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Cofree quiver settings

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Abstract

We give a complete classification of all quivers Q and dimension vectors α for which the representation space $\mathsf{Rep}(Q, \alpha)$ is cofree as a representation of its natural symmetry group GL_α.

Keywords: Quivers; Representations; Cofree; Invariant theory

1. Introduction

Consider a linear reductive complex algebraic group G and a representation $\phi : G \to GL(V)$. Such a representation is called *cofree* if its coordinate ring $\mathbb{C}[V]$ is a graded free module over the ring of invariants $\mathbb{C}[V]^G$. Cofree representations were studied amongst others by Popov in [7] and Schwarz in [8] and were classified by Schwarz for G a connected simple complex algebraic group.

A representation space of a quiver Q of dimension vector α is a natural example of the situation described in the previous paragraph. Recall that a quiver Q is a directed graph specified by a set of vertices Q_0, a set of arrows Q_1 and two maps, $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ specifying for each arrow its source and target vertex. A dimension vector $\alpha : Q_0 \to \mathbb{N}$ assigns to each vertex v a natural number α_v. Given a dimension vector α, the space of α-dimensional representations

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of Q is the linear space

$$\mathsf{Rep}(Q,\alpha) = \bigoplus_{a\in Q_1} M_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C}).$$

This space carries a natural action of the reductive group

$$\mathsf{GL}_\alpha = \prod_{v\in Q_0} \mathsf{GL}_{\alpha_v}(\mathbb{C})$$

by conjugation, that is, for $(g_v)_{v\in Q_0} \in \mathsf{GL}_\alpha$ and $(V_a)_{a\in Q_1} \in \mathsf{Rep}(Q,\alpha)$ we have

$$(g_v)_{v\in Q_0}.(V_a)_{a\in Q_1} = \left(g_{t(a)} V_a g_{s(a)}^{-1}\right)_{a\in Q_1}.$$

Thus for each pair (Q,α), which we call a quiver setting, we obtain an example of the situation described in the first paragraph. We call a quiver setting cofree if its representation space $\mathsf{Rep}(Q,\alpha)$ is cofree as a representation of GL_α. A quiver will be graphically depicted as

$$t(a) \xleftarrow{\ a\ } s(a)$$

and a quiver setting will be depicted by listing the dimension of each vertex in a circle where the vertex of the quiver was drawn:

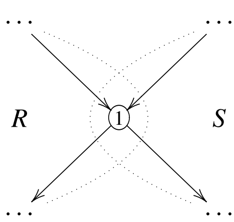
$$\alpha_{t(a)} \xleftarrow{\ a\ } \alpha_{s(a)}.$$

The objective of this paper is to classify all such cofree quiver settings. To understand the statement of the main classification theorem, we need to introduce some concepts concerning the structure of the quiver.

A sequence of consecutive arrows $a_1 \ldots a_n$ with $s(a_i) = t(a_i)$, is called a path of length n. It is called a cycle if $t(a_1) = s(a_n)$. In every vertex there exists a trivial cycle of length 0. A quiver is strongly connected if there is a cycle through every couple of vertices. The strongly connected components of a quiver are the maximal strongly connected subquivers, note that a quiver is not necessarily the union of all its strongly connected components as there may exist arrows that are not contained in a cycle.

Recall from [2] that a strongly connected quiver setting (Q,α) is called a connected sum of two subquivers settings (R,ρ) and (S,σ) if it is the union of R and S and R and S share only one vertex and no arrows. Moreover this shared vertex must have dimension 1. We will denote this by $(Q,\alpha) = (R,\rho)\#_v(S,\sigma)$ where v is the shared vertex.

$R^{\#}_v S :=$



We will call quiver setting prime if it is strongly connected and not a connected sum of two subquiver settings. The quiver consisting of one vertex without loops will not be considered prime in order to avoid ambiguities. The prime components of a strongly connected quiver setting are the maximal subquiver settings that are prime. It is easy to see that every strongly connected quiver is the connected sum of its prime components.

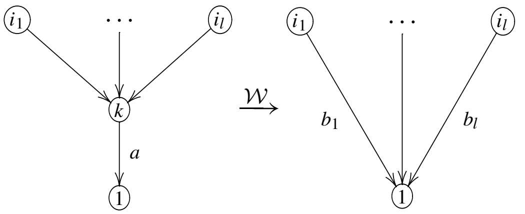
A path in Q will be called quasiprimitive if it does not run $n+1$ times through a vertex v with $\alpha_v = n$, i.e.

$$a_1 \dots a_n \text{ is quasiprimitive } \Leftrightarrow \forall v \in Q_0\colon \#\{i \mid s(a_i) = v\} \leqslant \alpha_v.$$

A quasiprimitive path from vertex v to vertex w will be denoted by a wavy arrow:

$$v \rightsquigarrow w.$$

A quasiprimitive cycle is a quasiprimitive path that is also a cycle. By reduction step $\mathcal{W}$ we mean the construction of a new quiver setting from a given quiver setting by removing a vertex (and connecting all arrows) in the situation illustrated below or its dual with the arrows reversed, where k is not smaller than the number of quasiprimitive cycles through (k).



We will also call this reduction step wedging.

The concepts above are very useful in simplifying the classification of the cofree quiver settings because

Lemma 1. *(See Section 2.4, Lemma 3.)*

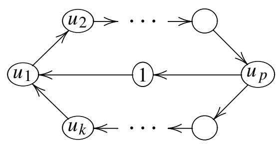
(1) *A quiver setting is cofree if and only if its strongly connected components are cofree.*
(2) *A strongly connected quiver setting is cofree if and only if its prime components are cofree.*
(3) *If (Q', α') is obtained from (Q, α) by applying reduction step $\mathcal{W}$ then (Q, α) is cofree if and only if (Q', α') is.*

With this lemma in mind it suffices to consider only prime settings that cannot be wedged (i.e. reduced using $\mathcal{W}$).

Theorem 1. *The prime quiver setting (Q, α) that are cofree and cannot be wedged are*

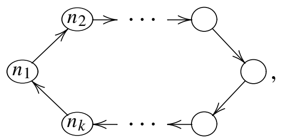
(1) *strongly connected quiver settings (P, ρ) for which*
   (a) *there is a vertex $v \in P_0$ such that $\rho(v) = 1$ and through which all cycles run,*
   (b) $\forall w \neq v \in P_0 : \rho(w) \geqslant \#\{v \rightsquigarrow w\} + \#\{v \leftsquigarrow w\} - 1,$

(2) *quiver settings* (P, ρ) *of the form*

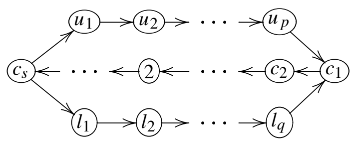


with $1 \leqslant p \leqslant k$, $1 \geqslant u_i$, *such that the minimal dimension of the cycle is reached either exactly once in the upper path* $u_1 \rightsquigarrow u_p$ *or never in the upper path but exactly once in the lower path* $u_p \rightsquigarrow u_k$,

(3) *cyclic quiver settings with arbitrary dimension vector*



(4) *quiver settings* (P, ρ) *consisting of two cyclic quivers, with* $p + s$ *and* $q + s$ *vertices, coinciding on* s *subsequent vertices* $(p, q$ *can be zero*$)$



with $u_i, l_j \geqslant 2$ *for all* $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ *and all* $c_k \geqslant 4$ *except for a unique vertex with dimension* 2.

Remark 1. Note that cases (iii) and (iv) exhaustively list all strongly connected cofree quiver settings with dimension vector without ones. This is because these quiver settings are always prime and cannot be wedged.

Remark 2. To determine whether a given quiver setting is cofree, one must use wedging and splitting in connected summands and then check whether the obtained quivers are in the list. Splitting in connected summands never changes the $\mathcal{W}$-reducibility condition for vertices of dimension higher than one (every quasiprimitive cycle is contained in just one of the summands). Wedging a vertex of dimension one can never make another vertex splittable (because it does not change the number of arrows arriving in or leaving the other vertices) or a higher-dimensional vertex $\mathcal{W}$-reducible. This means we can order the operations as follows: first wedge the higher-dimensional vertices, then split all possible vertices and finally wedge all possible one-dimensional vertices. These last wedges only occur in situations that are already reduced to case (iii) (note that some settings in (iii) can be wedged). This allows us to restate the theorem in the following way.

A strongly connected quiver setting is cofree if and only if after all possible wedging its prime components are in the list above.

Philosophically speaking, this means that we can see the cofree quiver settings as 'connected sums' of some basic types of cofree settings. However, these summands might be glued together in some path rather than in unique vertex (provided the dimensions in the path are big enough). The wedging operation is used to separate these basic settings to a real connected sum.

