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Travelling wave solutions of some classes of nonlinear evolution equations in (1 + 1) and (2 + 1) dimensions

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Abstract

The tanh method is proposed to find travelling wave solutions in (1+1) and (2+1) dimensional wave equations. It can be extended to solve a whole family of modified Korteweg–de Vries type of equations, higher dimensional wave equations and nonlinear evolution equations.

Keywords: Travelling wave solutions; Tanh series solution method; Nonlinear evolution equations; Solitary waves

1. Introduction

Recently, the tanh method (a technique to solve nonlinear evolution equations (NLEEs)) which straightforwardly leads to travelling wave solutions has been introduced by one of the authors [4–7]. An advantage of this method is that it avoids tedious algebra and guesswork. Furthermore, it can be used as a direct method as well as a perturbative one and it is applicable to a broad class of nonlinear wave and NLEEs. This technique can even generate solutions unobtainable by the inverse scattering transform method, a very powerful one in the context of integrable systems [10].

The tanh method has been extended to coupled NLEEs [10,11], and two-dimensional problems. It has been successfully applied to derive exact equations as well as approximate solutions [8,9].

In this paper, the method is extended here to solve a whole new class of NLEEs. First, a family of modified Korteweg–de Vries (mK–dV) type of equations is treated where the nonlinear term contains multiples of a square root; next, we investigate other equations. The starting point is the

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expectation to find solitary-wave-type solutions. A balancing procedure then allows determination of the exact power so that only some algebra will be needed to find the amplitude and speed of the travelling wave under consideration. In the same way $(2+1)$ NLEEs are equally treated. These results may be used for extensions of NLEEs in order to use them as a starting point for a numerical solution of those problems.

This paper is organized as follows: the tanh method for solving $(1+1)$ and $(2+1)$ dimensional (D) NLEEs is briefly discussed in Section 2. Section 3 presents exact solutions for some $(1+1)$ and $(2+1)$D nonlinear wave equations. In the last section, we conclude our results, giving some comments and discussions.

2. The tanh method for solving (1 + 1) and (2 + 1)D nonlinear evolution equations

Various forms of the tanh method have been introduced. First, a power series in tanh was used as an ansatz to obtain analytical solutions of travelling wave type for certain NLEEs [2,3]. In the next stage, a first improvement of the method through the change of the independent variable $\zeta = c(x - vt)$ and $\zeta = kx + nz - \omega t$ was made, respectively, for the $(1+1)$D and $(2+1)$D, typical for travelling waves, and the use of the hyperbolic function, $\tanh(\zeta)$, itself as a new variable [4]. As a result, the involved algebra was considerably reduced compared to that in the first stage. Later on, this technique was refined by incorporating the boundary conditions in the series expansion and by determining the velocity through asymptotics [5,6]. Hence, a large number of nonlinear wave and nonlinear evolution equations may be studied and solved in a simple and systematic way. The method is based on five steps as follows.

Consider the general form of an NLEE:

$$u_t = K(u). \tag{1}$$

The main steps of the method (more details may be found in Ref. [7]) are given in the following.

Step 1: When one is looking for travelling wave solutions, the first step is to introduce the wave variable $\zeta = c(x - vt)$ and $\zeta = kx + nz - \omega t$, respectively, for $(1+1)$D and the $(2+1)$D in Eq. (1), so that $u(x,t) = U(\zeta)$ or $u(x,z,t) = U(\zeta)$. The wave solution then has a characteristic width $h = |c|^{-1}$ and travels with unperturbed speed v. The parameters k and n represent the wave numbers in the x and z directions, respectively. The frequency ω is expected to be a function of the wave numbers k and n. Consequently, the new variable ζ implies the following changes:

$$\frac{\partial}{\partial t} \to -cv\frac{\mathrm{d}}{\mathrm{d}\zeta}, \quad \frac{\partial}{\partial x} \to c\frac{\mathrm{d}}{\mathrm{d}\zeta}, \text{ etc.,} \tag{2a}$$

or

$$\frac{\partial}{\partial t} \to -\omega\frac{\mathrm{d}}{\mathrm{d}\zeta}, \quad \frac{\partial}{\partial x} \to k\frac{\mathrm{d}}{\mathrm{d}\zeta}, \quad \frac{\partial}{\partial z} \to n\frac{\mathrm{d}}{\mathrm{d}\zeta} \text{ etc.} \tag{2b}$$

