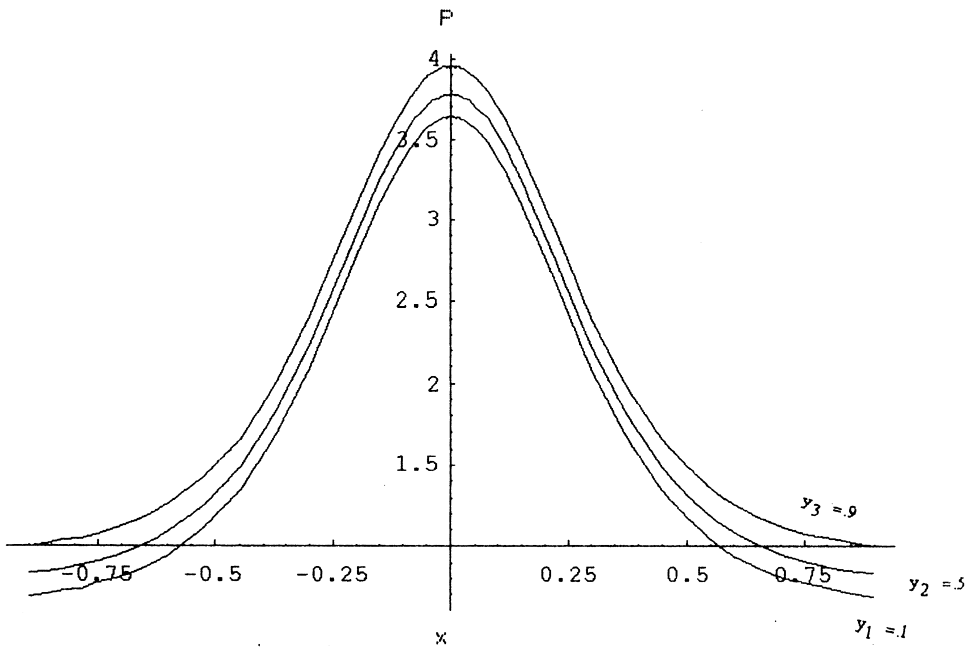

Fig. 4. The gas pressure for the ordinary form of Liouville's equation with a finite h for $\lambda = 3$, $h = 1$, $P_0 = 1$ and $u_0^2 = 8\pi$.

Table 1
The maximum absolute errors from the exact solution of the magnetic potential $\tilde{u}$ of the Liouville's equation at $y = 0.5$, $\lambda = 1$

x	CSCMs	FDM
− 0.8	1.29E−6	6.7E−3
− 0.6	1.52E−5	5.2E−3
− 0.4	1.42E−5	2.51E−3
− 0.2	4.17E−5	5.58E−4
0.0	3.45E−5	2.68E−3
0.2	9.078E−5	5.45E−3
0.4	5.41E−5	3.81E−3
0.6	1.59E−6	3.71E−3
0.8	2.87E−6	5.14E−3

Table 2
The maximum absolute errors of the gas pressure P for the Liouville's equation at $y = 0.5$, $\lambda = 1$, $h = 1$, $P_0 = 0$, $u_0^2 = 8\pi$

x	CSCMs	FDM
− 0.8	2.45E−7	2.55E−3
− 0.6	5.21E−6	6.11E−3
− 0.4	3.87E−6	4.54E−3
−0.2	2.11E−6	2.58E−4
0.0	3.41E−6	2.38E−3
0.2	4.58E−7	5.87E−3
0.4	1.68E−6	6.25E−3
0.6	5.58E−6	5.98E−3
0.8	1.91E−6	3.59E−3

In Table 1, the given numbers refer to the maximum absolute errors of the magnetic potential $\tilde{u}$ of the Liouville's equation (13) from exact solution (56) by applying CSCMs and the finite-difference method (FDM).

In Table 2, the given numbers refer to the maximum absolute errors of the gas pressure P of the Liouville's equation (13) and the finite-difference method (FDM).

Applicable for some relevant values of λ. The approximations of the magnetic potential $\tilde{u}$ of the Liouville's equation (13) are displayed in Figs. 5 and 6 and the approximations of the gas pressure P are given in Figs. 7 and 8. Note that $\tilde{u}$, $P \to 0$ as $y \to \infty$.