Wedging operates also by shortening some paths in quiver settings that have vertices with high dimensions. E.g. quiver settings that look like (ii) but have a longer path connecting the central vertex ① to the cycle are also cofree provided the extra vertices in this path all have dimension at least $\min\{u_1, \dots, u_k\} + 1$.

The classification is obtained starting from a classical result by Popov [7], which states that a representation ϕ is cofree if and only if it is coregular (that is, $\mathbb{C}[V]^G$ is isomorphic to a polynomial ring) and the codimension in V of the zero set $N_G(V)$ of elements of positive degree in $\mathbb{C}[V]^G$ is equal to $\dim \mathbb{C}[V]^G$. This result, in combination with the classification of coregular quiver representations by the first author in [1] and the study of the nullcone of quiver representations by the second author in [11] yields the complete classification presented.

The paper is organized in the following manner. In Section 2 we collect most of the definitions and background material needed for the rest of the paper. In Sections 3.1 and 3.2 the methods to obtain the classification are discussed.

2. Preliminaries

In this section, we collect all necessary material for the rest of the paper.

2.1. The quotient space, the defect and prime components

Definition 1. The *Euler form of a quiver Q* is the bilinear form on dimension vectors defined by

$$\chi_Q(\alpha, \beta) = \sum_{v \in Q_0} \alpha_v \beta_v - \sum_{a \in Q_1} \alpha_{s(a)} \beta_{t(a)}.$$

Definition 2. The quotient space of $\mathsf{Rep}(Q, \alpha)$ with respect to the natural action of GL_α classifies all isomorphism classes of semisimple representations and is denoted by $\mathsf{iss}(Q, \alpha)$. The quotient map with respect to this action will be denoted by

$$\pi : \mathsf{Rep}(Q, \alpha) \twoheadrightarrow \mathsf{iss}(Q, \alpha).$$

The fiber of π over $\pi(0)$ is called the *nullcone* of the quiver setting and is denoted by $\mathsf{Null}(Q, \alpha)$.

Definition 3. For a given quiver setting (Q, α), we define the *defect* as

$$\operatorname{def}(Q, \alpha) := \dim \mathsf{Null}(Q, \alpha) - \dim \mathsf{Rep}(Q, \alpha) + \dim \mathsf{iss}(Q, \alpha).$$

The defect is a positive number because the generic fiber of the quotient map $\mathsf{Rep}(Q, \alpha) \to \mathsf{iss}(Q, \alpha)$ has dimension $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha)$. It measures how much worse the nullcone is than the generic fiber.

We then have

Proposition 1. *(See Popov, [7].) A quiver setting (Q,α) is cofree if and only if it is coregular and* $\operatorname{def}(Q,\alpha)=0$.

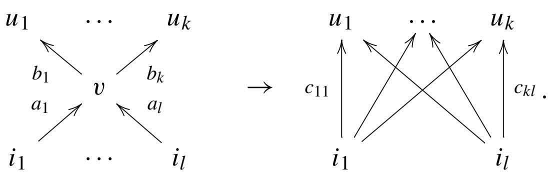
2.2. Reducing quiver settings

In [1], three different types of reduction moves on a quiver setting (Q,α) were introduced. These are

$\mathcal{R}_{\mathrm{I}}^{v}$: let v be a vertex without loops such that

$$\alpha_v \geqslant \sum_{w\to v}\alpha_w \text{or} \alpha_v \geqslant \sum_{v\to w}\alpha_w$$

(the sum is taken over the arrows, so some α_w may appear more than once).
Construct a new quiver setting $(\mathcal{R}_{\mathrm{I}}^{v}(Q),\mathcal{R}_{\mathrm{I}}^{v}(\alpha))$ by removing v and connecting all arrows running through v:

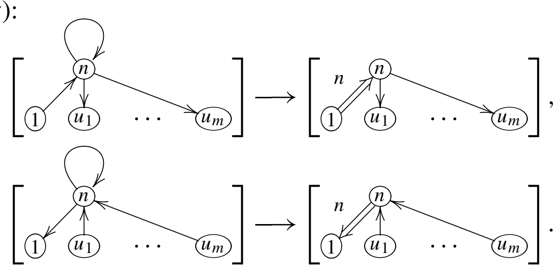


For this step we have $\mathsf{iss}(Q,\alpha)\cong\mathsf{iss}(\mathcal{R}_{\mathrm{I}}^{v}(Q),\mathcal{R}_{\mathrm{I}}^{v}(\alpha))$.

$\mathcal{R}_{\mathrm{II}}^{v}$: let v be a vertex with $\alpha(v)=1$ and n loops. Let $(\mathcal{R}_{\mathrm{II}}^{v}(Q),\alpha)$ be the quiver setting obtained by removing all these loops. We then have

$$\mathsf{iss}(Q,\alpha)\cong\mathsf{iss}\big(\mathcal{R}_{\mathrm{II}}^{v}(Q),\alpha\big)\times\mathbb{A}^n.$$

$\mathcal{R}_{\mathrm{III}}^{v}$: Let v be a vertex with one loop and $\alpha(v)=n$ such that there is a unique arrow leaving (arriving in) v and the target (source) of this arrow has dimension one. Let $(\mathcal{R}_{\mathrm{III}}^{v}(Q),\alpha)$ be the quiver setting obtained by removing the loop in v and adding $n-1$ additional arrows between v and its neighboring vertex with dimension 1 (all having the same orientation as the original arrow):



For this step we have

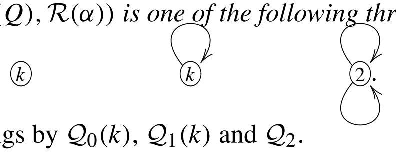
$$\mathsf{iss}(Q,\alpha)\cong\mathsf{iss}\big(\mathcal{R}_{\mathrm{III}}^{v}(Q),\alpha\big)\times\mathbb{A}^n.$$

Definition 4. Let (Q,α) be a quiver setting.

(1) If none of the above reduction steps can be applied to (Q,α) then this setting is called *reduced*.
(2) By $(\mathcal{R}(Q),\mathcal{R}(\alpha))$ we denote the quiver setting obtained after repeatedly applying all of the above reduction steps until no longer possible. This setting is called the *reduced quiver setting of* (Q,α).

We now have the following two results.

Theorem 2. *(See Bocklandt, [1].) Let (Q,α) be a strongly connected quiver setting, then (Q,α) is coregular if and only if $(\mathcal{R}(Q),\mathcal{R}(\alpha))$ is one of the following three settings*:



We will denote these settings by $\mathcal{Q}_0(k)$, $\mathcal{Q}_1(k)$ and $\mathcal{Q}_2$.

Theorem 3. *(See Van de Weyer, [11].) Let (Q,α) be a quiver setting, then*

$$\operatorname{def}(Q,\alpha)\geqslant \operatorname{def}\bigl(\mathcal{R}(Q),\mathcal{R}(\alpha)\bigr).$$

2.3. The Luna Slice Theorem

We will also use the Luna Slice Theorem, formulated for quiver representations (see [5]). Let (Q,α) be a quiver setting and let $S\in \mathsf{iss}(Q,\alpha)$ correspond to the following decomposition in simples

$$S=S_1^{\oplus e_1}\oplus\cdots\oplus S_k^{\oplus e_k},$$

with S_i a simple representation of dimension vector α_i (for $1\leqslant i\leqslant k$).

Define the quiver Q_S as the quiver with k vertices and $\delta_{ij}-\chi_Q(\alpha_i,\alpha_j)$ arrows from vertex i to vertex j. Define α_S as the dimension vector that assigns e_i to vertex i (for $1\leqslant i\leqslant k$).

Theorem 4. *(See Le Bruyn–Procesi, [5].) With notations as above,*

(1) *there exists an étale isomorphism between an open neighborhood of S in $\mathsf{iss}(Q,\alpha)$ and an open neighborhood of the zero representation in $\mathsf{iss}(Q_S,\alpha_S)$.*
(2) *there is an isomorphism as GL_α-varieties*

$$\pi^{-1}(S)\cong \mathsf{GL}_\alpha\times^{\mathsf{GL}_{\alpha_S}}\mathsf{Null}(Q_S,\alpha_S).$$

We define

Definition 5.

- An element $S=S_1^{\oplus e_1}\oplus\cdots\oplus S_k^{\oplus e_k}$ is said to be of *representation type* $(\alpha_1,e_1;\alpha_2,e_2,\ldots,\alpha_k,e_k)$, where α_i is the dimension vector of S_i for $1\leqslant i\leqslant k$.
- The quiver setting (Q_S,α_S) is called the *local quiver* of S.

