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Closed form evaluation of a class of improper integrals

In the paper [1] which is part of a series of papers in a project to prove all the integrals in the book [2], the following integral is calculated ($n \geq 0$):

$$I = \int_0^{+\infty} \frac{\ln t}{(1+t^2)^{n+1}}\, dt. \tag{1}$$

The calculation is based on properties of the gamma and the digamma function, and Feynman's trick of differentiating under the integral sign. In this paper we use a more straightforward method, based on properties of self-reciprocal polynomials, a topic that I first came across when still at school, and which isn't very well known.
The main result in this paper relies on a closed form evaluation of the following indefinite integral:

$$\int \frac{t^{2n}-1}{(1+t^2)^{n+1}}\, dt. \tag{2}$$

This integral pops up if we rewrite (1) like this:

$$\begin{aligned} I &= \int_0^1 \frac{\ln t}{(1+t^2)^{n+1}}\, dt + \int_1^{+\infty} \frac{\ln t}{(1+t^2)^{n+1}}\, dt \\ &= -\int_0^1 \frac{t^{2n}-1}{(1+t^2)^{n+1}} \ln t\, dt \end{aligned}$$

where we have used the substitution $t \to 1/t$ in the second integral in the sum. Note that both integrals in the sum converge, the first one is bounded by $\int_0^1 \ln t\, dt$, the second one by $\int_1^{+\infty} \frac{\ln t}{t^2}\, dt$.
We can write the result as a double integral:

$$I = \int_0^1 \frac{t^{2n}-1}{(1+t^2)^{n+1}} \left(\int_t^1 \frac{1}{y}\, dy \right) dt$$

and if we change the order of integration we get a form with essentially the integral (2):

$$I = \int_0^1 \frac{1}{v} \left(\int_0^v \frac{t^{2n}-1}{(1+t^2)^{n+1}}\, dt \right) dv.$$

To calculate the indefinite integral (2), we start by factoring the polynomial in the numerator:

$$t^{2n} - 1 = (t^2-1)(t^{2n-2} + t^{2n-4} + \ldots + t^2 + 1).$$

The second factor is what we call a self-reciprocal polynomial (see [3]). A polynomial $p(t) = \sum_{i=0}^m a_i t^i$ is called self-reciprocal or palindromic if the coefficients satisfy $a_i = a_{m-i}$ for all i. If $m = 2k$ is even, such a polynomial can be written in the form $p(t) = t^k q\left(t + \frac{1}{t}\right)$

with q a polynomial of degree k in $t+\frac{1}{t}$.
In our case we have:

$$t^{2n-2}+t^{2n-4}+\ldots+t^2+1=t^{n-1}F_{n-1}(t)$$

with $F_{n-1}(t)=t^{n-1}+t^{n-3}+\ldots+\frac{1}{t^{n-3}}+\frac{1}{t^{n-1}}$, and the following theorem tells us how we can write $F_{n-1}(t)$ as a polynomial in $t+\frac{1}{t}$:

Theorem 1. *For $n \geq 1$ we have:*

$$F_{n-1}(t)=\sum_{i=1}^{n}\binom{n-i}{n-2i+1}(-1)^{i-1}\left(t+\frac{1}{t}\right)^{n-2i+1}. \tag{3}$$

Proof Note that we assume that $\binom{m}{k}=0$ if $k<0$. This theorem is an immediate consequence of the relationship between $f_{m-1}(t), f_m(t)$ and $f_{m+1}(t)$ with

$$f_k(t)=t^k+\frac{1}{t^k}\ .$$

It is easy to see that for $m \geq 1$:

$$f_{m+1}(t)=\left(t+\frac{1}{t}\right)f_m(t)-f_{m-1}(t).$$

As a consequence we have that for $m \geq 1$:

$$F_{m+1}(t)=\left(t+\frac{1}{t}\right)F_m(t)-F_{m-1}(t).$$

To prove this, use the following table:

$$\begin{array}{l} F_0(t)=1 \\ F_1(t)= \quad f_1(t) \\ F_2(t)=1 \ + \ f_2(t) \\ F_3(t)= \quad f_1(t) \ + \ f_3(t) \\ F_4(t)=1 \ + \ f_2(t) \ + \ f_4(t) \\ F_5(t)= \quad f_1(t) \ + \ f_3(t) \ + \ f_5(t) \\ F_6(t)=1 \ + \ f_2(t) \ + \ f_4(t) \ + \ f_6(t) \\ \quad\vdots \end{array}$$

and the fact that $1+f_2(t)=\left(t+\frac{1}{t}\right)f_1(t)-1$.
Note that

$$F_0(t)=1=\sum_{i=1}^{1}\binom{1-i}{1-2i+1}(-1)^{i-1}\left(t+\frac{1}{t}\right)^{1-2i+1}$$

and

$$F_1(t)=t+\frac{1}{t}=\sum_{i=1}^{2}\binom{2-i}{2-2i+1}(-1)^{i-1}\left(t+\frac{1}{t}\right)^{2-2i+1}.$$

