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Reference:
Levrie Paul. - Closed form evaluation of a class of improper integrals
To cite this reference: https://hdl.handle.net/10067/1621870151162165141
Closed form evaluation of a class of improper integrals

In the paper [1] which is part of a series of papers in a project to prove all the integrals in the book [2], the following integral is calculated \((n \geq 0)\):

\[
I = \int_0^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}} \, dt. \tag{1}
\]

The calculation is based on properties of the gamma and the digamma function, and Feynman’s trick of differentiating under the integral sign. In this paper we use a more straightforward method, based on properties of self-reciprocal polynomials, a topic that I first came across when still at school, and which isn’t very well known.

The main result in this paper relies on a closed form evaluation of the following indefinite integral:

\[
\int \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \, dt. \tag{2}
\]

This integral pops up if we rewrite (1) like this:

\[
I = \int_0^1 \frac{\ln t}{(1 + t^2)^{n+1}} \, dt + \int_1^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}} \, dt
\]

\[
= -\int_0^1 \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \ln t \, dt
\]

where we have used the substitution \(t \to 1/t\) in the second integral in the sum. Note that both integrals in the sum converge, the first one is bounded by \(\int_0^1 \ln t \, dt\), the second one by \(\int_1^{+\infty} \frac{\ln t}{t^2} \, dt\).

We can write the result as a double integral:

\[
I = \int_0^1 \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \left(\int_t^1 \frac{1}{y} \, dy\right) \, dt
\]

and if we change the order of integration we get a form with essentially the integral (2):

\[
I = \int_0^1 \frac{1}{v} \left(\int_0^v \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \, dt\right) \, dv.
\]

To calculate the indefinite integral (2), we start by factoring the polynomial in the numerator:

\[
t^{2n} - 1 = (t^2 - 1)(t^{2n-2} + t^{2n-4} + \ldots + t^2 + 1).
\]

The second factor is what we call a self-reciprocal polynomial (see [3]). A polynomial \(p(t) = \sum_{i=0}^m a_i t^i\) is called self-reciprocal or palindromic if the coefficients satisfy \(a_i = a_{m-i}\) for all \(i\). If \(m = 2k\) is even, such a polynomial can be written in the form \(p(t) = t^k q(t + \frac{1}{t})\).
with \( q \) a polynomial of degree \( k \) in \( t + \frac{1}{t} \).

In our case we have:

\[
t^{2n-2} + t^{2n-4} + \ldots + t^2 + 1 = t^{n-1}F_{n-1}(t)
\]

with \( F_{n-1}(t) = t^{n-1} + t^{n-3} + \ldots + \frac{1}{t^{n-3}} + \frac{1}{t^{n-1}} \), and the following theorem tells us how we can write \( F_{n-1}(t) \) as a polynomial in \( t + \frac{1}{t} \):

**Theorem 1.** For \( n \geq 1 \) we have:

\[
F_{n-1}(t) = \sum_{i=1}^{n} \left( \frac{n-i}{n-2i+1} \right) (-1)^{i-1} \left( t + \frac{1}{t} \right)^{n-2i+1}.
\]  

**Proof** Note that we assume that \( \binom{m}{k} = 0 \) if \( k < 0 \). This theorem is an immediate consequence of the relationship between \( f_{m-1}(t), f_m(t) \) and \( f_{m+1}(t) \) with

\[
f_k(t) = t^k + \frac{1}{t^k}.
\]

It is easy to see that for \( m \geq 1 \):

\[
f_{m+1}(t) = \left( t + \frac{1}{t} \right) f_m(t) - f_{m-1}(t).
\]

As a consequence we have that for \( m \geq 1 \):

\[
F_{m+1}(t) = \left( t + \frac{1}{t} \right) F_m(t) - F_{m-1}(t).
\]

To prove this, use the following table:

\[
\begin{align*}
F_0(t) &= 1 \\
F_1(t) &= f_1(t) \\
F_2(t) &= 1 + f_2(t) \\
F_3(t) &= f_1(t) + f_3(t) \\
F_4(t) &= 1 + f_2(t) + f_4(t) \\
F_5(t) &= f_1(t) + f_3(t) + f_5(t) \\
F_6(t) &= 1 + f_2(t) + f_4(t) + f_6(t) \\
&\vdots
\end{align*}
\]

and the fact that \( 1 + f_2(t) = \left( t + \frac{1}{t} \right) f_1(t) - 1 \).

Note that

\[
F_0(t) = 1 = \sum_{i=1}^{1} \left( \frac{1-i}{1-2i+1} \right) (-1)^{i-1} \left( t + \frac{1}{t} \right)^{1-2i+1}
\]

and

\[
F_1(t) = t + \frac{1}{t} = \sum_{i=1}^{2} \left( \frac{2-i}{2-2i+1} \right) (-1)^{i-1} \left( t + \frac{1}{t} \right)^{2-2i+1}.
\]
Hence (3) is satisfied for $n = 1$ and $n = 2$. Using induction and a well-known identity for the binomial coefficients, the theorem follows easily.

