

Central Singularities of Quantum Spaces

Lieven Le Bruyn*

*Departement Wiskunde en Informatica, Universitaire Instelling Antwerpen,
B-2610 Wilrijk, Belgium*

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Let A be a positively graded, connected affine \mathbb{C} -algebra generated in degree one which is a finite module over a central subring R . Let \mathcal{A} be the structure sheaf over $\text{Proj}(R)$ and $\text{Spec } \mathcal{Z}$ the central Proj. If $\text{injdim}(A) < \infty$, then $\text{Spec } \mathcal{Z}$ has singularities unless the ramification locus of \mathcal{A} is pure of codimension one. If $\text{gldim}(A) < \infty$, then the codimension ≥ 2 parts of the ramification locus of \mathcal{A} and the singular locus of $\text{Spec } \mathcal{Z}$ coincide. © 1995 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, A will be a positively graded affine \mathbb{C} -algebra

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$$

which is connected (that is, $A_0 = \mathbb{C}$), and generated as a \mathbb{C} -algebra in degree one. Moreover, we will primarily be interested in the case when A is a finite module over a central graded affine subalgebra R . Examples of current interest include quadratic quantum algebras at roots of unity and Sklyanin or Odeskii–Feigin algebras associated to a torsion point on an elliptic curve [13, 16, 20].

The structure sheaf of the graded R -module A is a sheaf of coherent algebras \mathcal{A} over $Y = \text{Proj}(R)$. The center \mathcal{Z} of \mathcal{A} is a sheaf of coherent commutative \mathcal{O}_Y -algebras and we can associate to it a commutative scheme $X = \text{Spec } \mathcal{Z}$ together with an affine morphism

$$f: X = \text{Spec } \mathcal{Z} \rightarrow Y = \text{Proj}(R)$$

such that $f_* \mathcal{O}_X \simeq \mathcal{Z}$ and f_* induces an equivalence between \mathcal{O}_Y -modules and \mathcal{Z} -modules [5, Ex. II.5.17]. $\text{Spec } \mathcal{Z}$ is called the central Proj associated to A .

* Research associate NFWO (Belgium). E-mail address: lebruyn@wins.uia.ac.be.

For 3-dimensional Sklyanin algebras M. Artin proved [2] that $\mathbf{Spec} \mathcal{Z} \cong \mathbb{P}^2$. However, for 4-dimensional Sklyanin algebras it was shown by S. P. Smith and J. Tate [17] that $\mathbf{Spec} \mathcal{Z} \not\cong \mathbb{P}^3$ as $\mathbf{Spec} \mathcal{Z}$ has singularities.

In this paper we will study the singularities of the central Proj in some generality. The main result is

THEOREM 1. (1) *If $\mathrm{injd}(\mathcal{A}) < \infty$, then $\mathbf{Spec} \mathcal{Z}$ has always singularities unless the ramification locus of $f^*\mathcal{A}$ is pure of codimension one.*

(2) *If $\mathrm{gldim}(\mathcal{A}) < \infty$, the codimension ≥ 2 part of the ramification locus of $f^*\mathcal{A}$ coincides with the codimension ≥ 2 part of the singular locus of $\mathbf{Spec} \mathcal{Z}$.*

Sklyanin algebras have finite global dimension [20] and their ramification loci are pure of codimension one in dimension 3 and contain components of codimension two and three in dimension 4. So, the theorem is compatible with the results recalled above.

In proving the theorem we obtain a result which may be of independent interest. It follows from combining an old result of W. Vasconcelos [22] with recent work of J. T. Stafford and J. Zhang [19].