2.4. Cofreeness and quiver operations

The relation of all the concepts introduced earlier with being cofree can be expressed by the following lemma:

Lemma 2. *Suppose (Q, α) is cofree. If (Q', α') is*

(1) *a subquiver of (Q, α),*
(2) *a local quiver of (Q, α) or*
(3) *a quiver obtained by applying reduction moves to (Q, α),*

then (Q', α') is also cofree.

Proof. The representation space of as subquiver respectively local quiver is a proper GL_α-subrepresentation respectively a proper slice representation of $\mathsf{Rep}(Q, \alpha)$. By [9, Theorem 2.1] these are both cofree if the original is cofree. The third property is a direct consequence of Theorem 3. □

The lemma above only gives necessary conditions for (Q, α) to be cofree. We also have sufficient conditions:

Lemma 3.

(1) *A quiver setting is cofree if and only if its strongly connected components are cofree.*
(2) *A strongly connected quiver setting is cofree if and only if its prime components are cofree.*
(3) *If (Q', α') is obtained from (Q, α) by applying a reduction step $\mathcal{W}$ then (Q, α) is cofree if and only if (Q', α') is.*

Proof. The quotient space of a quiver Q is the product of the quotient spaces of its strongly connected components Q_i. The null cone and the representation variety are both the product of the null cones of the strongly connected components together with an affine space that is the product of $\mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}$ for all arrows that are not contained in a cycle of Q and hence not in strongly connected

$$\mathsf{Null}(Q, \alpha) = \prod_i \mathsf{Null}(Q_i, \alpha_i) \times \prod_{a \in Q \setminus \cup_i Q_i} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}),$$

$$\mathsf{Rep}(Q, \alpha) = \prod_i \mathsf{Rep}(Q_i, \alpha_i) \times \prod_{a \in Q \setminus \cup_i Q_i} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}).$$

For a strongly connected quiver Q the null cone, the quotient variety and the representation variety are all the product of the corresponding varieties of the prime components.

To prove the property for wedging requires some more work. We only have to prove that the condition is sufficient. The necessity follows from Lemma 2. Let π denote the projection $\mathsf{Null}(Q, \alpha) \to \mathsf{Null}(Q', \alpha')$. The dimension of the generic fiber of $V \in \mathsf{Null}(Q', \alpha')$ is

$$\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{Rep}(Q', \alpha') = \sum_r i_r(k-1) + k$$

and this occurs when at least one of the linear maps V_{b_i} is nonzero. If they are all zero the dimension is

$$\max\left(\sum_r i_r(k-1)+k, \sum_r i_r k\right) \tag{$*$}$$

the second entry comes from the dimension of the subvariety of the fiber where a is zero. This is the larger one if $\sum_r i_r > k$, and in that case the dimension of the subset

$$X := \left\{V \in \mathsf{Null}(Q', \alpha') \;\middle|\; \forall r \leqslant l\colon\ V_{b_r} = 0\right\}$$

is at most $\dim \mathsf{Null}(Q', \alpha') - \sum_r i_r + \#\{\text{cycles through } k\}$. Indeed, through every $V \in X$ we can draw an affine space

$$V + \left\{W \;\middle|\; \forall 1 \leqslant j \leqslant l\colon\ \mathsf{Span}(\mathsf{Im}\, V_{1 \rightsquigarrow i_j}) \subset \ker W_{b_j}\right\}$$

with dimension at least $\sum_r i_r - \#\{\text{cycles through } k\}$ and all these spaces are disjoint. This yields

$$\begin{aligned}
\dim \pi^{-1}(X) &\leqslant \dim X + \sum_r i_r k \\
&\leqslant \dim \mathsf{Null}(Q', \alpha') - \sum_r i_r + \#\{\text{cycles through } k\} + \sum_r i_r k \\
&\leqslant \dim \mathsf{Null}(Q', \alpha') + \sum_r i_r(k-1) + k \\
&\leqslant \dim \mathsf{Rep}(Q', \alpha') - \dim \mathsf{iss}(Q', \alpha') + \sum_r i_r(k-1) + k \\
&\leqslant \dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha). \qquad \square
\end{aligned}$$

3. Proof of Theorem 1

Throughout the remainder of the paper, we assume every path quasiprimitive unless stated otherwise. Cycles will also be assumed quasiprimitive. The advantage of working with quasiprimitive cycles is that there are only a finite number of them and this number equals $\dim \mathsf{iss}(Q, \alpha)$ if (Q, α) is coregular. This is because it holds for the basic settings $\mathcal{Q}_0(k)$, $\mathcal{Q}_1(k)$, $\mathcal{Q}_2(2)$ and because for the reduction moves $\mathcal{R}^v_{\mathrm{I}}$, $\mathcal{R}^v_{\mathrm{II}}$, $\mathcal{R}^v_{\mathrm{III}}$, the difference in dimension and the difference in number of cycles between the original and reduced setting is the same.

By Lemma 3, we can restrict ourselves to classifying prime quiver settings that cannot be wedged. By Theorem 2, we know that in order to classify all cofree quiver settings, we have to determine which of the quiver settings reducing to $\mathcal{Q}_0(1)$, $\mathcal{Q}_1(k)$ and $\mathcal{Q}_2$ are cofree (note that $\mathcal{Q}_0(k), k > 1$ and $\mathcal{Q}_1(k), k = 1$ are not possible). We will consider the different possible reduced settings in the following subsections.

First of all, note that the only quivers reducing to $\mathcal{Q}_1(k)$ are the cyclic quivers with smallest dimension in their dimension vector equal to k. The nilpotent representations of the cyclic quiver were studied extensively in [6] and it is known that there are only finitely many orbits of nilpotent representations of the cyclic quiver. But a classical result (e.g. [3, II.4.2, Satz 1]) then yields that the quotient map must be equidimensional, so we have a first result.

Theorem 5. 1(iii) *Let Q be the cyclic quiver with n vertices and let α be any dimension vector then (Q,α) is a cofree quiver setting.*

Next, we will determine which quiver settings reducing to setting $\mathcal{Q}_0(1)$ are cofree. These are quiver settings which contain at least one vertex of dimension 1. We split this task into two parts: first we determine all cofree quiver settings for which all cycles go through the same vertex of dimension 1, resulting in Theorem 1(i). After that we treat the more general case where there might be other cycles as well. Provided these settings are prime they all can be summarized in a unique type namely the one in Theorem 1(ii).

Finally, we study the quiver settings reducing to $\mathcal{Q}_2$. The underlying quiver is first shown to be equal to two cyclic quiver coinciding on a number of subsequent vertices. Then, a restriction on the possible dimension vectors is obtained using the description of the Hesselink stratification of the nullcone from [4]. This restriction is shown to be sufficient, leading to the prime components described in Theorem 1(iv).

3.1. *Quiver settings reducing to $\mathcal{Q}_0(1)$*

To every point $V \in \mathsf{Rep}(Q,\alpha)$ and a vertex v, we can assign a new dimension vector σ^v where σ^v_w is the dimension of the vector space

$$\mathsf{Span} \bigcup_{v \overset{p}{\rightsquigarrow} w} \mathsf{Im}\, V_p.$$

We will call σ^v the *relevant dimension vector with base* v. In this formula, the trivial path through v is not counted. When we do count the trivial path, we will denote this by a $\bar{\sigma}^v$ (i.e. $\bar{\sigma}^v_v = \alpha_v$ while σ^v_v can be smaller). When the base vertex is obvious we will omit the superscript. We will denote the set of representations with a given relevant dimension σ with base v by $\mathsf{Rel}_\sigma(Q,\alpha)$.

Theorem 6. (1(i)) *Suppose (Q,α) is a strongly connected quiver setting such that Q has a vertex v with dimension 1 through which all cycles run. The setting (Q,α) is cofree if and only if for every vertex $w \neq v$ the dimension*

$$\alpha_w \geqslant \#\{v \rightsquigarrow w\} + \#\{v \leftsquigarrow w\} - 1. \tag{1}$$

Proof. First of all note that these quiver settings are also coregular. To prove this, we put a partial order on the vertices different from v such that $w \leqslant w'$ if there is a path from v to w' through w. This is indeed a partial order: antisymmetry and transitivity follow from the fact that all cycles pass through v. A minimal vertex is the target of an arrow coming from v. Because such a minimal vertex has at least one arrow leaving, the inequality in the theorem implies that this minimal vertex is reducible by $\mathcal{R}_{\mathrm{I}}$. Proceeding in this way, we can reduce all vertices different from v.

