ELSEVIER Journal of Pure and Applied Algebra 102 (1995) 207–219

JOURNAL OF PURE AND APPLIED ALGEBRA

H-module endomorphism rings

Fred Van Oystaeyen[a,*], Yinhuo Zhang[b,1]

[a] Department of Mathematics, University of Antwerp UIA, B-2610 Antwerpen, Belgium
[b] Department of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

Communicated by E.M. Friedlander; received 1 March 1993

Abstract

We discuss the endomorphism ring extension of a module over an H-comodule algebra. A necessary and sufficient condition for such an endomorphism ring extension to be Hopf Galois is obtained. The (weak) H-stability for Hopf modules is introduced to study the endomorphism ring extension of a Hopf module. Moreover, the stable Clifford theory for graded rings and modules is extended to the Hopf case.

0. Introduction

For a Hopf algebra H, an algebra extension A/B is called an H-extension if A is an H-comodule algebra, and B is the subalgebra of invariants. If H is the usual group algebra then A is just the group graded algebra and B is the 1-component. The condition that A is strongly graded is equivalent to A/B being H-Galois. In this paper we pay attention to finite-dimensional Hopf algebras H and H-extensions A/B. In [3], Dade introduced the (weak) G-invariance for graded modules to the study of graded endomorphism rings of graded modules. In that case the endomorphism ring is strongly graded if and only if the graded module is weakly G-invariant, and is a crossed product if and only if the graded module is G-invariant. In case the graded ring R is strongly graded, the (weak) G-invariance of an induced graded module $N \otimes_{R_1} R$ may be reduced to the (weak) G-invariance of an R_1-module N. In [10], Schneider introduced the H-stability for a B-module in case A/B is a faithfully flat H-Galois extension, and generalized the theorem [3, 5.14] to the Hopf case, cf. [10, Theorem 3.6]. It is natural to ask which Hopf modules possess an 'H-Galois' endomorphism ring. In Section 2 we solve this problem by introducing the weakly H-stable Hopf modules and H-stable Hopf modules, cf. Corollary 2.7 and Proposition 2.9, extending the graded theory already existing. When A/B is H-Galois and M is

*Corresponding author.
[1] Supported by a grant of the European Liaison Committee.

SSDI 0022-4049(94)00076-U

a right A-module, then $\mathrm{End}_B M$ is an H-module algebra with invariant subalgebra $\mathrm{End}_A M$. Theorem 2.3 states that $\mathrm{End}_A(M \otimes_B A)$ is just the usual smash product $\mathrm{End}_B M \# H$. This allows to form a Morita context

$$\{\mathrm{End}_A(M \otimes_B A), \mathrm{End}_A M, \mathrm{End}_B M, \mathrm{End}_B M. [\,,\,], (\,,\,)\}.$$

From this fact, we derive our main Theorem 2.4 which states that $\mathrm{End}_B M / \mathrm{End}_A M$ is H^*-Galois if and only if $M \otimes_B A$ weakly divides M_A, and $\mathrm{End}_B M$ has trace one elements if and only if M_A divides (or weakly divides) $M \otimes_B A$. Moreover, a Higman criterion for existence of trace one elements is established, cf. Proposition 2.6.

In Section 3, a more extensive criterion for induced Hopf modules possessing an H-Galois endomorphism rings is provided. If we restrict Proposition 3.1 to graded rings then it states that $\mathrm{End}_R(N \otimes_{R_1} R)$ is strongly graded if and only if N is weakly G-invariant. Moreover, the stable Clifford theory for graded rings may be extended to the Hopf case.

1. Preliminaries

Throughout this paper, k is a fixed field, and algebras are defined over k. The index will be omitted where possible. H is always a finite-dimensional Hopf algebra over k with the comultiplication Δ, counit ε and antipode S. For full detail on Hopf algebras we refer to [11], and in particular we use the sigma notation: $\Delta h = \sum h_{(1)} \otimes h_{(2)}$, for $h \in H$.

An algebra A is a *right H-comodule algebra* if A is a right H-comodule module with comodule structure map ρ being an algebra map. Dually, an algebra A is said to be an H-module algebra if A is an H-module satisfying the compatibility condition: $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$, for $h \in H$, $a, b \in A$.

Note that in case H is a finite Hopf algebra, the H-module category coincides with the H^*-comodule category. That is to say, a vector space V is a left H-module if and only if it is a right H^*-comodule. So an H-module algebra is an H^*-comodule algebra, and vice versa.

An algebra extension A/B is called an *H-extension* if A is a right H-comodule algebra and B is its *invariant subalgebra* $A^{\mathrm{Co}H} = \{b \in A \mid \rho(b) = b \otimes 1\}$. By an *$H$-Galois extension A/B*, we mean an H-extension A/B with the bijective map

$$\beta : A \otimes_B A \to A \otimes H, \quad \beta(a \otimes b) = \sum ab_{(0)} \otimes b_{(1)}.$$

Moreover, if the antipode S is bijective, the other canonical map

$$\beta' : A \otimes_B A \to A \otimes H, \quad \beta'(a \otimes b) = \sum a_{(0)} b \otimes a_{(1)},$$

is also bijective, and $\beta' = \Phi\beta$, here $\Phi(a \otimes h) = \rho(a)(1 \otimes S(h))$ with inverse Φ^{-1} given by $\Phi^{-1}(a \otimes h) = (1 \otimes S^{-1}(h))\rho(a)$.