PROPOSITION 1. (1) *If $\mathrm{injd}(\mathcal{A}) < \infty$, then $f^*\mathcal{A}$ is a sheaf of Cohen–Macaulay modules over $\mathbf{Spec} \mathcal{Z}$.*

(2) *If $\mathrm{gldim}(\mathcal{A}) < \infty$, then $f^*\mathcal{A}$ is a sheaf of Cohen–Macaulay maximal orders over $\mathbf{Spec} \mathcal{Z}$ which is itself a normal Cohen–Macaulay scheme.*

The study of the ramification locus of $f^*\mathcal{A}$ turns out to be equivalent to the study of fat point modules of \mathcal{A} of a certain multiplicity. It is known that fat point modules are closely tied to finite dimensional simple \mathcal{A} -modules [15] and that the latter ones can be studied via invariant theory; see, for example, [1, 14, 6]. If our graded algebra \mathcal{A} is a quotient of a Noetherian graded algebra of finite global dimension having Hilbert series $(1 - t)^{-k}$ for some $k \in \mathbb{N}$, we give the following description of fat point modules

PROPOSITION 2. *With assumption as above, let $\mathrm{Mod}_m(\mathcal{A})$ be the variety of m -dimensional representations of \mathcal{A} . There is a natural $\mathrm{PGL}_m \times \mathbb{C}^*$ -action on it, the PGL_m -component encoding isomorphism as \mathcal{A} -modules and the \mathbb{C}^* -component coming from the gradation on \mathcal{A} . Then,*

(1) *If an orbit has a finite isotropy group of order l , it determines a fat point module of \mathcal{A} of multiplicity m/l .*

(2) *If \mathcal{A} is a finite module over a central subalgebra, every fat point module of \mathcal{A} arises in this way.*

In the last section we illustrate these results on a class of Auslander regular Clifford algebras associated to an n -dimensional linear system of quadrics in \mathbb{P}^{n-1} . The theory of quadratic forms describes their finite dimensional simple modules and hence via Proposition 2 their fat point modules and so via the main theorem the ramification of the central Proj's.

PROPOSITION 3. *If the n -dimensional linear system of quadrics in \mathbb{P}^{n-1} has no base points, then the associated Clifford algebra is a quadratic Auslander-regular algebra of $\text{gldim} = n$ satisfying the Cohen–Macaulay property. Moreover, if n is even the central Proj has singularities.*

This result connects with an unpublished result of M. Van den Bergh [21].

2. SINGULARITIES AND RAMIFICATION

Throughout this section, $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \dots$ will be a positively graded affine \mathbb{C} -algebra generated in degree 1 which is a finite module over a graded central subalgebra R .

If A has finite global or injective dimension, it follows from recent work of J. T. Stafford and J. Zhang [19] that A has excellent homological properties. To be precise, we say that A

- *satisfies the Auslander-condition* if for every finitely generated left A -module M , every integer $j \geq 0$ and every (right) A -submodule N of $\text{Ext}_A^j(M, A)$ we have that $j(N) \geq j$. Here, $j(N)$ is the grade number of N which is the least integer i such that $\text{Ext}_A^i(M, A) \neq 0$.

- *satisfies the Cohen–Macaulay property* if for every finitely generated left A -module M we have the equality $\text{GK dim}(M) + j(M) = \text{GK dim}(A)$ where GK dim denotes the Gelfand–Kirillov dimension.

The Stafford–Zhang result asserts that if $\text{gldim}(A) < \infty$ (resp. $\text{injdim}(A) < \infty$), A satisfies the Auslander-condition and the Cohen–Macaulay property. In short, we say that A is Auslander-regular (resp. Auslander–Gorenstein).

Let $Y = \text{Proj}(R)$ be the projective variety associated to R . It is covered by affine open subsets $X(c) = \text{Spec}(S_c)$ for every homogeneous element $c \in R$ where $S_c = (R_c)_0$ the degree zero part of the localization of R at the homogeneous multiplicative set $\{1, c, c^2, \dots\}$. The structure sheaf \mathcal{A} of the graded R -module A has sections $\Gamma(X(c), \mathcal{A}) = (A_c)_0$ which we will denote by Λ_c . Λ_c is an order with center Z_c (which contains $(Z(A)_c)_0$ but may be larger) and the Z_c 's determine a coherent sheaf \mathcal{Z} of \mathcal{O}_Y -algebras.