Step 2: At this level, the PDE (1) has become an ODE, which can be integrated as long as all terms contain derivatives in ζ. Usually, the integration constant is taken to be zero in view of the localized solutions one is looking for; the nonzero case can be treated as well.

Step 3: One introduces $y = \tanh\zeta$ as a new independent variable leading to the following change of derivatives:

$$\frac{d}{d\zeta} \to (1-y^2)\frac{d}{dy}, \tag{3}$$

$$\frac{d^2}{d\zeta^2} \to (1-y^2)\left[\frac{d}{dy}(1-y^2)\frac{d}{dy}\right], \text{ etc.} \tag{4}$$

Step 4: The following series expansion is proposed as a solution:

$$\left[\begin{array}{l} u(x,z,t) \\ u(x,t) \end{array}\right. = U(\zeta) = S(y) = \sum_{m=0}^{M} b_m y^m, \tag{5}$$

with

$$y = \tanh\zeta = \left[\begin{array}{l} \tanh(kx + nz - \omega t), \\ \tanh(c(x - vt)). \end{array}\right. \tag{6}$$

The parameter M is determined by balancing the linear term(s) of highest order with the nonlinear one(s). Normally, M is a positive integer so that an analytic solution in closed form may be obtained. This balancing procedure will be extended here to the case where M is not an integer.

Step 5: The boundary conditions are implemented, if needed. One usually starts with the assumption

$$U(\zeta) \to 0, \quad \frac{d^{\tilde{n}}U}{d\zeta^{\tilde{n}}} \to 0 \ (\tilde{n} = 1, 2, \ldots) \text{ as } \zeta \to \pm\infty. \tag{7}$$

The integration constants, if present, should all be set to zero. Hence $S(y) \to 0$ as $y \to 1$; note that the solution may also vanish for $\zeta \to -\infty$ or $y \to -1$. Since in most cases $M = 2$, the following forms, which incorporate automatically the boundary conditions, are then proposed as possible solutions:

$$\text{(i)} \quad S_1(y) = b_0(1-y)(1+b_1 y) = (1-y)T(y) \text{ with } b_1 \pm 1, \tag{8a}$$

$$\text{(ii)} \quad S_2(y) = b_0(1-y)^2, \tag{8b}$$

both forms represent shock wave type of solutions and,

$$\text{(iii)} \quad S_3(y) = b_0(1-y^2), \text{ solitary wave type of solution.} \tag{8c}$$

Note that in the first and third cases $S_1(y)$ and $S_3(y)$ vanish for $\zeta \to \infty$ as $\exp(-2\zeta)$ and $S_2(y)$ as $\exp(-4\zeta)$. Applying these asymptotics, the velocity of travelling wave solution can be determined a priori, which greatly simplifies the calculations [3].

3. Exact analytical solution for (1 + 1) and (2 + 1)D nonlinear wave equations by tanh method

In this section, we explore the method discussed in Section 2 to solve some $(1+1)$D and $(2+1)$D nonlinear wave equations.

