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The Wallman Compactification in Fuzzy Neighborhood Spaces

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1. INTRODUCTION

Most compactifications in the setting of fuzzy spaces have severe limitations. Thus, e.g., one typical condition which is usually assumed is weak inducedness (see H. W. Martin [17], U. Cerruti [3], and Liu Ying-Ming and Luo Maokang [7]). However, in the category FTS of fuzzy topological spaces [9], this condition is inadequate in the extreme since it is equivalent to being topologically generated. Consequently results obtained for such a class of spaces are via the functorial relations ω and ι simply equivalent to classical results in the category TOP of topological spaces.

A subcategory of FTS which does not reduce to TOP and which has received wide interest in the last years is that of fuzzy neighborhood spaces [5], denoted FNS. For a part of the work that has been done in this area we refer the reader to [1, 2, 4–6, 10–16, 18, 19, 22].

In this paper we construct a Wallman-type compactification (see, e.g., [20, 21]) in FNS which turns out to have nice properties. The tools underlying our construction are provided by the theory of convergence in FTS given in [8, 9].

2. PRELIMINARIES

We do not recall notations and definitions of concepts which by now are standard in the subject. For the definition of the subcategory FNS of FTS we refer to [10, 11]. For concepts and results involving prefilters and convergence in FTS we refer to [8, 9]. We recall that a fuzzy topological space is a fuzzy neighborhood space if and only if there exists a basis for the

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closed fuzzy sets consisting of two-valued fuzzy sets (where one of the two values has to be 1). This characterization of FNS was given in [25, Theorem 4.1], where no less than 20 equivalent characterizations are given for fuzzy neighborhood spaces. For characterizations of compactness and ultracompactness in FNS we refer to [11].

3. Wallman Compactification

The notion of saturated prefilter plays an important role in fuzzy neighborhood spaces.

We recall that an arbitrary prefilter $\mathbb{F}$ on a set X is called saturated if

$$(\forall \varepsilon \in I_0 : \mu_\varepsilon \in \mathbb{F}) \Rightarrow (\sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon) \in \mathbb{F}).$$

However, since in a fuzzy neighborhood space the fact that μ is closed does not imply that $\mu - \varepsilon$ is closed for any $\varepsilon \in I_0$, when we work with closed prefilters we have to adapt our usual formulation of saturated prefilters and saturation of prefilters.

3.1. Definition. A closed prefilter $\mathbb{F}$ on X is called saturated if $\forall \mu \in I^X$: $((\forall \varepsilon \in I_0 : (\mu + \varepsilon) \wedge 1 \in \mathbb{F}) \Rightarrow \mu \in \mathbb{F})$.

If $\mathbb{F}$ is a closed prefilter then

$$\tilde{\mathbb{F}} := \{\mu \text{ closed} \mid \forall \varepsilon \in I_0 : (\mu + \varepsilon) \wedge 1 \in \mathbb{F}\}$$

is called the saturation of $\mathbb{F}$.

The reader can easily verify that this formulation of saturation coincides with the original one if $\mathbb{F}$ is an arbitrary prefilter.

In the sequel by a 1-level prefilter we shall mean a prefilter $\mathbb{F}$ such that for all $\mu \in \mathbb{F}: \sup_{x \in X} \mu(x) = 1$. With the notations of [8, 9] this means that the characteristic value of $\mathbb{F}$, $c(\mathbb{F})$, has to equal 1.

The simple proof of the next proposition is left to the reader.

3.2. Proposition. *For any closed* 1*-level prefilter* $\mathbb{F}$ *we have*:

(1) $\mathbb{F}$ *is saturated if and only if* $\tilde{\mathbb{F}} = \mathbb{F}$.

(2) $\tilde{\mathbb{F}}$ *is the smallest closed saturated prefilter finer than* $\mathbb{F}$.