By [5, Ex. II.5.17] we can associate to \mathcal{Z} a commutative scheme $X = \mathbf{Spec} \mathcal{Z}$ together with an affine morphism

$$f: X = \mathbf{Spec} \mathcal{Z} \rightarrow Y = \mathbf{Proj}(R)$$

such that $f_* \mathcal{O}_X \simeq \mathcal{Z}$. The commutative scheme $X = \mathbf{Spec} \mathcal{Z}$ is called the central Proj of A , see, for example, [2, 17]. Using the above notation we have:

LEMMA 1. *If A is Auslander–Gorenstein (resp. Auslander-regular) satisfying the Cohen–Macaulay property, then so does $\Lambda_c = \Gamma(X(c), \mathcal{A})$.*

Proof. A_c is a strongly graded ring, that is, $(A_c)_{-1}(A_c)_1 = (A_c)_0 = \Lambda_c$. Let $\deg(c) = k$ and write $c = \sum_i d_i l_i$ with $d_i \in A_{k-1}$ and $l_i \in A_1$ which is possible as A is generated in degree 1. Then, $1 = c^{-1} \cdot c = \sum_i (c^{-1} d_i) l_i \in (A_c)_{-1}(A_c)_1$ and as $(A_c)_{-1}(A_c)_1$ is an ideal of $(A_c)_0 = \Lambda_c$ this proves the claim.

Using [12, Theorem I.3.4] we know that taking degree zero parts induces an equivalence of categories

$$(-)_0: A_c\text{-gr} \rightarrow \Lambda_c\text{-mod}$$

from graded left A_c -modules to left Λ_c -modules (the inverse functor is $A_c \otimes_{\Lambda_c} -$).

Finally, all required properties first go up under central localization at c to A_c and as they can be expressed in category theoretical terms the above equivalence induces them on Λ_c . ■

The following result rests on work of W. Vasconcelos [22]. Let Λ be an algebra which is a finite module over a local commutative ring S with maximal ideal m . Clearly, there are only a finite number of maximal ideals $\{P_1, \dots, P_k\}$ of Λ lying over m . We call Λ *moderated Gorenstein* if and only if

- $\text{injdim}(\Lambda) = \text{Kdim}(\Lambda) = n < \infty$.
- For all $1 \leq i \leq k$ we have that $\max \{p: \text{Ext}_{\Lambda}^p(\Lambda/P_i, \Lambda) \neq 0\} = n$.

Vasconcelos' result states that if Λ is a moderated Gorenstein algebra over a local domain S , Λ is a Cohen–Macaulay module over its center. For more details we refer to [22] or [10, IV.1].

PROPOSITION 4. *Let A be a positively graded affine algebra generated in degree one which is a finite module over a central subring. Then,*

- (1) *If $\text{injdim}(A) < \infty$, then $f^* \mathcal{A}$ is a sheaf of Cohen–Macaulay modules over its center $\mathbf{Spec} \mathcal{Z}$.*

(2) If $\text{gldim}(A) < \infty$, then $f^*\mathcal{A}$ is a sheaf of maximal Cohen-Macaulay orders over $\text{Spec } \mathcal{Z}$ which is a normal Cohen-Macaulay scheme.

Proof. Of course, it suffices to verify these claims locally in an open affine subset $X(c)$ of Y . We know already that Λ_c is an Auslander-Gorenstein ring (resp. Auslander-regular if $\text{gldim} < \infty$) satisfying the Cohen-Macaulay property. The same facts hold for $\Lambda_x = \Lambda_c \otimes_{S_c} (S_c)_x$ for every point $x \in X(c)$. We assume the finite global or injective dimension to be $n + 1$. From the Cohen-Macaulay property we deduce that

$$j(\Lambda_x/P_i) = n$$

for each of the maximal ideals P_i of Λ_x over m_x . As $\text{Ext}_{\Lambda_x}^k(\Lambda_x/P_i, \Lambda_x) = 0$ for all $k \geq n + 1$, we see that Λ_c is moderated Gorenstein and therefore locally a Cohen-Macaulay module over its center.

By results of T. Levasseur [11] and J. T. Stafford [18] we know that if A is Auslander-regular it is a domain and a maximal order. Whereas the maximal order property goes up to A_c it is in general not true that the degree zero part of a strongly graded maximal order is maximal, see [10, p. 105] for an example. However, here we can use the fact that Λ_c is a tame order by [10, IV.1.7] and tame domains are maximal orders. In particular, the center is a normal domain and the Cohen-Macaulayness follows from [10, IV.1.6]. ■

We want to apply the foregoing to the study of the singular locus of $\text{Spec } \mathcal{Z}$. In order to do so we need an extra condition on the ramification locus \mathcal{A} .

