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Product Innovation Incentives by an Incumbent Firm: A Dynamic Analysis*

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Abstract

We study in a dynamic framework how product innovation activities of a firm are influenced by its production capacity investments for an established product and vice versa. The firm initially has capacity to sell an established product, and it also has the option to undertake an R&D project, which upon completion allows the firm to introduce a new vertically and horizontally differentiated product to the market, thereby extending its product range. The breakthrough probability of detecting the new product depends on both the value of the firm's R&D stock and its current R&D investment. It is shown that the initial production capacity for the established product influences the intensity of R&D activities of the firm. In particular, there are constellations such that for large initial production capacity for the established product the firm never invests in R&D and the new product is never introduced. For small initial capacity the firm keeps investing in R&D implying that eventually the new product is always introduced. Finally, for an intermediate range of initial capacity levels the firm initially invests in product R&D, but then reduces these investments to zero. In this scenario the new product is introduced with a positive probability, which is however substantially smaller than 1. From a technical perspective this analysis gives the example of a new type of Skiba threshold phenomenon in the framework of a multi-mode optimization model.

Key Words: product innovation, multi-product firm, R&D, piecewise deterministic optimal control, Skiba curve.

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1 Introduction

The overall aim of this paper is to develop and exploit a dynamic framework of analysis that allows studying the optimal investment strategies of established incumbent firms under consideration of the uncertainty about future changes in the market structure. The considered changes in the market structure are due to changes in the range of products offered on the market which are triggered by product innovations of incumbent firms. As the ability to introduce new products is typically based on innovation effort of the innovating firm, such changes are endogenous results of firm strategies and industry dynamics.

In particular, this paper studies the problem of a firm undertaking R&D activities to come up with a new (vertically and horizontally) differentiated product and at the same time adjusts the capacity for the established product. The main feature of our setup is that the change in market structure induced by the new product introduction is explicitly modeled and that the hazard rate of innovation is a function of both the intensity of investment in R&D and the accumulated knowledge stock. This latter feature distinguishes the present paper from Dawid et al. (2013a) where innovation was considered as an exogenous stochastic process. In order to focus on the interplay between the stochastic dynamic R&D process and the production capacity adjustments we abstract in our setting from competition effects.

We develop a dynamic model, where an incumbent firm offers an established product. At some ex-ante unknown point in time the range of products is enlarged, because the firm obtains the option to introduce a new product, which is vertically and horizontally differentiated from the existing product. This option results from successfully finishing an R&D project, which requires continued R&D investments. The timing of the breakthrough is stochastic and cannot be perfectly predicted. Capacities cannot be (fully) transferred between the production of different products, and therefore the introduction of the new product reduces the value of the existing capacity. The firm's objective is to maximize its total discounted profits by optimally selecting its investments in production capacities for the different products it offers and, before the new product is introduced, by choosing its innovation effort.

The problem outlined above is of substantial real-world relevance since, as has been shown e.g. in Chandy and Tellis (2000), a large fraction of product innovations has been achieved by established incumbents. For such firms there are important feedback effects between their strategies on established markets (like capacity investments) and innovation strategies aiming at the introduction of new products that extend the product range. The main contribution of this paper is to consider the effect of an expected change in the market structure on investment behavior in the established market, as well as the feedback between capacity investments for the established product and innovation efforts, in a dynamic framework. In spite of the large literature on capacity investments and innovation incentives a rigorous integrated analysis of these effects is so far missing.

The considered market environment captures in a stylized way the dynamic emergence of new 'submarkets' in an established market and its effect for the established product before and after the occurrence of the new product. Real world examples resembling such a setting are numerous: In the TV industry major producers of standard

CRT television sets have started the production of flatscreens around the year 2000. Although the production processes for these two variants are based on very different technologies, for many years firms offered both products simultaneously. A second example along these lines is the (time-phased) introduction of hybrid cars by many car manufacturers, which opened a new submarket in this industry co-existing with the established ones. It should be noted that a common feature of these examples is that the introduction of the new products becomes possible due to the availability of a new technology, or more generally due to technological progress. In such situations the capacity devoted to the established product typically can hardly be transferred to the production of the new product.

The main research question to be addressed within the framework sketched above is how the incentive to invest in innovative activities is influenced by the (current) capacity for the established product. In particular, we explore two issues. First, we examine under which circumstances old market capacity might have long lasting effects in a sense that it prevents the firm from innovating at all. Second, we consider the effect of old market capacity on the distribution of innovation time in cases where the new product is developed. In the latter scenario we also characterize how the investment in established product capacity is adjusted in light of the anticipation of a future new product introduction.

Establishing the firm's optimal investment strategies requires solving a two-mode piecewise deterministic dynamic optimization problem, where the two modes correspond to the pre-innovation and post-innovation phase. Our approach to solving the model is to analytically characterize as much as possible the solution before using a numerical method of collocation to complete the analysis. Furthermore, we employ an innovative numerical method that allows us to delineate the basins of attraction of two different long-run steady states emerging in our model. We show that the initial values of old market capacity and knowledge stock determine whether the incumbent firm eventually introduces the new product or not. There are constellations such that for large initial production capacity for the established product the firm never invests in R&D and the new product is never introduced, whereas for small initial capacity the firm keeps investing in R&D as long as the innovation has not arrived. In this scenario the new product is always introduced. Finally, for an intermediate range of the production capacity the firm initially invests in product R&D, but then reduces these investments to zero. In this scenario the new product is introduced with a positive probability, which is however substantially smaller than 1.

Endogenizing the R&D activities of an incumbent gives rise to a number of new insights. For instance, we show that there is a non-monotonic relationship between the degree of horizontal differentiation of the new product and capacity accumulation on the established market prior to innovation. If the new product is a close substitute to the established one, then this does not only influence the degree of cannibalization after the new product introduction, but the firm is also less willing to invest in R&D during the innovation phase. This delays the expected arrival of the innovation, which fosters investment of the firm in the established product. The interplay of these effects gives rise to the non-monotonicity.

Our analysis is the first to establish a dependency of long run outcomes on initial conditions in a multi-mode framework. Compared to the established literature on Skiba phenomena in dynamic optimization problems (see e.g. Skiba (1978) or Haunschmied et al. (2005)) our analysis adds the observation that there are initial conditions such that each of the two (deterministic) long run steady states is reached with positive probability. Our approach allows to characterize this set of initial conditions and also the set of initial conditions for which there is only one possible long run outcome.

Another contribution of our paper is to put a new perspective on the extensive discussion on the link between firm size and innovativeness initiated by Schumpeter (1934). Endogenizing R&D activities of an intertemporally maximizing incumbent allows us to find that small firms with a sufficiently high knowledge stock are the most innovative ones (cf. Dolfsma and Van der Velde (2014)). This is in accordance with results obtained in Dawid et al. (2013b) in a static setting. Also Yin and Zuscovitch (1998) obtain that small firms have stronger incentives to perform product innovations, which extend their product range. They establish their results in a myopic adjustment framework considering both product and process innovation. Our approach differs in that we consider a fully dynamic model with capacity dynamics and a hazard rate depending on knowledge stock. Furthermore, since we are carrying out a global analysis, we are able to stress potential implications of firm size on long run innovation outcomes.

A dynamic global analysis examining the effects of initial characteristics on innovation paths of a firm has also been provided in Hinloopen et al. (2013). Employing a framework based on Cellini and Lambertini (2009) they consider an intertemporally optimizing monopolist, which at each point in time determines production quantity and investment in cost reducing process innovations. In their analysis they allow for scenarios where initial production costs are above the reservation price and show that in such a setting it can be optimal for the firm to invest in process innovation before entering the market with positive quantities. The size of the initial costs determines whether the firm eventually enters the market and also the long run level of process innovation effort. A crucial difference with our work is that in their paper R&D investments lead to immediate and known cost reductions, while in our framework an R&D project takes time to complete where the breakthrough probability depends on both current R&D investment and knowledge stock built up by earlier investment. Moreover, whereas Hinloopen et al. (2013) are concerned with process innovation, we concentrate on product innovation.

Analyzing the R&D strategies of multi-product firms is an important issue. Some previous contributions in this area study this problem within a static framework (see e.g. Lambertini (2003), Lin and Zhou (2013)), but recently also a dynamic perspective has been adopted. Product innovation incentives for dynamically optimizing firms have been studied by Lambertini and Mantovani (2009) with a focus on its interplay with process innovation activities. Contrary to this contribution, we explicitly take into account the stochastic nature of product R&D and the fact that new product introduction induces discrete changes in the demand structure. Also our focus on the relationship between firm size and innovativeness distinguishes our paper from Lambertini and Mantovani (2009).

The paper is organized as follows. The next section introduces the model and details its assumptions. Section 3 is devoted to analyses of the long run economic behavior before and after innovation. The dynamic analysis that follows in section 4 presents an innovative numerical procedure for the characterization of Skiba curves separating the basins of attraction in dynamic optimization problems and shows that the initial stocks of the monopolist with respect to production capacity and knowledge can have crucial and non-continuous effects on the firm's innovation activities and the overall probability that the new product is introduced. Subsequently, the effects of the characteristics of the new product, that emerges after innovation, on the incentives to innovate or to invest in the established product prior to innovation are examined. Some concluding remarks are given in section 5.

2 The Model

We consider at the onset a firm, which produces an established product 1. In addition the firm has the opportunity to undertake an R&D project with the aim to develop a new differentiated product called product 2. The completion time T of the R&D project is stochastic. More precisely, innovation arrives at a hazard rate λ, which is a function of current knowledge stock, $K_R(t)$, as well as the current investment $I_R(t)$ in R&D. Following Doraszelski (2003) we employ an additive form of the hazard rate given by

$$\lambda(I_R(t), K_R(t)) = \alpha I_R(t) + \beta K_R^{\psi}(t) \;\; \alpha \geq 0, \;\; \beta \geq 0, \;\; \psi > 0. \tag{1}$$

Between $t = 0$ and $t = T$ only product 1 is sold on the market. After time T the firm is able to produce both product 1 and product 2. In what follows we denote the time before the innovation, i.e. $t \in [0, T)$, as mode m_1. The time after the innovation is mode m_2, which thus runs from time $t = T$ onwards.

The firm has an initial production capacity $K_1(0) = K_1^{ini}$ for the established product and an initial knowledge stock K_R^{ini}. In mode m_1, the firm invests in capacity $K_1(t)$ for product 1 and the knowledge stock $K_R(t)$. There are also two types of investments in the second mode, namely investments in capacities $K_1(t)$ and $K_2(t)$ for products 1 and 2, respectively. The capital accumulation equations are standard:

$$\dot{K}_1(t) = I_1(t) - \delta_1 K_1(t) \qquad \delta_1 \geq 0, \;\; K_1(0) = K_1^{ini}, \tag{2}$$

$$\dot{K}_2(t) = I_2(t) - \delta_2 K_2(t) \qquad \delta_2 \geq 0, \;\; K_2(0) = 0, \tag{3}$$

$$\dot{K}_R(t) = I_R(t) - \delta_R K_R(t) \qquad \delta_R \geq 0, \;\; K_R(0) = K_R^{ini}. \tag{4}$$

The depreciation rates of capital δ_1 and δ_2 are assume to be strictly positive. As for $\delta_R > 0$, it represents organizational forgetting (see Doraszelski (2003) and references therein).

We allow the firm to intentionally scrap production capacities (i.e. $I_1, I_2 \in \mathbb{R}$) but make the sensible assumption that knowledge investments are non-negative, i.e.

$$I_R(t) \geq 0 \;\; \forall t \geq 0. \tag{5}$$

An important assumption of the model is that the firm cannot invest in production capacity of the second product before the product has been introduced to the market, which implies that $I_2(t) = 0$ in mode m_1. Furthermore, all stocks have to be non-negative:

$$K_i(t) \geq 0 \;\; \forall t \geq 0 \;\; i = 1, 2, R. \tag{6}$$

Concerning demand we rely on a standard linear model, where, following the literature on capital accumulation games (e.g. Dockner et al. (2000)), it is assumed that production capacities for both products are fully exploited. Prices are thus given by the linear inverse demand system

$$p_1(t) = 1 - K_1(t) - \eta K_2(t), \tag{7}$$
$$p_2(t) = 1 + \theta - \eta K_1(t) - K_2(t). \tag{8}$$

The parameter η $(0 < \eta < 1)$ determines the degree of horizontal differentiation, whereas the parameter θ determines the degree of vertical differentiation. By setting $\theta > 0$ one imposes that product 2 is of higher quality than product 1.

Investment costs are assumed to be linear quadratic for all stocks:

$$C_i(I_i(t)) = \mu_i I_i(t) + \frac{1}{2}\gamma_i I_i(t)^2 \quad i = 1, 2, R., \tag{9}$$

where the parameters μ_i and $\gamma_i,\ i = 1, 2, R$ are positive.