4.2. Sinh-Poisson equation model

Example 3. (*For the ordinary form of sinh-Poisson*) *equation*. Consider the ordinary form of sinh-Poisson equation

$$\frac{d^2 S}{dx^2} - \lambda^2 \sinh S(x) = 0, \quad -1 \leqslant x \leqslant 1 \tag{57}$$

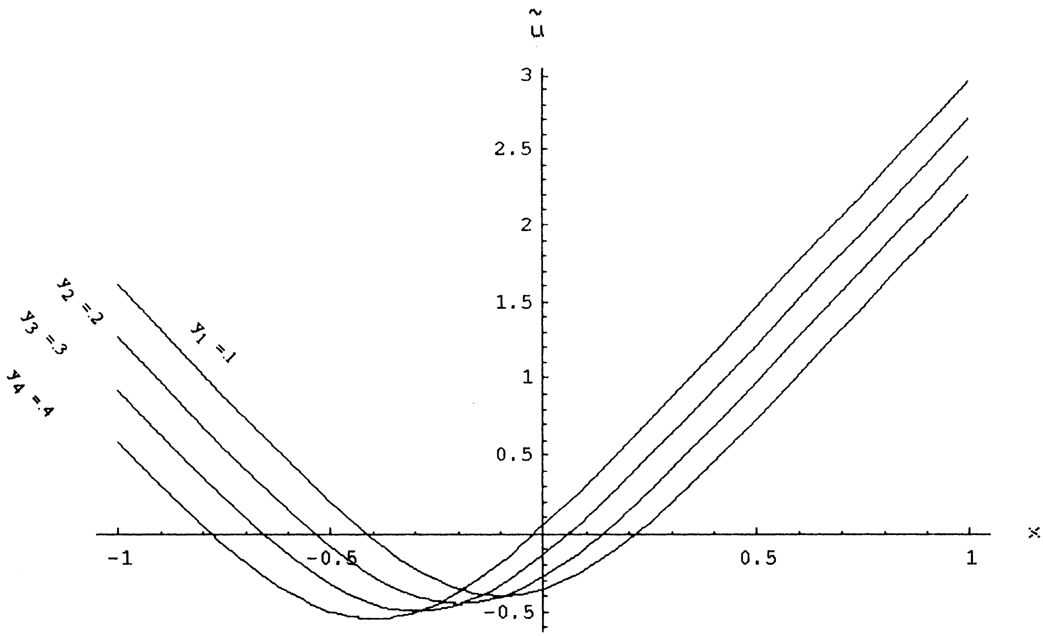

Fig. 5. The potential field lines $\tilde{u}$ for the Liouville's equation for $\lambda = 3$, $h = 1$.

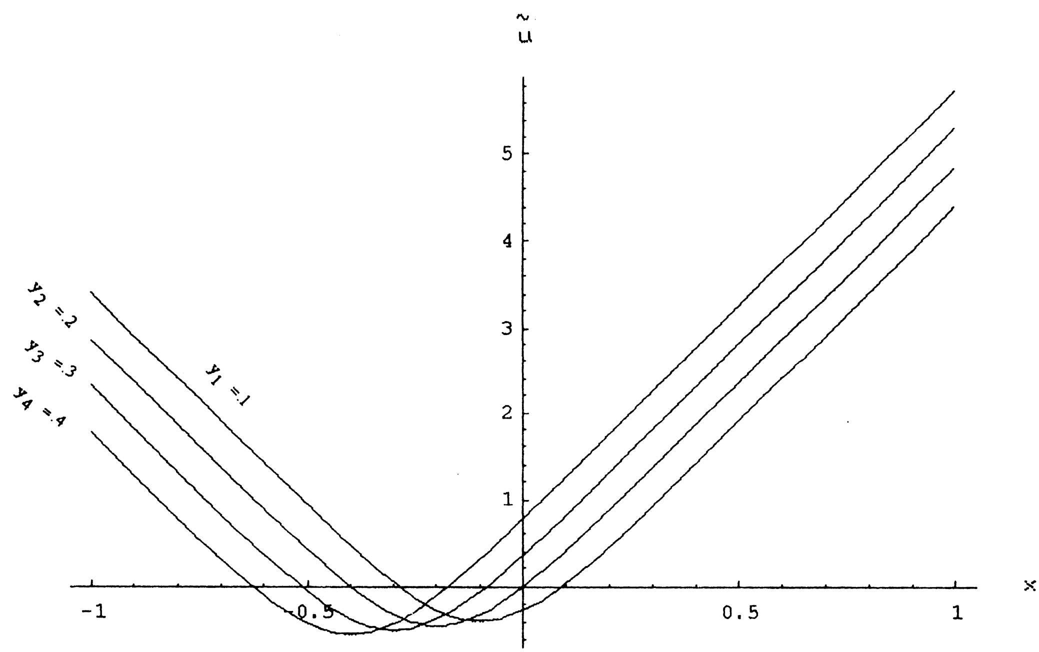

Fig. 6. The potential field lines $\tilde{u}$ for the Liouville's equation for $\lambda = 7$, $h = 1$.