Recall that a right A-module M is a *Hopf module* if M is also a right H-comodule satisfying the compatibility condition:

$$\rho_M(ma) = \sum m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}.$$

Denote by M_A^H the *Hopf module category*, and write $M_A^H(M,N)$ for the set of all *Hopf morphisms* from M to N, which are both right A-morphisms and right H-comodule morphisms. To a Hopf module M, we associate a B-module $M_0 = \{m \in M \mid \rho(m) = m \otimes 1\}$ which is called the *invariant module* of M. It is easy to see that $M_0 \simeq M_A^H(A,M)$. We call the functor $(-)_0$ (or $M_A^H(A,-)$) the *invariant functor* from M_A^H to Mod-B. Note that H is a finite Hopf algebra. We then may identify M_A^H with the module category Mod-$A \# H^*$. Because of this fact, the invariant functor may be regarded as $\mathrm{Hom}_{A\#H^*}(A,-)$.

It is well known that a finite-dimensional Hopf algebra is a Frobenious algebra over k, with the Frobenious isomorphism

$$\Theta : {}_{H^*}H_H \to {}_{H^*}H^*_H.$$

Here, $h^* \rightharpoonup h = \sum \langle h^*, h_{(2)}\rangle h_{(1)}$, and $h^* \leftharpoondown h = \sum \langle h^*_{(1)}, h\rangle h^*_{(2)}$. Set $T = \Theta(1)$ and $t = \Theta^{-1}(\varepsilon)$, then T, resp. t is a left, resp. right, integral of H^*, resp. H. Recall from [5] or [13] that an H-Galois extension A/B is a Frobenious extension with Frobenius system $(\tilde{T}, X_i^t, Y_i^t)$, i.e., for all $a \in A$, $\sum \tilde{T}(aX_i^t)\, Y_i^t = \sum X_i^t(Y_i^t a) = a$, where $\sum X_i^t \otimes Y_i^t = \beta^{-1}(1 \otimes t)$ and $\tilde{T}(a) = T \cdot a$.

An H-extension A/B is said to have *trace one elements* if there exists an element $c \in A$ such that $\tilde{T}(c) = T \cdot c = 1$. An H-Galois extension A/B with trace one elements is called a *strictly Galois extension*, this is equivalent to A/B being a faithfully flat Galois extension, cf. [7].

Consider an H^*-extension A/B, i.e. A is a left H-module algebra since H is a finite Hopf algebra. The smash product algebra $A \# H$ may be defined. The following bimodule structures on the vector space A were defined in [2]:

$${}_{A\#H}A_B, \text{ via } (a\#h \rightharpoonup x = a(h\cdot x),\ x \cdot b = xb, \text{ and}$$
$${}_BA_{A\#H}, \text{ via } b \cdot x = bx,\ x \leftharpoonup a\#h = S^{-1}(h^\lambda)\cdot(xa),$$

where $a, x \in A$, $h \in H$, $h^\lambda = \sum h_{(1)}\langle \lambda, h_{(2)}\rangle$, λ is the distinguished group-like element in H^*, cf. [9], satisfying

$$t'h = \langle \lambda, h\rangle t' \quad \forall h \in H,$$

here t' is a left integral of H.

These bimodules allow to form a Morita context:

$$\{A\#H,\ B, {}_{A\#H}A_{B}, {}_BA_{A\#H}, [\,,\,], (\,,\,)\},$$

together with bimodule maps:

$$[\,,\,] : A \otimes_B A \to A\#H,\ a \otimes b \mapsto at'b,$$

$$(\,,\,) : A \otimes_{A\#H} A \to B,\ a \otimes b \mapsto t'(ab).$$

It is well known that $[\,,\,]$ is surjective if and only if A/B is H^*-Galois, and $(,)$ is surjective if and only if A has trace 1 elements or equivalently A is a projective left (or right) $A \# H$-module.

2. H-module endomorphism rings

In this section we fix an H-Galois extension A/B, i.e. $\beta: A \otimes_B A \to A \otimes H$ is bijective. Denote by $\sum X_i^h \otimes Y_i^h$ the element $\beta^{-1}(1 \otimes h)$ in $A \otimes_B A$. Such elements enjoy the following properties, cf. [10, 3.4].

Lemma 2.1. *Let $b \in B$, $a \in A$, $h \in H$, and t the left integral of H.*

(a) $\sum bX_i^h \otimes Y_i^h = \sum X_i^h \otimes Y_i^h b$, *in particular*, $\sum aX_i^t \otimes Y_i^t = \sum X_i^t \otimes Y_i^t a$,

(b) $\sum a_{(0)} \sum X_i^{a_{(1)}} \otimes Y_i^{a_{(1)}} = 1 \otimes a$,

(c) $\sum X_i^h Y_i^h = \varepsilon(h) 1_A$,

(d) $\sum X_i^h \otimes Y_{i(0)}^h \otimes \sum Y_{i(1)}^h = \sum X_i^{h_{(1)}} \otimes Y_i^{h_{(1)}} \otimes h_{(2)}$,

(e) $\sum X_{i(0)}^h \otimes Y_i^h \otimes X_{i(1)}^h = \sum X_i^{h_{(2)}} \otimes Y_i^{h_{(2)}} \otimes S(h_{(1)})$,

(f) $\sum X_i^{hl} \otimes Y_i^{hl} = \sum X_i^l X_j^h \otimes Y_i^h Y_i^l$,

(g) $\sum X_i^{h_{(1)}} \otimes Y_i^{h_{(1)}} X_j^{h_{(2)}} \otimes Y_j^{h_{(2)}} = \sum X_i^h \otimes 1 \otimes Y_i^h$,

(h) $\sum X_{i(0)}^h Y_{i(0)}^h \otimes X_{i(1)}^h \otimes Y_{i(1)}^h = \sum 1 \otimes S(h_{(1)}) \otimes h_{(2)}$.