Characterizing the firm's optimal strategy boils down to solving a piecewise deterministic optimal control problem. The dynamics is deterministic in each mode of the problem, however there is a stochastic transition between the modes. The profit maximizing firm faces the following optimization problem:

$$\begin{aligned} max_{I_1,I_2,I_R} J = E_{\mathcal{P}_m} \Bigg\{ \int_0^\infty e^{-rt} & \Big[(1 - K_1 - \eta K_2)K_1 + (1 + \theta - \eta K_1 - K_2)K_2 \\ & -\mu_1 I_1 - \frac{\gamma_1}{2} I_1^2 - \mu_2 I_2 - \frac{\gamma_2}{2} I_2^2 - \mu_R I_R - \frac{\gamma_R}{2} I_R^2\Big] dt \Bigg| \\ & m(0) = m_1, \quad K_1(0) = K_1^{ini}, \quad K_2(0) = 0, \quad K_R(0) = K_R^{ini} \Bigg\}, \end{aligned} \tag{10}$$

subject to the state and mode dynamics:

$$\dot{K}_i(t) = I_i(t) - \delta^i K_i(t), \qquad i \in \{1, 2, R\}, \tag{11}$$
$$K_i(t) \geq 0, \quad \forall t \geq 0, \quad i \in \{1, 2\}, \tag{12}$$
$$I_R(t) \geq 0, \quad \forall t \geq 0, \tag{13}$$
$$I_2(t) = 0, \quad \text{for all } t \text{ s.t. } m(t) = m_1, \tag{14}$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{P}\left\{m(t + \Delta) = m_2 \mid m(t) = m_1\right\} = \alpha I_R(t) + \beta K_R^{\psi}(t). \tag{15}$$

It should be noted that the constraint $I_R \geq 0$ implies a non-negative K_R, such that no non-negativity constraint on K_R is needed.

3 Analysis

We analyze the problem backwards, i.e. in the first section the situation after the firm has innovated is examined. The second section considers the situation before the breakthrough in the innovation process has taken place.

3.1 Firm Investments after Product Innovation

In mode m_2 the firm faces a standard capital accumulation problem with two capital stocks and adjustment costs. Straightforward calculations show that there is at most one interior steady state for the problem given by

$$K_1^{ss} = \frac{(1-(r+\delta_1)\mu_1)(2+(r+\delta_2)\gamma_2\delta_2) - 2\eta(1+\theta-(r+\delta_2)\mu_2)}{(\delta_2\gamma_2(r+\delta_2)+2)(\delta_1\gamma_1(r+\delta_1)+2)-4\eta^2}$$
$$K_2^{ss} = \frac{(1+\theta-(r+\delta_2)\mu_2)(2+(r+\delta_1)\gamma_1\delta_1) - 2\eta(1-(r+\delta_1)\mu_1)}{(\delta_2\gamma_2(r+\delta_2)+2)(\delta_1\gamma_1(r+\delta_1)+2)-4\eta^2}$$

In what follows we assume that parameters are such that $K_1^{SS} > 0, K_2^{SS} > 0$. Furthermore, it is easy to see that under our assumptions the unique steady-state is (globally) asymptotically stable under the optimal investment pattern (see Appendix A.1). The following proposition characterizes the value function for mode m_2.

Proposition 1. *In mode m_2, the value function of the firm has the form*

$$V_{(m_2)} = aK_1^2 + bK_1 + dK_1K_2 + eK_2^2 + fK_2 + g, \tag{16}$$

where a, b, d, e, f, g satisfy the nonlinear system:

$$\begin{cases} ra &= \frac{2}{\gamma_1}a^2 + \frac{1}{2\gamma_2}d^2 - 2a\delta_1 - 1 \\ re &= \frac{2}{\gamma_2}e^2 + \frac{1}{2\gamma_1}d^2 - 2e\delta_2 - 1 \\ rd &= \frac{2}{\gamma_1}ad + \frac{2}{\gamma_2}ed - d(\delta_1+\delta_2) - 2\eta \\ rb &= \frac{2}{\gamma_1}a(b-\mu_1) + \frac{1}{\gamma_2}d(f-\mu_2) - \delta_1 b + 1 \\ rf &= \frac{2}{\gamma_2}e(f-\mu_2) + \frac{1}{\gamma_1}d(b-\mu_1) - \delta_2 f + \theta + 1 \\ rg &= \frac{1}{2\gamma_1}b^2 + \frac{1}{2\gamma_2}f^2 - \frac{\mu_1}{\gamma_1}b - \frac{\mu_2}{\gamma_2}f + \frac{1}{2\gamma_1}\mu_1^2 + \frac{1}{2\gamma_2}\mu_2^2. \end{cases} \tag{17}$$

The optimal investment functions are given by

$$I^*_{1,(m_2)}(K_1, K_2) = \frac{1}{\gamma_1}\left(b - \mu_1 + 2aK_1 + dK_2\right)$$
$$I^*_{2,(m_2)}(K_1, K_2) = \frac{1}{\gamma_2}\left(f - \mu_2 + dK_1 + 2eK_2\right).$$

Proof. See Appendix A.2 □

The system of nonlinear equations (17) has been numerically solved to determine the sextuple (a, b, d, e, f, g) for each constellation of parameters. Although in general (17) can have multiple solutions, the transversality condition $\lim_{t\to+\infty} e^{-rt}V_{(m_2)}(t) = 0$ allows to select a unique sextuple of coefficients characterizing the actual value function. This is in accordance with standard results (see Jun and Vives (2004)) implying that capital accumulation problems like the one here in mode m_2, have at most one interior stable steady state.

3.2 Firm Investments Prior to Product Innovation

Having characterized optimal behavior in mode m_2 we now move to the analysis of investment prior to the innovation. In mode m_1 the objective function (10) can be rewritten as a two-stage problem where T is the stochastic switching time from m_1 to m_2 :

$$\max_{I_1,I_R} J = E\left\{\int_0^T e^{-rt}\left[K_1(1-K_1)-\mu_1 I_1-\frac{\gamma_1}{2}I_1^2-\mu_R I_R-\frac{\gamma_R}{2}I_R^2\right]dt+e^{-rT}V_{(m_2)}(K_1(T),0)\right\},$$

subject to (11)-(15). In a slight abuse of notation we denote fixed points of the state dynamics in mode m_1 under optimal investment as *mode-1 steady states* in spite of the fact that there might be a positive probability that the system moves from mode m_1 to m_2 in finite time. In order to characterize the mode-1 steady states we employ the maximum principle. In particular, we follow Haurie and VanDelft (1991) in introducing an auxiliary state variable z (which can be interpreted as the accumulated hazard rate) with state dynamics

$$\dot{z}(t)=\alpha I_R(t)+\beta K_R^{\psi}(t),$$

and $z(0)=0$. After some transformation related to the auxiliary variable z the following Hamiltonian can be formulated for the problem (see Appendix A.3):

$$\begin{aligned}H(K_1,K_R,I_1,I_R,u_1,u_R,u_z) &= e^{-z}\Big[K_1(1-K_1)-\mu_1 I_1-\frac{1}{2}\gamma_1 I_1^2-\mu_R I_R-\frac{1}{2}\gamma_R I_R^2\\ &+u_1(I_1-\delta_1 K_1)+u_R(I_R-\delta_R K_R)+(\alpha I_R+\beta K_R^{\psi})V_{(m_2)}(K_1,0)+u_z(\alpha I_R+\beta K_R^{\psi})\Big],\end{aligned}$$

where u_1 and u_R are the co-state variables (after an appropriate transformation) associated with the dynamic constraints on K_1 and K_R, respectively, and u_z is the co-state associated with the auxiliary variable z. In this notation the transformed co-states u_1 and u_R are defined by $u_i=\tilde{u}_i e^z,\ i\in\{1,R\}$, where $\tilde{u}_i$ defines the standard co-state corresponding to the state variable i. To get an intuitive understanding of the rationale behind this transformation, it should be noticed that the standard co-states $\tilde{u}_i$ represent the (expected) marginal contribution of the corresponding state variable to the objective in mode m_1. This marginal contribution at a given time t is only realized if the mode is indeed m_1, which implies that the standard co-state would decrease over time even if the states stay constant. To control for this, the transformed co-states represent the marginal contribution conditional to the situation that mode m_1 still prevails at the corresponding time. Since the probability that mode m_1 prevails at t is given by $e^{-z(t)}$, the standard co-states have to be divided by $e^{-z(t)}$to obtain this conditional value. The transformed co-states obtained in this way no longer change over time in a steady-state of mode m_1. To account for the state and control constraints, let us form the Lagrangian L:

$$\begin{aligned}L(K_1,K_R,I_1,I_R,u_1,u_R,u_z,\omega_{K_1},\omega_{I_R}) &= e^{-z}\Big[K_1(1-K_1)-\mu_1 I_1-\frac{1}{2}\gamma_1 I_1^2-\mu_R I_R-\frac{1}{2}\gamma_R I_R^2\\ &+u_1(I_1-\delta_1 K_1)+u_R(I_R-\delta_R K_R)+(\alpha I_R+\beta K_R^{\psi})V_{(m_2)}(K_1,0)+u_z(\alpha I_R+\beta K_R^{\psi})\\ &+\omega_{K_1}K_1+\omega_{I_R}I_R\Big],\end{aligned} \quad (18)$$

where ω_{K_1} and ω_{I_R} are the (transformed) Kuhn-Tucker multipliers associated with the constraints on K_1 and I_R. Maximizing the Lagrangian yields the optimal controls

$$I_R = \frac{u_R - \mu_R + \alpha(V_{(m_2)}(K_1, 0) + u_z) + \omega_{I_R}}{\gamma_R}, \tag{19}$$

$$I_1 = \frac{u_1 - \mu_1}{\gamma_1}, \tag{20}$$

where the trajectories of the co-states have to satisfy the following co-state equations (see Appendix A.3 for a derivation):

$$\dot{u}_1 = (r + \delta_1 + \alpha I_R + \beta K_R^{\psi})u_1 - (1 - 2K_1) - (\alpha I_R + \beta K_R^{\psi}) \left(\frac{\partial V_{(m_2)}(K_1, 0)}{\partial K_1} \right) - \omega_{K_1}, \tag{21}$$

$$\dot{u}_R = (r + \delta_R + \alpha I_R + \beta K_R^{\psi})u_R - \beta\psi K_R^{\psi-1}(V_{(m_2)}(K_1, 0) + u_q), \tag{22}$$

$$\begin{aligned} \dot{u}_z = {} & (r + \alpha I_R + \beta K_R^{\psi})u_z + K_1(1 - K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2, \\ & - \mu_R I_R - \frac{1}{2}\gamma_R I_R^2 + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1, 0), \end{aligned} \tag{23}$$

and the transversality conditions

$$\lim_{t\to\infty} e^{-rt}u_1 K_1 = 0,$$
$$\lim_{t\to\infty} e^{-rt}u_R K_R = 0.$$

Furthermore, the multipliers have to satisfy the complementary slackness conditions

$$\begin{aligned} \omega_{K_1} \geq 0,\ \omega_{K_1} K_1 = 0, \\ \omega_{I_R} \geq 0,\ \ \omega_{I_R} I_R = 0. \end{aligned} \tag{24}$$

A mode-1 steady state is characterized by the condition that the time derivatives of all states and co-states except for z are zero. It should be noted that since the auxiliary variable expresses the accumulated hazard rate, it keeps increasing also if both states K_1, K_R have reached mode-1 steady state values, provided that the corresponding steady state value of K_R is positive. Based on these considerations it is straightforward to see that a mode-1 steady state satisfies

$$\begin{aligned} 0 = {} & I_1 - \delta_1 K_1, \\ 0 = {} & I_R - \delta_R K_R, \\ 0 = {} & 1 - 2K_1 + (\alpha I_R + \beta K_R^{\psi}) \frac{\partial V_{(m_2)}(K_1, 0)}{\partial K_1} - (r + \alpha I_R + \beta K_R^{\psi} + \delta_1)u_1 + \omega_{K_1}, \\ 0 = {} & (r + \alpha I_R + \beta K_R^{\psi} + \delta_R)u_R - \beta\psi K_R^{\psi-1}\left(V_{(m_2)}(K_1, 0) + u_z\right), \\ 0 = {} & K_1(1 - K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R I_R - \frac{1}{2}\gamma_R I_R^2 \\ & + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1, 0) + (r + \alpha I_R + \beta K_R^{\psi})u_z, \\ I_R = {} & \frac{1}{\gamma_R}(u_R - \mu_R + \alpha\left(V_{(m_2)}(K_1, 0) + u_z\right) + \omega_{I_R}), \\ I_1 = {} & \frac{1}{\gamma_1}(u_1 - \mu_1), \end{aligned} \tag{25}$$

together with (24). All vectors $(K_1^*, K_R^*, I_1^*, I_R^*, u_1^*, u_R^*, u_z^*, \omega_{K_1}^*, \omega_{I_R}^*)$ satisfying this system of equations are candidates for (locally asymptotically stable) steady states under the optimal policy provided that the transversality conditions hold as well. For an interior fixed point of the state-costate system (2), (4), (21), (22), (23), which implies $\omega_{K_1}^* = \omega_{I_R}^* = 0$, standard arguments establish that transversality is satisfied if and only if the Jacobian of the state-costate system at the steady state has at least two eigenvalues with negative real parts.

Based on these considerations we can establish the following conditions to be satisfied by a candidate for an interior steady state under the optimal policy.