\mathcal{A} is a sheaf of orders in a central simple algebra D which we assume to be of dimension e^2 over its center, that is, e is the p.i.-degree of the sheaf of algebras \mathcal{A} . Globalizing well known result we can view $\text{Spec } \mathcal{Z}$ as parametrizing isoclasses of e -dimensional semi-simple representations of \mathcal{A} . Hence, there is an open set $U \subset \text{Spec } \mathcal{Z}$ consisting of points corresponding to simple e -dimensional representations of \mathcal{A} . U is called the Azumaya-locus of \mathcal{A} because for every point $x \in U$ we have that \mathcal{A}_x is an Azumaya algebra over its center. Its complement V is called the ramification locus of \mathcal{A} .

If Λ is an order over its center S which is a normal domain, we say that Λ is a reflexive Azumaya algebra provided Λ_p is Azumaya for every height one prime p of S . Or geometrically, the ramification locus of Λ in $\text{Spec}(S)$ has codimension ≥ 2 . A reflexive Azumaya algebra is Azumaya if and only if it is a projective S -module; see, for example, [9]. Combining these facts with the foregoing results we have:

PROPOSITION 5. *Let A be a positively graded affine algebra, generated in degree one which is a finite module over a central subring R . Let $c \in R$ be an homogeneous element such that the ramification locus V_c of Λ_c is of codimension at least 2.*

- (1) *If $\text{injd}(\Lambda_c) < \infty$, then $V_c \subset f^{-1}(X(c)) \cap (\text{Spec } \mathcal{Z})_{\text{sing}}$.*
- (2) *If $\text{gldim}(\Lambda_c) < \infty$, then $V_c = f^{-1}(X(c)) \cap (\text{Spec } \mathcal{Z})_{\text{sing}}$.*

Proof. Let $x \in f^{-1}(X(c))$ be a smooth point of $\text{Spec } \mathcal{Z}$, then \mathcal{A}_x is a Cohen–Macaulay module over its center \mathcal{Z}_x which is a regular domain. Hence, \mathcal{A}_x is a projective \mathcal{Z}_x -module. By assumption, Λ_x is also a reflexive Azumaya algebra over the normal domain \mathcal{Z}_x hence it must be Azumaya. Hence $x \notin V_c$ proving the first part.

Conversely, if $x \notin V_c$ then \mathcal{A}_x is an Azumaya algebra over \mathcal{Z}_x . By assumption we know that $\text{gldim}(\mathcal{A}_x) = n$, but then also $\text{gldim}(\mathcal{Z}_n) = n$. Hence, x is a smooth point of $\text{Spec } \mathcal{Z}$. ■

From this, the main theorem follows immediately.

3. FAT POINT MODULES AND INVARIANTS

In order to study the singular locus of $\text{Spec } \mathcal{Z}$ it is important to determine the ramification locus of the sheaf of algebras \mathcal{A} , that is, we have to determine the open subset U of $\text{Spec } \mathcal{Z}$ corresponding to simple e -dimensional representations of \mathcal{A} where e is the p.i.-degree of \mathcal{A} . Here, the notion of a fat point module over the positively graded algebra A comes into the picture.

A *fat point module* F of A is a 1-critical graded A -module. If A is a quotient of a graded Noetherian algebra of finite global dimension with Hilbert series $(1 - t)^{-k}$ for some $k \in \mathbb{N}$, then the dimension of the homogeneous parts F_i is constant for i sufficiently large. This constant number is then called the multiplicity of F .

For every homogeneous central element $c \in R$ we can use the fact that Λ_c is strongly graded as in the foregoing section to see that there is a functorial bijection between the finite dimensional simple $(\Lambda_c)_0 = \Lambda_c$ -modules and the fat points of A which have no c -torsion. Under this bijection a fat point module F corresponds to $(F[c^{-1}])_0$ and the multiplicity of the fat point (i.e., the constant dimension of its homogeneous components) equals the dimension of the corresponding simple module.