When one uses the canonical map β' instead of β, the elements $\sum U_i^h \otimes V_i^h = \beta'^{-1}(1 \otimes h)$ have similar properties. Let M, N and K be in Mod-A. We define a left H-action on the set $\mathrm{Hom}_B(M,N)$ as follows:

$$(h \cdot f)(m) = \sum f(mX_i^{S(h)}) Y_i^{S(h)}$$

for $h \in H$, $m \in M$ and $f \in \mathrm{Hom}_B(M,N)$.

Lemma 2.2. (a) $\mathrm{Hom}_B(M,N)$ *is a left H-module with the invariant module* $\mathrm{Hom}_A(M,N)$.

(b) *For* $g \in \mathrm{Hom}_B(M,N)$, $f \in \mathrm{Hom}_B(N,K)$, $h \in H$,

$$h \cdot (f \circ g) = \sum (h_{(1)} \cdot f) \circ (h_{(2)} \cdot g).$$

(c) $\mathrm{End}_B M / \mathrm{End}_A M$ *is an H^*-extension.*

Proof. Straightforward. □

Recall from [10] that the endomorphism ring $\mathrm{End}_A(N \otimes_B A)$ of an induced Hopf module $M \otimes_B A$ is an H-comodule algebra with comodule structure map ρ, the following composite map:

$$\mathrm{End}_A(N \otimes_B A) \xrightarrow{\mathrm{Hom}(-,\rho)} \mathrm{Hom}_A(N \otimes_B A, N \otimes_B A \otimes H)$$

$$\simeq \mathrm{End}_A(N \otimes_B A) \otimes H.$$

In case A/B is strictly H-Galois, Schneider proved that $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B N$ is an H-crossed product if and only if N_B is H-stable, i.e., $N \otimes_B A \simeq N \otimes H$ as right B-modules and right H-comodules. This is a generalization of [3, 5.14]. However, any A-module M_A is H-stable as a B-module since

$$\Phi: M \otimes_B A \to M \otimes H, \qquad m \otimes a \mapsto \sum m a_{(0)} \otimes a_{(1)},$$

is a right B-module and right H-comodule isomorphism with inverse $\Phi^{-1}(m \otimes h) = \sum m X_i^h \otimes Y_i^h$. Thus $\mathrm{End}_A(M \otimes_B A)$ is certainly an H-crossed product when A/B is a strictly Galois extension. In fact, we may obtain the precise structure for $\mathrm{End}_A(M \otimes_B A)$ only assuming that A/B is a Galois extension.

Theorem 2.3. *Suppose that A/B is H-Galois, M is a right A-module. Then* $\mathrm{End}_A(M \otimes_B A) \cong \mathrm{End}_B M \# H$.

Proof. Define $\eta: \mathrm{End}_B M \# H \to \mathrm{End}_A(M \otimes_B A)$ as follows:

$\eta(f \# h)(m \otimes a) = \sum f(m X_i^h) \otimes Y_i^h a$, which is well defined. We show that η is an algebra isomorphism. In fact, η is the composite isomorphism of the following isomorphisms:

$$\begin{aligned}
\mathrm{End}_B M \otimes H &\simeq \mathrm{End}_B M \otimes H \\
&\simeq \mathrm{Hom}_B(M, M \otimes H) \\
&\simeq \mathrm{Hom}_B(M, M \otimes_B A) \quad (\text{by } \mathrm{Hom}(-, \Phi^{-1})) \\
&\simeq \mathrm{End}_A(M \otimes_B A),
\end{aligned}$$

where the first isomorphism send $f \otimes h$ to $\sum S^{-1}(h_{(1)}) \cdot f \otimes h_{(2)}$, and the second one is the canonical isomorphism. It is easy to see that the forgoing isomorphisms are H-colinear. It follows that $\mathrm{End}_B M = \mathrm{End}_A(M \otimes_B A)_0$, i.e. $\mathrm{End}_A(M \otimes_B A)/\mathrm{End}_B M$ is an H-extension. It remains to be proved that η is an algebra map. Notice that $\beta: A \otimes_B A \to A \otimes H$ is bijective. It induces a bijective map

$$A \otimes_B A \otimes_B A \xrightarrow{(\beta \otimes 1)\,(1 \otimes \beta)} A \otimes H \otimes H.$$

One easily checks

$$\begin{aligned}
&(\beta \otimes 1)(1 \otimes \beta)\left(\sum X_j^{h_{(2)}} X_k^{S(h_{(1)})} \otimes Y_k^{S(h_{(1)})} \otimes Y_j^{h_{(2)}}\right) \\
&\quad = (\beta \otimes 1)(1 \otimes \beta)\left(\sum 1 \otimes X_i^h \otimes Y_i^h\right),
\end{aligned}$$

which entails $\sum 1 \otimes X_i^h \otimes Y_i^h = \sum X_j^{h_{(2)}} X_k^{S(h_{(1)})} \otimes Y_k^{S(h_{(1)})} \otimes Y_j^{h_{(2)}}$.