(3) *If* $\mathbb{F}$ *is prime then* $\tilde{\mathbb{F}}$ *is prime.*

3.3. Proposition. *The collection of all closed saturated* 1*-level prefilters is upwards inductive.*

Proof. It suffices to note that if $(\mathbb{F}_j)_{j\in J}$ is an increasing chain then

$$\bigvee_{j\in J} \mathbb{F}_j := \left\{\mu \text{ closed } \forall \varepsilon\in I_0,\ \exists \xi \in \bigcup_{j\in J} \mathbb{F}_j : \xi \leqq (\mu+\varepsilon)\wedge 1\right\}$$

is a closed saturated 1-level prefilter. ∎

By Zorn's lemma [20] we can now consider the set Ψ of all maximal closed saturated 1-level prefilters.

3.4. Proposition. *If $\mathbb{F}\in\Psi$ then $\mathbb{F}$ is a prime prefilter.*

Proof. Suppose that $\mu \vee \nu \in \mathbb{F}$ and $\mu \notin \mathbb{F}$; then by maximality there exists $\xi_0\in\mathbb{F}$ such that for all $\xi \geqq \xi_0$, $\xi\in\mathbb{F}$

$$\sup_{x\in X} \mu \wedge \xi(x) < 1.$$

Since $\mu\vee\nu\in\mathbb{F}$ this however implies that for all $\xi\leqq\xi_0$, $\xi\in\mathbb{F}$ we have

$$\sup_{x\in X} \nu\wedge\xi(x) = 1,$$

which, since $\mathbb{F}$ is maximal and since $\{\xi\in\mathbb{F} \mid \xi\leqq\xi_0\}$ is a base for $\mathbb{F}$, implies $\nu\in\mathbb{F}$. ∎

From now on we suppose that X is weakly-T_1 and symmetric. The concept weak-T_1 was introduced in [24], and for fuzzy neighborhood spaces it means that for any $x\in X$ we have $\overline{1_x}(y)<1$ for all $y\neq x$. By a symmetric fuzzy neighborhood space we mean a space which is such that for all $x, y\in X$ we have $\overline{1_x}(y)=\overline{1_y}(x)$.

This is one of the R_0-type axioms which are introduced in [23], where it is denoted by R_0^6. We now define two sets of maximal closed saturated 1-level prefilters.

$$R(X) := \{\mathbb{F}\in\Psi \mid \forall x\in X\colon \operatorname{adh}\mathbb{F}(x)<1\}$$

and

$$R_c(X) := \{\mathbb{F}\in\Psi \mid \sup_{x\in X} \operatorname{adh}\mathbb{F}(x)<1\}.$$

Clearly $R_c(X)\subset R(X)$ and it follows from [11] that if X is ultracompact $R(X)=R_c(X)=\varnothing$ and if X is compact but not ultracompact then $R(X)\neq\varnothing$ and $R_c(X)=\varnothing$. Now we make the following identification between points of x and "fixed" maximal closed saturated 1-level prefilters. For $x\in X$ put

$$\mathbb{D}(x) := \{\mu \text{ closed} \mid \mu(x)=1\};$$

then since X is symmetric

$$X \to \Psi$$
$$x \to \mathbb{D}(x)$$

is an injection, and we have $\Psi = X \cup R(X)$.

It is worthwhile to note that the supposition that X be symmetric cannot be dropped. The fact that $\mathbb{D}(x)$ is indeed maximal depends on this property. Indeed, if we suppose that $\mathbb{F}$ is a closed saturated 1-level prefilter finer than $\mathbb{D}(x)$ then for any $\mu \in \mathbb{F}$ we have

$$\sup_{y \in X} (\mu \wedge \overline{1_x})(y) = 1.$$

When one elaborates this expression making use of symmetry (i.e., replacing $\overline{1_x}(y)$ by $\overline{1_y}(x)$) this implies $\bar{\mu}(x) = 1$, i.e., $\mu(x) = 1$ and so $\mu \in \mathbb{D}(x)$.