To calculate the defect, note that $V \in \mathsf{Null}(Q,\alpha)$ if and only if $\sigma_v = 0$, so

$$\mathsf{Null}(Q,\alpha) = \bigcup_{\sigma_v=0} \mathsf{Rel}_\sigma(Q,\alpha) \quad \text{and} \quad \dim \mathsf{Null}(Q,\alpha) = \max_{\sigma_v=0} \dim \mathsf{Rel}_\sigma(Q,\alpha).$$

We can calculate the dimension of $\mathsf{Rel}_\sigma(Q,\alpha) \subset \mathsf{Null}(Q,\alpha)$ as follows. If there exists a vertex w such that

$$\delta_w := \sum_{t(a)=w} \bar{\sigma}_{s(a)} - \sigma_w < 0,$$

$\mathsf{Rel}_\sigma(Q,\alpha)$ will be empty (recall that $\bar{\sigma}$ is the same dimension vector as σ except that $\bar{\sigma}_v = 1$).

If this is not the case

$$\begin{aligned}
\dim \mathsf{Rel}_\sigma(Q,\alpha) &= \sum_{w\in Q_0} \sigma_w(\alpha_w - \sigma_w) + \sum_{a\in Q_1} \big(\alpha_{s(a)}\alpha_{t(a)} - \bar{\sigma}_{s(a)}(\alpha_{t(a)} - \sigma_{t(a)})\big) \\
&= \dim \mathsf{Rep}(Q,\alpha) - \sum_{w\neq v} \Bigg(\sum_{t(a)=w} \bar{\sigma}_{s(a)} - \sigma_w\Bigg)(\alpha_w - \sigma_w) - \sum_{t(a)=v} \sigma_{s(a)} \\
&= \dim \mathsf{Rep}(Q,\alpha) - \sum_{w\neq v} \delta_w(\alpha_w - \sigma_w) - \sum_{t(a)=v} \sigma_{s(a)}.
\end{aligned}$$

The first term on the first line calculates the dimension of all possible choices of a σ_w-dimensional subspaces in an α_w-dimensional subspace for every w. The second term gives the dimension of the space of all possible maps R_a, $a \in Q_1$ mapping the correct subspaces onto each other.

Now we calculate the last term of the third line. For a given n we have that

$$\sum_{t(a)=v} \sigma_{s(a)} = \#\{\text{cycles of length} \leqslant n\} + \sum_{\substack{w \overset{p}{\rightsquigarrow} v \\ |p|=n, w\neq v}} \sigma_w - \sum_{w\neq v} \delta_w \cdot \#\big\{w \overset{p}{\rightsquigarrow} v,\ |p| < n\big\}.$$

We will prove this statement by induction. Denote the formula by $(*)$. For $n=1$ only the middle term of the right-hand side is nonzero and it is equal to the left-hand side. Suppose that the formula holds for n, we now want to prove it for $n+1$. We split every σ_w as $\sum_{t(a)=w} \sigma_{s(a)} - \delta_w$. If $s(a) = v$ then ap is a cycle of length $n+1$ and in this case $\sigma_{s(a)} = 1$. We will put these terms apart.

$$\begin{aligned}
\sum_{\substack{w \overset{p}{\rightsquigarrow} v \\ |p|=n, w\neq v}} \sigma_w &= \sum_{\substack{w \overset{p}{\rightsquigarrow} v \\ |p|=n, w\neq v}} \Bigg(\sum_{t(a)=s(p)} \sigma_{s(a)} - \delta_w\Bigg) \\
&= \sum_{\substack{v \overset{ap}{\rightsquigarrow} v \\ |ap|=n+1}} 1 + \sum_{\substack{w \overset{ap}{\rightsquigarrow} v \\ |ap|=n+1, w\neq v}} \sigma_w - \sum_{\substack{w \overset{p}{\rightsquigarrow} v \\ |p|=n, w\neq v}} \delta_w \\
&= \#\{\text{cycles of length } n+1\} + \sum_{\substack{w \overset{p}{\rightsquigarrow} v \\ |p|=n+1, w\neq v}} \sigma_{s(p)}
\end{aligned}$$

$$-\sum_{w\neq v}\delta_w\cdot\#\big\{w \overset{p}{\rightsquigarrow} v,\ |p|=n\big\}.$$

We can substitute this formula in $(*)$ and add the left term respectively the right terms to obtain the equation for $n+1$. Because every cycle contains v and the length of the paths is bounded, we get for $n\gg 1$ that the middle term becomes zero and hence

$$\sum_{t(a)=v}\sigma_{s(a)} = \#\{\text{cycles}\} - \sum_{w\neq v}\delta_w\cdot\#\big\{w \overset{p}{\rightsquigarrow} v\big\}$$
$$= \dim \mathsf{iss}(Q,\alpha) - \sum_{w\neq v}\delta_w\cdot\#\big\{w \overset{p}{\rightsquigarrow} v\big\}.$$

The last equality holds because (Q,α) reduces to a quiver with one vertex of dimension 1 and k loops where k is the number of cycles.

The formula for the dimension of $\mathsf{Rel}_\sigma(Q,\alpha)$ now becomes

$$\dim \mathsf{Rel}_\sigma(Q,\alpha) = \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha) - \sum_{w\neq v}\delta_w\Bigg(\sum_{t(a)=w}(\alpha_w-\sigma_w) - \#\big\{w \overset{p}{\rightsquigarrow} v\big\}\Bigg).$$

Note that if $\delta_w>0$ then $\sigma_w\leqslant \#\{v \rightsquigarrow w\}-1$, so, if we suppose that

$$\alpha_w \geqslant \#\{v \rightsquigarrow w\} + \#\{v \leftsquigarrow w\} - 1$$

then

$$(\alpha_w-\sigma_w) - \#\{v \rightsquigarrow w\}\geqslant 0.$$

We now have that $\dim \mathsf{Rel}_\sigma(Q,\alpha)\leqslant \dim \mathsf{Rep}(Q,\alpha)-\dim \mathsf{iss}(Q,\alpha)$ and therefore $\operatorname{def}(Q,\alpha)=0$ and (Q,α) is cofree.

On the other hand, if there exists a vertex w such that

$$\alpha_w < \#\{v \rightsquigarrow w\} + \#\{v \leftsquigarrow w\} - 1$$

we can construct a relevant dimension vector σ such that the corresponding δ is only nonzero for such w (which is always possible because the dimensions of the other vertices are always big enough). The dimension of $\mathsf{Rel}_\sigma(Q,\alpha)$ is then bigger than $\dim \mathsf{Rep}(Q,\alpha)-\dim \mathsf{iss}(Q,\alpha)$ so (Q,α) is not cofree. $\square$

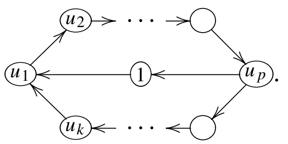
The theorem above provides us an interesting corollary which we will use in the next proofs.

Corollary 1. *If (Q,α) is cofree and v,w are vertices of dimension one, then v and w are contained in at most one cycle. If $\alpha=1$ then (Q,α) is the connected sum of cycles.*

Proof. If this were not the case one could find a subsetting of (Q, α) that reduces to  contradicting the previous theorem and Lemma 2. If $\alpha = 1$ the prime components of Q are these cycles. $\square$

For the second step, we first determine the general form of an unwedgeable prime cofree quiver setting, and then we determine which ones of this general form are indeed cofree.

Theorem 7. (1(ii) Part 1) *If* (Q, α) *is a cofree, prime, unwedgeable setting that reduces to* $\mathcal{Q}_0(1)$ *and if it is not of the form* 1(i)*, then* (Q, α) *looks like*



Proof. Because (Q, α) is coregular we can find a sequence of reduction steps $(\mathcal{R}_{\mathrm{I}}, \ldots, \mathcal{R}_{\mathrm{III}})$ that deletes all vertices except this vertex of dimension 1. The order of this sequence matters because some vertex might become reducible only after some other reductions have been made. However, it is possible to order the reduction steps in the following way:

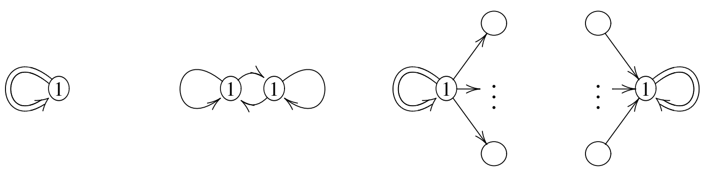
(1) first delete vertices of dimension bigger than one without loops using $\mathcal{R}_{\mathrm{I}}$-moves,
(2) delete vertices of dimension bigger than one with loops using an $\mathcal{R}_{\mathrm{III}}$- and a $\mathcal{R}_{\mathrm{I}}$-move,
(3) finally, delete the remaining vertices of dimension one using $\mathcal{R}_{\mathrm{II}}$- and a $\mathcal{R}_{\mathrm{I}}$-moves.