3.1. *The* (1 + 1)D *wave equations in the form of a modified Korteweg–de Vries (mK–dV) equation*

In this subsection we solve the unidirectional nonlinear wave equation in a plasma which consists of cold ions and nonisothermal electrons in the form of a mK–dV equation of the type given in [1,12]. We investigate the general equation

$$\frac{\partial \phi}{\partial t} + F(\phi)\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0, \tag{9}$$

which is used to expeditiously highlight the different features of the solitary wave. Solutions that lead to the spiky or bursting solitary waves along with the double layers are the outcome in their recent observations. The term $F(\phi)$ is in general nonlinear in the potential ϕ. We consider the following forms of $F(\phi)$:

(1) If $F(\phi)$ has the form

$$F(\phi) = \phi^{(r+1)/2}, \quad r \geqslant 0. \tag{10}$$

Eq. (9) becomes

$$\frac{\partial \phi}{\partial t} + \phi^{(r+1)/2}\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0, \quad r \geqslant 0. \tag{11}$$

To solve (11), we take the transformation

$$\phi(x,t) = v^2(x,t). \tag{12}$$

Eq. (11) becomes

$$vv_t + v^{(r+2)}v_x + 3v_x v_{xx} + vv_{xxx} = 0. \tag{13}$$

We look for travelling wave solution of Eq. (13) by using the tanh method described in Section 2. Therefore we introduce

$$\zeta = c(x - vt), \quad v(x,t) = U(\zeta). \tag{14}$$

Eq. (13) becomes

$$-vU\frac{dU}{d\zeta} + U^{(r+2)}\frac{dU}{d\zeta} + 2c^2\frac{dU}{d\zeta}\frac{d^2U}{d\zeta^2} + c^2\frac{d}{d\zeta}\left[U\frac{d^2U}{d\zeta^2}\right] = 0. \tag{15}$$

For the boundary conditions, we assume that

$$U(\zeta) \to 0, \quad \frac{dU}{d\zeta} \text{ and } \frac{d^2U}{d\zeta^2} \to 0 \text{ as } \zeta \to \pm\infty \tag{16}$$

since we look for solitary waves. Integrating Eq. (15), we get

$$-\frac{v}{2}U^2 + \frac{1}{r+3}U^{(r+3)} + c^2\left(\frac{dU}{d\zeta}\right)^2 + c^2 U\frac{d^2U}{d\zeta^2} = 0. \tag{17}$$

Introducing $y = \tanh\zeta$ as a new independent variable, and putting

$$v(x,t) = U(\zeta) = S(y), \tag{18}$$

we get

$$-\frac{v}{2}S^2+\frac{1}{r+3}S^{(r+3)}+c^2S(1-y^2)\left[-2y\frac{\mathrm{d}S}{\mathrm{d}y}+(1-y^2)\frac{\mathrm{d}^2S}{\mathrm{d}y^2}\right]+c^2(1-y^2)^2\left(\frac{\mathrm{d}S}{\mathrm{d}y}\right)^2=0. \quad (19)$$

If y^M is the highest power, we get from the balancing procedure (the linear term of the highest order is compared with the nonlinear ones in Eq. (19)) $M=2/(r+1)$, so that we assume that $S(y)$ is proportional to $(1-y^2)^{1/(r+1)}$, a solitary wave solution given by

$$S(y)=a(1-y^2)^{1/(r+1)}. \quad (20)$$

Substituting (20) into (19), we get

$$-\frac{v}{2}a^2(1-y^2)^{2/(r+1)}+\frac{a^{(r+1)}}{(r+3)}(1-y^2)^{(r+3)/(r+1)}+\frac{4a^2c^2}{(r+1)^2}y^2(1-y^2)^{2/(r+1)}$$
$$+a^2c^2(1-y^2)^{(r+2)/(r+1)}\left[\frac{4y^2}{(r+1)^2}(1-y^2)^{-r/(r+1)}-\frac{2}{(r+1)}(1-y^2)^{1/(r+1)}\right]=0, \quad (21)$$

or

$$-\frac{v}{2}+\frac{a^{(r+1)}}{(r+3)}(1-y^2)+c^2\left[\frac{(2r+10)}{(r+1)^2}y^2-\frac{2}{(r+1)}\right]=0. \quad (22)$$

Comparing the coefficients of each power of y on both sides, and solving the resulting two equations we end up with

$$v=\frac{16c^2}{(r+1)^2} \quad \text{and} \quad a=\left[\frac{(r+3)(2r+10)}{(r+1)^2}c^2\right]^{1/(r+1)}. \quad (23)$$