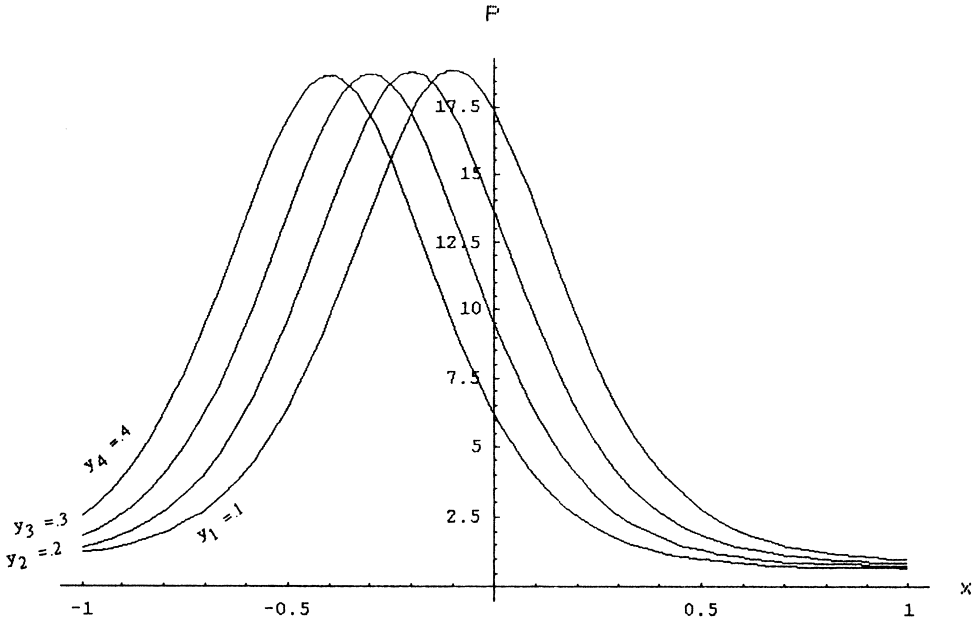

Fig. 7. The gas pressure for Liouville's equation for $\lambda = 3$, $h = 1$, $P_0 = 1$ and $u_0^2 = 8\pi$.

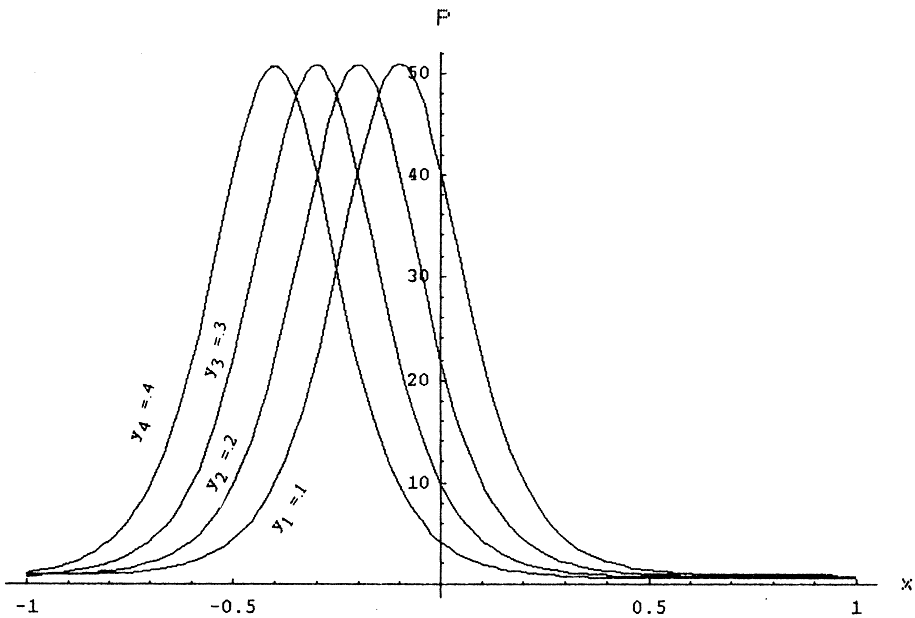

Fig. 8. The gas pressure for Liouville's equation for $\lambda = 7$, $h = 1$, $P_0 = 1$ and $u_0^2 = 8\pi$.