This ordering is possible because the reduction moves in (ii) and (iii) do not change the reducibility conditions of the vertices that are reduced in (i). The quiver setting one obtains after applying all moves in (i) will be called the skeleton of (Q, α) and will be denoted by $(Q^{\mathrm{I}}, \alpha^{\mathrm{I}})$. The skeleton only consists of vertices of dimension 1 and vertices of higher dimensions with a unique loop because these can be reduced using $\mathcal{R}_{\mathrm{III}}$.

We can separate the vertices of higher dimension in (Q, α) in two classes: those that are contained in a cycle of higher dimension and those that are not.

Take v to be a vertex in the second class. Every arrow in the skeleton corresponds to a unique path in (Q, α) by definition of $\mathcal{R}_{\mathrm{I}}$, so we can look at the subquiver P_v^{I} of Q^{I} whose arrows correspond to paths through v in Q. If S is the set of source vertices for these arrows and T the set of target vertices, then there is an arrow between every vertex of S and every vertex of T. Indeed, two paths $p = p_1 v p_2$ and $q = q_1 v q_2$ can be combined to $p_1 v q_2$ and $q_1 v p_2$, which connect the source of p with the target of q and vice versa.

As there can be only one cycle between every two vertices in Q^{I} of dimension 1 (Corollary 1), the only possibilities for P_v^{I} are

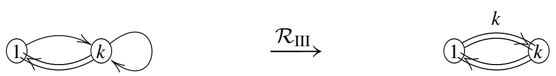


where the dimensions on the empty vertices are arbitrary.

Now we claim that the last two settings are impossible because (Q,α) is unwedgeable. Suppose that there is a v such P_v^{I} is of the last type (the other type can be treated analogously) and look at the path from v to ①.

The vertex on this path closest to ① is also reducible of the same type as v, so without loss of generality we can assume that v has a unique arrow connecting it with ①. Because we have assumed that (Q,α) is unwedgeable the number of cycles through v must be bigger than α_v. This contradicts the fact that (Q,α) is cofree. Indeed, look at the subquiver $(Q',\alpha') \subset (Q,\alpha)$ spanned by all cycles through v. Not all cycles of this quiver will necessarily go through ①, but the extra cycles will have no vertex of dimension 1. If this were the case, then not all arrows in such a cycle are contained in the same cycle through v. This means that (as all cycles through v go through ①) there are at least two cycles through two vertices of dimension one and this contradicts the hypothesis that (Q,α) is cofree (Corollary 1).

These extra cycles will thus be reducible using $\mathcal{R}_{\mathrm{III}}$-moves. These reductions can be done in (Q',α') before one has to reduce v. Indeed, let c be such a cycle there is a unique path from ① to this cycle and a unique path back, otherwise there is a subquiver in (Q',α') reducing to the form

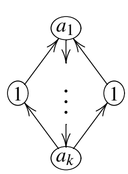


or its dual. Those two are not cofree by Theorem 1(i). The unique path back contains of course v and reducing v only affects the reducibility conditions on this path. Therefore the path starting from ① to the cycle and the cycle itself can be reduced before reducing v. If we do this to all such cycles we obtain a quiver from (Q',α') for which all quasiprimitive cycles go through ①. But because v was not wedgeable this quiver does not satisfy the condition of Theorem 1(i) and hence it is not cofree (contradicting the fact that (Q,α) is cofree).

Now we can assume that P_v^{I} is either of the first class or one of the first two possibilities we considered for the second class. We use the condition that (Q,α) is prime to determine the shape of Q in each of these cases.

For the first class we already know that there is a unique path from the cycle to a unique vertex of dimension 1 and back. These paths both consist of one arrow because the vertices in (Q,α) are unwedgeable. Hence, the primeness of (Q,α) implies that Q consists only of the cycle and this 1-dimensional vertex.

If v is in the second class we know by primeness and the previous paragraph that $(Q^{\mathrm{I}},\alpha^{\mathrm{I}})$ consists only of vertices with dimension one. This means that $(Q^{\mathrm{I}},\alpha^{\mathrm{I}})$ is a connected sum of cycles. If we are not in the case of Theorem 1(i), the primeness of (Q,α) ensures that least one vertex of the second possibility take this vertex to be v. The quiver setting spanned by the paths corresponding to the arrows in P_v^{I} is of the form



Note again that the paths from the one-dimensional vertices to a_1 (and from a_k to these vertices) have length one because (Q,α) is supposed to be unwedgeable. There are no extra arrows from or to the a_i so all arrows from the rest of the quiver pass through the one-dimensional vertices.

Because (Q,α) is prime and Q^{I} is a connected sum of cycles Q is equal to the setting above. Note that such a setting is a special case of our general setting where the one of the one-dimensional vertices is the central vertex and the other is contained in the cycle. □

Theorem 8. (1(ii) Part 2) *A quiver setting (Q,α) of the form above is cofree if and only if there is either exactly one vertex in the upper path $u_1 \rightsquigarrow u_p$ which attains the minimal dimension* $\min\{u_1,\dots,u_k\}$, *or there is no such vertex in the upper path but there is exactly one such vertex in the lower path $u_p \leftsquigarrow u_k$.*

We will denote this condition by (M).

Proof. Let (Q,α) be a setting of the previous form. Denote the vertex u_1 with i. For every positive integer n we denote by p_n the path of length n starting in i not running through 1. By p_{-n} we mean the path of length n ending in i not running through 1. The notation $i+n$ will stand for the target of p_n and $i-n$ will mean the source of p_{-n}. We denote the arrow entering the cycle by a_i and the arrow leaving by a_u. Let m be the first nonnegative integer such that α_{i+m} is minimal and let k be the length of the cycle c not through 1.

The diagram below summarizes all these definitions:

$$\begin{tikzcd}
 & i+n \arrow[r, dotted] & i+m \arrow[dr, dotted] & \\
i+k=i \arrow[ur, squiggly, "p_n"] & & 1 \arrow[ll, "a_i"'] & i+p \arrow[l, "a_u"'] \arrow[dl, dotted] \\
 & i-n \arrow[ul, squiggly, "p_{-n}"] & \cdots \arrow[l, dotted] &
\end{tikzcd} \quad \text{and } \alpha_{i+m} < \alpha_{i+n} \text{ if } 0 \leqslant n < m.$$

To every nilpotent representation $V \in \mathsf{Rep}(Q,\alpha)$ we can assign a sequence

$$\sigma_n := \begin{cases} \alpha_{i-|n|}, & n \leqslant 0, \\ \dim \mathsf{Im}\, V_{p_n}, & n > 0 \end{cases}$$

and two natural numbers $s, t \in k\mathbb{N} \times \mathbb{N}$:

- s is the largest number such that $\mathsf{Im}\, V_{a_i} \subset \mathsf{Im}\, V_{p_s}$ and is hence a multiple of k (if $V_{a_i} = 0$ we chose $s \in k\mathbb{N}$ the smallest number such that $V_{p_s} = 0$).
- $t \geqslant s$ is the largest number such that $\mathsf{Im}\, V_{p_t a_i} \neq 0$ (if $V_{a_i} = 0$ we chose $t = 0$).

Now let $\mathsf{Rel}_{\sigma,s,t}\, Q$ be the set of all representations with a given σ, s, t. If this set is nonempty, $(\sigma, s, t) \in \mathbb{N}^{\mathbb{Z}} \times k\mathbb{N} \times \mathbb{N}$ satisfies the following relations:

S1 $\forall n \in \mathbb{N}$: $\sigma_n \leqslant \alpha_{i+n}$,
S2 $\forall n \in \mathbb{N}$: $\sigma_{-n} = \alpha_{i-n}$,
S3 $\forall n \in k\mathbb{N}$: $\sigma_n = 0 \Rightarrow s \leqslant n$,
S4 $\sigma_j - \sigma_{j+n} \geqslant \lfloor \frac{n}{k} \rfloor$ if $j, n \geqslant 0$ and $\sigma_j > 0$,
S5 $\sigma_{s+t} \neq 0 \Rightarrow \sigma_{s+t} > \sigma_{s+t+1}$ and $\sigma_{s+t} = 0 \Rightarrow t = 0$.