Therefore, the study of the open set U is equivalent to the study of fat point modules of A of multiplicity e .

There is a close though subtle connection between fat point modules for A and finite dimensional simple representations of A . Much of what follows is at least implicit in [15, Sects. 1 and 2] but we give a slightly different account.

Let $\text{Mod}_m(A)$ be the affine variety parametrizing m -dimensional A -modules. If X_0, \dots, X_n is a basis for A_1 , then $\text{Mod}_m(A)$ is the subvariety of $M_m(\mathbb{C})^{\oplus n+1}$ determined by the entries of the matrices we obtain by substituting generic m by m matrices for the X_i in the defining equations of A ; see, for example, [6, II.2.7]. There is a natural PGL_m -action on $\text{Mod}_m(A)$, the orbits of which are the isomorphism classes of m -dimensional A -modules. Because A is a graded algebra, there is an additional \mathbb{C}^* action on $\text{Mod}_m(\mathbb{C})$ induced by sending X_i to λX_i for $\lambda \in \mathbb{C}^*$.

PROPOSITION 6. *Let A be a basic affine graded \mathbb{C} -algebra generated in degree one and assume that A is a quotient of a Noetherian graded algebra B with Hilbert series $(1-t)^{-k}$ and finite global dimension. If a $\text{PGL}_m \times \mathbb{C}^*$ -orbit in $\text{Mod}_m(A)$ has finite isotropy group of order l , then this orbit determines a fat point module of A of multiplicity m/l . Conversely, we recover the m -dimensional representations in the orbit as the finite dimensional simple quotients of the fat point module.*

Proof. Observe first that the isotropy group can only be finite for simple m -dimensional A -modules. Let S be a simple m -dimensional A -module in the orbit, then $P = \text{Ann}(S)$ is a maximal ideal of A . Let P_g be the largest graded ideal of A contained in P , then $\text{GKdim}(A/P_g) \leq 1$ (for if $\dim((A/P_g)_i) \rightarrow \infty$, A/P_g has homogeneous elements in $\text{Ann}(S)$ contradicting maximality of P_g). But then, A/P_g is a p.i.-ring with unique graded prime ideal $M = (A/P_g)_+$ whence inverting an homogeneous central element we obtain a graded central simple algebra which is the full graded ring of quotients $Q^g(A/P_g)$. Using Tsen's theorem and the characterization of graded central simple algebras in [12] we see that $Q^g(A/P_g)$ is a graded matrix ring $\text{END}(V)$

$$Q^g(A/P_g) \cong M_m(\mathbb{C}[x, x^{-1}]) (e_1 \cdot 0, e_2 \cdot 1, \dots, e_d \cdot (d-1))$$

(that m is the p.i.-degree of A/P_g is [12, I.2.10]) where x is a central element of degree 1 and where e_i is the number of base vectors in degree $0 \leq i \leq d$ in the graded vector space V over the graded field $\mathbb{C}[x, x^{-1}]$. For more details on this we refer again to [12, I.4, I.5, and II.6].

The positive part of V , $F = V_{\geq 0}$ is a graded A -module with periodic Hilbert series with periods (e_1, e_2, \dots, e_d) . As A is a quotient of a graded regular algebra we know that this Hilbert series must eventually become constant, say of multiplicity e . Therefore, all $e_i \neq 0$ and all are equal to e

which is then m/d . The fact that F determines a fat point module of A of multiplicity e and that all simple A -quotients of it lie in the \mathbb{C}^* -orbit of S follow (or see [15, Sects. 1, 2]).

In case A is a finite module over a central subalgebra, every fat point module of A has finite dimensional simple quotients and hence the above argument can be applied and we obtain Proposition 2. The method also enables us to relate the p.i.-degree of A (say N) and that of \mathcal{A} , e . If d is the greatest common divisor of the degrees of central homogeneous elements in A , then $e = N/d$. Finally, observe that the special form of graded quotient rings of maximal graded ideals of A in case it is a quotient of a suitable ring B can sometimes be used as a criterium to show that no such B exists.

