ELSEVIER Journal of Pure and Applied Algebra 107 (1996) 337–351

JOURNAL OF PURE AND APPLIED ALGEBRA

Induction functors and stable Clifford theory for Hopf modules

Fred Van Oystaeyen [a,*], Yinhuo Zhang [b,1]

[a] *Department of Mathematics, University of Antwerp UIA, B-2610, Belgium*

[b] *Department of Mathematics, Fudan University, Shanghai 200433, PR China*

Abstract

In this paper, we introduce a revised induction functor to study the Hopf modules and the invariant modules for a Hopf comodule algebra. The simple Hopf modules and the stable Clifford theory are discussed.

0. Introduction

Throughout, H is an arbitrary Hopf algebra with a bijective antipode S over a field k. The object we will study is a k-algebra extension A/B that is an H-extension with a total integral. In Section 1 we recall the definitions and review some basic facts about comodule algebras and Hopf modules. Our aim in Section 2 is to establish an equivalence between Mod-B and a quotient category of $\mathbf{M}_A^H$(cf. Theorem 2.4). This may be used to obtain a characterization of Hopf Galois extensions, e.g., A/B is H-Galois if and only if A is a generator in $\mathbf{M}_A^H$. In Section 3 we introduce a modified induction functor $-\bar{\otimes}_B A$ which together with the invariant functor $(-)_0$ determines an equivalence between Mod-B and a full subcategory of $\mathbf{M}_A^H$ (cf. Theorem 3.10).

In Section 4 we discuss the relation between simple B-modules and simple Hopf modules and again we establish an equivalence between the corresponding categories. Finally, in Section 5 we are able to extend Dade's stable Clifford theory for group graded rings [5] to the case of H-comodule algebras.

* Corresponding author.

[1] The author is supported by a grant of the European Liaison Committee.

SSDI 0022-4049(95)00073-9

1. Preliminaries

H is a Hopf algebra, over a fixed field k, with comultiplication Δ, counit ε and antipode S. We refer to [13] for full detail on Hopf algebras, and in particular we use the sigma notation: $\Delta h = \sum h_{(1)} \otimes h_{(2)}$, for $h \in H$.

Definition 1.1. An algebra A is called a *right H-comodule algebra* if A is a right H-comodule with comodule structure ρ being an algebra map.

An algebra extention A/B is called an *H-extension* if A is a right H-comodule algebra and B is its *invariant subalgebra* $A^{\mathrm{Co}H} = \{b \in A \mid \rho(b) = b \otimes 1\}$. Denote by $G(H)$ the group of all *group-like elements* in H, i.e., $g \in H, \Delta(g) = g \otimes g$. For an H-comodule algebra A, we consider $\mathscr{S}(A) = \{a \in A | \rho(a) \in A \otimes G(H)\}$, to be a graded subalgebra of type $G(H)$, and it is called the *semi-invariant subalgebra* of A.

By an H-Galois extension A/B, we mean an H-extension A/B with the bijective map

$$\beta : A \otimes_B A \longrightarrow A \otimes H, \quad \beta(a \otimes b) = \sum ab_{(0)} \otimes b_{(1)}.$$

If the antipode S is bijective, another canonical map,

$$\beta' : A \otimes_B A \longrightarrow A \otimes H, \quad \beta'(a \otimes b) = \sum a_{(0)} b \otimes a_{(1)}$$

is also bijective. Recall that a right A-module M is a *Hopf module* if M is also a right H-comodule satisfying the compatibility condition: $\rho_M(ma) = \sum m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$. Denote by $\mathbf{M}_A^H$ the *Hopf module category*, and write $\mathbf{M}_A^H(M,N)$ for the set of all *Hopf morphisms* from M to N, which are both right A-homomorphisms and right H-comodule morphisms. To a Hopf module M, we associate a B-module $M_0 = \{m \in M \mid \rho(m) = m \otimes 1\}$ which is called the *invariant module* of M. It is not hard to see that $M_0 \simeq \mathbf{M}_A^H(A,M)$. We call the functor $(-)_0$ (or $\mathbf{M}_A^H(A,-)$) the *invariant functor* from $\mathbf{M}_A^H$ to Mod-B. For an H-comodule algebra A, a *total integral* $\varphi : H \longrightarrow A$ is an H-comodule map with $\varphi(1) = 1$.

Example 1.2. Let $H = kG$, G a group, and R a graded algebra of type G. In each homogeneous component R_σ, take an element $x_\sigma \in R_\sigma$ and $x_1 = 1$. Define $\varphi : H \longrightarrow R$ given by $\sigma \mapsto x_\sigma$, then φ is a total integral.

Note that a total integral generalizes the notion of an integral for a finite dimensional cosemisimple Hopf algebra [6] as well as the notion of a trace-1 element for an H-module algebra [3]. In the sequel, we always assume that H has a *bijective antipode* S. The property of having a total integral has the following nice characterizations [6, 11, 14]:

Proposition 1.3. *The following are equivalent for an H–extension A/B:*

(a) *There is a total integral $\varphi : H \longrightarrow A$;*

(b) *A is a relative injective comodule;*
(c) *The canonical structure map* $\rho : A \longrightarrow A \otimes H$ *splits in* $\mathbf{M}_A^H$;
(d) *A is coflat as an H-comodule;*
(e) *The invariant functor* $(-)_0$ *is exact;*
(f) *A is a projective object in* $\mathbf{M}_A^H$.