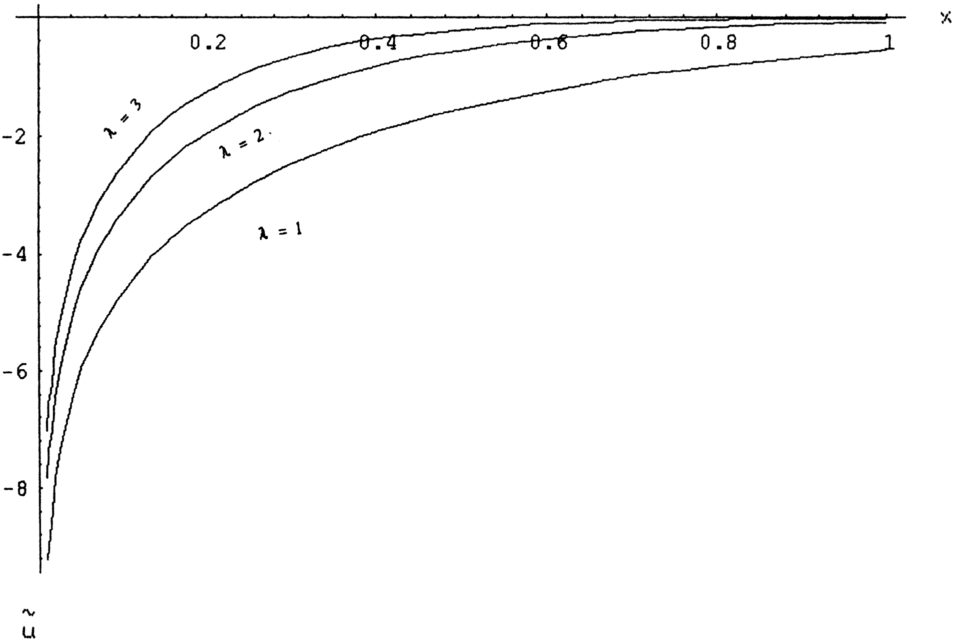

Fig. 9. The potential field lines $\tilde{u}$ for the ordinary form of sinh-Poisson equation.

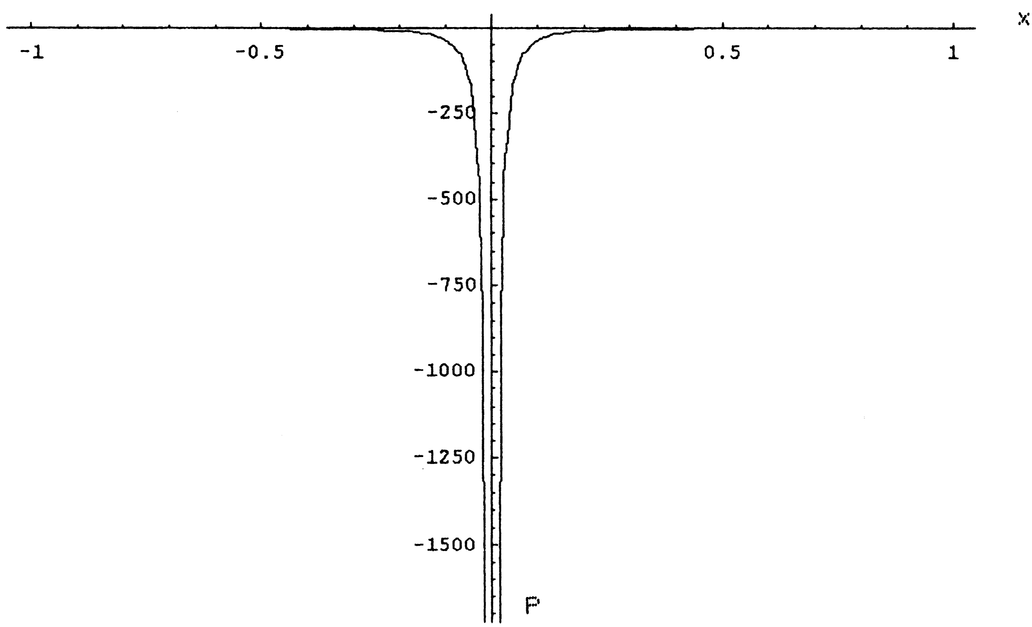

Fig. 10. The gas pressure for the ordinary form of sinh-Poisson equation with a finite h for $\lambda = 1$, $h = 1$, $P_0 = 1$ and $u_0^2 = 4\pi$.