In the important special case that A is (a quotient of) a quadratic algebra, the foregoing proposition can be rephrased in quiver-terms as

COROLLARY 1. *A fat point module of a quadratic algebra A (as above) of multiplicity e corresponds to the \mathbb{C}^* -orbit of a simple representation of dimension vector (e, e, \dots, e) of a circular quiver of length d on the vertices $\{v_1, \dots, v_d\}$ with arrows*

$$\begin{array}{ccccccc}
 & X_0^{(i-1)} & & X_0^{(i)} & & & \\
 & \rightarrow & & \rightarrow & & & \\
 \cdots & \bullet_{v_{i-1}} & \vdots & \bullet_{v_i} & \vdots & \bullet_{v_{i+1}} & \cdots \\
 & X_n^{(i-1)} & & X_n^{(i)} & & & \\
 & \rightarrow & & \rightarrow & & &
 \end{array}$$

with relations of the form

$$\sum a_{kl} X_k^{(i-1)} X_l^{(i)} = 0$$

whenever $\sum a_{kl} X_k X_l$ is a quadratic relation of A .

4. AN EXAMPLE

In this section we will introduce a large class of Auslander-regular algebras which are finite modules over their centers and study their fat point module strata.

Let $R = \mathbb{C}[Y_1, \dots, Y_n]$ be the commutative polynomial ring and let $A(M)$ be the Clifford algebra over R associated to a symmetric $n \times n$ matrix M with all its entries linear terms in the Y_i . That is, let

$$M = M_1 Y_1 + \cdots + M_n Y_n$$

with all $M_i \in M_n(\mathbb{C})$ symmetric matrices, then $A(M)$ is generated by X_i, Y_i , $1 \leq i \leq n$, with defining relations

$$X_i X_j + X_j X_i = \sum_{k=1}^n (M_k)_{ij} Y_k$$

and the Y_i central. Observe that one defines a gradation on $A(M)$ by giving $\deg(X_i) = 1$ and $\deg(Y_i) = 2$.

The algebra $A(M)$ is associated to an n -dimensional linear system

$$\mathcal{Q} = \mathbb{C}Q_1 + \cdots + \mathbb{C}Q_n$$

of quadrics $Q_i \subset \mathbb{P}^{n-1}$ where Q_i is the zero set of $\sum_{kl} (M_i)_{kl} Y_k Y_l$. A base point of the linear system of quadrics \mathcal{Q} is a common zero of the quadrics Q_i , $1 \leq i \leq n$. Observe that if we fix a basis for \mathcal{Q} we can associate to the system a symmetric $n \times n$ matrix M with linear entries in R and hence the Clifford algebra $A(M)$.

PROPOSITION 7. *Let $\mathcal{Q} = \mathbb{C}Q_1 + \cdots + \mathbb{C}Q_n$ be an n -dimensional linear system of quadrics in \mathbb{P}^{n-1} and $A(M)$ the corresponding Clifford algebra. Equivalent are*

- (1) *$A(M)$ is a quadratic Auslander-regular algebra of $\text{gldim} = n$ satisfying the Cohen–Macaulay property.*
- (2) *The system of quadrics \mathcal{Q} has no base points in \mathbb{P}^{n-1} .*

Proof. The fact that $A(M)$ is generated in degree one and has $\text{gldim}(A(M)) = n$ if and only if \mathcal{Q} has no base points is merely translating [3, Sect. 1] to current terminology. As $A(M)$ is a finite module over the central subring R the result follows from the Stafford–Zhang result [19]. ■

Remark 1. Because a generic n -dimensional system of quadrics in \mathbb{P}^{n-1} has no base points we recover a result of M. Van den Bergh [21] who proved Auslander regularity of $A(M)$ for generic M by a deformation argument. The above result has the advantage in applications of clarifying what we mean by a “generic” M .