Proposition 2. *Let $(K_1^*, K_R^*, I_1^*, I_R^*, u_1^*, u_R^*, u_z^*)$ be an interior fixed point of the state-costate dynamics in mode m_1. Then*

(a) The eigenvalues of the Jacobian are given by

$$\begin{aligned}\xi_{1,2,3,4} &= \frac{r+\lambda^*}{2} \pm \frac{1}{2}\sqrt{2M + (r+\lambda^*)^2 \pm 2\sqrt{M^2 - 4Q}},\\ \xi_5 &= r + \lambda^*,\end{aligned}$$

where

$$\begin{aligned}M = {} & \alpha^2 \frac{1}{\gamma_1}\frac{1}{\gamma_R}(u_1^* - (2aK_1^* + b))^2 - \frac{1}{\gamma_R}(r + \lambda^* + \delta_R)u_R^*\left((V^* + u_z^*)^{-1} + (K_R^*)^{-1}(1-\psi)\right)\\ & - \frac{1}{\gamma_1}(2 - 2a\lambda^*)) - (r+\lambda^*)(\delta_1 + \delta_R) - \delta_1^2 - \delta_R^2 - \frac{\alpha}{\gamma_R}u_R^*(r + \lambda^* + 2\delta_R) - \beta\psi\frac{u_R^*}{\gamma_R}(K_R^*)^{\psi-1},\end{aligned}$$

$$\begin{aligned}Q = {} & -\frac{1}{\gamma_1}\frac{1}{\gamma_R}(u_1^* - (2aK_1^* + b))^2(\beta\psi(K_R^*)^{\psi-1} + \alpha\delta_R + (r+\lambda^*)\alpha)(\beta\psi(K_R^*)^{\psi-1} + \alpha\delta_R)\\ & + \left[\delta_1(r + \lambda^* + \delta_1) + \frac{1}{\gamma_1}(2 - 2a\lambda^*)\right]\left[\frac{1}{\gamma_R}(r + \lambda^* + \delta_R)u_R^*(u_R^*(V^* + u_z)^{-1}\right.\\ & \left. + (K_R^*)^{-1}(1-\psi)) + ((r + \lambda^* + \delta_R)\delta_R + \frac{1}{\gamma_R}u_R(\beta\psi(K_R^*)^{\psi-1} + (r+\lambda^*)\alpha + 2\alpha\delta_R))\right],\end{aligned}$$

where $V^* = V_{(m_2)}(K_1, 0)$ and $\lambda^* = \alpha\delta_R K_R^* + \beta(K_R^*)^2$.

(b) The Jacobian has at least two eigenvalues with negative real part if and only if

$$M < 0, \tag{26}$$

and

$$0 < Q \leq \frac{M^2}{4}. \tag{27}$$

If at least one of the two inequalities is violated, (K_1^*, K_R^*) is not a candidate for being a steady state under the optimal policy.

Proof. See Appendix A.4 □

In what follows we will consider numerical examples to illustrate our findings and determine the optimal investment paths. The default parameter setting for this purpose is

$$\begin{aligned} &\alpha = 1, \beta = 0.1, \psi = 2, \mu_1 = 0.2, \mu_2 = 0.4, \mu_R = 1.029, \gamma_1 = \gamma_2 = 0.15, \\ &\gamma_R = 0.02, \delta_1 = \delta_2 = 0.1, \delta_R = 0.3, \eta = 0.9, \theta = 0.1, r = 0.04, \end{aligned} \tag{28}$$

but we will also explore robustness of our findings with respect to parameter changes, in particular for those parameters determining the degree of horizontal and vertical differentiation. Whereas several parameter values, like the discount rate and depreciation rate are standard values, some other choices should be motivated. In our default scenario we assume that the hazard rate can be significantly influenced by current investment, even if the specific knowledge stock of the firm is not large. This is reflected by $\alpha = 1$ and $\beta = 0.1$. Such a scenario corresponds to what is referred to a 'science-based industry' in the literature on industry evolution (see Nelson and Winter (1982)), i.e. an industry where breakthroughs are mainly based on publicly available knowledge rather than on accumulated firm-specific knowledge. In the opposite case, i.e. the hazard rate mainly depends on the knowledge stock, we speak of a 'cumulative industry'. This would occur when the value of α is close or even equal to zero, whereas β has a significantly positive value. Choosing $\psi = 2$ implies that the hazard rate is a convex function of the knowledge stock, which captures the empirical observation of economies of scale in innovation processes. Furthermore, by setting $\mu_2 = 0.4 > \mu_1 = 0.2$, it is assumed that capacity buildup is more expensive for a newly introduced product than for the well established one. Since the focus of the analysis lies on the interplay between the established and the (anticipated) new product, we are considering a situation where the new product is a relatively close substitute to the established one ($\eta = 0.9$) and vertically differentiated ($\theta = 0.1$). Finally, it is assumed that knowledge depreciates faster than physical capital ($\delta_R > \delta_1 = \delta_2$). The values of the parameters determining the costs of knowledge investment (μ_R, γ_R) are crucial in determining whether positive R&D investment occurs and whether a unique steady state or several fixed points occur in mode m_1.

Straightforward calculations show that under this parameter setting the system (25) has three solutions given by

$$\begin{aligned} &K_1^{*,1} = 0.477, \quad K_R^{*,1} = 0.781, \quad u_1^{*,1} = 0.207, \quad u_R^{*,1} = 0.204, \quad u_z^{*,1} = -6.079 \\ &K_1^{*,2} = 0.484, \quad K_R^{*,2} = 0.171, \quad u_1^{*,2} = 0.207, \quad u_r^{*,2} = 0.082, \quad u_z^{*,2} = -5.961 \\ &K_1^{*,3} = 0.485, \quad K_R^{*,3} = 0, \quad u_1^{*,3} = 0.207, \quad u_R^{*,3} = 0, \quad u_z^{*,3} = -6. \end{aligned}$$

The first two of these steady states are interior ones, which means that Proposition 2 can be applied to determine whether they are candidates for being a steady state under the optimal policy. Calculating M and Q for these two steady states shows that only the first one can be a steady state under optimal investment[1]. In this steady state there is a strictly positive hazard rate, which implies that for any initial condition

[1] The values of M and Q are given by $M = -17.99,\ Q = 65.54$ for the first steady state, which implies that all conditions of Proposition 2 are satisfied, whereas for the second fixed point we obtain $M = -6.31,\ Q = -84.14$ implying that the condition $Q > 0$ is violated.

where the optimal path leads towards $(K_1^{*,1}, K_R^{*,1})$ the new product will be introduced with probability 1 and therefore mode m_2 will prevail in the long run. On the other hand, also the third fixed point can serve as a steady state under the optimal policy. In this case the hazard rate is zero in the long run, i.e. if a firm is in this steady state the probability of future innovation is zero. Based on the (local) method of the maximum principle, for a given initial situation it cannot be determined whether the firm under its optimal policy will converge either to the first or the third fixed point. Therefore, we rely on (numerical) dynamic programming methods to characterize the optimal investment paths of the firm and their implications for the probability of innovation.

4 Dynamics

After having explained the numerical procedure applied to solve the model in section 4.1, the resulting investment behavior is presented in section 4.2. Section 4.3 includes a comparative statics analysis regarding the key parameters of our model.

4.1 The Numerical Procedure

We employ a combination of a collocation method using Chebychev polynomials and homotopy to obtain an approximation of the value function of the problem and the optimal investment strategies in mode m_1. In particular, these methods are used to obtain (approximate) solutions to the Hamilton-Jacobi-Bellman (HJB) equation for the problem in mode m_1, which is given by

$$rV_{(m_1)} = \max_{I_1,\, I_R} \Bigg[K_1(1-K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 {I_1}^2 + \frac{\partial V_{(m_1)}}{\partial K_1}(I_1 - \delta_1 K_1) - \mu_R I_R - \frac{1}{2}\gamma_R {I_R}^2 + \frac{\partial V_{(m_1)}}{\partial K_R}(I_R - \delta_R K_R) + (\alpha I_R + \beta K_R^{\psi})(V_{(m_2)}(K_1, 0) - V_{(m_1)}(K_1, K_R)) \Bigg]. \quad (29)$$

The last term on the right hand side of the HJB-equation has to be added because this is a piece-wise deterministic problem and it captures the effect of the future jump to mode m_2 on the value function in mode m_1 (see e.g. Dockner et al. (2000)). From the HJB-equation optimal investment levels can be derived:

$$I^*_{1,(m_1)}(K_1, K_R) = \frac{1}{\gamma_1}\left(\frac{\partial V_{(m_1)}}{\partial K_1} - \mu_1\right)$$
$$I^*_{R,(m_1)}(K_1, K_R) = \frac{1}{\gamma_R}\left(\frac{\partial V_{(m_1)}}{\partial K_R} - \mu_R + \alpha(V_{(m_2)}(K_1, 0) - V_{(m_1)})\right), \quad (30)$$

and the Bellman equation can be rewritten as

$$(r + \alpha I_R^* + \beta K_R^{\psi})V_{(m_1)} - \frac{\partial V_{(m_1)}}{\partial K_1}(I_1^* - \delta_1 K_1) - \frac{\partial V_{(m_1)}}{\partial K_R}(I_R^* - \delta_R K_R) = F(K_1, K_R, I_1^*, I_R^*), \quad (31)$$

with

$$F(K_1, K_R, I_1^*, I_R^*) = K_1(1-K_1) - \mu_1 I_1^* - \tfrac{1}{2}\gamma_1 I_1^{*2} - \mu_R I_R^* - \tfrac{1}{2}\gamma_R I_R^{*2} + (\alpha I_R^* + \beta K_R^{\psi})V_{(m_2)}(K_1, 0).$$

Since no closed form solutions for this non-linear partial differential equation are available, we numerically approximate the solution by polynomial approximations of $V_{(m_1)}$. To this end we generate a set of n_1 Chebychev nodes $\mathcal{N}_{K_1}$ in $[0, \bar{K}_1]$ and a set of n_R Chebychev nodes $\mathcal{N}_{K_R}$ in the interval $[0, \bar{K}_R]$ for appropriately chosen values of $\bar{K}_1$ and $\bar{K}_R$ (see e.g. Judd (1998) for the definition of Chebychev nodes and Chebychev polynomials). We further define the set of interpolation nodes in the state space X as

$$\mathcal{N} = \{(K_1, K_R) | K_1 \in \mathcal{N}_{K_1}, K_R \in \mathcal{N}_{K_R}\}.$$

Note that the cardinality of $\mathcal{N}$ is $n_1\ n_R$. Our goal is to calculate a polynomial approximation of $V_{(m_1)}$, which (approximately) satisfies (31) on the set of interpolation nodes $\mathcal{N}$. It is well known that the choice of Chebychev interpolation nodes avoids large oscillations of the interpolating polynomial between the interpolation nodes (as could occur e.g. for equi-distant nodes) and implies that the interpolating polynomials approximately solve the HJB equations on the entire state space. The set of basis functions for the polynomial approximation is determined as $\mathcal{B} = \{B_{j,k}, j = 1, .., n_1, k = 1, .., n_R\}$ with

$$B_{j,k}(K_1, K_R) = T_{j-1}\left(-1 + \frac{2K_1}{\bar{K}_1}\right) T_{k-1}\left(-1 + \frac{2K_R}{\bar{K}_R}\right),$$

where $T_j(x)$ denotes the j-the Chebychev polynomial. Since Chebychev polynomials are defined on $[-1, 1]$, the state variables have to be transformed accordingly.
The value function is approximated by

$$V_{(m_1)}(K_1, K_R) \approx \hat{V}(K_1, K_R) = \sum_{j=1}^{n_1} \sum_{k=1}^{n_R} C_{j,k} B_{j,k}(K_1, K_R),\ (K_1, K_R) \in X, \tag{32}$$

where $C = \{C_{j,k}\}$ with $j = 1, \cdots, n_1, k = 1, \cdots, n_R$ is the set of $n_1 n_R$ coefficients to be determined. To calculate these coefficients we set up a system of non-linear equations derived from the condition that $\hat{V}_{(m_1)}$ satisfies the HJB equation (31) on the set of interpolation nodes $\mathcal{N}$. This system consists of $n_1 n_R$ equations with $n_1 n_R$ unknowns (i.e. the coefficients $C_{j,k}$). We denote this system by $G(C) = 0$, where G is a mapping $\mathbb{R}^{n_1 n_R} \to \mathbb{R}^{n_1 n_R}$. It is solved by a recursive algorithm, where based on an initial guess $\tilde{C}^0 = \{C^0_{j,k}, j = 1, .., n_1, k = 1, .., n_R\}$ of the coefficients in iteration $l \geq 1$ the coefficients $\tilde{C}^{l-1}$ are used to calculate approximations of the value functions and their partial derivatives at each node in $\mathcal{N}$. These approximations are inserted for all terms that occur in (31) where the value function or its derivatives appear in a non-linear form. Inserting the approximation (32) with C replaced by $\tilde{C}^l$ for all terms in (31), where the value function and its derivatives occur in a linear way, yields a linear system of equations for the coefficients $\tilde{C}^l$, which even for large values of $n_1 n_R$ can be solved efficiently using standard methods as long as the coefficient matrix is well conditioned. The solution of this linear system gives the new set of coefficient values $\tilde{C}^l$. To complete the iteration, the new approximations of the value functions and their derivatives are inserted into all (including the non-linear) corresponding terms in (31) and the resulting absolute value of the left hand side of this equation relative to the corresponding value function is determined for all nodes in $\mathcal{N}$. If the maximum of this relative error is below a given threshold ϵ the algorithm is stopped, we set $C = \tilde{C}^l$

and the current approximation of the value function is used to calculate the optimal investment functions.

Unfortunately, no general conditions can be given that guarantee the existence of a stable fixed point of the described algorithm, which corresponds to the (approximate) solution of the HJB-equation. Also, starting with an appropriate initial guess for the coefficients is often crucial for convergence to a meaningful fixed point, even if there exists such a stable fixed point. Furthermore, the described method faces a fundamental problem if the optimal control has jumps in the considered state space. The value function exhibits a kink along the manifold where the optimal control jumps (i.e. although the value function is everywhere continuous it is not differentiable at points where the control jumps) and such kinks cannot be captured by the polynomial approximation used in the collocation method, which by definition is smooth on the entire considered state space. In particular, the optimal control derived from the polynomial approximation of the value function must always be continuous on the entire state space.