The necessity of S1–S3 follows straight from the definition, S4 is needed otherwise the map from the vertex $i+j$ to itself is not nilpotent. Property S5 expresses the fact that the map $\mathsf{Im}\, V_{p_t} \to \mathsf{Im}\, V_{p_{t+1}}$ cannot be an isomorphism because $\mathsf{Im}\, V_{p_t a_i}$ is annihilated.

The codimension of $\mathsf{Rel}_{\sigma,s}\, Q$ in $\mathsf{Rep}(Q,\alpha)$ can be written as

$$\operatorname{codim} \mathsf{Rel}_{\sigma,s}\, Q = \sum_{j\geqslant 1} \underbrace{(\sigma_{j-k}-\sigma_j)(\sigma_{j-1}-\sigma_j)}_{\text{(i)}} + \underbrace{\alpha_i - \sigma_s}_{\text{(ii)}} + \underbrace{\sigma_{t+1}}_{\text{(iii)}} + \underbrace{\#([s,s+t]\cap p+k\mathbb{N})}_{\text{(iv)}},$$

where

(1) counts the conditions needed to ensure that the σ_{j-1}-dimensional space $\mathsf{Im}\, V_{p_{j-1}}$ maps onto a σ_j-dimensional subspace of $\mathsf{Im}\, V_{p_{j-k}}$,
(2) counts the conditions needed to map ① inside $\mathsf{Im}\, V_{p_s}$,
(3) counts the number of conditions to annihilate $\mathsf{Im}\, V_{p_t a_i}$ inside $\mathsf{Im}\, V_{p_{t+1}}$,
(4) checks whether the images of ① under the paths to $i+p$ are contained in the kernel of a_u.

By introducing the factors

$$\delta_n := \sigma_{n-1} - \sigma_n,$$
$$f_{js} := \begin{cases} \sigma_{j-k} - \sigma_j - 1, & j \in [s+1, s+t+1], \\ \sigma_{j-k} - \sigma_j, & j \notin [s+1, s+t+1], \end{cases}$$

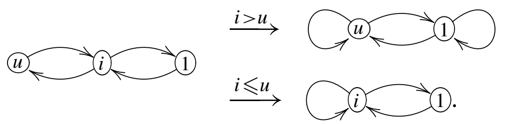
we can rewrite the expression in a more manageable way:

$$\begin{aligned} \operatorname{codim} \mathsf{Rel}_{\sigma,s}\, Q &= \sum_{j\geqslant 1} (\sigma_{j-k}-\sigma_j)\delta_j + \alpha_i - (\delta_{s+1}+\cdots+\delta_{t+1}) + \#[s,s+t]\cap p+k\mathbb{N} \\ &= \sum_{j\geqslant 0} f_{js}\delta_j + \alpha_i + \#[s,s+t]\cap p+k\mathbb{N}. \end{aligned}$$

Because there are only a finite number of (σ,s,t) satisfying S1–S5, the codimension of the nullcone is the minimum of all possible $\operatorname{codim} \mathsf{Rel}_{\sigma,s}\, Q$. To prove that (Q,α) is cofree one has to show that the codimension is maximal:

$$\operatorname{codim} \mathsf{Null}(Q,\alpha) = \dim \mathsf{iss}(Q,\alpha) = \begin{cases} 2\alpha_{i+m}, & m \leqslant p, \\ 2\alpha_{i+m}+1, & m > p. \end{cases}$$

The difference in the two cases comes from the fact that the R_{I}-moves reduce these settings differently:



We will now prove the sufficiency of condition (M): if it holds then for every (σ,s,t) satisfying S1–S5 we have $\operatorname{codim} \mathsf{Rel}_{\sigma,s}(Q,\alpha) \geqslant 2\alpha_{i+m}(+1)$.

Before we estimate the codimension we first need to calculate a useful inequality

$$\Delta_{ts} := \sum_{f_{js}\leqslant t,\, j\in[1,m]} \delta_j = \sigma_{j_1-1} + \underbrace{-\sigma_{j_1} + \sigma_{j_2-1}}_{\leqslant 0} + \underbrace{\sigma_{j_2} + \cdots}_{\leqslant 0} - \sigma_{j_\mu} \leqslant \alpha_i - \underbrace{(\alpha_{i+m} - t - x_s)}_{\leqslant \sigma_m \leqslant \sigma_{j_\mu}},$$

where the last term, x_s, is equal to 1 if $s=0$ and 0 otherwise. Note that this inequality can only become an equality if $\forall j \in [1,m]: \delta_j \neq 0 \Rightarrow f_{js} \leqslant t$, so two Δ's cannot both reach their upper bound and therefore

$$\Delta_{ts} + \Delta_{t+1,s} \leqslant 2(\alpha_i - \alpha_{i+m} + t + x_s).$$

If $j > m$ then by S1, S2 and S4 we can conclude that $\sigma_{j-k} \geqslant \sigma_{j-1}$, so $f_{js} \geqslant \delta_j - 1$ and equality is only achieved inside $[s+1, s+t+1]$. If $j_1 < j_2 \in [s+1, s+t+1] \cap [m+1, \infty[$ and $j_2 \leqslant j_1 + k$ then $f_{j_2 s} \geqslant \delta_{j_2} + \cdots + \delta_{j_1} - 1$, so there are at most $\#[s+1, s+t+1] \cap m+1+k\mathbb{N} = \#[s, s+t] \cap m+k\mathbb{N}$ j's bigger than m such that $f_{js} = 0$ and $\delta_j > 0$. Moreover for these j we have that $\delta_j = 1$.

Now we can construct a lower bound for the codimension:

$$\begin{aligned}
\operatorname{codim} \mathsf{Rel}_{\sigma,s} Q &= \sum_{j\geqslant 1} f_{js}\delta_j + \alpha_i + \#[s, s+t] \cap p + k\mathbb{N} \\
&\geqslant \sum_{j\geqslant 1} \delta_j - \sum_{f_{js}=0} \delta_j - 2 \sum_{f_{js}=-1} \delta_j + \alpha_i + \#[s, s+t] \cap p + k\mathbb{N} \\
&= \sum_{j\geqslant 1} \delta_j - \#[s, s+t] \cap m + k\mathbb{N} - \Delta_{0s} - \Delta_{-1s} + \alpha_i + \#[s, s+t] \cap p + k\mathbb{N} \\
&= \sum_{j\geqslant 1} \delta_j - \Delta_{0s} - \Delta_{-1s} + \alpha_i + \underbrace{\begin{cases} 1, & p < t \bmod k < m, \\ -1, & m < t \bmod k < p, \\ 0, & \text{else} \end{cases}}_{:=q_t} \\
&\geqslant \begin{cases} \alpha_i - 2(\alpha_i - \alpha_{i+m}) + \alpha_i + q_t, & s = 0, \\ \alpha_i - (\alpha_i - \alpha_{i+m}) - 0 + \alpha_i + q_t, & s > 0 \end{cases} \\
&\geqslant 2\alpha_{i+m} + q_t \geqslant \mathsf{iss}(Q, \alpha) - 1.
\end{aligned}$$

For this lower bound to be reached we must have that

L1 there are as many as possible $\ell > m$ in $[s+1, s+t+1]$ such that $\delta_\ell \neq 0$ and $f_{\ell s} = 0$,
L2 $\forall j \in \mathbb{N}$: $f_{js} \leqslant 1$ and $\forall j \in [s+1, s+t+1]$: $f_{js} \leqslant 0$ and hence $\delta_j \leqslant 1$ if $j \notin [1,m]$,
L3 $\Delta_{0s} + \Delta_{-1s}$ reaches its upper limit,
L4 if $s > 0$ then $\alpha_i = \alpha_{i+m}$ so $m = 0$,
L5 $q_t = 0$ if $p < m$ and $q_t = -1$ if $m < p$.

If $s > 0$, the conditions L1 and L2 imply that the distance between two consecutive ℓ's such that $\delta_l \neq 0$ is exactly k. Take ℓ to be the representative between 0 and k. By S5 also $t = \ell \bmod k$, so by L5 either $m > \ell > p$ or $p \leqslant m > \ell$. If we assume condition (M) then $\alpha_{i+\ell} > \alpha_{i+m}$, so

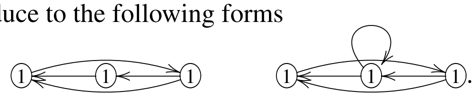
$$f_{\ell s} = \begin{cases} \alpha_{i+\ell} - \sigma_{\ell-1} + \delta_\ell - 1 > \alpha_{i+\ell} - \alpha_{i+m} > 0, & s = 0, \\ \alpha_{i+\ell} - \sigma_{\ell-1} + \delta_\ell > \alpha_{i+\ell} - \alpha_{i+m} + 1 > 1, & s > 0 \end{cases}$$

which is in contradiction with L1 if $s = 0$ or L2 if $s > 0$. The lower bound cannot be reached so the setting is cofree.