Now

$$\begin{aligned}
\eta((f \# h)(g \# l))(m \otimes 1) &= \eta(f(h_{(1)} \cdot g) \# h_{(2)} l)(m \otimes 1) \\
&= \sum f(h_{(1)} \cdot g)(m X_i^{h_{(2)} l} \otimes Y_i^{h_{(2)} l}) \\
&= \sum f(g(m X_i^{h_{(2)} l} X_j^{S(h_{(1)})}) Y_j^{S(h_{(1)})}) \otimes Y_i^{h_{(2)} l}
\end{aligned}$$

$$= \sum f(g(mX_i^l X_j^{h_{(2)}} X_k^{S(h_{(1)})}) Y_k^{S(h_{(1)})}) \otimes Y_j^{h_{(2)}} Y_i^l$$
$$= \sum f(g(mX_i^l) X_j^h) \otimes Y_j^h Y_i^l$$
$$= \eta(f \# h) \circ \eta(g \# l)(m \otimes 1). \qquad \square$$

For two modules M_A, N_A, write $M_A < N_A$ to indicate that M_A is a direct summand of some direct sum of a finite number of copies of N_A, i.e. M_A weakly divides N_A. A Hopf algebra H is said to be *unimodular* if there exists a non-zero two-sided integral in H. In this case, all one sided integrals are two sided integrals. For example, if G is a finite group, then kG and kG^* are unimodular.

Theorem 2.4. *Suppose that A/B is H-Galois, and H is unimodular. Let M be a right A-module, then*
(a) $\mathrm{End}_B M/\mathrm{End}_A M$ *is H^*-Galois if and only if* $M \otimes_B A < M$.
(b) $\mathrm{End}_B M$ *has trace* 1 *elements if and only if* $M_A < M \otimes_B A$.
Moreover, if both (a) *and* (b) *hold, i.e.* $M \otimes_B A \sim M_A$, *then* $(\mathrm{End}_B M \otimes_{\mathrm{End}_A M} -, -_{\mathrm{End}_A(M \otimes_B A)} \mathrm{End}_B M)$ *defines a Morita equivalence between* Mod-$\mathrm{End}_A M$ *and* Mod-$\mathrm{End}_A(M \otimes_B A)$.

Proof. Set $F = \mathrm{End}_B M$, and $G = \mathrm{End}_A M$. We identify $E = \mathrm{End}_A(M \otimes_B A)$ with $F \# H$ by Theorem 2.3. Because of Lemma 2.2, ${}_E F_G$ and ${}_G F_E$ are bimodules. Now ${}_E\mathrm{Hom}_A(M, M \otimes_B A)_G$ and ${}_G\mathrm{Hom}_A(M \otimes_B A, M)_E$ are bimodules in a natural way. We prove that δ and σ, defined by

$\delta : {}_E F_G \to {}_E \mathrm{Hom}_A(M, M \otimes_B A)_G$, $\delta(f)(m) = \sum f(mX_i^t) \otimes Y_i^t$, and
$\sigma : {}_G F_E \to {}_G \mathrm{Hom}_A(M \otimes_B A, M)_E$, $\sigma(f)(m \otimes a) = f(m)a$,

are bimodule isomorphism, where t is the non-zero integral. It is easy to see that δ is an (F, G)-homomorphism and σ is a (G, F)-isomorphism. Following [5] δ is bijective. It is enough to show that δ is left H-linear and σ is right H-linear. We calculate, for all $h \in H, f \in F$, $m \in M$, and $a \in A$,

$$\delta(h \cdot f)(m) = \sum h \cdot f(mX_i^t) \otimes Y_i^t$$
$$= \sum f(mX_i^t X_j^{S(h)}) X_j^{S(h)} \otimes Y_i^t$$
$$= \sum f(mX_i^t) X_j^h \otimes Y_j^h Y_i^t$$
$$= h \cdot \delta(f)(m).$$

It follows that $\delta(h \cdot f) = h \cdot \delta(f)$, where the third equality holds because of

$$\sum X_i^t X_j^{S(h)} \otimes Y_j^{S(h)} \otimes Y_i^t = \sum X_i^t \otimes X_j^h \otimes Y_j^h X_i^t$$

in $A \otimes_B A \otimes_B A$. Indeed, the following composite map induced by β is bijective.

$$\bar{\beta} : A \otimes_B A \otimes_B A \xrightarrow{\beta \otimes 1} A \otimes H \otimes_B A \xrightarrow{1 \otimes \tau} A \otimes_B A \otimes H \xrightarrow{\beta \otimes 1} A \otimes H \otimes H,$$

where τ is the twisted map. Now

$$\bar{\beta}(\sum X_i^t X_j^{S(h)} \otimes Y_j^{S(h)} \otimes Y_i^t) = 1 \otimes t \otimes S(h)$$

and

$$\begin{aligned}\bar{\beta}(\sum X_i^t \otimes X_j^h \otimes Y_j^h Y_i^t) &= \sum X_i^t X_{j(0)}^h Y_{j(0)}^h Y_{i(0)}^t \otimes Y_{j(1)}^h Y_{i(1)}^h \otimes X_{j(1)}^h \\ &= \sum X_i^t Y_{i(0)}^t \otimes h_{(2)} Y_{i(1)}^t \otimes S(h_{(1)}) \\ &= \sum 1 \otimes h_{(2)} t \otimes S(h_{(1)}) \\ &= 1 \otimes t \otimes S(h),\end{aligned}$$

where the second equality holds because of Lemma 2.1(h), and the last one follows from the unimodularity of H. For σ, notice that the right H-action on F is defined as $f \cdot h = S^{-1}(h^{\lambda}) \cdot f = S^{-1}(h) \cdot f$ because of $\lambda = \varepsilon$ when H is unimodular. Thus

$$\begin{aligned}\sigma(f \cdot h)(m \otimes a) &= \sigma(S^{-1}(h) \cdot f)(m \otimes a) \\ &= f(m X_i^{S(S^{-1}(h))}) Y_i^{S(S^{-1}(h))} a \\ &= f(m X_i^h) X_i^h a \\ &= (\sigma(f) \cdot h)(m \otimes a).\end{aligned}$$