Lemma 1.4 ((cf. [6, 10]). *Let* A/B *be an H-extension with a total integral. Then for any B-module N, the canonical map* $N \longrightarrow (N \otimes_B A)_0$, $n \mapsto n \otimes 1$, *is an isomorphism.*

Proposition 1.5. *With assumptions as above, the Hopf module category* $\mathbf{M}_A^H$ *is a Grothendieck category.*

Proof. $\mathbf{M}_A^H$ is clearly a preadditive category. Any Hopf morphism in $\mathbf{M}_A^H$ has kernel and cokernel because these exist both when the morphism is viewed as an A-module and as an H-comodule morphism. It is easily verified that any Hopf morphism α factorizes as $\alpha = \beta\gamma$, where β is a kernel and γ is a cokernel. Moreover, every finite family of Hopf modules has an obvious direct sum (or product). Consquently $\mathbf{M}_A^H$ is certainly an abelian category. In order to establish that $\mathbf{M}_A^H$ is a Grothendieck category, we should check that it satisfies the condition AB5: $(\sum_i M_i) \cap N = \sum_i M_i \cap N$, and that it has a generator. Since AB5 holds for A-modules we only have to check that any intersection $M_i \cap N$ is again a Hopf module. However, the latter follows from the fact that $M_i \cap N$ is a kernel of the Hopf morphism: $M_i \oplus N \longrightarrow M_i + N$. Hence $\mathbf{M}_A^H$ allows arbitrary direct sums. Finally, let us show that $\mathbf{M}_A^H$ has a generator. Define a Hopf module $A \otimes H$ with comodule structure stemming from the factor H and the A-module structure given by $(x \otimes h)a = \sum x a_{(0)} \otimes h a_{(1)}$. Let ψ be the inverse map of $\rho : A \longrightarrow A \otimes H$ in $\mathbf{M}_A^H$, i.e., $\psi\rho = 1_A$. Define a map η_M in $\mathbf{M}_A^H$ for any Hopf module M as follows:

$$(A \otimes H)^{(M)} \simeq M \otimes A \otimes H \xrightarrow{\eta_M} M,$$

given by $\eta_M(m \otimes a \otimes h) = \sum m_{(0)} \psi(a \otimes S(m_{(1)})h)$. It is not difficult to check that η_M is a Hopf morphism. Furthermore, $\eta_M \sum (m_{(0)} \otimes 1 \otimes m_{(1)}) = \sum m_{(0)} \psi(1 \otimes S(m_{(1)}) m_{(2)}) = m$. So η_M is epimorphic. It follows that $A \otimes H$ is a generator. □

Note that the above proposition is generalized in [1]. The condition 'existence of a total integal' is not necessary.

The forgetful functor $U : \mathbf{M}_A^H \longrightarrow$ Mod-A associates to M the underlying right A-module $\underline{M}$. This functor has a right adjoint functor F : Mod-$A \longrightarrow \mathbf{M}_A^H$, which associates to N the Hopf module $N \otimes H$ with comodule structure coming from H and A-module structure as follows: $(n \otimes h)a = \sum n a_{(0)} \otimes h a_{(1)}$, $n \in N$, $h \in H$, and $a \in A$. Note that $F(\underline{M}) = \underline{M} \otimes H$ is isomorphic to the Hopf module $M \otimes H$ having the A-module structure coming from M and the comodule structure as follows: $\rho(m \otimes h) = \sum m_{(0)} \otimes h_{(1)} \otimes h_{(2)} m_{(1)}$. The latter is called the *shifted Hopf module* associated to M, which is a generalization of the shifted graded module. The isomorphism $\underline{M} \otimes H \simeq M \otimes H$ is given by the following map:

$$\underline{M} \otimes H \longrightarrow M \otimes H, \quad m \otimes h \mapsto \sum m_{(0)} \otimes hS^{-1}(m_{(1)}).$$

So we see that $A \otimes H$ in $\mathbf{M}_A^H$ has two definitions as a Hopf module.

Proposition 1.6. *(U,F) is an adjoint pair of functors.*

Proof. For $M \in \mathbf{M}_A^H, N \in$ Mod-A, we establish the following abelian group homomorphisms:

$$\gamma : \mathrm{Hom}_A(M,N) \longrightarrow \mathbf{M}_A^H(M,N \otimes H)$$

given by $\gamma(f)(m) = \sum f(m_{(0)}) \otimes m_{(1)}$, $f \in \mathrm{Hom}_A(M,N)$, and

$$\delta : \mathbf{M}_A^H(M,N \otimes H) \longrightarrow \mathrm{Hom}_A(M,N)$$

given by $\delta(g)(m) = (1 \otimes \varepsilon)g(m)$, $g \in \mathbf{M}_A^H(M,N \otimes H)$. It is routine to verify that γ and δ are well defined and inverse to each other. □

Corollary 1.7. (a) *If M is projective in $\mathbf{M}_A^H$ then $\underline{M}$ is a projective A-module.*
(b) *If N is an injective A-module, then $N \otimes H$ is an injective Hopf module.*

Remark 1.8. In case H is a finite dimensional Hopf algebra, then $\mathbf{M}_A^H = Mod - A\#H$. We have an induction functor $- \otimes_A A\#H$ and a coinduction functor $\mathrm{Hom}_{-A}(A\#H, -) : Mod - A \longrightarrow \mathbf{M}_A^H$. In [15, Theorem 3.2] we observed that these two functors are isomorphic. On the other hand, $(U, \mathrm{Coind.})$ is again an adjoint pair of functors. By the uniqueness of the right adjoint of U we must have $F \sim$ Coind. In the more general situation we did consider above, the functor F may then be viewed as a generalization of the coinduction functor in the finite dimensional case.

Corollary 1.9. *Suppose that H is a finite dimensional Hopf algebra such that either H is cosemisimple or H is unimodular with an involutive antipode S, i.e., $S^2 = S$ such that A has a total integral. Then a Hopf module M is projective (resp. injective) in $\mathbf{M}_A^H$ if and only if $\underline{M}$ is projective (resp. injective) in* Mod $- A$.

Proof. The assumption entails that $A\#H^*/A$ is separable [15]. So the 'if part' holds. It only remains to show that M is injective in $\mathbf{M}_A^H$ implies that $\underline{M}$ is injective in Mod-A. But this follows easily from the isomophism

$$\mathrm{Hom}_{-A}(N,\underline{M}) \simeq \mathrm{Hom}_{A\#H^*}(N \otimes_A A\#H^*, M). \qquad \square$$

Again consider $H = kG$, a group Hopf algebra with a finite group G, R a graded algebra of type G. A graded module M is gr-projective, resp. gr-injective if and only if M is R-projective, resp. R-injective.