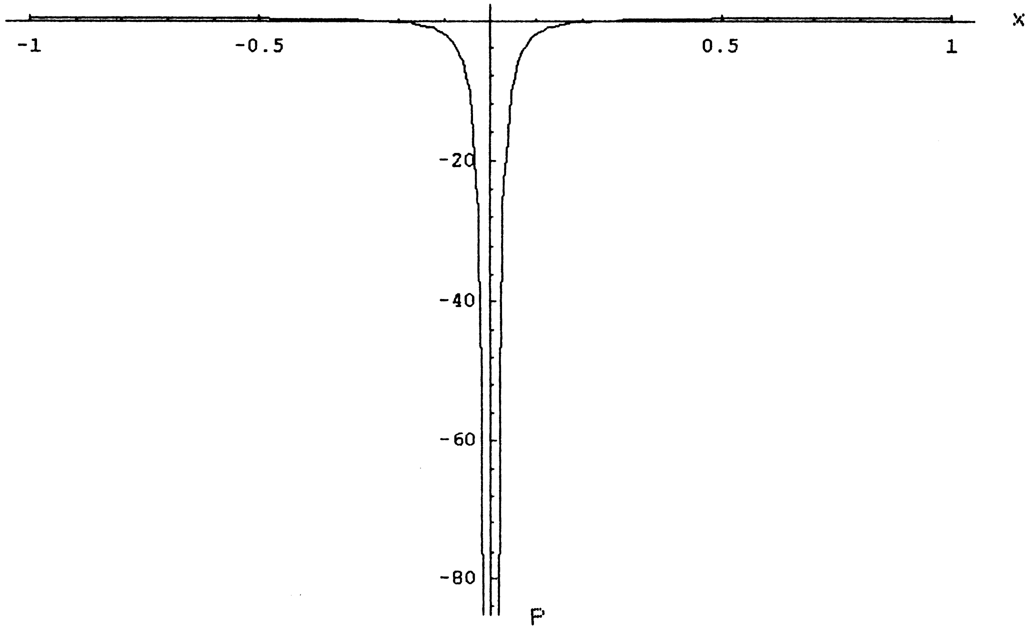

Fig. 11. The gas pressure for the ordinary form of sinh-Poisson equation with a finite h for $\lambda = 3$, $h = 1$, $P_0 = 1$ and $u_0^2 = 4\pi$.

Table 3
The maximum absolute errors of the magnetic potential $\tilde{u}$ from the exact solution of the sinh-Poisson equation at $y = 0.5$, $\lambda = 1$

x	CSCMs	FDM
− 0.8	1.54E − 4	8.21E − 4
− 0.6	8.99E − 4	2.84E − 3
− 0.4	4.65E − 3	2.64E − 3
− 0.2	2.21E − 3	1.44E − 3
0.0	1.71E − 3	1.49E − 3
0.2	1.82E − 3	2.21E − 3
0.4	1.91E − 3	3.25E − 3
0.6	3.12E − 3	3.98E − 3
0.8	3.81E − 3	4.11E − 3

with boundary conditions

$$S(-1) = -4\coth^{-1}(e^{-2\lambda}),\qquad S(1) = -4\coth^{-1}(e^{2\lambda}), \tag{58}$$

which integrates readily to give

$$S(x) = -4\coth^{-1}(e^{2\lambda x}). \tag{59}$$

Table 4
The maximum absolute errors of the gas pressure P for the sinh-Poisson equation at $y=0.5$, $\lambda=1$, $h=1$, $P_0=0$, $u_0^2=4\pi$

x	CSCMs	FDM
− 0.8	2.49E − 4	8.19E − 4
− 0.6	1.51E − 3	2.58E − 3
− 0.4	1.12E − 3	1.12E − 3
− 0.2	1.11E − 3	1.51E − 3
0.0	8.88E − 4	9.21E − 3
0.2	8.01E − 4	9.01E − 3
0.4	7.66E − 4	8.48E − 3
0.6	6.68E − 4	6.88E − 3
0.8	5.02E − 4	6.05E − 3