It follows that $\sigma(f \cdot h) = \sigma(f) \cdot h$, and hence σ is right H-linear.

Next, we show that the canonical maps

$$\begin{aligned}&F \otimes_G F \xrightarrow{\delta \otimes \sigma} \mathrm{Hom}_A(M, M \otimes_B A) \otimes_G \mathrm{Hom}_A(M \otimes_B A, M) \\ &\qquad \xrightarrow{\text{can}'} E = F \# H, \\ &F \otimes_E F \xrightarrow{\sigma \otimes \delta} \mathrm{Hom}_A(M \otimes_B A), M) \otimes_E \mathrm{Hom}_A(M, M \otimes_B A) \\ &\qquad \xrightarrow{\text{can}''} G,\end{aligned}$$

are exactly the maps $[\,,\,]$ and $(\,,\,)$, respectively, in our Morita context

$$\{F \# H, G, {}_{F \# H}F_G, {}_G F_{F \# H}, [\,,\,], (\,,\,)\},$$

where $[f, g] = ftg \in F \# H$, and $(f, g) = t \cdot (f \circ g) \in G$. In fact for all $f, g \in F$,

$$\begin{aligned}\delta(f) \circ \sigma(g)(m \otimes a) &= \delta(f)(g(m)a) \\ &= \delta(f)(g(m))a \\ &= \sum f(g(m) X_i^t) Y_i^t a \\ &= ftg(m \otimes a),\end{aligned}$$

consequently, $\delta(f) \circ \sigma(g) = [f, g] = ftg \in F \# H$. Similarly, $\sigma(f) \circ \sigma(g) = t \cdot (f \circ g) \in G$.

It is well known that *can'* is surjective if and only if $M \otimes_B A < M$, and *can''* is surjective if and only if $M_A < M_A \otimes_B A$. Now (a) and (b) follow both $\delta \otimes \sigma$ and $\sigma \otimes \delta$ being bijective and [2, 1.5] or [13, 2.1]. □

Remark 2.5. From the proof of the foregoing theorem, Theorem 2.4 (b) holds for any finite Hopf algebra (not necessarily unimodular). In fact we have a more extensive Higman Criterion for existence of trace 1 elements.

Proposition 2.6. *Suppose that A/B is an H-Galois extension and M is a right A-module. Then the H-module algebra $\mathrm{End}_B M$ has a trace one element if and only if there exists an induced module $N \otimes_B A$ such that M_A divides $N \otimes_B A$. In this situation, M_A divides $M \otimes_B A$.*

Proof. 'If part': Suppose that $N \otimes_B A$ is an induced module such that M_A divides $N \otimes_B A$. Let T and t be the integrals as in Section 1. Then $(\tilde{T}, X_i^t, Y_i^t)$ is a Frobenius system, i.e. for all $a \in A$,

$$\sum T\cdot(aX_i^t)\, Y_i^t = \sum X_i^t T\cdot(Y_i^t a) = a.$$

Let π be the canonical map

$$N \otimes_B A \to N \otimes_B A, \qquad n \otimes a \mapsto n(T\cdot a) \otimes 1.$$

Then $\pi \in \mathrm{End}_B(N \otimes_B A)$, and

$$\begin{aligned}(t\cdot\pi)(n\otimes a) &= \sum \pi(n \otimes aX_i^t) \otimes Y_i^t \\ &= \sum nT\cdot(aX_i^t) \otimes Y_i^t \\ &= \sum n \otimes T\cdot(aX_i^t)\, Y_i^t \\ &= n \otimes a,\end{aligned}$$

namely, $t\cdot\pi = I_{N\otimes_B A} \in \mathrm{End}_A(N \otimes_B A)$. By assumption, there are $\theta \in \mathrm{Hom}_A(M, N \otimes_B A)$ and $\psi \in \mathrm{Hom}_A(N \otimes_B A, M)$ such that $\psi \circ \theta = I_M \in \mathrm{End}_A M$. It follows that $\phi = \psi \circ \pi \circ \theta$ is the desired trace 1 element in $\mathrm{End}_B M$ since $t\cdot\phi = \sum (t_{(1)}\cdot\psi)\circ(t_{(2)}\cdot\pi)\circ(t_{(3)}\cdot\theta) = \psi\circ(t\cdot\pi)\circ\theta = I_M$ by Lemma 2.2.