In order to deal with this problem, a combination of the collocation method described above and a homotopy method is used to generate two 'local value functions'. These local value functions provide for different initial states the maximal discounted payoff among all paths converging to a given steady state candidate. The actual value function is then given by the upper envelope of these two local value functions. If the two intersect in the considered state space, then indeed both steady states are locally stable fixed points under the optimal policy and the boundary between the basins of attraction of the two fixed points is given by the manifold where the two local value functions intersect. Along this manifold, which in the literature is referred to as Skiba-curve or DNS-curve (see Haunschmied et al. (2005)) the optimal control then typically is discontinuous.

In order to obtain the local value function around the fixed point with positive knowledge stock the collocation method is applied to a parameter setting, where, deviating from (28), the cost of R&D investment is reduced to $\mu_R = 0.7$. For this setting all investments are strictly positive on the relevant state space and the collocation method yields a very good approximation of the value function. For all initial states the optimal trajectories converge to the fixed point with positive knowledge stock. Using this solution as a starting point, a homotopy method is used to gradually increase the parameter μ_R from $\mu_R = 0.7$ to the desired value of $\mu_R = 1.029$ adjusting the coefficient vector C according to

$$\frac{\partial C}{\partial \mu_R} = -[G'(C)]^{-1} \frac{\partial G(C)}{\partial \mu_R}.$$

We denote the obtained approximation of the value function for $\mu_R = 1.029$ by $\hat{V}^l_{(m_1)}$, where the superscript l indicates that it was obtained by approaching the target value of μ_R from below. Since the homotopy method only takes into account the local marginal changes in the optimal investment policy as the parameter μ_R is increased, we conclude that $\hat{V}^l_{(m_1)}(K_1, K_R)$ provides an accurate approximation of the actual value function only for those states (K_1, K_R) where no jump in the optimal policy occurs if μ_R is increased from 0.7 to 1.029. Put differently, $\hat{V}^l_{(m_1)}(K_1, K_R)$ provides an accurate approximation of the actual value function for all states where the optimal trajectory converges to the positive knowledge steady state[2].

The local value function for the fixed point with $K_R = 0$ is obtained analogously. The collocation method is applied for $\mu_R = 1.1$, which is a parameter value where the optimal investment is zero on the entire considered state space. Hence for this parameter setting all optimal trajectories converge to the steady state with a knowledge stock of zero. Then, again using homotopy, the effect of decreasing the parameter μ_R from 1.1 to 1.029 is calculated. The resulting approximation of the local value function is denoted by $\hat{V}^h_{(m_1)}$. By the same arguments as above $\hat{V}^h_{(m_1)}(K_1, K_R)$ provides an accurate approximation of the actual value function for all states where the optimal trajectory converges to the steady state with $K_R = 0$.

The actual approximation of the value function is then given by $V_{(m_1)}(K_1, K_R) = max[\hat{V}^l_{(m_1)}(K_1, K_R), \hat{V}^h_{(m_1)}(K_1, K_R)]$ and the investment functions are obtained from (30) using this approximation of the value function. The intersection line between $\hat{V}^l_{(m_1)}$ and $\hat{V}^h_{(m_1)}$ determines the Skiba curve where optimal controls jump.

4.2 Optimal Dynamics of Investment and Innovation

Using our default parameter setting and applying the numerical procedure described in the previous section[3], we realize that indeed both candidates for a stable steady state have a basin of attraction of positive size. The value function of the problem for mode m_1 is depicted in Figure 1, where the bold black line indicates the Skiba curve which separates the basins of attraction of the two steady states. As expected the value function increases as the knowledge stock is increased, whereas, due to our assumptions that capacities are always fully used an increase of the capacity for the established product can have negative effects if the capacity is already substantially above the steady state output levels.

4.2.1 Optimal Investment Functions

The corresponding investment functions are given in Figure 2. Panel (a) shows how investment in established production capacity depends on the current established product capacity level and on knowledge. It shows a strong negative dependence of this investment on the established product capital stock. This is understandable because a higher established product capacity reduces the output price for the established product (note that capacity is fully used so higher capacity means higher sales, which reduces price), so that profitability of established product investment goes down with this capacity. Also, it can be seen that the investment function I_1 does not display any noticeable discontinuity along the Skiba curve.

[2]It should be noted that the approximated value function generated by this method does not exactly satisfy the HJB equation, but that there is always a remaining error. For a 'local value function' constructed in the way above this error remains small in the part of the state space close to the considered stable steady state, whereas it becomes relatively large in the parts of the state space where the optimal trajectory converges to an alternative steady state.

[3]The parameters in the numerical method were chosen as $\bar{K}_1 = 1, \bar{K}_R = 1.2$ and $n_1 = n_R = 8$. For reasons of clarity in most figures we only depict parts of this considered state space. Also, in the calculations concerning the welfare analysis in Section 4.2.3 the considered range of K_R was increased to $\bar{K}_R = 1.6$.

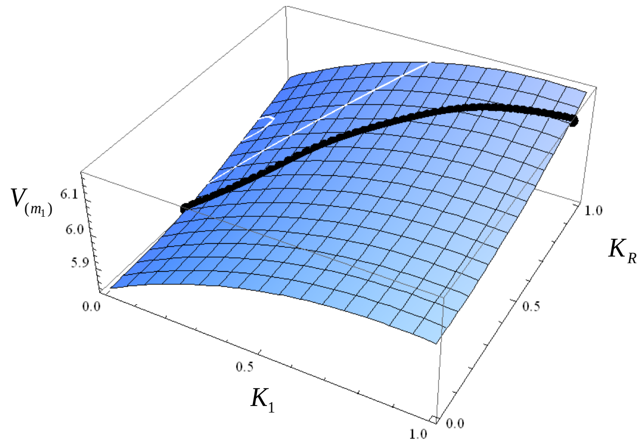
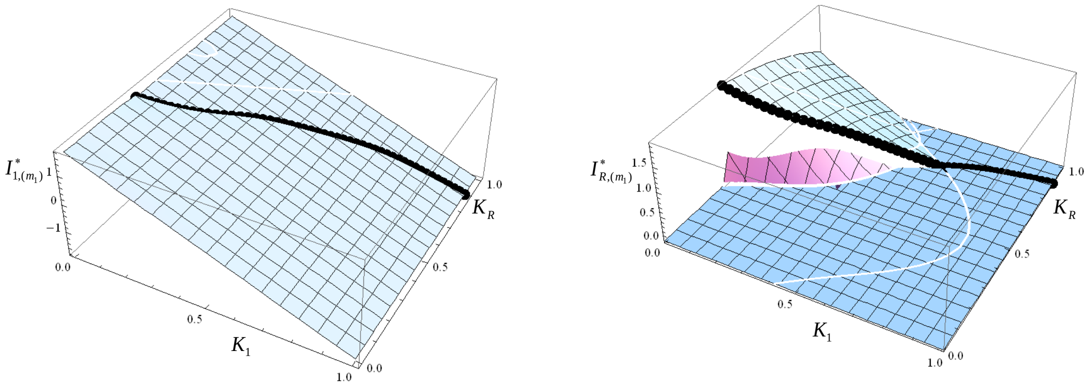


Figure 1: Value function of the problem in mode m_1. The black line indicates where the two local value functions intersect.



(a)

(b)

Figure 2: Optimal investment functions in mode m_1

Quite a different picture emerges if we consider R&D investment. Figure 2(b) shows the function I_R exhibits a substantial downward jump along a large part of the Skiba curve if the initial state is moved from the basin of attraction of the steady state with positive knowledge stock into the basin of attraction of the zero knowledge steady state. It can also be seen that close to the Skiba curve R&D investment is positive even in the basin of attraction of the zero knowledge steady state. For a considerable part of the state space optimal R&D investment is zero. In those areas where investment is positive it decreases with the established product capacity. This is because the higher the established product capacity the lower the output price of the innovative product. Moreover, introducing the innovative product on the market results in a larger reduction of established product revenue the higher the established product capacity is. Second, we see that R&D investment non-monotonically depends on the knowledge stock, where R&D investment is especially large for intermediate levels of knowledge. Two countervailing effects are responsible for this non-monotonicity. First, since the hazard rate is a convex function of the knowledge stock, the marginal effect

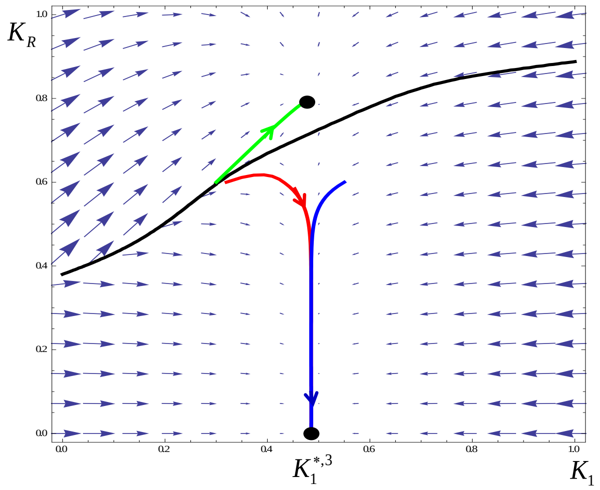


Figure 3: Optimal dynamics in mode m_1 and basins of attraction of the two long-run steady states

of an increase of the knowledge stock on the hazard rate becomes larger the larger the hazard rate already is. This leads to an increase of investment incentives as the knowledge stock increases. Second, if K_R is large the hazard rate is already large, implying that the expected waiting time of innovation is small. Since the marginal return to R&D investments are zero in mode m_2 this reduces incentives for R&D investments. As can be seen in Figure 2(b) the first of these two effects dominates in the basin of attraction of the zero investment steady state- leading to a positive dependence of I_R from K_R, whereas the second effect is stronger in the part of the state space where optimal trajectories converge to the steady state with positive knowledge stock.

4.2.2 Three Innovator Scenarios

Figure 3 shows the vector field of the state-dynamics under optimal firm behavior, where the co-existing stable steady states and the Skiba curve separating their basins are indicated by black dots and a black line respectively. The jump in the optimal control along the Skiba curve, which influences the state-dynamics, can be clearly seen. To highlight the dependence of the qualitative characteristics of optimal trajectories from initial conditions three trajectories are highlighted. For all three trajectories the initial knowledge stock is given by $K_R^{ini} = 0.6$. Those three trajectories represent three qualitatively different scenarios, which we now discuss.

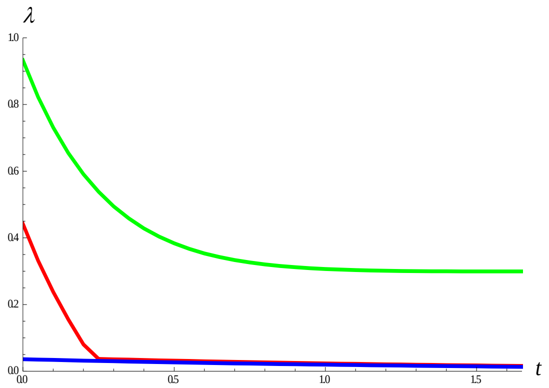
Dedicated Innovator Scenario
The green trajectory results when the initial established product capacity level is small ($K_1^{ini} = 0.3$). Then the firm invests to increase both established product capital and knowledge. Initially, investments are so large that instantaneous profits are negative in the beginning of the trajectory, as shown in Figure 4(b). When the established product capacity is low, the price of the established product is high, which explains

why the firm wants to grow in the established product market. The firm is also eager to innovate as long as the established product sales are low, because, first, introducing the innovative product on the market does not lead to a large reduction of established product revenue, and, second, a low established product capacity implies that the output price for the innovative product is high. The firm converges to a steady state where both established product capacity and the knowledge stock are positive. Note that it is beforehand not known up to which point in time this trajectory is realized because of the positive innovation probability for all t.

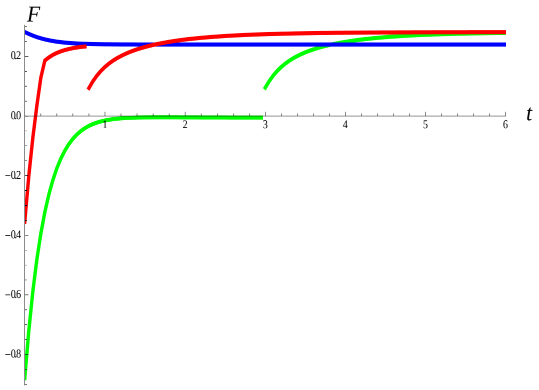
For the considered parameter setting R&D costs in mode m_1 stay at such a high level that the firm makes losses for its entire time in mode m_1. As soon as the breakthrough in the innovation project is realized, the firm starts investing in innovative product capacity, and, at the same time strongly reduces its investment in the capacity for the old product. Technically speaking, mode m_1 passes into mode m_2. Inspection of the investment trajectories (not shown here) reveals that the firm scraps parts of its old market capacity (i.e. $I_1 < 0$) directly after the innovation has been introduced. Also the firm of course stops investing in new knowledge. The reduction of I_1 and I_R outweighs the costs of the buildup of capacity for the new product and instantaneous profits jump up at the point in time when mode m_2 is entered and keeps increasing from that point on. Figure 4(a) shows that the probability of innovation is especially large in the beginning, that is when R&D investments are at the highest level (cf. Figure 2(b)). The cumulative probability that the firm eventually innovates is equal to one here, as is confirmed in Figure 5, which shows how the cumulative innovation probability depends on the initial level of K_1 and in and Figure 6(a), where the distribution of innovation times for the initial condition $K_1^{ini} = 0.3$ is shown. The mean innovation time is $\mathbb{E}[T] = 2.61$ and the probability that the jump to mode m_2 occurs before $t = 15$ is more than 99%. Summarizing, we observe that if the firm has a relatively small established product capacity and a sufficiently large knowledge stock when the innovation option becomes available (i.e. $t = 0$), it is dedicated to pursue the innovation project till completion regardless of the actual innovation time. It should be noted that even with small established product capacity a certain initial knowledge stock is necessary to become a dedicated innovator. In particular for a small firm which is new to the market, and therefore has not built up sufficient knowledge, a dedicated R&D investment strategy is not optimal. Hence, our model provides a theoretical foundation for recent empirical observations by Dolfsma and Van der Velde (2014).