2. Quotient categories of Hopf modules

In this section H is a Hopf algebra over the field k with a bijective antipode S. A/B is an H-extension with a total integral φ. For $M \in \mathbf{M}_A^H$, the submodules M_i such that $(M_i)_0 = 0$ have a sum $\sum M_i = \tau(M)$ for which $(\tau(M))_0 = 0$ because the invariant functor $(-)_0$ is exact and preserves direct sums. $\tau(M)$ is a uniquely largest Hopf submodule having the property that it has zero for its invariants. Since τ is respected by any Hopf morphisms τ is a subfunctor of the identity functor I on $\mathbf{M}_A^H$. Recall that a subfunctor τ of I is a *radical* if $\tau(M/\tau(M)) = 0$ for any Hopf module $M \in \mathbf{M}_A^H$. The exactness of $(-)_0$ ensures that τ is an idempotent radical in the Grothendieck category $\mathbf{M}_A^H$ (see [12] or [7] for some detail on radicals, localizations, torsion theory or quotient categories).

To such a radical τ, we can associate a torsion theory $(\mathscr{T}, \mathscr{F})$, where the torsion class is $\mathscr{T} = \{M \in \mathbf{M}_A^H \mid \tau(M) = M\}$, and $\mathscr{F} = \{M \in \mathbf{M}_A^H \mid \tau(M) = 0\}$ is then called the torsion free class in $\mathbf{M}_A^H$. Recall that a torsion theory is *hereditary* if the torsion class is closed under subobjects, and this just means the associated radical being left exact. We have:

Lemma 2.1. *$(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory.*

Corollary 2.2. *If M is a simple Hopf module then M is either in $\mathscr{T}$ or else in $\mathscr{F}$.*

In case H is a finite dimensional Hopf algebra, $\mathbf{M}_A^H$ is a module category. In this case, $\mathscr{T}$ is cogenerated by an injective torsion free module, and $\mathscr{F}$ is closed under injective envelopes. Recall that a nonempty subclass of objects of a Grothendieck category is called a *Serre subcategory* if it is closed under subobjects, quotient objects, and extensions, If, in addition, it is closed under arbitrary direct sums, then we say that it is a *localizing subcategory*.

Lemma 2.3. *$\mathscr{T}$ is a localizing subcategory of $\mathbf{M}_A^H$.*

Proof. Note that A is a finitely generated projective object in $\mathbf{M}_A^H$ and the functor $(-)_0$ is isomorphic to $\mathbf{M}_A^H(A, -)$. □

With respect to a localizing subcategory of a Grothendieck category one can form a quotient category. Hence we may form the quotient category $\mathbf{M}_A^H/\mathscr{T}$, denoted by $\overline{\mathbf{M}}_A^H$, which is defined as follows: The objects of $\overline{\mathbf{M}}_A^H$ are those of $\mathbf{M}_A^H$ and the morphisms are defined by

$$\overline{\mathbf{M}}_A^H(M,N) = \varinjlim_{M',N'} (M', N/N'),$$

where $M' \subseteq M$, $M/M' \in \mathscr{T}$, and $N' \subseteq N$, $N' \in \mathscr{T}$. In view of [7], $\overline{\mathbf{M}}_A^H$ is also a Grothendieck category. We denote by $X : \mathbf{M}_A^H \longrightarrow \overline{\mathbf{M}}_A^H$, and $Y : \overline{\mathbf{M}}_A^H \longrightarrow \mathbf{M}_A^H$ the canonical functors. It is well known that X is an exact functor and Y is the right adjoint of X. Moreover, Y is a left exact functor. Let

$$\phi : X \circ Y \longrightarrow \mathrm{I}_{\overline{\mathbf{M}}_A^H}, \qquad \psi : \mathrm{I}_{\mathbf{M}_A^H} \longrightarrow Y \circ X$$

be the natural transformations of functors X and Y. Then ϕ is an isomorphism. Furthermore, if $M \in \mathbf{M}_A^H$, then we have the exact sequence

$$0 \longrightarrow \ker(\psi_M) \longrightarrow M \xrightarrow{\psi_M} (Y \circ X) \longrightarrow \operatorname{coker}(\psi_M) \longrightarrow 0,$$

where $\ker(\psi_M)$ and $\operatorname{coker}(\psi_M)$ are in $\mathscr{T}$. Define functors

$$W : \text{Mod-}B \longrightarrow \overline{\mathbf{M}}_A^H,\ W = X \circ (- \otimes_B A),$$
$$V : \overline{\mathbf{M}}_A^H \longrightarrow \text{Mod-}B,\ V = (-)_0 \circ Y.$$

Theorem 2.4. *Suppose that A/B is an H-extension with a total integral. Then the above functors W and V define an equivalence between* Mod-B *and the quotient category $\overline{\mathbf{M}}_A^H$.*

Proof. For $N \in$ Mod-B, we have

$$\begin{aligned}(V \circ W)(N) &= (-)_0 \circ Y \circ X \circ (- \otimes_B A)(N)\\ &\simeq (-)_0 \circ \mathrm{Id}(N \otimes_B A)\\ &= (N \otimes_B A)_0 \simeq N.\end{aligned}$$

So $V \circ W \simeq \mathrm{Id}_B$. On the other hand, let M be in $\overline{\mathbf{M}}_A^H$, we have the natural map $\mu_{Y(M)} : Y(M)_0 \otimes_B A \longrightarrow Y(M)$, where both $\ker(\mu_{Y(M)})$ and $\operatorname{coker}(\mu_{Y(M)})$ are torsion Hopf modules because of exactness of $(-)_0$ and Lemma 1.4. It follows that $X(\mu_{Y(M)})$ is an isomorphism, i.e.,

$$X(Y(M)_0 \otimes_B A) \simeq X \circ Y(M)$$

in $\overline{\mathbf{M}}_A^H$. The composite map $\phi_M \circ X(\mu_{Y(M)}) : M \longrightarrow M$ must be an isomorphim since $\phi(M)$ is. But it is easily seen that the latter map is nothing but the desired transformation $W \circ V(M) \longrightarrow M$. So $W \circ V \simeq Id_{\overline{\mathbf{M}}_A^H}$. □

The above theorem is a generalization of the result obtained in the finite dimensional case in [2] (H is semisimple) and in [16] (with a trace-1 element).