'Only if part': Suppose $\phi \in \mathrm{End}_B M$ is a trace 1 element. Define maps $\theta : M \to M \otimes_B A$, $m \mapsto \sum \phi(mX_i^t) \otimes Y_i^t$ and $\psi : M \otimes_B A \to M$, $m \otimes a \mapsto ma$. It is easy to see that both θ and ψ are A-linear, and $\psi \circ \theta = I_M \in \mathrm{End}_A M$. It follows that M divides $M \otimes_B A$. □

Let R be a strongly graded ring of type G, G a finite group, M a right R-module. Then there exists a G-action by automorphisms on $\mathrm{End}_{Re} M$, cf. [4] such that $(\mathrm{End}_{Re} M)^G = \mathrm{End}_R M$. Theorem 2.4 states that $\mathrm{End}_{Re} M/\mathrm{End}_R M$ is Galois if and only if $M \otimes_{Re} R$ weakly divides M_R, and Proposition 2.6 states that the trace map $t = \sum_{\sigma \in G} : \mathrm{End}_{Re} M \to \mathrm{End}_R M$ is surjective if and only if there is R_e-module N such that M_R divides (or weakly divides) the induced R-module $M \otimes_{Re} R$.

Corollary 2.7. *Let H^* be an unimodular Hopf algebra, A/B an H-extension. Let $\mathscr{S} = A \# H^*$, M a Hopf module, i.e. a right $\mathscr{S}$-module.*

(a) $\mathrm{End}_A M/\mathrm{End}_{\mathscr{S}} M$ *is H-Galois if and only if* $M \otimes_A \mathscr{S} < M_{\mathscr{S}}$.

(b) $\mathrm{End}_A M$ *has a trace* 1 *element if and only if there is an induced module* $N \otimes_A \mathscr{S}$ *such that* $M_{\mathscr{S}} < N \otimes_A \mathscr{S}$.

(c) *If* $M_{\mathscr{S}} \sim M \otimes_A \mathscr{S}$ *then* $\mathrm{End}_{\mathscr{S}} M$ *is Morita equivalent to* $\mathrm{End}_A M \# H^*$.

A Hopf module satisfying Corollary 2.7(a) is said to be a *weakly stable Hopf module.*

Example 2.8 (*Dade* [3, 4.6]). Let R be a graded ring of type G, G a finite group, and M a graded R-module. Then $\mathrm{End}_R M$ is strongly graded ring if and only if $M \otimes_R \mathscr{S} < M_{\mathscr{S}}$, $\mathscr{S} = R \# G^*$. However, $M \otimes_R \mathscr{S} \simeq M \otimes G \simeq \oplus_{\lambda \in G} M(\lambda)$ (the direct sum of shifted graded modules of M) as graded modules. So $M \otimes_R \mathscr{S} < M_{\mathscr{S}}$ happens if and only if M is weakly G-invariant.

Consider again an H-extension A/B and smash product $\mathscr{S} = A \# H^*$. For a Hopf module $M_{\mathscr{S}}$, the induced module $M \otimes_A \mathscr{S}$ is a right $\mathscr{S}$-module in the usual way, and a right H-comodule with comodule structure stemming from M. On the other hand, $M \otimes H$ is a right $\mathscr{S}$-module stemming from M, and a right H-comodule module via the structure of H. A Hopf module M is said to be *H-stable* if $M \otimes_A \mathscr{S} \simeq M \otimes H$ as right $\mathscr{S}$-modules and right H-comodules, where the structures are as defined above.

Proposition 2.9. *Suppose that H^* is a unimodular Hopf algebra and A/B is an H-extension. If the Hopf module M is H-stable, then* $\mathrm{End}_A M/\mathrm{End}_{A\#H^*} M$ *is a cleft extension (namely an H-crossed product extension).*

Proof. It is enough to show that $\mathrm{End}_A M/\mathrm{End}_{\mathscr{S}} M$ has the normal basis property i.e. $\mathrm{End}_A M \simeq \mathrm{End}_{\mathscr{S}} M \otimes H$ as left $\mathrm{End}_{\mathscr{S}} M$-modules and right H-comodules (cf. [6] or [1]) since the H-extension in Galois by Corollary 2.7. However,

$$\begin{aligned}\mathrm{End}_A M &\simeq \mathrm{Hom}_{\mathscr{S}}(M, M \otimes_A \mathscr{S})\\ &\simeq \mathrm{Hom}_{\mathscr{S}}(M, M \otimes H)\\ &\simeq \mathrm{Hom}_{\mathscr{S}}(M, M) \otimes H\\ &= \mathrm{End}_{\mathscr{S}} M \otimes H\end{aligned}$$

as left $\mathrm{End}_{\mathscr{S}} M$-modules and right H-comodules since M is H-stable. □

Corrollary 2.10 (Schneider [10, 3.6]). *Suppose that H^* is unimodular and that A/B is a strict H-Galois extension, N is a right H-stable B-module in sense of Schneider, i.e.* $N \otimes_B A \simeq N \otimes H$ *as right B-modules and right H-comodules. Then* $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B N$ *is an H-crossed product extension.*

Proof. It is sufficient to check that $N \otimes_B A$ is H-stable. In fact, suppose that $\phi: N \otimes_B A \simeq N \otimes H$ is a right B-module and a right H-comodule isomorphism. ϕ induces the following right $\mathscr{S}$-module and right H-comodule isomorphism:

$$(N \otimes_B A) \otimes_B A_{\mathscr{S}} \xrightarrow{\phi \otimes 1} (N \otimes H \otimes\ ) \otimes_B A_{\mathscr{S}} \simeq A \otimes_B A_{\mathscr{S}} \otimes H.$$