To prove the necessity of (M), we use the technique of local quivers. Consider a representation that is the direct sum of $\alpha_{i+m} - 1$ copies of a simple representation W with dimension vector 1 on the cycle and zero on the central vertex plus some simple representations S_v corresponding to the vertices with the correct multiplicities:

$$W^{\oplus \alpha_{i+m}-1} \oplus S_{①} \bigoplus_{j=0}^{k-1} S_{i+j}^{\oplus \alpha_{i+j} - \alpha_{i+m}+1}.$$

The subquiver of the local quiver spanned by all vertices except the one coming from W looks the same as the original quiver only the dimensions on the cycle have been reduced by $\alpha_{i+m} - 1$. If condition (M) does not hold there are either two one-dimensional vertices in the upper or there is none in the upper path but two in the lower path. In these cases we can use reduction moves to see that these settings reduce to the following forms

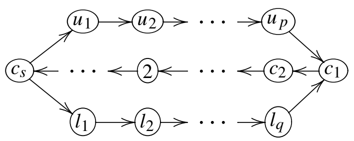


Both settings are not cofree. □

3.2. *Quiver settings reducing to $\mathcal{Q}_2$*

Finally, we turn our attention towards all quivers reducing to $\mathcal{Q}_2$. Recall the following lemma from [10].

Lemma 4. *Let (Q, α) be a quiver setting reducing to $\mathcal{Q}_2$, then (Q, α) consists of two cyclic quivers coinciding on s subsequent vertices, one of these of dimension* 2:



with $u_i, l_j, c_k \geqslant 2$ for all $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ and $1 \leqslant k \leqslant s$. Such a setting will be denoted by $\mathcal{Q}_2(p, q, s)$. The numbers p and q can be zero, s must be bigger than 1.

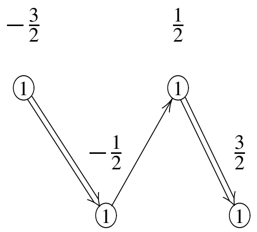
When considering the situations of the lemma above, the following result will prove useful.

Lemma 5. *The quiver settings*

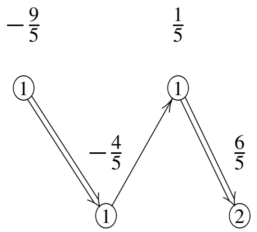
with $3 \geqslant d \geqslant 2$ are not equidimensional.

Proof. We will prove this for the first setting, which we will denote by (Q, α). The proof for the second setting is completely analogous. We will construct a stratum in the Hesselink stratification of $\mathsf{Null}(Q, \alpha)$ of dimension strictly greater than $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha)$, so $\operatorname{def}(Q, \alpha) > 0$. Using the notations and conventions from [4], consider the following level quivers with corresponding coweight:

if $d = 2$:



if $d = 3$:



For $d = 2$ this level quiver with corresponding coweight determines a Hesselink stratum of dimension 9, whereas $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha) = 7$. For $d = 3$ the level quiver determines a Hesselink stratum of dimension 14 whereas $\dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha) = 13$. $\square$

An immediate consequence of this lemma is

Lemma 6. *The setting $\mathcal{Q}_2(p, q, s)$ is not cofree if at least* 2 *vertices in the path* $c_1 \rightsquigarrow c_s$ *have dimension* 2 *or* 3.

Proof. Consistently applying reduction step $\mathcal{R}_\mathrm{I}$ to the vertex with greatest dimension in either of these settings reduces the quiver setting to one of the two settings from Lemma 5. Now because these settings are not cofree, the original quiver setting cannot be cofree either. $\square$

We will now show that these are the only situations reducing to $\mathcal{Q}_2$ that are not cofree. For this, we need another two lemmas.

Lemma 7. *Consider the map*

$$\pi : \mathsf{Hom}(U, V) \times \mathsf{Hom}(V, W) \twoheadrightarrow \mathsf{Hom}(U, W) : (X, Y) \mapsto Y \circ X$$

where U and V are vector spaces of dimension at least 2*, W is a vector space of dimension exactly* 2 *and* $\dim U - \dim V + 1 \geqslant 0$*, then*

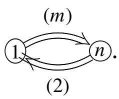
(1) $\dim \pi^{-1}(0) = \dim U \dim V + \dim V - \dim U + 1$;
(2) *for $Z \in \mathsf{Hom}(U, W)$ with* $\mathrm{rk}(Z) = 1$ *we have that* $\dim \pi^{-1}(Z) = \dim U \dim V + \dim V - \dim U$;

(3) *for* $Z \in \mathsf{Hom}(U, W)$ *with* $\mathrm{rk}(Z) = 2$ *we have that* $\dim \pi^{-1}(Z) = \dim U \dim V + 2\dim V - 2\dim U$.

If $\dim U - \dim V + 1 < 0$ *we have for all* $Z \in \mathsf{Hom}(U, W)$ *that*

$$\dim \pi^{-1}(Z) = \dim U \dim V + 2\dim V - 2\dim U.$$

Proof. We let $\dim U = m$ and $\dim V = n$ so we may identify the above situation with the quotient map of the representation space of the following quiver setting (Q, α):

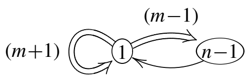


In order to prove (1), we have to compute the dimension of the nullcone of this quiver setting. This was done in Theorem 1(i) and from this we obtain

$$\dim \mathsf{Null}(Q, \alpha) = mn + 2n - 2m + m + 2 - n - 1$$

if $m + 2 - n - 1 \geqslant 0$.

To show that (2) holds, we first note that if Z has rank one, it has representation type $((1, 1), 1; (0, 1), n - 1)$. Its local quiver setting (Q_Z, α_Z) then corresponds to the quiver setting

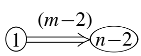


and the dimension of its fiber has to be

$$\begin{aligned}\dim \pi^{-1}(Z) &= \dim \mathsf{Null}(Q_Z, \alpha_Z) + \dim GL_\alpha - \dim GL_{\alpha_Z}\\ &= (m-1)(n-1) + n - 1 - (m-1) + (m-n) + 1 + n^2 - 1 - (n-1)^2\\ &= mn - m + n\end{aligned}$$

where the dimension of the nullcone is again due to Theorem 1(i).

Finally, (3) holds by a similar computation. If Z has rank 2 it has to be of representation type $((1, 2), 1; (0, 1); n - 2)$. Then its local quiver (Q_Z, α_Z) becomes



and the dimension of its fiber becomes

$$\begin{aligned}\dim \pi^{-1}(Z) &= \dim \mathsf{Null}(Q_Z, \alpha_Z) + \dim GL_\alpha - \dim GL_{\alpha_Z}\\ &= (m-2)(n-2) + 1 + n^2 - 1 - (n-2)^2\\ &= mn - 2m + 2n.\end{aligned}$$

If $\dim U - \dim V + 1 < 0$ in each of the situations the local quiver is cofree, proving the last claim of the lemma. □

Lemma 8. *Let $(\tilde{A}_n, \alpha)$ be a cyclic quiver setting and denote the arrows of this cyclic quiver by a_0 through a_n such that $s(a_0)$ has minimal dimension. If $\alpha(s(a_0)) \geqslant 2$ then $\mathsf{Null}(\tilde{A}_n, \alpha)$ has no irreducible component C contained in*

$$N_n = \left\{ V \in \mathsf{Null}(\tilde{A}_n, \alpha) \;\middle|\; V(a_n)V(a_{n-1}) \dots V(a_0) = 0 \right\}.$$

Proof. Let σ denote the relevant dimension vector with base $s(a_0)$ then we see that $N_n = \bigcup_{\sigma_{s(a_0)}=0} \mathsf{Rel}_\sigma$. So

$$\begin{aligned}
\operatorname{codim} N_n &= \min_{\sigma_{s(a_0)}=0} \sum_{j=0}^{n} (\alpha_{t(a_j)} - \sigma_{t(a_j)})(\sigma_{s(a_j)} - \sigma_{t(a_j)}) \\
&\geqslant \min_{\sigma_{s(a_0)}=0} \sum_{j=0}^{n} (\alpha_{s(a_0)} - \sigma_{t(a_j)})(\sigma_{s(a_j)} - \sigma_{t(a_j)}) \\
&> \min_{\sigma_{s(a_0)}=0} \sum_{j=0}^{n} (\sigma_{s(a_j)} - \sigma_{t(a_j)}) = \alpha_{s(a_0)}.
\end{aligned}$$

For the least inequality we used that if $(\sigma_{s(a_j)} - \sigma_{t(a_j)}) > 0 \Leftrightarrow (\alpha_{s(a_0)} - \sigma_{t(a_j)}) > 0$ and because $\alpha_{s(a_0)} > 1$ at least one of the $(\alpha_{s(a_0)} - \sigma_{t(a_j)}) > 1$.