No Innovator Scenario

The blue trajectory corresponds to a scenario where the initial established product capacity level is large ($K_1^{ini} = 0.55$). The innovation incentive is low now due to the cannibalization effect: introducing the innovative product on the market would reduce the price of the established product, which decreases revenue due to the large sales level. For this reason the firm does not invest in R&D at any time here. Due to the initial positive knowledge stock the hazard rate is initially positive but very small. It decreases over time to zero (see also Figure 4(a) and Figure 5). Consequently, the cumulative innovation probability of the firm is very small and with a probability close to one the firm stays in mode m_1 and converges to a steady state without any knowledge



(a)



(b)

Figure 4: Dynamics of (a) the hazard rate in mode m_1 and (b) instantaneous profits for different initial conditions. The assumed realization of the innovation time in panel (b) for $K_1^{ini} = 0.3$ (green line) is $T = 2.9$ and for $K_1^{ini} = 0.32$ (red line) is $T = 0.7$. For $K_1^{ini} = 0.55$ (blue line) it is assumed that no innovation occurs.

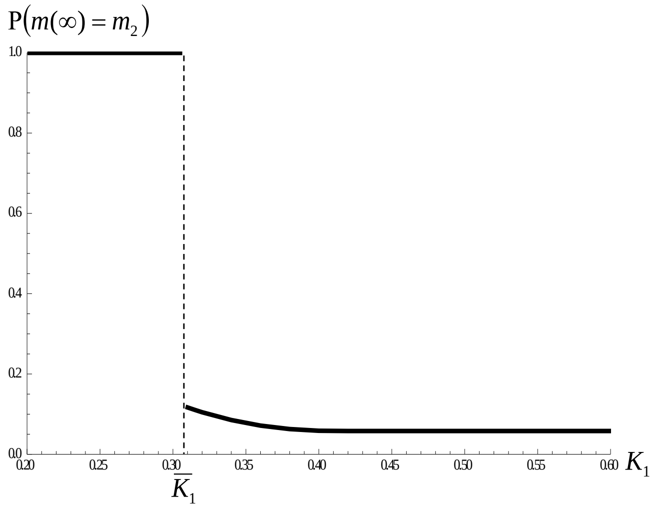
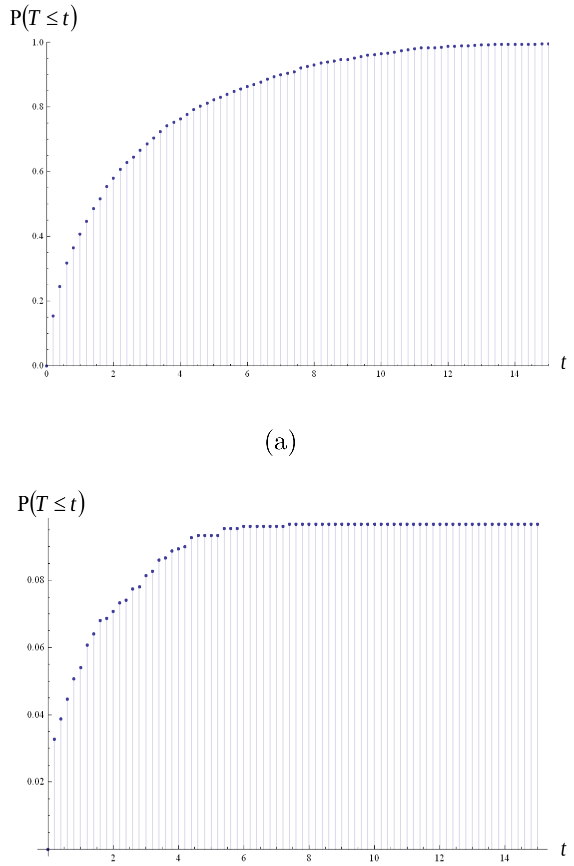


Figure 5: Probability that new product is introduced for $K_R(0) = 0.6$.

and a positive established product capacity level being denoted by $K_1^{*,3}$. Due to the missing investments in knowledge and the small investments in capacity for the established market the instantaneous profits are positive throughout the whole planning period. However, the long run instantaneous profits stay below the level of that of the successful innovator (compare the green and blue lines in Figure 4(a)). In this sense having a strong initial position on the established market has a negative impact on the instantaneous profits of the firm in the long run. Due to the discounting the value function is however larger for $K_1(0) = 0.55$ (blue trajectory) compared to $K_1(0) = 0.3$ (green trajectory).

Temporary Innovator Scenario
The red trajectory arises when the initial established product capacity level is of intermediate size ($K_1^{ini} = 0.32$). Initially the firm grows on the established product market while it invests in R&D to increase knowledge. The initial investment outlays are such that the instantaneous profits are negative (see Figure 4(b)). However, at some point established product sales have become so large that the firm no longer wants to pursue innovation. Consequently, the firm stops investing in R&D and the trajectory converges to a steady state with established production capacity being equal to $K_1^{*,3}$ and zero knowledge stock. As discussed in the previous paragraph, the instantaneous payoff at this steady state is positive. Along the trajectory it can happen that the innovation breakthrough takes place and the firm starts producing the innovative product. From Figure 4(a) we obtain that this probability especially is positive at the beginning of the planning period, i.e. when the firm actively invests in R&D. But, as can be derived from Figures 5 and 6(b), the cumulative probability that the innovation breakthrough eventually happens is substantially less than one. Intuitively, the temporary innovator gives the innovation a chance while it is still relatively small on the established market, but then abandons the innovation project once it has built up sufficient capacity for the established product. This interpretation is confirmed by



(a)

(b)

Figure 6: Distribution of innovation times for (a) $K_1(0) = 0.3$ and (b) $K_1(0) = 0.32$ with $K_R(0) = 0.6$.

considering the distribution of the innovation time under optimal firm behavior for this initial condition. Whereas the probability of a finite innovation time is less than 10%, the expected innovation time conditional on being finite is given by $\mathbb{E}_{T<\infty}[T] = 1.33$, which is only about half the expected innovation time of the dedicated innovator.

Whereas in the parameter setting considered here increasing the capacity for the established product induces a downward jump of the innovation probability, this probability remains positive, although small. In case the Skiba curve intersects with the $K_R = 0$ line increasing the initial value of K_1 can induce a transition from a case where mode m_2 is reached with probability 1 to a case where the second mode is reached with probability zero. In Appendix A.5 we demonstrate that such a case occurs in our model for a value of $\mu_R = 0.975$, which is slightly smaller than our default value. Assuming that the firm has no initial knowledge (i.e. $K_R^{ini} = 0$) then implies that for $K_1^{ini} = 0.1$ the innovation is introduced with probability 1, whereas for $K_1^{ini} = 0.2$ investment in R&D and the hazard rate are constant zero along the optimal trajectory such that the firm stays in m_1 without innovation for sure.

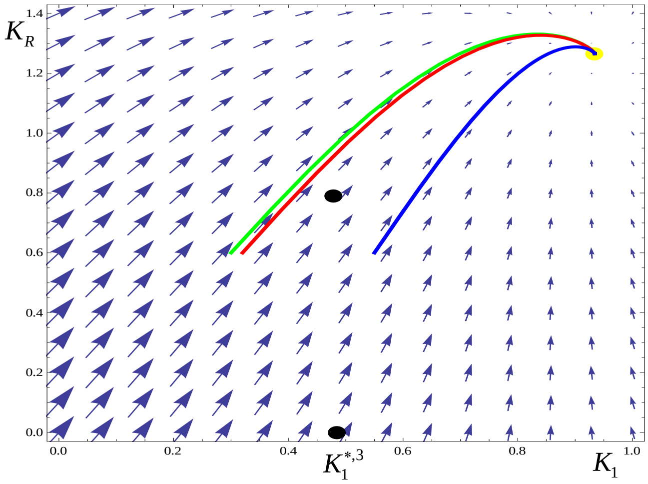


Figure 7: Phase diagram under welfare maximizing investment policies

4.2.3 Welfare Analysis

The observation, that for a certain subset of initial conditions the monopolist refrains from R&D investment and thereby reduces the probability of product innovation virtually to zero, raises the question whether such a 'non-innovation strategy' is also desirable from a welfare perspective. In order to address this question we characterize the welfare maximizing investment strategies. In particular, we consider an analogous dynamic optimization problem with two modes to that of the monopolist given in (10), where the instantaneous objective function has been replaced by the welfare function[4] and the objective now reads

$$J(I_1, I_2, I_R) = E_{\mathcal{P}_m}\left\{\int_0^\infty e^{-rt}\left[K_1 + (1+\theta)K_2 - \frac{1}{2}\left(K_1^2 + K_2^2\right) - \eta K_1 K_2 \right.\right.$$
$$\left.\left. -\mu_1 I_1 - \frac{\gamma_1}{2}I_1^2 - \mu_2 I_2 - \frac{\gamma_2}{2}I_2^2 - \mu_R I_R - \frac{\gamma_R}{2}I_R^2\right] dt\right\} \quad (33)$$

Using the numerical methods lined out above we obtain the value function and the corresponding welfare maximizing investment functions for physical capital and knowledge. Inserting these investment functions into the state dynamics in mode m_1 yields for our default parameter constellation the phase diagram shown in Figure 7.

Comparing this figure with Figure 3, where the phase diagram under optimal investment of the firm is shown, clearly indicates that for all initial conditions giving rise to the No-Innovator or Temporary Innovator scenario, optimal firm behavior induces (with positive probability) a long run outcome qualitatively different from the welfare maximizing one. For all initial values of the firm's capacity for the established product and knowledge stock, under the welfare maximizing investment strategy a positive knowledge stock is built in the long run, which implies that the new product is introduced with probability one. The long run hazard rate under the efficient

[4] It should be noted that the linear inverse demand can be derived from a quadratic utility function of a representative consumer. The welfare function is given by the difference between this utility function and investment costs (note that production costs are set to zero), which is equivalent to the sum of consumer and producer surplus.

path is also higher than the hazard rate in the Dedicated Innovator scenario. The intuition for this observation is straight forward. The firm does not take into account the additional consumer surplus that is generated due to the introduction of the new product. Hence, its incentives to invest in knowledge generation are below the efficient level. Furthermore, as has to be expected, also the incentives for the firm to invest in physical capital are below the efficient level.

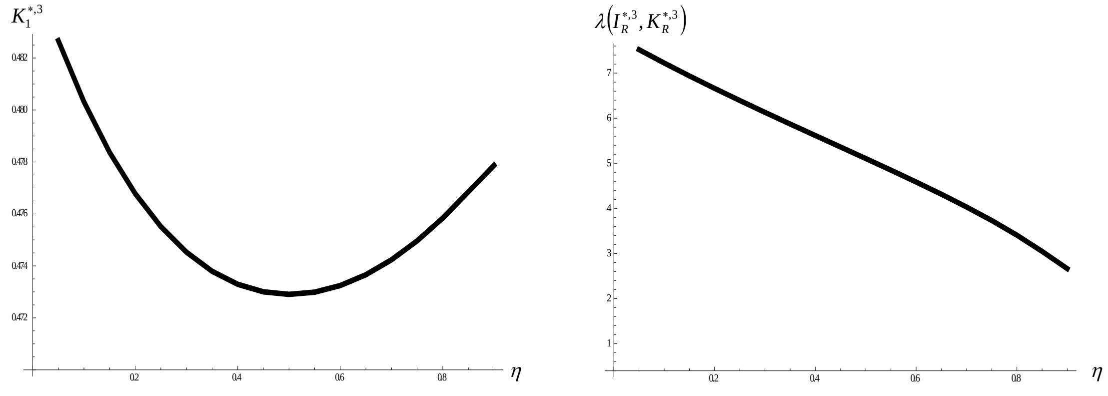
4.3 Effects of Product Differentiation on Innovation and Investment

Having discussed the dependence of the optimal behavior of the firm from the initial values of the state variables K_1 and K_R we now explore how optimal behavior depends on key parameters of the model. In particular, we are interested in the effects of changes of the degree of vertical and horizontal differentiation on the R&D activities and physical capacity investments prior to the innovation. To keep this analysis as transparent as possible we concentrate on the effects of the differentiation parameters on the steady state values of the established product capacity K_1 and the hazard rate λ in the positive innovation steady state of mode m_1. It is evident that the differentiation parameters have no influence on the (locally) stable steady state on the $K_R = 0$ line, since in this steady state the new product is never introduced.