Corollary 2.5. *With assumptions as above, the following statements are equivalent:*

(a) *A/B is H-Galois;*
(b) *Any Hopf module is torsion free;*
(c) *A is a generator in $\mathbf{M}_A^H$;*
(d) *The functor $(-)_0$ is separable;*
(e) *$(-)_0$ and $- \otimes_B A$ define an equivalence between $\mathbf{M}_A^H$ and* Mod-B;
(f) *$(\mathscr{T}, \mathscr{F})$ is a perfect torsion theory (in the sense of* [12]).

Proof. For details on separable functors we refer to [9]. The implication (e) $\Rightarrow$ (d) is then obvious. Since any separable functor is clearly faithful, (d) implies (c). (c) $\Rightarrow$ (b) because of $M_0 \simeq \mathbf{M}_A^H(A,M) \neq 0$ for any Hopf module $M \in \mathbf{M}_A^H$. (b) $\Rightarrow$ (e) follows from the foregoing theorem.

(e) $\Rightarrow$ (a) is really obvious since $A \otimes H$ is a Hopf module with invariant B-module A. The following proof of (a) $\Rightarrow$ (e) is essentially the same as the one given in [11; 3.5]. For any Hopf module $M \in \mathbf{M}_A^H$, we get an exact sequence:

$$M \otimes A \otimes A \simeq M \otimes A \otimes H \xrightarrow{\eta} M \longrightarrow 0,$$

where η is mentioned in Proposition 1.5. Since $(-)_0$ is exact, we obtain an exact sequence: $M \otimes A \longrightarrow M_0 \longrightarrow 0$, which in turn yields a commutative diagram

$$\begin{array}{ccccc} M \otimes A \otimes_B A & \longrightarrow & M & \longrightarrow & 0 \\ \updownarrow & & \uparrow can & & \\ (M \otimes A) \otimes_B A & \longrightarrow & M_0 \otimes_B A & \longrightarrow & 0. \end{array}$$

It follows that can is surjective and hence $M_0 \neq 0$. The equivalence of (b) and (f) follows from the definition of perfect torsion theories. □

Remark 2.6. In the above corollary, the condition (b) can be weakened to the following form: (b'). Any simple Hopf module is torsion free.

In fact, in this case, if M is a simple Hopf module, then $M_0 \neq 0$, so that A generates M in $\mathbf{M}_A^H$. Since any finitely generated Hopf module M in $\mathbf{M}_A^H$ contains a maximal Hopf submodule and A is projective in $\mathbf{M}_A^H$, it follows that A generates any finitely generated Hopf module, and hence A is a generator in $\mathbf{M}_A^H$.

3. Induction

As in Section 2, $\tau(M)$ is the largest torsion Hopf submodule of Hopf module M. Since $(-)_0$ is exact, we have $\tau(M/\tau(M)) = 0$ for any Hopf module M. This allows us to revise the definition of induced Hopf modules.

Definition 3.1. A Hopf module induced by a B-module N is not $N \otimes_B A$ but rather its factor module $N \bar{\otimes}_B A = N \otimes_B A / \tau(N \otimes_B A)$.

If $n \in N$, $a \in A$, then $n \bar{\otimes} a$ denotes the image in $N \bar{\otimes} A$ of the element $n \otimes a \in N \otimes A$, and $\rho(n \bar{\otimes} a) = \sum n \bar{\otimes} a_{(0)} \otimes a_{(1)}$. It is easy to see that $n \bar{\otimes} bx = nb \bar{\otimes} x$, for all $b \in B$, and $N \bar{\otimes}_B A$ is generated by such elements as $n \bar{\otimes} x$, $n \in N$, $x \in A$.

Lemma 3.2. *For any B-module N, the canonical map* $N \longrightarrow (N \bar{\otimes}_B A)_0$ *is an isomorphism in* Mod-B.

Proof. By definition of $N\bar{\otimes}A$, we have an exact sequence in $\mathbf{M}_A^H$:

$$0 \longrightarrow \tau(N \otimes_B A) \longrightarrow N \otimes_B A \longrightarrow N \bar{\otimes}_B A \longrightarrow 0,$$

Using the exact functor $(-)_0$, we get an exact sequence in Mod-B:

$$0 \longrightarrow \tau(N \otimes_B A)_0 \longrightarrow (N \otimes_B A)_0 \longrightarrow (N \bar{\otimes}_B A)_0 \longrightarrow 0.$$

But $\tau(N \otimes_B A)_0 = 0$. So it follows that

$$(N \bar{\otimes}_B A)_0 \simeq (N \otimes_B A)_0 \simeq N$$

in Mod-B. □

For any Hopf morphism $\phi : M \longrightarrow N$ in $\mathbf{M}_A^H$, ϕ maps $\tau(M)$ into $\tau(M)$. Hence it induces a Hopf morphism $\bar{\phi} : M/\tau(M) \longrightarrow N/\tau(N)$. So the induction functor $-\bar{\otimes}_B A$ is an additive functor from Mod-B to $\mathbf{M}_A^H$. One easily verifies that the additive functor $-\bar{\otimes}_B A$: Mod-$B \longrightarrow \mathbf{M}_A^H$ preserves arbitrary direct sums of modules. For convenience, we denote $M/\tau(M)$ by $\overline{M}$.