Because of the equidimensionality all irreducible components of $\mathsf{Null}(\tilde{A}_n, \alpha)$ have the same codimension: $\dim \mathsf{iss}(Q, \alpha) = \alpha_{s(a_0)}$. Therefore N_n cannot contain one of them. □

These last lemmas now allow us to prove

Theorem 9. *A quiver setting (Q, α) reducing to $\mathcal{Q}_2$ is cofree if and only if it is of the form $\mathcal{Q}_2(p, q, s)$ with all $u_i, l_i \geqslant 2$ and with exactly one $c_i = 2$ and all the rest not smaller than* 4.

Proof. We will prove the statement by induction on s.

First assume $s = 1$. In this case the quiver (Q, α) is a connected sum of two cyclic quiver settings $(\tilde{A}_p, \alpha_p)$ and $(\tilde{A}_q, \alpha_q)$ in a vertex with dimension 2 which we denote by v. We will denote the arrows of the first cyclic quiver by $a_0, \dots, a_p$ and of the second cyclic quiver by $b_0, \dots, b_q$ with $s(a_0) = s(b_0) = v$ and $t(a_i) = s(a_{i+1 \bmod p+1})$ respectively $t(b_j) = s(a_{j+1 \bmod q+1})$. We have an embedding

$$\mathsf{Null}(Q, \alpha) \subset \mathsf{Null}(\tilde{A}_p, \alpha_p) \times \mathsf{Null}(\tilde{A}_q, \alpha_q).$$

Any maximal irreducible component of $\mathsf{Null}(Q, \alpha)$ has to be a proper closed subset of an irreducible component of $\mathsf{Null}(\tilde{A}_p, \alpha_p) \times \mathsf{Null}(\tilde{A}_q, \alpha_q)$. Indeed, if we denote a representation in such an irreducible component of $\mathsf{Null}(Q, \alpha)$ as (V, W), we know that

$$\mathsf{Tr}\big(V(a_p) \dots V(a_0) W(b_q) \dots W(b_0)\big) = 0.$$

Now by Lemma 8, in all irreducible components of $\mathsf{Null}(\tilde{A}_p, \alpha_p)$ and $\mathsf{Null}(\tilde{A}_q, \alpha_q)$ there are elements V, W satisfying

$$V(a_p)\dots V(a_0) \neq 0 \quad \text{and} \quad W(b_q)\dots W(b_0) \neq 0.$$

This means in any irreducible component of $\mathsf{Null}(\tilde{A}_p, \alpha_p) \times \mathsf{Null}(\tilde{A}_q, \alpha_q)$ we get a representation (V, W) such that

$$V(a_p)\dots V(a_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad W(b_q)\dots V(b_0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This yields

$$\mathsf{Tr}\big(V(a_p)\dots V(a_0)W(b_q)\dots W(b_0)\big) = 1.$$

But then we have

$$\dim \mathsf{Null}(Q, \alpha) \leqslant \dim \mathsf{Null}(\tilde{A}_p, \alpha_p) + \dim \mathsf{Null}(\tilde{A}_q, \alpha_q) - 1$$

and hence

$$\dim \mathsf{Null}(Q, \alpha) = \dim \mathsf{Rep}(Q, \alpha) - \dim \mathsf{iss}(Q, \alpha).$$

Now assume $s > 1$ and let f be the unique vertex in $c_1 \rightsquigarrow c_s$ for which $\alpha_f = 2$ and suppose that all the other $c_i \geqslant 4$.

If v is a vertex $c_1 \rightsquigarrow c_s$ in next to f then we can apply an $\mathcal{R}_{\mathrm{I}}$ move to v to reduce to a quiver of the form $\mathcal{Q}_2(p, q, s-1)$. We will now show that such a move keeps the defect invariant.

Suppose that v is on the right of f (the other case is similar) so there is an arrow $a : v \to f$. Consider the projection map corresponding to the reduction move $\mathcal{R}_{\mathrm{I}}^v$

$$\pi : \mathsf{Null}(Q, \alpha) \twoheadrightarrow \mathsf{Null}\big(\mathcal{R}_{\mathrm{I}}(Q), \mathcal{R}_{\mathrm{I}}(\alpha)\big).$$

By Lemma 7 we know that if

$$\sum_{x \in Q_1, t(x) = v} \alpha\big(s(x)\big) - \alpha(v) + 1 \leqslant 0$$

the dimension of any fiber of π is the same so the defect does not change.

Now assume $\sum_{x \in Q_1, t(x) = v} \alpha(s(x)) - \alpha(v) + 1 > 0$ and let C be a maximal irreducible component of $\mathsf{Null}(Q, \alpha)$. By the dimension formula for morphisms we have for any element in $Z \in \pi(C)$ that

$$\dim(C) \leqslant \dim \pi(C) + \dim \pi^{-1}(Z).$$

We now can consider 3 cases

- If $\pi(C)$ contains an element Z such that

$$\mathrm{rk}\big((Z(x))_{x\in Q_1,t(x)=f}\big)=2$$

then this fiber is generic so $\operatorname{codim} C=\operatorname{codim}\pi(C)$.
- If all $Z\in\pi(C)$ have $\mathrm{rk}((Z(x))_{x\in Q_1,t(x)=f})\leqslant 1$ and we have an element with $\mathrm{rk}((Z(x))_{x\in Q_1,t(x)=f})=1$, then

$$\pi(C)\subset L_1\times \mathsf{Rep}(\overline{Q},\bar{\alpha})$$

with $(\overline{Q},\bar{\alpha})$ the quiver setting with all arrows x with $t(x)=f$ removed and L_1 the set of all linear maps from a vector space of dimension

$$\sum_{x\in Q_1,t(x)=v}\alpha\big(s(x)\big)$$

to a vector space of dimension 2 that have rank at most 1. By [3, II.4.1, Lemma 1], we have that L_1 is irreducible of dimension $\sum_{x\in Q_1,t(x)=v}\alpha(s(x))+1$. Now any Z in $\pi(C)$ has to satisfy $tr(X)=tr(Y)=0$ for X the cycle along the first cyclic quiver and Y the cycle along the second cyclic quiver. This means

$$\dim\pi(C)\leqslant \dim L_1+\dim\mathsf{Rep}(\overline{Q},\bar{\alpha})-2.$$

But then

$$\begin{aligned}\dim C&\leqslant \dim\pi(C)+\dim\pi^{-1}(Z)\\&=\dim L_1+\dim\mathsf{Rep}(\overline{Q},\bar{\alpha})-2\\&\quad+\alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha\big(s(x)\big)+\alpha(v)-\sum_{x\in Q_1,t(x)=v}\alpha\big(s(x)\big)\\&=\dim\mathsf{Rep}(Q,\alpha)-\alpha(v)-1\\&\leqslant\dim\mathsf{Rep}(Q,\alpha)-5\\&=\dim\mathsf{Rep}(Q,\alpha)-\dim\mathsf{iss}(Q,\alpha).\end{aligned}$$

- If all Z in $\pi(C)$ satisfy $(Z(x))_{x\in Q_1,t(x)=f}=0$ then

$$\pi(C)\subset\{0\}\times\mathsf{Rep}(\overline{Q},\bar{\alpha}).$$

But then

$$\begin{aligned}\dim C&\leqslant \dim\pi(C)+\dim\pi^{-1}(Z)\\&\leqslant\dim\mathsf{Rep}(\overline{Q},\bar{\alpha})\\&\quad+\alpha(v)\sum_{x\in Q_1,t(x)=v}\alpha\big(s(x)\big)+\alpha(v)-\sum_{x\in Q_1,t(x)=v}\alpha\big(s(x)\big)+1\end{aligned}$$

$$
\begin{aligned}
&\leqslant \dim \mathsf{Rep}(Q,\alpha) - \alpha(v) - \sum_{x\in Q_1, t(x)=v} \alpha\big(s(x)\big) + 1\\
&\leqslant \dim \mathsf{Rep}(Q,\alpha) - 5\\
&= \dim \mathsf{Rep}(Q,\alpha) - \dim \mathsf{iss}(Q,\alpha).
\end{aligned}
$$

Finally note that all other possibilities were already shown to be not cofree in Lemma 6. $\square$

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