We use this theorem to rewrite (2) as an integral with variable of integration $u = t + \frac{1}{t}$:

$$
\int \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} dt = \int \frac{(t^2 - 1)t^{n-1}F_{n-1}(t)}{(1 + t^2)^{n+1}} dt
$$

$$
= \int \frac{F_{n-1}(t)}{(t + \frac{1}{t})^{n+1}} \left( 1 - \frac{1}{t^2} \right) dt
$$

$$
= \int \frac{F_{n-1}(t)}{(t + \frac{1}{t})^{n+1}} d\left(t + \frac{1}{t}\right)
$$

$$
= \sum_{i=1}^{n} \left[ \frac{n - i}{n - 2i + 1} \right] (-1)^{i-1} \int u^{-2i} du
$$

leading immediately to:

**Theorem 2.** For $n \geq 1$ we have:

$$
\int_{0}^{v} \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} dt = \sum_{i=1}^{n} \left[ \frac{n - i}{n - 2i + 1} \right] \frac{(-1)^{i}}{2i-1} \left( \frac{v}{v^2 + 1} \right)^{2i-1}.
$$

Note that this theorem also holds for $n = 0$.

The same method can be applied to prove more generally:

**Theorem 3.** For $n \geq k \geq 0$ we have:

$$
\int_{0}^{v} \frac{t^{2n-k} - t^k}{(1 + t^2)^{n+1}} dt = \sum_{i=1}^{n-k} \left[ \frac{n - k - i}{n - k - 2i + 1} \right] \frac{(-1)^{i}}{k+2i-1} \left( \frac{v}{v^2 + 1} \right)^{k+2i-1}.
$$

We are now ready to calculate the integral we started off with. We have rewritten it as a double integral, and now use Theorem 2:

$$
I = \int_{0}^{1} \frac{1}{v} \left( \int_{0}^{v} \frac{t^{2n-1}}{(1 + t^2)^{n+1}} dt \right) dv = \sum_{i=1}^{n} \left[ \frac{n - i}{n - 2i + 1} \right] \frac{(-1)^{i}}{2i-1} \int_{0}^{1} \frac{v^{2i-2}}{(v^2 + 1)^{2i-1}} dv.
$$

This new integral is easy to calculate using the substitution $v = \tan z$:

$$
\int_{0}^{1} \frac{v^{2i-2}}{(v^2 + 1)^{2i-1}} dv = \int_{0}^{\frac{\pi}{4}} (\sin z \cos z)^{2i-2} dz = \frac{1}{2^{2i-1}} \int_{0}^{\frac{\pi}{2}} \sin^{2i-2} w dw
$$

leading to a Wallis integral, for which we know that:

$$
\int_{0}^{\frac{\pi}{2}} \sin^{2m} w dw = \frac{1}{2^m} \binom{2m}{m} \frac{\pi}{2}.
$$

Combining all this, we get our final result:
Theorem 4. For \( n \geq 0 \) we have:

\[
\frac{2}{\pi} \int_{0}^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}} \, dt = \sum_{i=1}^{n} \frac{(-1)^i}{2i - 1} \left( \frac{n - i}{n - 2i + 1} \right) \left( \frac{2i - 2}{i - 1} \right) \frac{1}{2^{4i-3}}.
\]

Some remarks. 1. If we compare this result with the one found in [1], we get the following combinatorial identity:

\[
(A_n) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{2i - 1} \left( \frac{n - i}{n - 2i + 1} \right) \left( \frac{2i - 2}{i - 1} \right) \frac{1}{2^{4i-3}} = \left( \frac{2n}{n} \right) \frac{1}{2^{2n}} \sum_{i=1}^{n} \frac{1}{2i - 1}
\]

which I haven’t been able to prove directly. But I’ve tried: see the next remark.

2. If you want to simplify expressions like the one above, it’s always a good idea to consult Sloane’s On-Line Encyclopedia of Integer Sequences (OEIS) [4]. If we look for the sequence of the numerators of \( A_n \), which starts with 1, 1, 23, 11, 563, 1627, \ldots, we immediately find Sloane’s sequence A002549, and as it turns out, the value of the integral in Theorem 4 is precisely the coefficient of \( x^n \) in the Maclaurin series of the function \( f(x) = \log(1 - x) \sqrt{1 - x} \). Note that this series can be found by multiplying the series for \( \log(1 - x) \) with the binomial series for \( \sqrt{1 - x} \). The result is:

\[
\log(1 - x) \sqrt{1 - x} = \sum_{n=1}^{\infty} a_n x^n \quad \text{with} \quad a_n = -\sum_{i=1}^{n} \left( \frac{2i - 2}{i - 1} \right) \frac{1}{2^{2i-1}(n - i + 1)}.
\]

This doesn’t help much but leads to the conjecture that \( A_n \) can be written in yet another way:

\[
A_n = \sum_{i=1}^{n} \left( \frac{2i - 2}{i - 1} \right) \frac{1}{2^{2i-1}(n - i + 1)}.
\]

3. It is possible to prove the equality of these three expressions using the sophisticated machinery of Zeilberger’s algorithm, which is explained in the very nice (and freely downloadable) book A=B [5] and which is implemented in most computer algebra systems. Zeilberger’s algorithm finds a linear recurrence relation for a given sequence. It turns out that for the three sequences above the recurrence is essentially the same.

References


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