We will now study the fat point module strata of the coherent sheaf of algebras \mathcal{A} over $Y = \text{Proj}(R)$ which is a weighted projective $n - 1$ space. To a point $(y_1, \dots, y_n) \in Y$ corresponds a maximal graded ideal p of R such that $R/p \simeq \mathbb{C}[y]$ with y a degree two element. We can specialize the symmetric matrix M at p and obtain a symmetric matrix $M(p)$ over $\mathbb{C}[y]$. The relevant stratification of Y will be given by the sets

$$Y_k = \{p \in Y | \text{rank } M(p) = k\}.$$

The following result is based on standard results on quadratic forms over fields and principal ideal domains, see [6; 7, Chap. 5; 8, Sects. II.2 and II.4] for more details.

PROPOSITION 8. *Over every $p \in Y$ there is a unique maximal graded prime ideal P of $A(M)$ such that $A(p) = A(M)/P \otimes_{\mathbb{C}[y]} \mathbb{C}[y, y^{-1}]$ has the following structure*

- *If $p \in Y_{2l}$, then $A(p) \simeq M_{2^l}(\mathbb{C}[y, y^{-1}](2^{l-1} \cdot 0, 2^{l-1} \cdot 1))$, that is, $A(p)$ is a graded central simple algebra of p.i.-degree 2^l with center $\mathbb{C}[y, y^{-1}]$.*
- *If $p \in Y_{2l-1}$, then $A(p) \simeq M_{2^{l-1}}(\mathbb{C}[x, x^{-1}](2^{l-1} \cdot 0))$, that is, $A(p)$ is a graded central simple algebra of p.i.-degree 2^{l-1} with center $\mathbb{C}[x, x^{-1}]$ where x is a degree 1 element.*

Proof. Over $\mathbb{C}[y, y^{-1}]$ every symmetric $n \times n$ matrix of rank k is equivalent to one of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is an invertible symmetric $k \times k$ matrix with determinant $\in \mathbb{C}^* y^k$. Hence, every prime ideal P of $A(M)$ lying over p corresponds to a prime factor of the Clifford algebra of the quadratic form associated to A over $\mathbb{C}[y, y^{-1}]$. If $k = 2l$ is even such Clifford algebras are orders in central simple algebras of p.i.-degree 2^l . If $k = 2l - 1$ we use the fact that $\det(A)$ is not a square in $\mathbb{C}[y, y^{-1}]$ and hence the Clifford algebra is an order in a central simple algebra of p.i.-degree 2^{l-1} with center $\mathbb{C}[y, y^{-1}, \sqrt{\det(A)}]$. Clearly, over \mathbb{C} all graded central simple algebras are graded matrix rings and as we have a central element of degree ≤ 2 they are easy to determine. ■

Combining these observations with the results of the foregoing sections we have

PROPOSITION 9. *Let \mathcal{Q} be an n -dimensional linear system of quadrics in \mathbb{P}^{n-1} without base point. If $A(M)$ is the associated Clifford algebra over R . Then,*

- (1) *A point $p \in Y = \text{Proj}(R)$ corresponds to a fat point module of $A(M)$ of multiplicity 2^{l-1} if $p \in Y_k$ and $k = 2l$ or $k = 2l - 1$.*
- (2) *The ramification locus for $f^*\mathcal{A}$ is $f^{-1}(Y_{n-2})$ if n is even and is $f^{-1}(Y_{n-1})$ if n is odd.*
- (3) *If n is even, the singular locus of $\text{Spec } \mathcal{Z}$ is $f^{-1}(Y_{n-2})$.*

Remark 2. If $n = 4$, then points on X_4 or X_3 determine fat point modules of multiplicity 2. X_2 which is the zero set of all 3×3 minors of M determines the fat point modules of multiplicity 1 (that is, the point-variety of $A(M)$ in the terminology of [2]). For generic M one can verify that $\text{codim}(X_2) = 3$ so there can only be a finite number of points and $\text{Spec } \mathcal{Z}$ is a rational threefold with a finite number of isolated singularities. For generic M , the fact that $A(M)$ has a finite number of point-modules was obtained by M. Van den Bergh [21] who further shows that there are precisely 20 points.

Sklyanin algebras at torsion points seem to have some properties in common with Clifford algebras. Therefore, it is tempting to conjecture that the central Proj of Sklyanin algebras will be singular in even dimensions with a component of the singular locus equal to the zero set of the two central elements in degree $n/2$ given by [13]. We leave this as a suggestion for further work.

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