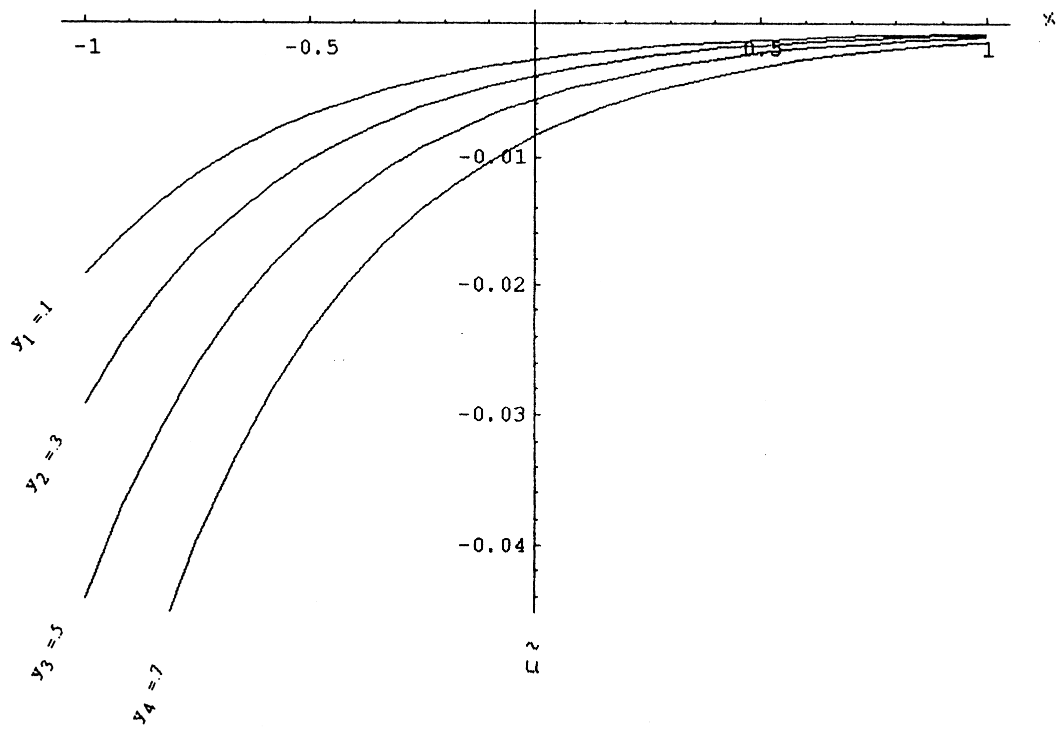

Fig. 12. The potential field lines $\tilde{u}$ for the sinh-Poisson equation for $\lambda=3$.

From Eq. (28), the magnetostatic potential take the form

$$\tilde{u}=-4\coth^{-1}(e^{2\lambda x}). \tag{60}$$

Applicable for some relevant values of λ. The approximations of the magnetic field $\tilde{u}$ of Eq. (60) are displayed in Fig. 9 and the approximations of the gas pressure are given in Figs. 10 and 11.

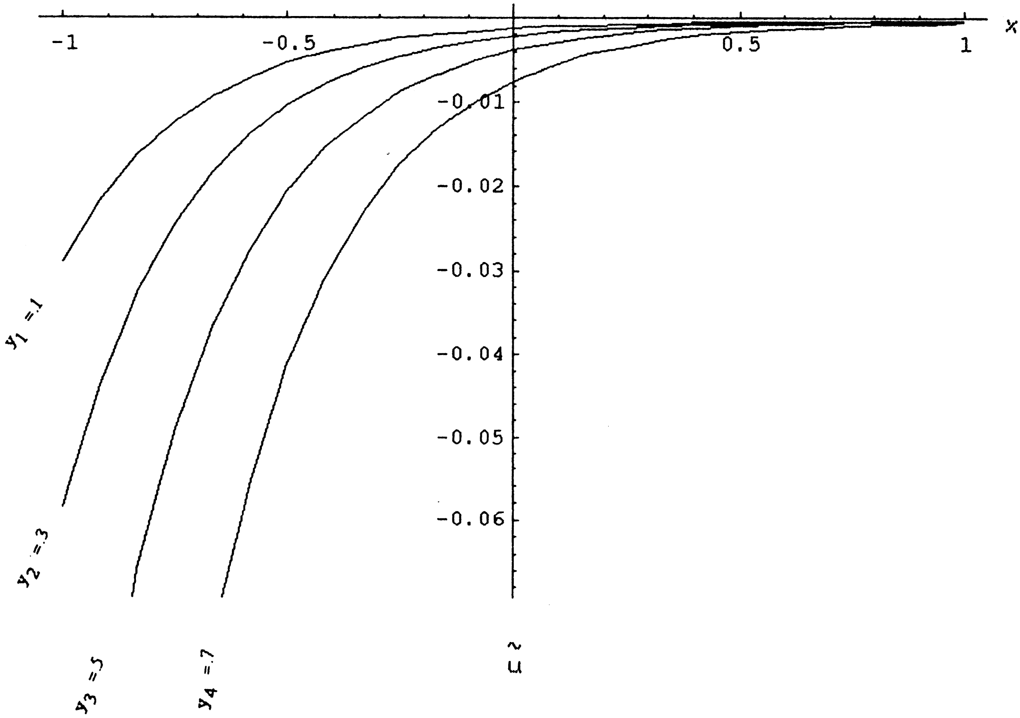

Fig. 13. The potential field lines $\tilde{u}$ for the sinh-Poisson equation for $\lambda = 5$.