In Figure 8(a) we see that the steady-state level of the capacity of the established product depends non-monotonously on the parameter η, which governs the degree of horizontal differentiation. In particular, for values of η close to 1, i.e. in cases where the new product is only weakly horizontally differentiated, an increase of η implies an increase of the capacity of the established product. At first sight this is quite counter-intuitive, since one might expect that the closer the new product is to the established one the larger is the expected cannibalization between the two products and the smaller should the capacity for the established product be. For an exogenous hazard rate this reasoning would be correct, but in our setting with endogenous hazard rate an additional effect has to be considered. As can be seen in Figure 8(b) the hazard rate decreases substantially if η moves closer to 1. Incentives to invest in R&D are lower the closer the new product is to the established one. The decrease in the hazard rate implies that the expected waiting time till the arrival of the new product increases and this has a positive effect on the incentives to invest in capacity for the established product. For large values of η this second effect dominates and the hence there is a positive relationship between η and the steady state capacity of the established product.

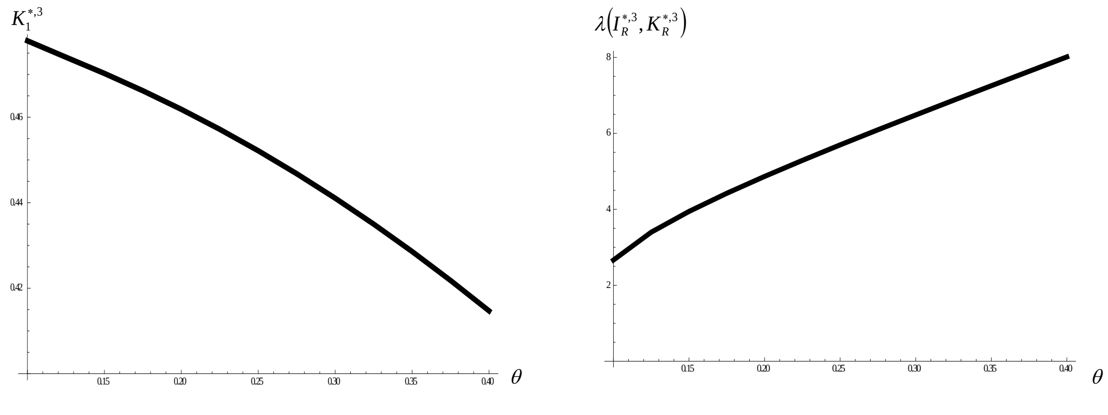
Considering the effect of the degree of vertical differentiation no such tradeoff arises. As the parameter θ capturing the degree of vertical differentiation increases the hazard rate goes up (see Figure 9 (b)) and hence the expected time till innovation becomes smaller. This decreases incentives to invest in K_1. Furthermore, if the quality of the new product becomes larger then this negatively affects the demand for the established product after the innovation, which also reduced incentives to invest in capacity for this product. Therefore, an increase in the degree of vertical differentiation always leads to lower investment in established product capacity.

We have carried out extensive additional sensitivity checks for other model pa-



(a) (b)

Figure 8: Steady state level of physical capital (a) and hazard rate (b) in mode m_1 for different degrees of horizontal differentiation



(a) (b)

Figure 9: Steady state level of physical capital (a) and hazard rate (b) in mode m_1 for different degrees of vertical differentiation

rameters in order to ensure the robustness of our findings, but since these analyses generated economically very intuitive results we abstain from presenting them here in detail.

5 Conclusions

To our knowledge, this paper is the first attempt to analyze in a fully dynamic framework the incentives of an established incumbent to engage in a risky R&D project with the aim to extend its own product range. Using an innovative numerical method we identify three distinctive scenarios depending on the initial levels of the established product capacity and the knowledge stock. Of specific interest is the insight that there arises a hysteresis effect in a sense that the initial conditions crucially determine the probability that the innovation takes place, even if an infinite time horizon is considered. A noteworthy implication of our analysis is that starting a product innovation project and terminating it after a relatively short period of time can be optimal from the firm perspective. This insight is especially remarkable since we abstract from competition in the current setting, which implies that the termination of the project is not triggered by the success of an innovating competitor. Rather it is driven by the interaction of the growth of the established product capacity with the stochastic nature of the R&D project. Our welfare analysis indicates that there is scope for policy intervention with the aim to stimulate investment in the risky R&D project.

An extension of this endogenous innovation framework to a duopoly model is the next step. It will be interesting to explore how the effects identified in this paper interact with strategic considerations arising in a duopoly setup.

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A Appendix

A.1 Location and Stability of the Steady State in Mode m_2:

The expression of the Hamiltonian in mode m_2 is:

$$\begin{aligned}H^{m_2}(K_1, K_2, I_1, I_2, u_{1,m_2}, u_2) = {} & K_1(1 - K_1 - \eta K_2) + K_2(1 + \theta - \eta K_1 - K_2) \\ & - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_2 I_2 - \frac{1}{2}\gamma_2 I_2^2 \\ & + u_{1,m_2}(I_1 - \delta_1 K_1) + u_{2,m_2}(I_2 - \delta_2 K_2)\end{aligned}$$

where u_{1,m_2} and u_{2,m_2} are respectively the costate variabes associated with the dynamic constraints on K_1 and K_2.

First order conditions for the optimization of the Hamiltonian are:

$$\frac{\partial H^{(m_2)}}{\partial K_1} = u_{1,m_2} \Longrightarrow \dot{u_{1,m_2}} - (r + \delta_1)u_{1,m_2} = 2K_1 + 2\eta K_2 - 1 \tag{34}$$

$$\frac{\partial H^{m_2}}{\partial K_2} = u_{2,m_2} \Longrightarrow \dot{u_{2,m_2}} - (r + \delta_2)u_{2,m_2} = 2K_2 + 2\eta K_1 - (1 + \theta) \tag{35}$$

$$\frac{\partial H^{m_2}}{\partial I_1} = 0 \Longrightarrow -\mu_1 - \gamma_1 I_1 + u_{1,m_2} = 0 \tag{36}$$

$$\frac{\partial H^{m_2}}{\partial I_2} = 0 \Longrightarrow -\mu_2 - \gamma_2 I_2 + u_{2,m_2} = 0 \tag{37}$$

$$\frac{\partial H^{m_2}}{\partial u_{1,m_2}} = 0 \Longrightarrow I_1 - \delta_1 K_1 = \dot{K_1} \tag{38}$$

$$\frac{\partial H^{m_2}}{\partial u_{2,m_2}} = 0 \Longrightarrow I_2 - \delta_2 K_2 = \dot{K_2} \tag{39}$$

We assume that (I_1, I_2) is a (piecewise) continuously differentiable function. Solving for I_1 and I_2 in (36) and (37) and substituting in (38) and (39) yields:
$I_1 = \frac{1}{\gamma_1}\left(u_{1,m_2} - \mu_1\right) \quad I_2 = \frac{1}{\gamma_2}\left(u_{2,m_2} - \mu_2\right)$

With further substitution, one gets the canonical system

$$\begin{aligned}\dot{K_1} &= -\delta_1 K_1 + \tfrac{1}{\gamma_1}(u_{1,m_2} - \mu_1) \\ \dot{K_2} &= -\delta_2 K_2 + \tfrac{1}{\gamma_2}(u_{2,m_2} - \mu_2) \\ \dot{u}_{1,m_2} &= (r + \delta_1)u_{1,m_2} + 2K_1 + 2\eta K_2 - 1 \\ \dot{u}_{2,m_2} &= (r + \delta_2)u_{2,m_2} + 2\eta K_1 + 2K_2 - (1 + \theta)\end{aligned} \tag{40}$$

Direct calculations show that there is a unique fixed point of this system given by

$$
\begin{aligned}
K_1^{ss} &= \frac{(1-(r+\delta_1)\mu_1)(2+(r+\delta_2)\gamma_2\delta_2) - 2\eta(1+\theta-(r+\delta_2)\mu_2)}{(\delta_2\gamma_2(r+\delta_2)+2)(\delta_1\gamma_1(r+\delta_1)+2) - 4\eta^2} \\
K_2^{ss} &= \frac{(1+\theta-(r+\delta_2)\mu_2)(2+(r+\delta_1)\gamma_1\delta_1) - 2\eta(1-(r+\delta_1)\mu_1)}{(\delta_2\gamma_2(r+\delta_2)+2)(\delta_1\gamma_1(r+\delta_1)+2) - 4\eta^2} \\
u_{1,m_2}^{ss} &= \frac{2K_1^{ss} + 2\eta K_2^{ss} - 1}{r+\delta_1} \\
u_{2,m_2}^{ss} &= \frac{2\eta K_1^{ss} + 2K_2^{ss} - (1+\theta)}{r+\delta_2}.
\end{aligned}
$$

Concerning stability of the steady state, Grass et al. (2008) provide necessary and sufficient conditions that the stable manifold around the steady state has dimension 2, which corresponds to an asymptotically stable steady state in the state-space. Denoting by J the Jacobian of the dynamics at the steady-state and defining K as

$$
K = \begin{vmatrix} \frac{\partial \dot{K}_1}{\partial K_1} & \frac{\partial \dot{K}_1}{\partial u_1} \\ \frac{\partial \dot{u}_1}{\partial K_1} & \frac{\partial \dot{u}_1}{\partial u_1} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{K}_2}{\partial K_2} & \frac{\partial \dot{K}_2}{\partial u_2} \\ \frac{\partial \dot{u}_2}{\partial K_2} & \frac{\partial \dot{u}_2}{\partial u_2} \end{vmatrix} + 2 \begin{vmatrix} \frac{\partial \dot{K}_1}{\partial K_2} & \frac{\partial \dot{K}_1}{\partial u_2} \\ \frac{\partial \dot{u}_1}{\partial K_2} & \frac{\partial \dot{u}_1}{\partial u_2} \end{vmatrix}
$$

the conditions in Grass et al. (2008) (page 345) read

$$
\begin{aligned}
K &< 0 \\
0 &< detJ < (\tfrac{K}{2})^2.
\end{aligned} \tag{41}
$$

For our model we obtain

$$
\begin{aligned}
K &= \begin{vmatrix} -\delta_1 & \frac{1}{\gamma_1} \\ 2 & r+\delta_1 \end{vmatrix} + \begin{vmatrix} -\delta_2 & \frac{1}{\gamma_2} \\ 2 & r+\delta_2 \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 \\ 2\eta & 0 \end{vmatrix} \\
&= \frac{-2}{\gamma_1} + \frac{-2}{\gamma_2} - \delta_1(r+\delta_1) - \delta_2(r+\delta_2)
\end{aligned}
$$

and

$$
\det J = \begin{vmatrix} -\delta_1 & 0 & \frac{1}{\gamma_1} & 0 \\ 0 & -\delta_2 & 0 & \frac{1}{\gamma_2} \\ 2 & 2\eta & r+\delta_1 & 0 \\ 2\eta & 2 & 0 & r+\delta_2 \end{vmatrix} = \left(\frac{2}{\gamma_1} + \delta_1(r+\delta_1)\right)\left(\frac{2}{\gamma_2} + \delta_2(r+\delta_2)\right) - \frac{4\eta^2}{\gamma_1\gamma_2}
$$

Hence, conditions (41) reduce to:

$$
\frac{4\eta^2}{\gamma_1\gamma_2} < \left(\frac{2}{\gamma_1} + \delta_1(r+\delta_1)\right)\left(\frac{2}{\gamma_2} + \delta_2(r+\delta_2)\right)
$$

Taking into account $\eta < 1$ as well as $\delta_i > 0,\ r > 0$ shows that this condition is always satsified, which implies that under our assumption the unique steady state is stable.

A.2 Proof of Proposition 1

The general form of the value function depend on the mode m of the system. The value function is of the form $V(K_1, K_R, m) = V_{(m_1)}$ in mode m_1 (because $K_2 = 0$ in mode m_1) and $V(K_1, K_2, m) = V_{(m_2)}$ in mode m_2 (because K_R is irrelevant in mode m_2).