Hence (3) is satisfied for $n = 1$ and $n = 2$. Using induction and a well-known identity for the binomial coefficients, the theorem follows easily.
We use this theorem to rewrite (2) as an integral with variable of integration $u = t + \frac{1}{t}$:

$$\begin{aligned}
\int \frac{t^{2n}-1}{(1+t^2)^{n+1}}\,dt &= \int \frac{(t^2-1)t^{n-1}F_{n-1}(t)}{(1+t^2)^{n+1}}\,dt \\
&= \int \frac{F_{n-1}(t)}{(t+\frac{1}{t})^{n+1}}\left(1-\frac{1}{t^2}\right)\,dt \\
&= \int \frac{F_{n-1}(t)}{(t+\frac{1}{t})^{n+1}}d\left(t+\frac{1}{t}\right) \\
&= \sum_{i=1}^{n} \binom{n-i}{n-2i+1}(-1)^{i-1}\int u^{-2i}\,du
\end{aligned}$$

leading immediately to:

Theorem 2. *For $n \geq 1$ we have:*

$$\int_0^v \frac{t^{2n}-1}{(1+t^2)^{n+1}}\,dt = \sum_{i=1}^{n} \binom{n-i}{n-2i+1}\frac{(-1)^i}{2i-1}\left(\frac{v}{v^2+1}\right)^{2i-1}.$$

Note that this theorem also holds for $n = 0$.
The same method can be applied to prove more generally:

Theorem 3. *For $n \geq k \geq 0$ we have:*

$$\int_0^v \frac{t^{2n-k}-t^k}{(1+t^2)^{n+1}}\,dt = \sum_{i=1}^{n-k} \binom{n-k-i}{n-k-2i+1}\frac{(-1)^i}{k+2i-1}\left(\frac{v}{v^2+1}\right)^{k+2i-1}.$$

We are now ready to calculate the integral we started off with. We have rewritten it as a double integral, and now use Theorem 2:

$$I = \int_0^1 \frac{1}{v}\left(\int_0^v \frac{t^{2n}-1}{(1+t^2)^{n+1}}\,dt\right)\,dv = \sum_{i=1}^{n} \binom{n-i}{n-2i+1}\frac{(-1)^i}{2i-1}\int_0^1 \frac{v^{2i-2}}{(v^2+1)^{2i-1}}dv.$$

This new integral is easy to calculate using the substitution $v = \tan z$:

$$\int_0^1 \frac{v^{2i-2}}{(v^2+1)^{2i-1}}dv = \int_0^{\frac{\pi}{4}} (\sin z\cos z)^{2i-2}dz = \frac{1}{2^{2i-1}}\int_0^{\frac{\pi}{2}} \sin^{2i-2} w\,dw$$

leading to a Wallis integral, for which we know that:

$$\int_0^{\frac{\pi}{2}} \sin^{2m} w\,dw = \frac{1}{2^{2m}}\binom{2m}{m}\frac{\pi}{2}.$$

Combining all this, we get our final result:

Theorem 4. *For $n \geq 0$ we have:*

$$\frac{2}{\pi}\int_0^{+\infty} \frac{\ln t}{(1+t^2)^{n+1}}\,dt = \sum_{i=1}^{n} \frac{(-1)^i}{2i-1}\binom{n-i}{n-2i+1}\binom{2i-2}{i-1}\frac{1}{2^{4i-3}}.$$

Some remarks. 1. If we compare this result with the one found in [1], we get the following combinatorial identity:

$$(A_n =) \sum_{i=1}^{n} \frac{(-1)^{i-1}}{2i-1}\binom{n-i}{n-2i+1}\binom{2i-2}{i-1}\frac{1}{2^{4i-3}} = \frac{\binom{2n}{n}}{2^{2n}}\sum_{i=1}^{n}\frac{1}{2i-1}$$

which I haven't been able to prove directly. But I've tried: see the next remark.
2. If you want to simplify expressions like the one above, it's always a good idea to consult Sloane's On-Line Encyclopedia of Integer Sequences (OEIS) [4]. If we look for the sequence of the numerators of A_n, which starts with $1, 1, 23, 11, 563, 1627, \ldots$, we immediately find Sloane's sequence A002549, and as it turns out, the value of the integral in Theorem 4 is precisely the coefficient of x^n in the Maclaurin series of the function $f(x) = \frac{\log(1-x)}{\sqrt{1-x}}$. Note that this series can be found by multiplying the series for $\log(1-x)$ with the binomial series for $\frac{1}{\sqrt{1-x}}$. The result is:

$$\frac{\log(1-x)}{\sqrt{1-x}} = \sum_{n=1}^{\infty} a_n x^n \quad \text{with} \quad a_n = -\sum_{i=1}^{n}\binom{2i-2}{i-1}\frac{1}{2^{2i-1}(n-i+1)}.$$

This doesn't help much but leads to the conjecture that A_n can be written in yet another way:

$$A_n = \sum_{i=1}^{n}\binom{2i-2}{i-1}\frac{1}{2^{2i-1}(n-i+1)}.$$

3. It is possible to prove the equality of these three expressions using the sophisticated machinery of Zeilberger's algorithm, which is explained in the very nice (and freely downloadable) book A=B [5] and which is implemented in most computer algebra systems. Zeilberger's algorithm finds a linear recurrence relation for a given sequence. It turns out that for the three sequences above the recurrence is essentially the same.

References

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