In the sequel we shall now always put

$$\hat{X} := X \cup R(X) \qquad \text{and} \qquad \hat{X}^c := X \cup R_c(X).$$

For any $\mu \in I^X$ which is closed and any $\mathbb{F} \in \hat{X}$ we define

$$I(\mu, \mathbb{F}) := \{\varepsilon \in I \mid (\mu + \varepsilon) \wedge 1 \in \mathbb{F}\}$$

and

$$\hat{\mu} : \hat{X} \longrightarrow I$$
$$\mathbb{F} \longrightarrow 1 - \inf I(\mu, \mathbb{F}).$$

The following proposition contains some basic properties which we shall require in the sequel

3.5. Proposition. *Let $\mu, \nu \in I^X$ be closed and let α be constant; then:*

(1) $\mu \leqq \nu \Rightarrow \hat{\mu} \leqq \hat{\nu}$
(2) $\hat{\alpha} = \alpha$
(3) $\mu \mathbin{\hat{\vee}} \nu = \hat{\mu} \vee \hat{\nu}$
(4) $\mu \mathbin{\hat{\wedge}} \nu = \mu \wedge \hat{\nu}$
(5) $\hat{\mu}|_X = \mu$
(6) $\inf_{\mathbb{F} \in \hat{X}} \hat{\mu}(\mathbb{F}) = \inf_{x \in X} \mu(x)$.

Proof. Parts (1), (2), (4), and (5) follow at once from the definition, (3) follows from Proposition 3.4, and (6) follows from (2). ∎

We now define $\hat{\Delta}$ to be the fuzzy topology on $\hat{X}$ generated by the basis

$$\{\hat{\mu} \mid \mu \text{ closed in } X\},$$

and $\hat{\Delta}^c$ to be the subspace fuzzy topology on $\hat{X}^c$.

We first prove that $(\hat{X}, \hat{\Delta}) \in |FNS|$. In order to do so we require the following two propositions. Let Γ stand for the collection of all closed fuzzy sets in X which attain only two values, one of which is 1.

3.6. PROPOSITION. *Any closed fuzzy set in X can be uniformly approximated from above by a finite infimum of elements in Γ.*

Proof. Let ξ be closed in X, let $\varepsilon \in I_0$, and choose $n \in \mathbb{N}_0$ such that $2/n \leqq \varepsilon$. For each $k \in \{1, ..., n-2\}$ put

$$X_k := \left\{ x \in X \,\middle|\, \frac{k}{n} \leqq \xi(x) < \frac{k+1}{n} \right\}.$$

By [25], Γ is a basis for the closed fuzzy sets in X, and therefore for each $k \in \{1, ..., n-2\}$ and each $x \in X_k$ there exists $1_{A_x} \vee \alpha_x \in \Gamma$ such that $\xi \leqq 1_{A_x} \vee \alpha_x$ and $\alpha_x < \xi(x) + 1/n$. Now if we put

$$\theta_k := 1_{\bigcap_{x \in X_k} A_k} \vee \left(\frac{k+2}{n}\right)$$

then $\theta_k \in \Gamma$, $\xi \leqq \theta_k$, and for all $x \in X_k$, moreover, $\theta_k(x) \leqq \xi(x) + \varepsilon$. Clearly then $\inf_{k=1}^{n-2} \theta_k$ fulfills the requirements. ∎

3.7. PROPOSITION. *If $1_Y \vee \alpha \in \Gamma$ and $\mathbb{F} \in \hat{X}$ then*

$$1_Y \mathbin{\hat{\vee}} \alpha(\mathbb{F}) = \begin{cases} 1 & \text{if } 1_Y \vee \alpha \in \mathbb{F} \\ \alpha & \text{if } 1_Y \vee \alpha \notin \mathbb{F}. \end{cases}$$