Substituting (23) into (20) and using (18), the solution of Eq. (13) is given by

$$v(x,t)=\left[\frac{(r+3)(2r+10)}{(r+1)^2}c^2\right]^{1/(r+1)}\operatorname{sech}^{2/(r+1)}\left[c\left(x-\frac{16c^2}{(r+1)^2}t\right)\right]. \quad (24)$$

From (12) and (24), the solution of Eq. (11) is given by [13]

$$\phi(x,t)=\left[\frac{(r+3)(2r+10)}{(r+1)^2}c^2\right]^{2/(r+1)}\operatorname{sech}^{4/(r+1)}\left[c\left(x-\frac{16c^2}{(r+1)^2}t\right)\right],\ \ r\geqslant 0. \quad (25)$$

Example 1. Setting $r=0$ in case (1) we obtain the solution of the equation

$$\frac{\partial\phi}{\partial t}+\phi^{1/2}\frac{\partial\phi}{\partial x}+\frac{\partial^3\phi}{\partial x^3}=0 \quad (26)$$

as

$$\phi(x,t)=900c^4\sec h^4\left[c(x-16c^2t)\right] \quad (27)$$

(2) If $F(\phi)$ has the form:

$$F(\phi)=\mathrm{i}\phi^{(r+1)/2}, \quad r\geqslant 0. \quad (28)$$

Eq. (9) becomes

$$\frac{\partial\phi}{\partial t}+\mathrm{i}\phi^{(r+1)/2}\frac{\partial\phi}{\partial x}+\frac{\partial^3\phi}{\partial x^3}=0,\quad r\geqslant 0. \tag{29}$$

Similar to case (1) the solution of Eq. (29) is given by

$$\phi(x,t)=\left[\frac{(r+3)(2r+10)}{\mathrm{i}(r+1)^2}c^2\right]^{2/(r+1)}\operatorname{sech}^{4/(r+1)}\left[c\left(x-\frac{16c^2}{(r+1)^2}t\right)\right],\quad r\geqslant 0. \tag{30}$$

Example 2. Setting $r=0$ in case (2) we obtain the solution of the equation

$$\frac{\partial\phi}{\partial t}+\mathrm{i}\phi^{1/2}\frac{\partial\phi}{\partial x}+\frac{\partial^3\phi}{\partial x^3}=0 \tag{31}$$

as

$$\phi(x,t)=-900c^4\operatorname{sech}^4[c(x-16c^2t)]. \tag{32}$$

3.2. *Exact analytical solution for* $(2+1)$D *nonlinear wave equations by tanh method*

In this subsection, we explore the method discussed in Section 2 to solve two examples in the form of $(2+1)$D nonlinear wave equations.

Example 3. Consider the sinh-Poisson equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\phi}{\partial t^2}=\lambda^2\sinh\phi. \tag{33}$$

To solve (33), we introduce

$$\zeta=kx+nz-\omega t,\quad \phi(x,z,t)=U(\zeta). \tag{34}$$

Eq. (33) becomes

$$(k^2+n^2-\omega^2)\frac{\mathrm{d}^2U}{\mathrm{d}\zeta^2}=\lambda^2\sinh U. \tag{35}$$

For the boundary conditions, we assume that

$$U(\zeta)\to 0\quad\text{and}\quad\frac{\mathrm{d}^2U}{\mathrm{d}\zeta^2}\to 0\ \text{as}\ \zeta\to\pm\infty. \tag{36}$$

Multiplying Eq. (35) by $\mathrm{d}U/\mathrm{d}\zeta$ and integrating with respect to ζ, we get

$$(k^2+n^2-\omega^2)\left(\frac{\mathrm{d}U}{\mathrm{d}\zeta}\right)^2=2\lambda^2\sinh^2\left(\frac{1}{2}U\right)+C, \tag{37}$$

where C is an arbitrary constant. Put

$$W=\exp\left(\frac{U}{2}\right)\quad\text{and}\quad C=0. \tag{38}$$

Eq. (37) becomes

$$\frac{dW}{d\zeta} = \frac{\lambda}{2\sqrt{(k^2+n^2-\omega^2)}}(W^2-1), \tag{39}$$

with the condition

$$W \to 1 \quad \text{and} \quad \frac{dW}{d\zeta} \to 0 \text{ as } \zeta \pm \infty. \tag{40}$$