Proposition 3.3 (Lifting property). *The following adjunction map is isomorphic for* $M \in \mathbf{M}_A^H, N \in$ *Mod-B* :

$$\mathbf{M}_A^H(N \bar{\otimes}_B A, \overline{M}) \simeq \mathrm{Hom}_B(N, M_0).$$

Proof. Define a map $\eta : \mathbf{M}_A^H(N \bar{\otimes}_B A, \overline{M}) \longrightarrow \mathrm{Hom}_B(N, M_0)$ given by $\eta(f)(n) = f(n\bar{\otimes}1)$, which is well defined because of $(\overline{M})_0 \cong M_0$. On the other hand, define a map

$$\zeta : \mathrm{Hom}_B(N, M_0) \longrightarrow \mathbf{M}_A^H(N \bar{\otimes}_B A, \overline{M})$$

given by $\zeta(g)(n\bar{\otimes}a) = \overline{g(n)a}$, which is well defined because $g \otimes A : N \otimes_B A \longrightarrow M$ is a Hopf morphism, and $\zeta(g)$ is induced by $g \otimes A$. Now

$$\zeta \circ \eta(f)(n\bar{\otimes}a) = \overline{\eta(f)(n)a} = \overline{f(n\bar{\otimes}1)a} = f(n\bar{\otimes}a).$$

So $\zeta \circ \eta = \mathrm{Id}$. Similarly, $\eta \circ \zeta = \mathrm{Id}$. □

Remark 3.4. It is well known that the Frobenius reciprocity theorem $\mathbf{M}_A^H(N \otimes_B A, M) \simeq \mathrm{Hom}_B(N, M_0)$ holds for any $N \in$ Mod-B and $M \in \mathbf{M}_A^H$. The above proposition just states that Hopf morphism f in $\mathbf{M}_A^H(N \bar{\otimes}_B A, \overline{M})$ is induced by some Hopf mor phism $\bar{f}$ in $\mathbf{M}_A^H(N \otimes_B A, M)$. In other words, f can be lifted to a Hopf morphism in $\mathbf{M}_A^H(N \otimes_B A, M)$.

Definition 3.5. We say that a Hopf module M is 0-generated if $M = M_0A$.

Proposition 3.6 (Universal property). *Suppose that M is a 0-generated Hopf module. For any morphism* $\phi \in \mathrm{Hom}_B(M_0, N)$ *there is a unique Hopf morphism* $\psi \in$

$\mathbf{M}_A^H(M, N \bar{\otimes}_B A)$ such that the following diagram commutes:

$$\begin{array}{ccc} M_0 & \xrightarrow{\subseteq} & M \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ N & \xrightarrow{-\bar{\otimes} 1} & N \bar{\otimes}_B A \end{array}$$

Proof. The natural morphism $\theta : M_0 \otimes_B A \longrightarrow M$ is a Hopf morphism, and its $\ker(\theta)$ is a torsion Hopf submodule of $M_0 \otimes_B A$. Since M is 0-generated, θ is an epimorphism. It follows that the statements hold if and only if there is a unique Hopf morphim λ : $M_0 \otimes_B A \longrightarrow N \bar{\otimes}_B A$ such that $\lambda(\ker(\theta)) = 0$, and the following diagram commutes:

$$\begin{array}{ccc} M_0 & \xrightarrow{-\otimes 1} & M_0 \otimes_B A \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ N & \xrightarrow{-\bar{\otimes} 1} & N \bar{\otimes}_B A \end{array}$$

It is obvious that there exists exactly one Hopf morphism λ in $\mathbf{M}_A^H$ such that the above diagram commutes. It is the composite map

$$\lambda : M_0 \otimes_B A \xrightarrow{\phi \otimes A} N \otimes_B A \longrightarrow N \bar{\otimes}_B A.$$

Now it remains to show that $\ker(\theta) \subseteq \ker(\lambda)$. This is true because $\ker(\theta) \subseteq \tau(M_0 \otimes_B A) \subseteq \ker(\lambda)$ since $N \bar{\otimes}_B A$ is torsion free. □

Corollary 3.7. *For any 0-generated Hopf module M, we get an abelian group isomorphism:*

$$\mathrm{Hom}_B(M_0, N) \simeq \mathbf{M}_A^H(M, N \bar{\otimes}_B A)$$

given by $\phi \mapsto \psi$ as in Proposition 3.6, and the inverse is the restriction in view of Lemma 3.2.

Proposition 3.8. *For any 0-generated Hopf Module M, $\overline{M} \cong M_0 \bar{\otimes}_B A$.*

Proof. Firstly, we suppose that M is torsion free, i.e. $\tau(M) = 0$. By Proposition 3.6, there is a unique Hopf morphism ψ such that the following diagram commutes:

$$\begin{array}{ccc} M_0 & \xrightarrow{\subseteq} & M \\ \updownarrow & & \downarrow{\scriptstyle \psi} \\ M_0 & \xrightarrow{-\bar{\otimes} 1} & M_0 \bar{\otimes}_B A \end{array}$$

On the other hand, since M is torsion free, the following diagram is commutative by the lifting property

$$\begin{array}{ccc} M_0 & \xrightarrow{-\bar{\otimes} 1} & M_0 \bar{\otimes}_B A \\ \updownarrow & & \downarrow{\scriptstyle \theta} \\ M_0 & \xrightarrow{\subseteq} & M \end{array}$$

Combining the foregoing two commutative diagrams, we get a Hopf morphism $\theta \circ \psi$ that restricts to the identity on M_0. It follows that $\theta \circ \psi$ is the identity map since M is 0-generated. In this case ψ must be injective. However, by constraction of ψ in proposition 3.6, ψ is surjective and hence an isomorphism.

For an arbitray 0-generated Hopf module M, $\overline{M} = M/\tau(M)$ is torsion free and 0-generated too. By the above argument, we obtain that $\overline{M} \simeq \overline{M}_0 \bar{\otimes}_B A \simeq M_0 \bar{\otimes}_B A$. □

Corollary 3.9. *For the 0-generated Hopf module M, $\mathrm{End}_B M_0 \cong \mathrm{End}_A^H(\overline{M})$. In particular, $\mathrm{End}_B N \cong \mathrm{End}_A^H(N \bar{\otimes}_B A)$ for any B-module N.*

Theorem 3.10. *Denote by $\mathbf{M}^0$ the full subcategory of all 0-generated Hopf modules which are τ-torsion free. Then the functors $(-)_0$ and $- \bar{\otimes}_B A$ form an equivalence between $\mathbf{M}^0$ and Mod-B.*

Remark 3.11. (a) Comparing the foregoing theorem to Theorem 2.4, we arrive at an equivalence between the quotient Hopf module category $\overline{\mathbf{M}}_A^H$ and the full Hopf subcategory $\mathbf{M}^0$.