Proof. The case $1_Y \vee \alpha \in \mathbb{F}$ is clear. If $1_Y \vee \alpha \notin \mathbb{F}$ then since $((1_Y \vee \alpha) + 1 - \alpha) \wedge 1 = 1$ it already follows that $1_Y \mathbin{\hat{\vee}} \alpha(\mathbb{F}) \geqq \alpha$. Suppose that $1_Y \mathbin{\hat{\vee}} \alpha(\mathbb{F}) > \alpha$; then there exists $\beta \in I_1$ such that $1_Y \vee \beta \in \mathbb{F}$. Then for any $\mu \in \mathbb{F}$ we have

$$\begin{aligned} \sup_{x \in X} \overline{1_Y} \wedge \mu(x) &\geqq \sup_{x \in Y} \mu(x) \\ &= \sup_{x \in X} \mu \wedge (1_Y \vee \beta)(x) \\ &= 1, \end{aligned}$$

which by maximality of $\mathbb{F}$ implies that $\overline{1_Y} \in \mathbb{F}$. Since $\overline{1_Y} \leqq 1_Y \vee \alpha$ this however implies $1_Y \vee \alpha \in \mathbb{F}$, which is a contradiction. ∎

3.8. THEOREM. *$(\hat{X}, \hat{\Delta}) \in |FNS|$ and $(\hat{X}^c, \hat{\Delta}^c) \in |FNS|$.*

Proof. By [25] we need only prove the first assertion. By Proposition 3.7 and again by [25] it suffices to show that

$$\hat{\Gamma} := \{\hat{\theta} \mid \theta \in \Gamma\}$$

is a basis for the closed fuzzy sets in $\hat{X}$.

Now if ξ is closed in X and $\varepsilon \in I_0$ then by Proposition 3.6 we can find a finite collection $(\theta_k)_{k=1}^n \subset \Gamma$ such that $\xi \leqq \inf_{k=1}^n \theta_k \leqq \xi + \varepsilon$, and then it follows from Proposition 3.7(1) and (4) that

$$\hat{\xi} \leqq \inf_{k=1}^{n} \hat{\theta}_k \leqq \hat{\xi} + \varepsilon$$

and we are done. ∎

Second we now prove that $(\hat{X}, \hat{\Delta})$ and $(\hat{X}^c, \hat{\Delta}^c)$ are "compactifications" of (X, Δ).

3.9. THEOREM. *$(\hat{X}, \hat{\Delta})$ is a weakly-T_1 ultracompactification of (X, Δ) in FNS, and likewise $(\hat{X}^c, \hat{\Delta}^c)$ is a weakly-T_1 compactification of (X, Δ) in FNS.*

Proof. That X is dense in $\hat{X}$ and hence in $\hat{X}^c$ follows at once from Proposition 3.5(6).

Let $\mathbb{F}$ be a prefilter on $\hat{X}$ such that $c(\mathbb{F}) = 1$. Then with the properties proved in Proposition 3.5(1), (2), (4), and (6) once again applied, it is immediately verified that

$$\mathbb{F}^* := \{\mu \text{ closed} \mid \hat{\mu} \in \mathbb{F}\}$$

is a prefilter on X such that $c(\mathbb{F}^*) = 1$. If there exists $x_0 \in X$ such that adh $\mathbb{F}^*(x_0) = 1$ then obviously adh $\mathbb{F}(\mathbb{D}(x_0)) = 1$, too. If such $x_0 \in X$ does not exist, take $\mathfrak{C} \in \hat{X}$ such that $\bar{\mathbb{F}}^* \subset \mathfrak{C}$; then adh $\mathbb{F}(\mathfrak{C}) = \inf_{\hat{\mu} \in \mathfrak{C}} \hat{\mu}(\mathbb{F}) = 1$.

It then follows from [11] that $\hat{X}$ is indeed ultracompact. If $\mathbb{F}$ is a prefilter on $\hat{X}^c$ such that $c(\mathbb{F}) = 1$ then the same argument, considering cases according to whether $\sup_{x \in X}$ adh $\mathbb{F}^*(x) = 1$ or not, shows that $\hat{X}^c$ is compact.