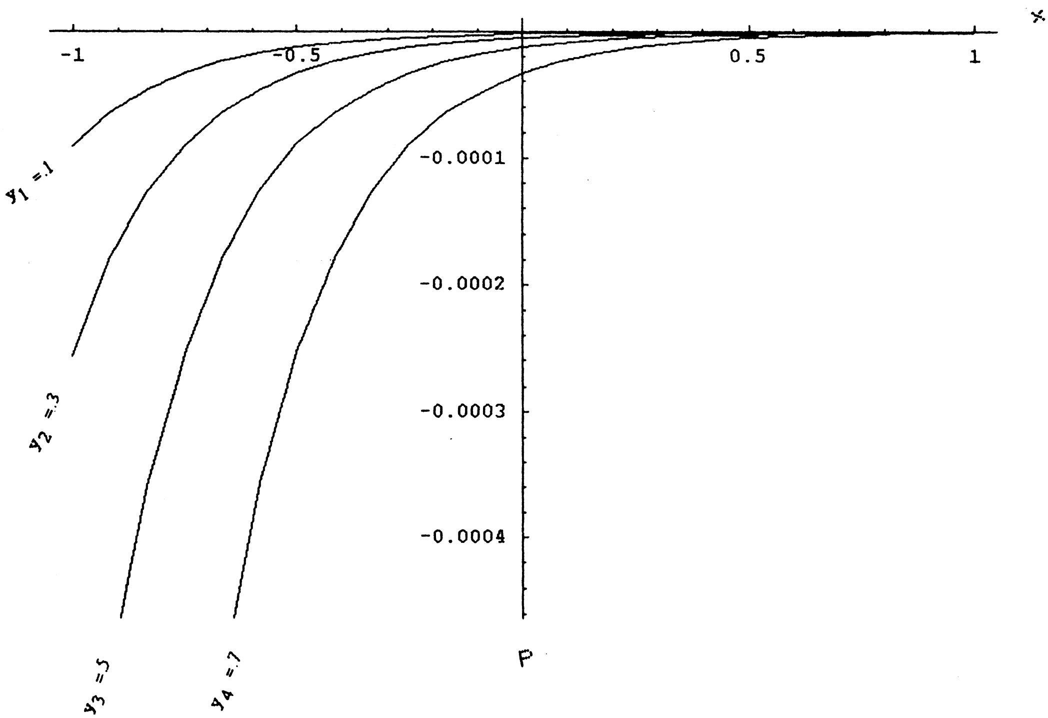

Fig. 14. The gas pressure for the sinh-Poisson equation for $\lambda = 3$, $h = 1$, $P_0 = 1$ and $u_0^2 = 4\pi/9$.

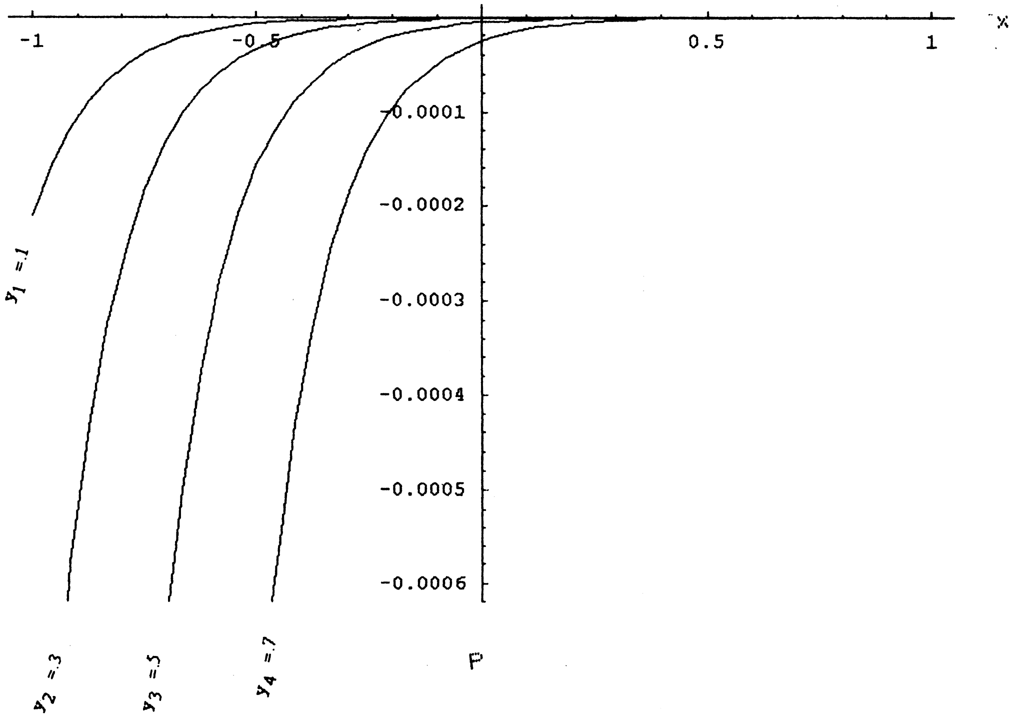

Fig. 15. The gas pressure for the sinh-Poisson equation for $\lambda = 5$, $h = 1$, $P_0 = 1$ and $u_0^2 = 4\pi/25$.

Example 4. Consider the sinh-Poisson BVP:

$$v_{xx} + v_{yy} = \lambda^2 \sinh(v), \quad -1 \leqslant x \leqslant 1, \quad y \geqslant 0. \tag{61}$$

This problem has the exact solution

$$v(x, y) = \tilde{u}(x, y) = -4\coth^{-1}[\mathrm{e}^{(\lambda(x+y))/\sqrt{2}+6}], \tag{62}$$

In Table 3, the given numbers refer to the maximum absolute errors of the magnetic potential $\tilde{u}$ of the sinh-Poisson equation (25) from exact solution (62) by applying CSCMs and the finite-difference method (FDM).