(b) Theorem 3.10 shows that A/B is H-Galois if and only if any Hopf module is 0-generated.

4. Simple Hopf modules

In this section, we discuss the relation between simple Hopf modules and simple B-modules for a fixed H-extension A/B which has a total integral φ. Let $G(H)$ be the group of group-like elements in H. There is a group action of $G(H)$ as automorphisms on the Hopf module category $\mathbf{M}_A^H$ as follows: for $\sigma \in G(H)$, $M \in \mathbf{M}_A^H$, $M^\sigma = M$ as underlying A-module, and the comodule structure is given by $\rho(m) = \sum m_{(0)} \otimes \sigma m_{(1)}$. To each element $\sigma \in G(H)$, we associate an equivalent functor

$$(-)^\sigma : \mathbf{M}_A^H \longrightarrow \mathbf{M}_A^H, \quad M \mapsto M^\sigma,$$

which has inverse $(-)^{\sigma^{-1}}$. We also associate a σ-invariant functor

$$(-)_\sigma : \mathbf{M}_A^H \longrightarrow \mathrm{Mod}-B, \qquad M \mapsto M_\sigma = \{m \in M \mid \rho(m) = m \otimes \sigma\}.$$

When $\sigma = 1$, $(-)_\sigma$ is the usual invariant functor $(-)_0$. It is easy to see $(-)_\sigma \simeq \mathbf{M}_A^H(A^\sigma, -)$. Since A is projective in $\mathbf{M}_A^H$, each A^σ, $\sigma \in G(H)$, is also projective in $\mathbf{M}_A^H$ and hence $(-)_\sigma$ is exact. Therefore, the semi-invariant functor

$$\mathscr{S}(-) : \mathbf{M}_A^H \longrightarrow \mathscr{S}(A) - gr, \qquad M \longrightarrow \mathscr{S}(M) = \bigoplus_{\sigma \in G(H)} M_\sigma,$$

is exact. On the other hand, there is an induction functor,

$$- \otimes_{\mathscr{S}(A)} A : \mathscr{S}(A) - \mathrm{gr} \longrightarrow \mathbf{M}_A^H, \qquad N \longrightarrow N \otimes_{\mathscr{S}(A)} A,$$

where the comodule structure of $N \otimes_{\mathscr{S}(A)} A$ comes from both N and A, i.e., $\rho(n_\sigma \otimes a) = \sum n_\sigma \otimes a_{(0)} \otimes \sigma a_{(1)}$, and the A-module structure is just the obvious one. Moreover, $(- \otimes_{\mathscr{S}(A)} A, \mathscr{S}(-))$ is an adjoint pair of functors. Now some modification of the proof of Lemma 1.4 yields the following result:

Lemma 4.1. (a) *For any B-module N, $(N \otimes_B A^\sigma)_\sigma \cong N$*; (b) *For any $\mathscr{S}(A)$-graded module M, $\mathscr{S}(M \otimes_{\mathscr{S}(A)} A) \cong M$ in $\mathscr{S}(A)$-gr.*

Proof. (a) is easy to check. We only give the proof of (b). Define the σ-trace map $T_\sigma : A \longrightarrow A_\sigma$, given by $T_\sigma(a) = \sum a_{(0)}\varphi(S(\sigma^{-1}a_{(1)}))$, where φ is the total integral. It is easy to verify that T_σ is well defined for any $\sigma \in G(H)$. Let

$$v : M \longrightarrow \mathscr{S}(M \otimes_{\mathscr{S}(A)} A), \quad m \mapsto m \otimes 1$$

be the canonical map of graded $\mathscr{S}(A)$-modules. We establish its inverse map ξ as follows:

$$\xi : \mathscr{S}(M \otimes_{\mathscr{S}(A)} A) \longrightarrow M, \quad \xi\left(\sum m_i \otimes a_i\right) = \sum_i \sum_{\mu \in G(H)} m_{i,\mu} T_{\mu^{-1}\sigma}(a_i),$$

where $\sum m_i \otimes a_i \in \mathscr{S}(M \otimes_{\mathscr{S}(A)} A)_\sigma$. Now

$$\xi v(m) = \xi(m \otimes 1) = \sum_{\mu \in G(H)} m_\mu T_{\mu^{-1}\mu}(1) = \sum_{\mu \in G(H)} m_\mu = m.$$

For $\sum m_i \otimes a_i \in \mathscr{S}(M \otimes_{\mathscr{S}(A)} A)_\sigma$, we have

$$\begin{aligned} v\xi\left(\sum m_i \otimes a_i\right) &= \sum_i \sum_{\mu \in G(H)} m_{i,\mu} T_{\mu^{-1}\sigma}(a_i) \otimes 1 \\ &= \sum_i \sum_\mu m_{i,\mu} \otimes a_{i(0)}\phi(S(\sigma^{-1}\mu a_{i(1)})) \\ &= \sum_i m_i \otimes a_i \varphi(S(\sigma^{-1}\sigma \cdot 1)) \\ &= \sum_i m_i \otimes a_i, \end{aligned}$$

where, the last but one equality holds since

$$\rho\left(\sum m_i \otimes a_i\right) = \sum_i \sum_\mu m_{i,\mu} \otimes a_{i(0)} \otimes \mu a_{i(1)} = \sum m_i \otimes a_i \otimes \sigma.$$

It implies that v is an isomophism. □

Lemma 4.2. (a) *Any simple torsion free Hopf module is 0-generated.*

(b) *If $M_\sigma \neq 0$, $\sigma \in G(H)$, then the simple Hopf module M is isomorphic to $(M_\sigma \bar{\otimes}_B A)^\sigma$ in $\mathbf{M}_A^H$.*

Proof. (a) M is torsion free forces $M_0 \neq 0$. $M_0 A$ is a nonzero Hopf submodule of M. It follows that $M = M_0 A$ is 0-generated.