Since A/B is H-Galois, the following diagram is commutative

$$\begin{tikzcd} \text{Mod-}A \arrow[r, "- \otimes_A \mathscr{S}"] \arrow[d, "\text{restr.}"'] & \text{Mod} - \mathscr{S} \\ \text{Mod-}B \arrow[ur, "- \otimes_B A"] & \end{tikzcd}$$

It follows that $\psi: (N \otimes_B A) \otimes_B A \simeq (N \otimes_B A) \otimes_A \mathscr{S}$ inherit the structure of a right $\mathscr{S}$-module and of a right H-comodule from $N \otimes_B A$. The composite map $(\phi \otimes 1) \circ \psi^{-1}$ is just the desired isomorphism. Consequently, $N \otimes_B A$ is H-stable. Because of the foregoing proposition $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_{\mathscr{S}}(N \otimes_B A)$ is an H-crossed product extension. However, $\mathrm{End}_{\mathscr{S}}(N \otimes_B A) \cong \mathrm{End}_B N$ as rings since A/B is strictly Galois. Therefore, $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B^N$ is an H-crossed product extension. □

Remark 2.11. (1) In the situation of Corollary 2.10, the Hopf module $N \otimes_B A$ is H-stable exactly when N is H-stable as a B-module in the sense of Schneider [10].

(2) In the graded case, the H-stability is nothing but the G-invariance of graded modules. So perhaps the converse of Proposition 2.9 is true.

3. Stable Clifford theory for Hopf Galois extensions

In this section we discuss the endomorphism rings of induced Hopf modules and extend the stable Clifford theory for graded rings to the case of Hopf extensions. In [10] stable modules were introduced in the study of the endomorphism rings of induced modules as H-crossed products. The H-stability coincides with classical G-invariance in the graded case. We introduce the weakly stable modules in order to study when the endomorphism rings of induced Hopf modules are Galois extensions.

Let A/B be an H-extension. A B-module N_B is said to be *weakly H-stable* if $N \otimes_B A < N_B$ as B-modules. In case A has a trace 1 element $c \in C_A(B)$, the centralizer of B in A, then N_B is weakly H-stable if and only if $N \otimes_B A \sim N_B$. In the sequel, H^* is always unimodular.

Proposition 3.1. *Let A/B be an H-extension with trace one elements and N a right B-module. Put $\mathscr{S} = A \# H^*$.*

(a) $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B N$ *is an H-extension.*

(b) *The extension ~~in~~ (a) is H-Galois if and only if $N \otimes_B \mathscr{S} < N \otimes_B A$ as right $\mathscr{S}$-modules.*

Proof. Since A has trace 1 elements, for any B-module N_B the canonical map $N \to (N \otimes_B A)_0$, $n \mapsto n \otimes 1$ is an isomorphism. It follows that

$$\begin{aligned}\mathrm{End}_{\mathscr{S}}(N \otimes_B A) &\simeq \mathrm{Hom}_B(N, \mathrm{Hom}_{\mathscr{S}}(A, N \otimes_B A))\\ &\simeq \mathrm{Hom}_B(N \otimes_B A)_0)\\ &\simeq \mathrm{End}_B N\end{aligned}$$

is a ring isomorphism. By Lemma 2.2 $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B N$ is an H-extension.

(b) Follows from Corollary 2.7. □

Note that if we take $N = B$, Proposition 3.1(b) just states that A/B *is* H-Galois if and only if $\mathscr{S} < A_{\mathscr{S}}$. This exactly means that $A_{\mathscr{S}}$ is a generator.

Corrollary 3.2. *Suppose that A/B is an strict H-Galois extension, N is a right B-module. Then $\mathrm{End}_A(N \otimes_B A)/\mathrm{End}_B N$ is H-Galois if and only if N is weakly stable.*

Proof. By assumption the induction functor $- \otimes_B A$ defines a Morita equivalence between Mod-B and the Hopf module category Mod-$A \# H^*$. For B-modules N_B, M_B, $N < M$ if and only if $N \otimes_B A < M \otimes_B A$ as Hopf modules. Since A/B is H-Galois, we obtain an $A \# H^*$-bimodule isomorphism

$$[\,,\,]: A \otimes_B A \to A \# H^* \qquad [a, b] = aTb,$$

where T is the integral as mentioned in Section 1. Set $\mathscr{S} = A \# H^*$. Then $N \otimes_B A < N_B$ if and only if $(N \otimes_B A) \otimes_B A_{\mathscr{S}} < N \otimes_B A_{\mathscr{S}}$ in Mod-$\mathscr{S}$. However,

$$(N \otimes_B A) \otimes_B A_{\mathscr{S}} \simeq N \otimes_B (A \otimes_B A)_{\mathscr{S}} \simeq N \otimes_B \mathscr{S}.$$

It follows that N is weakly stable if and only if $N \otimes_B \mathscr{S} < N \otimes_B A$ in Mod-$\mathscr{S}$. Now the statement follows from Proposition 3.1. □

Note that when A/B is strictly H-Galois, a Hopf module is weakly stable if and only if its invariant B-module is weakly stable.

Example 3.3. Suppose that R is graded ring of type G. G is a finite group. Let N be a R_e-module. Then $\mathrm{End}_R(N \otimes_{Re} R)$ is strongly graded if and only if $N \otimes_{Re} R \# G^* < N \otimes_{Re} R$ as graded modules. In case R is strongly graded, then this occurs if and only if $N \otimes_{Re} R < N$ as R_e-modules, namely N is weakly G-invariant.