Finally it now remains to show that $\hat{X}$ is weakly-T_1. Let $\mathbb{F}, \mathbb{H} \in \hat{X}$; then since

$$\overline{1_{\{\mathbb{F}\}}} = \inf\{\hat{\nu} \mid \nu \in \mathbb{F}\}$$

it follows from $\overline{1_{\{\mathbb{F}\}}}(\mathbb{H}) = 1$ that $\mathbb{F} \subset \mathbb{H}$, which by maximality implies $\mathbb{F} = \mathbb{H}$. This by [24] proves that $\hat{X}$ is weakly-T_1. ∎

We refer to $\hat{X}$ as the Wallman ultracompactification of X and to $\hat{X}^c$ as the Wallman compactification of X.

In order to justify naming the compactifications of the previous section Wallman compactifications we now have to show that in the topologically generated case our construction indeed coincides with the usual Wallman compactification.

3.10. PROPOSITION. *If (X, Δ) is topological, i.e., $\Delta = \omega(\mathcal{T})$ for some topology $\mathcal{T}$ on X, then there is a bijection between Ψ, the set of all maximal closed saturated* 1*-level prefilters, and $\mathcal{W}$, the set of all maximal closed filters in $(X, \mathcal{T})$, which is given by*

$$\Psi \xrightarrow{\iota_c} \mathcal{W}$$

$$\mathbb{F} \longrightarrow \{\mu^{-1}[\varepsilon, 1] \mid \mu \in \mathbb{F}, \varepsilon \in I_1\}$$

with inverse

$$\mathcal{W} \xrightarrow{\omega_c} \Psi$$

$$\mathcal{U} \longrightarrow \{\mu \text{ closed} \mid \forall \varepsilon \in I_0, \exists U \in \mathcal{U} : 1_U \leqq \mu + \varepsilon\}.$$

Proof. It is quite clear that if $\mathbb{F} \in \Psi$ (resp. $\mathcal{U} \in \mathcal{W}$) then $\iota_c(\mathbb{F})$ is a closed filter (resp. $\omega_c(\mathcal{U})$) is a closed saturated 1-level prefilter. Now if $\mathbb{F} \in \Psi$ and $\mathcal{U} \in \mathcal{W}$ are such that $\iota_c(\mathbb{F}) \subset \mathcal{U}$ then from

$$\mathbb{F} \subset \omega_c(\iota_c(\mathbb{F})) \subset \omega_c(\mathcal{U})$$

and the maximality of $\mathbb{F}$ it follows that $\mathbb{F} = \omega_c(\mathcal{U})$. Thus if $U \in \mathcal{U}$ then

$$U = 1_U^{-1}[\tfrac{1}{2}, 1] \in \iota_c(\mathbb{F}),$$

which proves $\iota_c(\mathbb{F}) = \mathcal{U}$, and thus ι_c, as a map, is well defined. If $\mathcal{U} \in \mathcal{W}$ and $\mathbb{F} \in \Psi$ is such that $\omega_c(\mathcal{U}) \subset \mathbb{F}$ then from

$$\mathcal{U} \subset \iota_c(\omega_c(\mathcal{U})) \subset \iota_c(\mathbb{F})$$

and the maximality of $\mathcal{U}$ it follows that $\mathcal{U} = \iota_c(\mathbb{F})$. Consequently $\omega_c(\mathcal{U}) = \omega_c(\iota_c(\mathbb{F})) = \mathbb{F}$ and we are done. ∎

3.11. THEOREM. *If $(X, \mathcal{T})$ is a T_1 topological space then the ultra-compactification $(\hat{X}, \widehat{\omega(\mathcal{T})})$, the compactification $(X^c, \widehat{\omega(\mathcal{T})}^c)$, and the embedding of the Wallman compactification $(\mathcal{W}, \omega(\mathcal{T}))$ coincide.*

Proof. When the foregoing proposition is applied, this follows at once from the observations that if $F \subset X$ is closed then $1_F \circ \omega_c = 1_{\hat{F}}$ and (consequently) $1_{\hat{F}} \circ \iota_c = 1_F$. ∎

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