(b) It is easy to see $(M^{\sigma^{-1}})_0 = M_\sigma$. Because of (a) and Proposition 3.8, we get $M^{\sigma^{-1}} \simeq M_\sigma \bar{\otimes}_B A$. It follows that $M \simeq (M^{\sigma^{-1}})^\sigma \simeq (M_\sigma \bar{\otimes}_B A)^\sigma$. □

Proposition 4.3. *Let M be a simple Hopf module in $\mathbf{M}_A^H$.*
(a) *$\mathscr{S}(M)$ is either a simple $\mathscr{S}(A)$-graded module or zero.*
(b) *M_σ is either zero or a simple B-module for any $\sigma \in G(H)$.*

Proof. (a) Suppose $\mathscr{S}(M) \neq 0$. Take any nonzero graded submodule N of $\mathscr{S}(M)$. Since $(- \otimes_{\mathscr{S}(A)} A, \mathscr{S}(-))$ is an adjoint pair of functors, we obtain the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{-\otimes 1} & N \otimes_{\mathscr{S}(A)} A \\ \downarrow & & \downarrow \theta \\ \mathscr{S}(M) & \xrightarrow{\subseteq} & M \end{array}$$

where θ is the natural map. Now the simplicity of M entails that θ is an epimorphism. Using the exact functor $\mathscr{S}(-)$ and Lemma 4.2 we obtain an exact sequence: $N \longrightarrow \mathscr{S}(M) \longrightarrow 0$. It follows that $\mathscr{S}(M) = \theta(N \otimes 1) = N$. Therefore, $\mathscr{S}(M)$ is a simple graded module. (b) Substituting $(-)_0$ for $\mathscr{S}(-)$, we obtain (b). □

Lemma 4.4. *If N is a simple B-module, then $N \bar{\otimes}_B A$ is a torsion free simple Hopf module.*

Proof. Suppose K is a nonzero Hopf submodule of $N \otimes_B A$ and properly contains the torsion submodule $\tau(N \otimes_B A)$. Then $K_0 \neq 0$. But $K_0 \subseteq (N \otimes_B A)_0 = N \otimes 1$. this implies $K_0 = N \otimes 1$. It follows that $N \otimes_B A = (N \otimes 1)A = K_0 A \subseteq K$; So $K = N \otimes_B A$. This means $N \bar{\otimes}_B A = N \otimes_B A / \tau(N \otimes_B A)$ is simple. □

Denote by **Sim**(A,H) (resp. **SSim**(A,H)) the full, weakly additive (resp. full additive) subcategory of $\mathbf{M}_A^H$ consisting of all torsion free simple (resp. semisimple) Hopf modules, and **Sim**(B)(resp. **SSim**(B)) the full weakly additive (resp. full additive) subcategory of Mod-B having for its objects all simple (resp. semisimple) B-modules. Combining the above two lemmas and Proposition 4.3, we obtain the following equivalence:

Theorem 4.5. *The restriction functor $(-)_0$ and the induction functor $- \bar{\otimes}_B A$ form an equivalence between subcategory* **Sim**(A,H)*(resp.* **SSim**(A,H)*) and subcategory* **Sim**(B)*(resp.* **SSim**(B)*).*

When H is a finite Hopf algebra, we may establish

Proposition 4.6. *Suppose that H is a finite dimensional Hopf algebra, and M is a simple A-module. There exists a simple Hopf module Σ containing M.*

Proof. Using the functor F as in Proposition 1.6, we obtain a Hopf module $F(M) = M \otimes H$. Let t be the right integral of H. We claim that $M \otimes H$ as a Hopf module is generated by $M \otimes t$. In fact, we have the Hopf isomorphism $t \otimes H^* \simeq H$ via $t \leftharpoonup h^* = \sum h^*(t_{(1)})t_{(2)}$ since H is finite. Thus $(M \otimes t) \leftharpoonup H^* = M \otimes H$, i.e., $M \otimes t$ generates $M \otimes H$. Now $M \otimes H$ is a finitely generated Hopf module since M is finitely generated A-module. There exists a maximal Hopf submodule N such that $M \otimes H/N$ is a simple Hopf module. The composition map $\delta \circ \pi : M \longrightarrow M \otimes H \longrightarrow M \otimes H/N = \Sigma$ is clearly an A-module map. Since $\delta(M)$ is not contained in N in view of the above claim, the simplicity of M entails that $\delta \circ \pi$ must be injective. □

5. Stable Clifford theory

Fix an H-Galois extension A/B with a total integral φ. A Hopf module M is said to be *weakly H-stable* if $F(M) = M \otimes H$ is a direct summand of some direct sum of copies of M in $\mathbf{M}_A^H$, denoted by $F(M) < M$. In case $H = kG$, the weakly H-stability is nothing but the *weakly G-invariance*. Note that A/B is strictly Galois. Each Hopf module M is induced by its invariant B-module M_0, i.e., $M \cong M_0 \otimes_B A$. So M is weakly H-stable if and only if $M < M_0$ as B-modules since $F(M) \cong M \otimes_B A$ in $\mathbf{M}_A^H$. In this situation, The weakly H-stability is weaker than the *H-stability* defined in [11]. Let M be a Hopf module in $\mathbf{M}_A^H$. Recall from [16] that the endomorphism ring extension $\mathrm{End}_A(M)/\mathbf{M}_A^H(M)$ is an H-Galois extension if and only if M is a weakly H-stable Hopf module in case H is a unimodular Hopf algebra. In this section, we develop a stable Clifford theory which generalizes the classical stable Clifford theory for graded modules [4]. Throughout this section we fix a simple Hopf module Σ which is weakly H-stable. The existence of the latter is ensured by the classical case.