The Hamilton-Jacobi-Bellman equation can be written as follows in mode m_2:

$$rV_{(m_2)} = \max_{I_1,\, I_2} \Big[K_1(1 - K_1 - \eta K_2) + K_2(1 + \theta - \eta K_1 - K_2) - \mu_1 I_1 - \frac{1}{2}\gamma_1 {I_1}^2 + \frac{\partial V_{(m_2)}}{\partial K_1}(I_1 - \delta_1 K_1) - \mu_2 I_2 - \frac{1}{2}\gamma_2 {I_2}^2 + \frac{\partial V_{(m_2)}}{\partial K_2}(I_2 - \delta_2 K_2) \Big] \tag{42}$$

First order conditions for optimal investment levels are :

$$\begin{aligned} I_1 &= \frac{1}{\gamma_1}\left(\frac{\partial V_{(m_2)}}{\partial K_1} - \mu_1\right) \\ I_2 &= \frac{1}{\gamma_2}\left(\frac{\partial V_{(m_2)}}{\partial K_2} - \mu_2\right) \end{aligned} \tag{43}$$

Relations (42) and (43) work out to yield:

$$rV_{(m_2)} = \left(-\frac{\mu_1}{\gamma_1} - \delta_1 K_1\right)\frac{\partial V_{(m_2)}}{\partial K_1} + \frac{1}{2\gamma_1}\left(\frac{\partial V_{(m_2)}}{\partial K_1}\right)^2 + \left(-\frac{\mu_2}{\gamma_2} - \delta_2 K_2\right)\frac{\partial V_{(m_2)}}{\partial K_2} + \frac{1}{2\gamma_2}\left(\frac{\partial V_{(m_2)}}{\partial K_2}\right)^2 + \frac{1}{2}\frac{\mu_1^2}{\gamma_1} + \frac{1}{2}\frac{\mu_2^2}{\gamma_2} + K_1(1 - K_1 - \eta K_2) + K_2(1 + \theta - K_2 - \eta K_1) \tag{44}$$

Owing to the quadratic form of the objective function and the linear form of the state dynamics, we postulate a linear quadratic form for the value function $V_{(m_2)}$:

$$V_{(m_2)} = aK_1^2 + bK_1 + dK_1K_2 + eK_2^2 + fK_2 + g$$

When this functional form is plugged back into (44), one gets by comparison of coefficients the following system:

$$\begin{cases} ra &= \frac{2}{\gamma_1}a^2 + \frac{1}{2\gamma_2}d^2 - 2a\delta_1 - 1 \\ re &= \frac{2}{\gamma_2}e^2 + \frac{1}{2\gamma_1}d^2 - 2e\delta_2 - 1 \\ rd &= \frac{2}{\gamma_1}ad + \frac{2}{\gamma_2}ed - d(\delta_1 + \delta_2) - 2\eta \\ rb &= \frac{2}{\gamma_1}a(b - \mu_1) + \frac{1}{\gamma_2}d(f - \mu_2) - \delta_1 b + 1 \\ rf &= \frac{2}{\gamma_2}e(f - \mu_2) + \frac{1}{\gamma_1}d(b - \mu_1) - \delta_2 f + \theta + 1 \\ rg &= \frac{1}{2\gamma_1}b^2 + \frac{1}{2\gamma_2}f^2 - \frac{\mu_1}{\gamma_1}b - \frac{\mu_2}{\gamma_2}f + \frac{1}{2\gamma_1}\mu_1^2 + \frac{1}{2\gamma_2}\mu_2^2 \end{cases}$$

A.3 Transformation of Lagrangian and Derivation of Co-State Equations

Using standard control theory the Lagrangian of the optimal control problem to solve is:

Rewriting the objective function we obtain

$$
\begin{aligned}
J &= E\left\{\int_0^T e^{-rt}[K_1p_1 - C_1(I_1) - C_R(I_R))]dt + e^{-rT}V_{(m_2)}(K_1(T), K_2(T))\right\} \\
&= \int_0^\infty e^{-rt}\mathbb{P}(m(t) = m_1)\left[K_1p_1 - C_1(I_1) - C_R I_R)] + \lambda(t)V_{(m_2)}(K_1, 0)\right] dt \\
&= \int_0^\infty e^{-rt}e^{-z}\left[K_1p_1 - C_1(I_1) - C_R(I_R))] + \lambda(t)V_{(m_2)}(K_1, 0)\right]
\end{aligned}
$$

The corresponding current value Hamiltonian has the following expression:

$$
\begin{aligned}
H(K_1, K_R, z, \tilde{u}_1, \tilde{u}_R, \tilde{u}_z) = e^{-z}\Big[&K_1(1-K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R I_R - \frac{1}{2}\gamma_R I_R^2 \\
&+(\alpha I_R + \beta K_R^\psi)V_{(m_2)}(K_1, 0)\Big] + \tilde{u}_1(I_1 - \delta_1 K_1) \\
&+ \tilde{u}_R(I_R - \delta_R K_R) + \tilde{u}_z(\alpha I_R + \beta K_R^\psi)
\end{aligned}
$$

where $\tilde{u}_1$ and $\tilde{u}_R$ are respectively the costate variabes associated with the dynamic constraints on K_1 and K_R, and $\tilde{u}_z$ the costate associated with the auxiliary variable z.

The corresponding Lagrangian reads

$$
\begin{aligned}
L(K_1, K_R, z, \tilde{u}_1, \tilde{u}_R, \tilde{u}_z, \tilde{\omega}_{K_1}, \omega_{I_R}) = e^{-z}\Big[&K_1(1-K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R I_R - \frac{1}{2}\gamma_R I_R^2 \\
+(\alpha I_R + \beta K_R^\psi)V_{(m_2)}(K_1, 0)\Big] &+ \tilde{u}_1(I_1 - \delta_1 K_1) + \tilde{u}_R(I_R - \delta_R K_R) + \tilde{u}_z(\alpha I_R + \beta K_R^\psi) \\
+ \tilde{\omega}_{K_1}K_1 + \tilde{\omega}_{I_R}I_R&
\end{aligned}
$$

and after the transformations $u_i = e^z\tilde{u}_i, i \in \{1, R, z\}$ and $\omega_i = e^z\tilde{\omega}_i, i \in \{K_1, I_R\}$ we obtain the transformed Lagrangian given in (18).

First order conditions imply:

$$
\frac{\partial L}{\partial I_1} = -\mu_1 e^{-z} - \gamma_1 I_1 e^{-z} + \tilde{u}_1 = 0 \tag{45}
$$

$$
\Rightarrow I_1 = \frac{\tilde{u_1} - \mu_1 e^{-z}}{\gamma_1 e^{-z}} = \frac{u_1 - \mu_1}{\gamma_1}
$$

$$
\frac{\partial L}{\partial I_R} = -\mu_R e^{-z} - \gamma_R I_R e^{-z} + \alpha V_{(m_2)}e^{-z} + \tilde{u}_R + \alpha\tilde{u}_z + \tilde{\omega}_{I_R} = 0 \tag{46}
$$

$$
\Rightarrow I_R = \frac{\tilde{u}_R + \alpha\tilde{u}_z + \tilde{\omega}_{I_R} - \mu_R e^{-z} + \alpha V_{(m_2)}e^{-z}}{\gamma_R e^{-z}} = \frac{u_R + \alpha u_z + \omega_{I_R} - \mu_R + \alpha V_{(m_2)}}{\gamma_R}
$$

$$\dot{\tilde{u}}_1 = r\tilde{u}_1 - \frac{\partial L}{\partial K_1}$$
$$\dot{\tilde{u}}_R = r\tilde{u}_R - \frac{\partial L}{\partial K_R}$$
$$\dot{\tilde{u}}_z = r\tilde{u}_z - \frac{\partial L}{\partial z}$$

Note that

$$\begin{aligned}\dot{u_1} &= \dot{\tilde{u}}_1 e^z + \tilde{u}_1 e^z \dot{z} \\ &= \dot{\tilde{u}}_1 e^z + \tilde{u}_1 e^z (\alpha I_R + \beta K_R^{\psi}),\end{aligned}$$

so that we obtain for the first co-state equation

$$\begin{aligned}\dot{u_1} &= (r\tilde{u}_1 - \frac{\partial L}{\partial K_1})e^z + \tilde{u}_1 e^z (\alpha I_R + \beta K_R^{\psi}) \\ &= ru_1 - (1-2K_1) + \delta_1 \tilde{u}_1 e^z - (\alpha I_R + \beta K_R^{\psi})\frac{\partial V_{(m_2)}(K_1,0)}{\partial K_1} - \tilde{\omega}_{K_1} e^z + u_1(\alpha I_R + \beta K_R^{\psi}) \\ &= (r + \delta_1 + \alpha I_R + \beta K_R^{\psi})u_1 - (1-2K_1) - (\alpha I_R + \beta K_R^{\psi})\frac{\partial V_{(m_2)}(K_1,0)}{\partial K_1} - \omega_{K_1}\end{aligned}$$

Likewise,

$$\dot{u_R} = \dot{\tilde{u}}_R e^z + \tilde{u}_R e^z (\alpha I_R + \beta K_R^{\psi}),$$

so that

$$\begin{aligned}\dot{u_R} &= \left(r\tilde{u}_R - \frac{\partial L}{\partial K_R}\right) e^z + u_R(\alpha I_R + \beta K_R^{\psi}) \\ &= ru_R - \beta\psi K_R^{\psi-1} V_{(m_2)}(K_1,0) + \delta_R \tilde{u}_R e^z - \psi\beta K_R^{\psi-1}\tilde{u}_z e^z + u_R(\alpha I_R + \beta K_R^{\psi}) \\ &= (r + \delta_R + \alpha I_R + \beta K_R^{\psi})u_R - \psi\beta K_R^{\psi-1}(V_{(m_2)}(K_1,0) + \ u_z)\end{aligned}$$

And, analogously

$$\begin{aligned}\dot{u_z} &= (r\tilde{u}_z - \frac{\partial L}{\partial z})e^z + u_z(\alpha I_R + \beta K_R^{\psi}) \\ &= ru_z + e^{-z}\left[K_1(1-K_1) - \mu_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R - \frac{1}{2}\gamma_R I_R^2 + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1,0)\right] e^z \\ &\quad + u_z(\alpha I_R + \beta K_R^{\psi}) \\ &= ru_z + K_1(1-K_1) - \mu_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R - \frac{1}{2}\gamma_R I_R^2 + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1,0) \\ &\quad + u_z(\alpha I_R + \beta K_R^{\psi}) \\ &= (r + \alpha I_R + \beta K_R^{\psi})u_z + K_1(1-K_1) - \mu_1 - \frac{1}{2}\gamma_1 I_1^2 - \mu_R - \frac{1}{2}\gamma_R I_R^2 \\ &\quad + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1,0)\end{aligned}$$

A.4 Proof of Proposition 2

In order to characterize the properties of the steady state, we study the Jacobian of the system of state-costate equations in $(K_1, K_R, z, u_1, u_R, u_z)$:

$$
\begin{aligned}
\dot{K}_1 &= -\delta_1 K_1 + I_1 \\
\dot{K}_R &= -\delta_R K_R + I_R \\
\dot{z} &= \alpha I_R + \beta K_R^{\psi} \\
\dot{u}_1 &= (r + \delta_1 + \alpha I_R + \beta K_R^{\psi})u_1 - (1 - 2K_1) - (\alpha I_R + \beta K_R^{\psi})\left(\frac{\partial V_{(m_2)}(K_1, 0)}{\partial K_1}\right) \\
\dot{u}_R &= (r + \delta_R + \alpha I_R + \beta K_R^{\psi})u_R - \beta\psi K_R^{\psi-1}(V_{(m_2)}(K_1, 0) + u_z) \\
\dot{u}_z &= (r + \alpha I_R + \beta K_R^{\psi})u_z + K_1(1 - K_1) - \mu_1 I_1 - \frac{1}{2}\gamma_1 I_1^2 \\
&\quad - \mu_R I_R - \frac{1}{2}\gamma_R I_R^2 + (\alpha I_R + \beta K_R^{\psi})V_{(m_2)}(K_1, 0),
\end{aligned}
$$

where I_1, I_1,and $V_{(m_2)}(K_1, 0)$ are replaced, as needed, by their respective expressions given by (19),(20) and (16). Given that there are three co-state equations in this system standard arguments show that a stable steady state requires three eigenvalues with positive real parts. Furthermore, given that we need two state variables to converge to the steady state (remember that z does not converge in mode m_1) we also need two eigenvalues with negative real parts. In what follows we show that the eigenvalues have these signs if and only if the conditions given in the Proposition hold.

Denote $V_1 = \frac{\partial V_{(m_2)}(K_1,0)}{\partial K_1} = 2aK_1 + b$

$$
\begin{aligned}
Y &= u_R\,(V + u_z)^{-1} + K_R^{-1}(1 - \psi) \\
Z &= (u_1 - V_1) \\
R &= r + \lambda \\
X &= \beta\psi K_R^{\psi-1} \\
F &= K_1(1 - K_1) - \mu_1 I_1 - \tfrac{1}{2}\gamma_1 I_1^2 - \mu_R I_R - \tfrac{1}{2}\gamma_R I_R^2 \\
F_1 &= \tfrac{\partial F}{\partial K_1} = 1 - 2K_1
\end{aligned}
$$

$$
J = \begin{bmatrix}
-\delta_1 & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 \\
\frac{\alpha}{\gamma_R}V_1 & -\delta_R & 0 & 0 & \frac{1}{\gamma_R} & \frac{\alpha}{\gamma_R} \\
\frac{\alpha^2}{\gamma_R}V_1 & X & 0 & 0 & \frac{\alpha}{\gamma_R} & \frac{\alpha^2}{\gamma_R} \\
2 - 2a\lambda + \frac{\alpha^2}{\gamma_R}V_1 Z & XZ & 0 & R + \delta_1 & \frac{\alpha}{\gamma_R}Z & \frac{\alpha^2}{\gamma_R}Z \\
\frac{\alpha^2}{\gamma_R}(u_R - X)V_1 & X\,(V + u_z)\,Y & 0 & 0 & \begin{matrix} R + \delta_R \\ + \frac{\alpha}{\gamma_R}u_R \end{matrix} & \begin{matrix} \frac{\alpha^2}{\gamma_R}u_R \\ -X \end{matrix} \\
\begin{matrix} 1 - 2K_1 \\ + \alpha V_1(\lambda - \frac{u_R}{\gamma_R}) \end{matrix} & X\,(V + u_z) & 0 & -\frac{u_1}{\gamma_1} & -\frac{1}{\gamma_R}u_R & R - \frac{\alpha}{\gamma_R}u_R
\end{bmatrix}
$$