In the sequel we fix $A/B, M, E$ satisfying

(Γa) A/B is strictly H-Galois,

(Γb) M is a weakly H-stable Hopf module,

(Γc) E/E_0, $E = \mathrm{End}_A M$, $E_0 = \mathrm{End}_{\mathscr{S}} M, \mathscr{S} = A \# H^*$, is H-Galois.

In a similar way to [3] we define full subcategories of Mod-A.

$\mathrm{Mod}(A|M_0)$ whose objects are those right A-modules which divides some direct sum of copies of B-module M_0, and

$\mathrm{Mod}(A|\mathit{weak}\, M_0) = \{N \in \mathrm{Mod}\text{-}A \mid N < M_0$ as B-modules$\}$.

Consider E_0 as the regular right E_0-module. In a similar way we define the full additive subcategories $\mathrm{Mod}(E|E_0)$ and $\mathrm{Mod}(E|\mathit{weak}\, M_0)$ of Mod-E, we have

$\mathrm{Mod}(E \mid E_0) = \{N \in \mathrm{Mod}\text{-}E \mid N$ is projective as a right E_0-module$\}$.

$\mathrm{Mod}(E \mid \mathit{weak}\, E_0) = \{N \in \mathrm{Mod}\text{-}E \mid N$ is finitely generated projective right E_0-module$\}$.

We may state the stable Clifford theory for the Hopf module M as a generalization of [3, Theorem 7.4]. We omit the proof which is similar to the one given in [3].

Theorem 3.4. *Suppose that (Γa)–(Γc) hold. Then the restrictions of the additive functor* $-\otimes_E M$ *and* $\mathrm{Hom}_A(M,-)$ *define an equivalence between the additive categories* $\mathrm{Mod}(E \mid \mathit{weak}\, E_0)$ *and* $\mathrm{Mod}(A \mid \mathit{weak}\, M_0)$. *In case* M_0 *is finitely generated B-module, then the restrictions of those functors define an equivalence between the additive categories* $\mathrm{Mod}(E|E_0)$ *and* $\mathrm{Mod}(A|M_0)$.

Corrollary 3.5. *Suppose that A/B is strictly H-Galois and* N_B *is a weakly H-stable B-module, so that* $E = \mathrm{End}_A(N \otimes_B A)$ *is H-Galois. Then the restrictions of the functors* $-\otimes_E (N \otimes_B A)$ *and* $\mathrm{Hom}_A(A,-)$ *define an equivalence between the additive categories* $\mathrm{Mod}(E|\mathit{weak}\, E_0)$ *and* $\mathrm{Mod}(A \mid \mathit{weak}\, N)$. *If N is a finitely generated B-module, then those functors define an equivalence between* $\mathrm{Mod}(E|E_0)$ *and* $\mathrm{Mod}(A|N)$. *Here* $E_0 = \mathrm{End}_B N$.

In the situation of (3.5), if, in addition, N_B is irreducible, then E_0 is a division ring by Schur's Lemma. So $\mathrm{Mod}(A|N) = \{K \in \mathrm{Mod}\text{-}A \mid K$ is N-primary as B-module$\}$, and $\mathrm{Mod}(E \mid E_0) = \mathrm{Mod}\text{-}E$. we obtain the following.

Corrollary 3.6. *With assumptions as above, the functor* $-\otimes_E (N \otimes_B A)$ *and the restriction of the functor* $\mathrm{Hom}_A(N \otimes_B A,-)$ *define an equivalence between the abelian category* Mod-E *and the full additive subcategory* $\mathrm{Mod}(A|N)$ *of* Mod-A *whose objects are those A-modules which are N-primary as B-modules.*

References

[1] J.R. Blattner and S. Montgomery, Crossed product and Galois extensions of Hopf algebras, Pacific J. Math. 137 (1989) 671–711.

[2] M. Cohen, D. Fischmann and S. Montgomery, Hopf Galois extensions, smash products and Morita equivalences, J. Algebra 133 (1990) 351–372.

[3] E.C. Dade, Group-graded rings and modules, Math. Z. 174 (1980) 241–262.

[4] E.C. Dade, The equivalence of various generalizations of group rings and modules, Math. Z. 181 (1982) 335–344.

[5] Y. Doi, Hopf extensions of algebras and Mascke-type theorems, Israel J. Math. 72 (1990) 99–108.

[6] Y. Doi and M. Takeuchi, Cleft comodules and Hopf modules, Comm. Algebra 12 (1984) 1155–1169.

[7] H.K. Kreimer and M. Takeuchi, Hopf Algebras and Galois extensions, Indiana Univ. Math. J. 30 (1981) 675–692.
[8] C. Năstăsescu and F. Van Oystaeyen, Graded Ring Theory (North-Holland Mathematics Library, 1982).
[9] D.E. Radford, The order of the antipode of finite dimensional hopf algebra is finite, Amer. J. Math. 98 (1976) 333–355.
[10] H.-J. Schneider, Hopf Galois extensions, Isreal J. Math. 72 (1990) 196–231.
[11] M.E. Sweedler, Hopf Algebras (Benjamin, New York, 1969).
[12] F. Van Oystaeyen and Y. Zhang, Inductions and coinductions for Hopf extensions, preprint.
[13] Y. Zhang, Hopf Frobenius extensions of algebras, Comm. Algebra 20 (7) (1992) 1907–1915.