Lemma 5.1. *Σ is torsion free, and semisimple as B-module which is $(\Sigma)_0$-primary, i.e., a direct sum of copies of $(\Sigma)_0$.*

Proof. Since Σ is irreducible, there is a set I such that $\Sigma \otimes H \simeq \Sigma^{(I)}$ in $\mathbf{M}_A^H$. It follows that

$$\Sigma \simeq (\Sigma \otimes H)_0 \simeq (\Sigma^{(I)})_0 \simeq (\Sigma_0)^{(I)}$$

since $(-)_0$ is exact and commutes with direct sums. □

Define a subcategory of Mod-A, say $\mathrm{Mod}(A|\Sigma)$ as follows: $\mathrm{Mod}(A|\Sigma) = \{M \in$ Mod-A such that there exists an A-linear epimorphism ϕ, a set I and $\Sigma^{(I)} \xrightarrow{\phi} M \longrightarrow 0\}$. This is a full additive subcategory of Mod-A. In fact, it is a Grothendieck category with a generator Σ, and having all objects being B-semisimple.

Lemma 5.2. $\mathrm{Mod}(A|\Sigma)$ *is an abelian category.*

Proof. We show that $\mathrm{Mod}(A|\Sigma)$ coincides with the category $\mathrm{Mod}(A|\Sigma_0) = \{M \in \text{Mod-}A \mid M \text{ is } \Sigma_0\text{-primary as } B\text{-module}\}$. The latter is clearly an abelian subcategory of Mod-A. It is easy to see that any object in $\mathrm{Mod}(A|\Sigma)$ is in $\mathrm{Mod}(A|\Sigma_0)$. Conversely, given an object $M \in \mathrm{Mod}(A|\Sigma_0)$, then $M \simeq \Sigma_0^{(I)}$ as B-modules for some set I. Take any irreducible B-submodule $N \subseteq M$ such that N is isomorphic to Σ_0 by definition of M. There is a B-isomorphism $\psi : \Sigma_0 \longrightarrow N \subseteq M$. Since $\Sigma \simeq \Sigma_0 \otimes_B A$ naturally, ψ can be extended to an A-morphism $\bar{\psi} : \Sigma \longrightarrow M$. It follows that

$$N = \psi(\Sigma_0) \subseteq \bar{\psi}(\Sigma) = \psi(\Sigma_0)A = NA \subseteq M.$$

Now M is Σ_0-primary as B-module. This entails that $M = \mathrm{Hom}_A(\Sigma, M)\Sigma$, which is equivalent to M being in $\mathrm{Mod}(A|\Sigma)$. Therfore, $\mathrm{Mod}(A|\Sigma)$ is an abelian subcategory. □

Lemma 5.3. *Σ is a finitely generated projective object in* $\mathrm{Mod}(A|\Sigma)$.

Proof. For any exact sequence $M \xrightarrow{\delta} N \longrightarrow 0$ in $\mathrm{Mod}(A|\Sigma)$ and morphism $\phi : \Sigma \longrightarrow N$, we look for a morphism $\psi : \Sigma \longrightarrow M$ such that $\delta \circ \psi = \phi$. Let ϕ restrict to Σ_0 which is a B-monomorphism. Since M, N are B-semisimple, δ splits as a B-homomophism. Thus $\phi|_{\Sigma_0}$ can be extended to a B-morphism $\psi' : \Sigma_0 \longrightarrow M$ such that $\delta \circ \psi' = \phi|_{\Sigma_0}$. Now ψ' can be extended to an A-morphism $\psi : \Sigma \simeq \Sigma_0 \otimes_B A \longrightarrow M$, which is the desired one since Σ is generated by Σ_0 as an A-module. It follows that Σ is projective in $\mathrm{Mod}(A|\Sigma)$. □

The Cifford theory with respect to the simple Hopf module Σ may be stated as follows. Let E denote the endomorphism ring $\mathrm{End}_A(\Sigma)$.

Theorem 5.4. *The category* $\mathrm{Mod}(A|\Sigma)$ *is a Grothendieck category with a small projective generator Σ. The functors* $\mathrm{Hom}_A(\Sigma, -)$ *and* $- \otimes_E \Sigma$ *define an equivalence between* $\mathrm{Mod}(A|\Sigma)$ *and module category* Mod-E.

Remark 5.5. As stated in the beginning of this section, in case H is a finite Hopf algebra the endomorphism ring $\mathrm{End}_A(\Sigma)$ is an H-Galois extension of the division algebra $E_0 = \mathbf{M}_A^H(\Sigma) = \mathrm{End}_B(\Sigma_0)$. We may expect that in the situation of Theorem 5.4, E is an H-Galois extension of E_0.

References

[1] S. Caenepeel and S. Raianu, Induction functors for Doi-Koppinen Hopf modules, preprint.

[2] S. Caenepeel, S. Raianu and F.Van Oystaeyen, Induction and coinduction for Hopf algebras: applications, J. Algebra 165 (1994) 204–222.

[3] M. Cohen and D. Ficshmann, Semisimple extensions and elements of trace 1, J. Algebra 149 (1992) 419–437.

[4] E.C. Dade, Group-graded rings and modules, Math. Z. 174 (1980) 241–262.

[5] E.C. Dade, Clifford theory for group-graded rings II, J. Reine Angew. Math. 387 (1988) 148–181.

[6] Y. Doi, Algebras with total integrals, Comm. Algebra 13 (1985) 22137–2159.

[7] P. Gabriel, Des categories abeliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[8] C. Năstăsescu, Some constructions over graded rings: applications, J. Algebra 120 (1989) 119–138.
[9] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, Separable functors applied to graded rings, J. Algebra 123 (1989) 397–413.
[10] H-J. Schneider, Principal homogeneous spaces, Isreal J. Math. 72 (1990) 167–195.
[11] H-J. Schneider, Hopf Galois extensions, Isreal J. Math. 72 (1990) 196–231.
[12] B. Stenström, Rings of Quotients (1975).
[13] M.E. Sweedlar, Hopf Algebras (1969).
[14] M. Takeuchi, Formal schemes over fields, Comm. Algebra 14 (1977) 1483–1528.
[15] F. Van Oystaeyen, Y. Xu and Y.H. Zhang, Inductions and coinductions for Hopf extensions, to appear.
[16] F. van Oystaeyen and Y.H. Zhang, H-module endomorphism rings, J. Pure Appl. Algebra, to appear.