Introducing $y = \tanh\zeta$ as a new independent variable, we propose the following series expansion as a solution:

$$W(\zeta) = S(y) = \sum_{m=0}^{M} a_m y^m. \tag{41}$$

Substituting (41) into (39), we get

$$(1-y^2)\frac{dS}{dy} - \frac{\lambda}{2\sqrt{(k^2+n^2-\omega^2)}}(S^2-1) = 0. \tag{42}$$

Balancing the linear term of the highest order with the nonlinear ones in Eq. (42) yields $M = 1$. Taking into consideration the boundary conditions (40) we propose the following unique form as a possible solution:

$$S(y) = a_0 + a_1 y. \tag{43}$$

Substituting (43) into (42), we get

$$a_1(1-y^2) - \frac{\lambda}{2\sqrt{(k^2+n^2-\omega^2)}}(a_0^2 + 2a_0a_1y + a_1^2y^2 - 1) = 0. \tag{44}$$

Comparing the coefficients of each power of y on both sides, we have
coeff. of y^0:

$$a_1 - \frac{\lambda}{2\sqrt{(k^2+n^2-\omega^2)}}(a_0^2-1) = 0, \tag{i}$$

coeff. of y:

$$a_0a_1 = 0 \;\Rightarrow\; a_0 = 0. \tag{ii}$$

coeff. of y^2:

$$-a_1 - \frac{\lambda}{2\sqrt{(k^2+n^2-\omega^2)}}a_1^2 = 0. \tag{iii}$$

From (i)–(iii), we get

$$a_0 = 0, \quad a_1 = 1 \quad \text{and} \quad \omega = \pm\sqrt{(k^2+n^2-\tfrac{1}{4}\lambda^2)}. \tag{45}$$

Substituting (45) into (43) and using (41), the solution of Eq. (39) is given by

$$W(\zeta) = \tanh\zeta, \quad \zeta = kx + nz \mp \sqrt{(k^2+n^2-\tfrac{1}{4}\lambda^2)}t. \tag{46}$$

From (38) and (34), the solution of Eq. (33) is given by

$$\phi(x,z,t) = 2\ln\left(\tanh\left[kx + nz \mp \sqrt{(k^2 + n^2 - \tfrac{1}{4}\lambda^2)t}\right]\right). \tag{47}$$

Example 4. Consider the sine-Poisson equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial t^2} = \lambda^2 \sin\phi. \tag{48}$$

Similar to Example 3, the solution of the sine-Poisson equation (48) is given by

$$\phi(x,z,t) = 4\tan^{-1}\left\{A\exp\left(2\left[kx + nz \mp \sqrt{(k^2 + n^2 - \tfrac{1}{4}\lambda^2)t}\right]\right)\right\}, \tag{49}$$

where A is an arbitrary constant.

4. Conclusion and discussion

It is shown that the tanh method in its present form is a powerful technique for investigating nonlinear wave equations, in particular those where diffusions are involved. The mean feature of this approach is based on the hypothesis that the travelling-wave solutions we are looking for may be found and expressed in terms of tanh. This hyperbolic function is then used as an independent variable. Moreover, the embedding of the boundary conditions within the proposed solutions (whenever possible) and the a priori determination of the velocity through asymptotes have a strong impact on the ease of use of the method, so that tedious algebra is avoided. Closed-form solutions are then derived in an elegant and straightforward way since we deal with polynomial expressions. This paper gives the solution of Eq. (9) for $(1+1)$D for general forms of $F(\phi)$ and the sinh-Poisson equation as well as the sine-Poisson equation for $(2+1)$D nonlinear wave equation.

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