In Table 4, the given numbers refer to the maximum absolute errors of the gas pressure P of the sinh-Poisson equation (25) and the finite-difference method (FDM).

Applicable for some relevant values of λ. The approximations of the magnetic potential $\tilde{u}$ of the sinh-Poisson equation (25) are displayed in Figs. 12 and 13 and the approximations of the gas pressure are given in Figs. 14 and 15. Note that $\tilde{u}, P \to 0$ as $y \to \infty$.

5. Conclusion

In this paper we have investigated nonlinear isothermal, magnetostatic atmospheric models with one ignorable coordinate of a Cartesian coordinate system xyz. The underlying elliptic equation governing the force balance are reduced to Liouville's equation and sinh-Poisson's equation, respectively.

The CSCMs is developed and applied to the nonlinear elliptic boundary value problem in a bounded domain, moreover some explicit examples are carried out and the results are tabulated and displayed graphically. However, its relations with real solar atmospheres are also reported. The accuracy of the results is compared with the obtained exact analytical solutions. We point out the numerical results are in highly good agreement with the exact solutions. We note also that if we put $v_x = v_y = 1$ in the CSCMs, we obtain the results of the FDM method.

References

[1] T. Amari, J.J. Aly, Two-dimensional isothermal magnetostatic equilibria in a gravitational field, Astron. Astrophys. 208 (1989) 361.

[2] J. Birn, H. Goldstein, K. Schindler, A theory of the onset of solar eruptive processes, Solar Phys. 57 (1978) 81.

[3] C. Canuto, M.Y. Hussaini, A. Quarterani, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, Berlin, 1988.

[4] S.E. El-gendi, Chebyshev solution of differential, integral and integro-differential equations, Comput. J. 12 (1969) 282.

[5] D. Gottlieb, S.A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, SIAM, Philadelphia, 1977.

[6] D.B. Haidvogel, Resolution of downstream boundary layers in the Chebyshev approximation to viscous flow problems, J. Comput. Phys. 33 (1979) 313.

[7] J. Heyvaerts, J.M. Larsy, M. Schatzman, P. Witomsky, A mathematical model of solar flares, Quart. Appl. Math. XLI (1983) 1.

[8] A. Karageorghis, T.N. Phillips, Chebyshev spectral collocation method for laminar flow through a channal contraction, J. Comput. Phys. 84 (1989) 114.

[9] A.H. Khater, Analytical solutions for some nonlinear magnetostatic atmosphere, Astrophys. Sp. Sci. 162 (1989) 151.

[10] A.H. Khater, D.K. Callebaut, R.S. Ibrahim, Bäcklund transformations and painlevé analysis: exact solutions for the nonlinear isothermal magnetostatic atmospheres, Phys. Plasmas 4 (1997) 2853.

[11] A.H. Khater, D.K. Callebaut, R.S. Ibrahim, O.H. El-Kalaawy, Bäcklund transformations and exact soliton solutions for nonlinear evolution equations of the ZS\AKNS system, Chaos, Solitons & Fractals 8 (1998) 1847.

[12] A.H. Khater, D.K. Callebaut, E.S. Kamel, Painlevé analysis and exact solution for an isothermal magnetic atmosphere, Solar Phys. 178 (1998) 285.

[13] A.H. Khater, O.H. El-Kalaawy, D.K. Callebaut, Bäcklund transformations and exact solutions for nonlinear elliptic equation modelling isothermal magnetostatic atmosphere, IMA J. Appl. Math., in press, 1998.

[14] A.H. Khater, R.S. Ibrahim, A.B. Shamardan, D.K. Callebaut, Bäcklund transformations and painlevé analysis: exact solutions for a Grad-Shafranov type magnetohydrodynamic equilibrium, IMA J. Appl. Math. 58 (1997) 51.

[15] B.C. Low, J.A. Hundhausen, E.G. Zweibel, Nonlinear periodic solutions for the isothermal magnetostatic atmosphere, Phys. Fluids 26 (1983) 2731.

[16] S.A. Orszag, Spectral methods for problems in complex geometries, J. Comput. Phys. 37 (1980) 70.

[17] G.M. Webb, Isothermal magnetostatic atmospheres. II. similarity solutions with current proportional to the magnetic potential cubed, Astrophys. J. 327 (1988) 933.

[18] G.M. Webb, G.P. Zank, Application of the sine-Poisson equation in solar magnetostatics, Solar Phys. 127 (1990) 229.

[19] W. Zwingmann, Theoretical study of onset conditions for solar eruptive processes, Solar Phys. 111 (1987) 309.