The characteristic polynomial $P(\xi) = |J - \xi \mathbb{1}_6|$, where $\mathbb{1}$ denotes the identity matrix, is the following determinant:

$$P(\xi) = \begin{bmatrix} -\delta_1 - \xi & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 \\ \frac{\alpha}{\gamma_R}V_1 & -\delta_R - \xi & 0 & 0 & \frac{1}{\gamma_R} & \frac{\alpha}{\gamma_R} \\ \frac{\alpha^2}{\gamma_R}V_1 & X & -\xi & 0 & \frac{\alpha}{\gamma_R} & \frac{\alpha^2}{\gamma_R} \\ 2 - 2a\lambda + \frac{\alpha^2}{\gamma_R}V_1 Z & XZ & 0 & R + \delta_1 - \xi & \frac{\alpha}{\gamma_R}Z & \frac{\alpha^2}{\gamma_R}Z \\ \frac{\alpha^2}{\gamma_R}(u_R - X)V_1 & X\left(V + u_z\right)Y & 0 & 0 & R + \delta_R + \frac{\alpha}{\gamma_R}u_R - \xi & \frac{\alpha^2}{\gamma_R}u_R - X \\ 1 - 2K_1 + \alpha V_1(\lambda - \frac{u_R}{\gamma_R}) & X\left(V + u_z\right) & 0 & -\frac{u_1}{\gamma_1} & -\frac{1}{\gamma_R}u_R & R - \frac{\alpha}{\gamma_R}u_R - \xi \end{bmatrix}$$

After subtracting α times the fifth column from the sixth column and given that the third column is all zeros except for $-\xi$, the characteristic polynomial can be rewritten as

$P(\xi) = -\xi(R - \xi)J_1 + \xi(\alpha\xi - R\alpha - \alpha\delta_R - X)J_2$

with

$$J_1 = \begin{Vmatrix} -\delta_1 - \xi & 0 & \frac{1}{\gamma_1} & 0 \\ \frac{\alpha}{\gamma_R}V_1 & -\delta_R - \xi & 0 & \frac{1}{\gamma_R} \\ 2 - 2a\lambda + \frac{\alpha^2}{\gamma_R}V_1 Z & XZ & R + \delta_1 - \xi & \frac{\alpha}{\gamma_R}Z \\ \frac{\alpha^2}{\gamma_R}(u_R - X)V_1 & X\left(V + u_z\right)Y & 0 & R + \delta_R + \frac{\alpha}{\gamma_R}u_R - \xi \end{Vmatrix}$$

and

$$J_2 = \begin{Vmatrix} -\delta_1 - \xi & 0 & \frac{1}{\gamma_1} & 0 \\ \frac{\alpha}{\gamma_R}V_1 & -\delta_R - \xi & 0 & \frac{1}{\gamma_R} \\ 2 - 2a\lambda + \frac{\alpha^2}{\gamma_R}V_1 Z & XZ & R + \delta_1 - \xi & \frac{\alpha}{\gamma_R}Z \\ F_1 + V_1(\lambda - \frac{u_R}{\gamma_R}) & X\left(V + u_z\right) & -\frac{u_1}{\gamma_1} & -\frac{1}{\gamma_R}u_R. \end{Vmatrix}$$

This implies that $\xi = 0$ is a root of the characteristic equation. For the further determination of stability this eigenvalue can be left aside since it corresponds to the artificial state $z(t)$. Recall the following relations that define the steady-state:

$$\begin{aligned} I_1 &= \delta_1 K_1 \\ I_R &= \delta_R K_R \\ F_1 &= (R + \delta_1)u_1 - \lambda V_1 \\ \beta\psi K_R^{\psi-1}(V + u_z) &= (R + \delta_R)u_R \\ -Ru_z &= F + (R - r)V \end{aligned}$$

Then:

$$X\left(V + u_z\right) = (R + \delta_R)u_R \text{ and } F_1 + \lambda V_1 = (R + \delta_1)u_1$$

so that

$$J_2 = \begin{Vmatrix} -\delta_1 - \xi & 0 & \frac{1}{\gamma_1} & 0 \\ \frac{\alpha}{\gamma_R}V_1 & -\delta_R - \xi & 0 & \frac{1}{\gamma_R} \\ 2 - 2a\lambda + \frac{\alpha^2}{\gamma_R}V_1 Z & XZ & R + \delta_1 - \xi & \frac{\alpha}{\gamma_R}Z \\ (R + \delta_1)u_1 - \alpha\frac{u_R}{\gamma_R} & (R + \delta_R)u_R & -\frac{u_1}{\gamma_1} & -\frac{1}{\gamma_R}u_R \end{Vmatrix}$$

After some lengthy calculations one gets:

$J_2 = \frac{1}{\gamma_R}(R-\xi)\left[-u_R(-\delta_1-\xi)(R+\delta_1-\xi)+\frac{1}{\gamma_1}u_R(2-2a\lambda)+\frac{1}{\gamma_1}u_1 Z(\alpha(-\delta_R-\xi)-X)\right]$

Now,

$$J_1 = \left\| \begin{array}{|l|l|l|l|} \hline -\delta_1-\xi & 0 & \frac{1}{\gamma_1} & 0 \\ \hline \frac{\alpha}{\gamma_R}V_1 & -\delta_R-\xi & 0 & \frac{1}{\gamma_R} \\ \hline 2-2a\lambda+\frac{\alpha^2}{\gamma_R}V_1 Z & XZ & R+\delta_1-\xi & \frac{\alpha}{\gamma_R}Z \\ \hline (\frac{\alpha^2}{\gamma_R}u_R - X)V_1 & (R+\delta_R)u_R Y & 0 & R+\delta_R+\frac{\alpha}{\gamma_R}u_R-\xi \\ \hline \end{array} \right\|$$

and after similar calculations one obtains:
$J_1 = (-\delta_1-\xi)(R+\delta_1-\xi)\left[(-\delta_R-\xi)(R+\delta_R+\frac{\alpha}{\gamma_R}u_R-\xi)-\frac{1}{\gamma_R}(R+\delta_R)u_R Y\right]$

$+\frac{1}{\gamma_1}(2-2a(R-r))\left[\frac{1}{\gamma_R}(R+\delta_R)u_R Y-(R+\delta_R+\frac{\alpha}{\gamma_R}u_R-\xi)(-\delta_R-\xi)\right]$

$+\frac{1}{\gamma_1}\frac{1}{\gamma_R}V_1 Z(X-\alpha(-\delta_R-\xi))(\alpha(R+\delta_R-\xi)+X)$

Finally,

$$\begin{aligned} P(\xi) = -\xi(R-\xi)\Bigg[&(-\delta_1-\xi)(R+\delta_1-\xi)\left[(-\delta_R-\xi)(R+\delta_R+\frac{\alpha}{\gamma_R}u_R-\xi)-\frac{1}{\gamma_R}(R+\delta_R)u_R Y\right] \\ &+\frac{1}{\gamma_1}(2-2a\lambda)\left(\frac{1}{\gamma_R}(R+\delta_R)u_R Y-(-\delta_R-\xi)(R+\delta_R+\frac{\alpha}{\gamma_R}u_R-\xi)\right) \\ &+\frac{1}{\gamma_1}\frac{1}{\gamma_R}V_1 Z(X-\alpha(-\delta_R-\xi))(\alpha(R+\delta_R-\xi)+X) \\ &-(\alpha\xi-R\alpha-\alpha\delta_R-X)\frac{1}{\gamma_R}\Big[-u_R(-\delta_1-\xi)(R+\delta_1-\xi) \\ &+\frac{1}{\gamma_1}u_R(2-2a(R-r))+\frac{1}{\gamma_1}u_1 Z(\alpha(-\delta_R-\xi)-X)\Big]\Bigg]. \end{aligned}$$

This implies that one eigenvalue is given by $\xi_5 = R$. The remaining four eigenvalues are given by the roots of the term in the largest brackets, which is a polynomial of degree four in ξ:

$$\xi^4 - 2R\xi^3 + M_2\xi^2 - M_3\xi + Q,$$

where

$$\begin{aligned} M_2 = {}& \alpha^2\frac{1}{\gamma_1}\frac{1}{\gamma_R}(u_1-V_1)^2-\frac{1}{\gamma_1}(2-2a(R-r))+\frac{1}{\gamma_R}(R+\delta_R)u_R(u_R(V+u_z)^{-1}+K^{-1}(1-\psi)) \\ &+R\delta_1+R\delta_R-R^2+\delta_1^2+\delta_R^2+\frac{\alpha}{\gamma_R}u_R(R+2\delta_R)+\beta\psi\frac{u_R}{\gamma_R}K_R^{\psi-1}, \end{aligned}$$

$$\begin{aligned} Q = {}& -\frac{1}{\gamma_1}\frac{1}{\gamma_R}(u_1-V_1)^2(\beta\psi K_R^{\psi-1}+\alpha\delta_R+R\alpha)(\beta\psi K_R^{\psi-1}+\alpha\delta_R) \\ &+\left[\begin{array}{c} \delta_1(R+\delta_1) \\ +\frac{1}{\gamma_1}(2-2a(R-r)) \end{array}\right]\left[\begin{array}{c} \frac{1}{\gamma_R}(R+\delta_R)u_R(u_R(V+u_z)^{-1}+K^{-1}(1-\psi)) \\ +((R+\delta_R)\delta_R+\frac{1}{\gamma_R}u_R(\beta\psi K_R^{\psi-1}+R\alpha+2\alpha\delta_R)) \end{array}\right] \end{aligned}$$

and

$$M_3 = RM_2 - R^3. \tag{47}$$

Denote

$$M = M_2 - R^2, \tag{48}$$

then the quartic equation

$$\xi^4 - 2R\xi^3 + M_2\xi^2 - M_3\xi + Q = 0$$

is equivalent to

$$p^4 + \left(M - \frac{1}{2}R^2\right)p^2 + Q + \frac{1}{16}R^4 - \frac{1}{4}MR^2 = 0$$

by virtue of the change of variable $p = \xi - \frac{R}{2}$ and by taking into account the above relations (47) and (48) among M, M_2 and M_3.

It follows that the four roots of the quartic equation in ξ are

$$\xi_{1,2,3,4} = \frac{R}{2} \pm \frac{1}{2}\sqrt{R^2 - 2M \pm 2\sqrt{M^2 - 4Q}}.$$

Since I_R is constrained to be non-negative, $\xi_5 = R = r + \alpha I_R + \beta K_R^{\psi}$ is positive and real. To obtain a stable steady state, one needs two of the other four eigenvalues to have positive real parts and the remaining two to have negative real parts. Following arguments analagous to those in Dockner (1985) and Dockner and Feichtinger (1991) it is straight-forward to conclude that two of the eigenvalues $\Xi_{1,2,3,4}$ have positive and two have negative real part if and only if

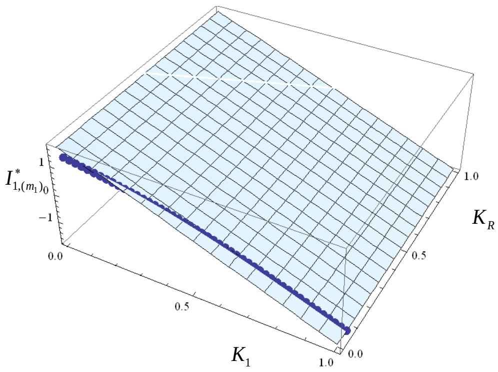
$$M < 0$$

and

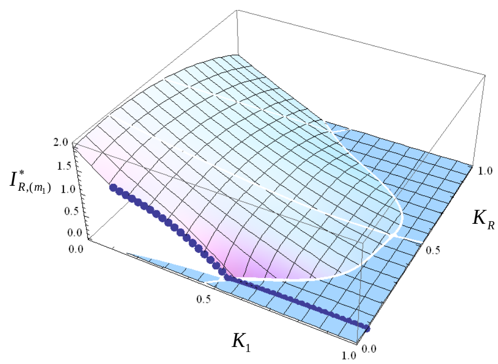
$$0 < Q \leq \frac{M^2}{4}.$$

This concludes the proof.

A.5 The Case With $\mu_R = 0.975$



(a)



(b)

Figure 10: Optimal investment functions in mode m_1 for $\mu_R = 0.975$.

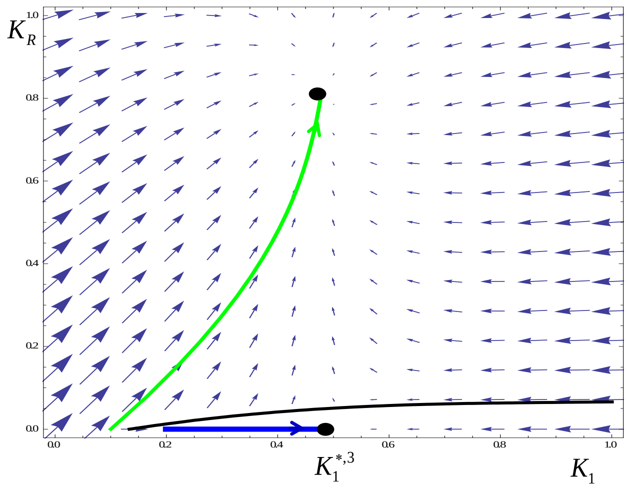


Figure 11: Optimal dynamics in mode m_1 and basins of attraction of the two long-run steady state for $\mu_R = 0.975$ and the initial conditions $(K_1^{ini}, K_R^{ini}) = (0.1, 0)$ (green line) and $(K_1^{ini}, K_R^{ini}) = (0.2, 0)$ (blue line).