Supported approach spaces

**Reference:**
Colebunders E., Lowen Robert.- Supported approach spaces
Quaestiones mathematicae - ISSN 1727-933X - 46:s:[1](2023), p. 161-190
Full text (Publisher's DOI): https://doi.org/10.2989/16073606.2023.2247724
To cite this reference: https://hdl.handle.net/10067/2020110151162165141

# SUPPORTED APPROACH SPACES

E. COLEBUNDERS, R. LOWEN

We dedicate this paper to the memory of our dear friend and colleague Horst Herrlich.


## Abstract

In this paper we work in the category of approach spaces with contractions [14], the objects of which are sets endowed with a numerical *distance* between sets and points. Approach spaces are to be considered a simultaneous generalization of both quasi-metric and topological spaces. Especially the fundamental notion of distance is reminiscent of the closure operator in a topological space and of the point-to-set distance in a quasi-metric space.

The embedding of the category of topological spaces with continuous maps and of quasi-metric spaces with non-expansive maps is extremely nice. Every approach space has both a quasi-metric coreflection as well as a topological coreflection. Different approach spaces though can have the same topological as well as the same quasi-metric coreflection, in other words, in general these coreflections do not determine the approach space.

In this paper we investigate approach spaces for which these coreflections, do determine the approach space. We will call such spaces *supported*. We prove that in the setting of compact approach spaces many examples of supported approach spaces can be found. Thus, compact spaces that are base-regular, which is a weakening of regularity, are always supported.

An important feature of supported approach spaces is the behaviour of contractions. On a supported domain contractivity is characterized by the combination of continuity for the topological coreflection and non-expansiveness for the quasi-metric coreflection. This result implies that a supported approach space actually is the infimum of its quasi-metric and its topological coreflection. In the course of our study we also give several more examples of both supported and non-supported approach spaces.


Keywords: Approach space, uniform approach space, quasi-metric coreflection, topological coreflection, contraction, closed and open expansiveness, compactness, regularity, weak adjointness, supported, split.
Mathematics Subject Classification: 54A05, 54C05, 54C10, 54D30, 54E35, 54E40, 54E99.

## 1. Introduction

In this paper we will work in the category App of approach spaces with contractions [14]. The objects of App are sets $(X, \delta)$ endowed with a numerical *distance* $\delta(x, A)$ between sets and points (see (1) for the exact formulation of the axioms) and a map $f : (X, \delta_X) \to (Y, \delta_Y)$ is a *contraction* if $\forall x \in X,\ \forall A \subseteq X,\ \delta_Y(f(x), f(A)) \leq \delta_X(x, A)$.

For many applications the context of approach spaces with contractions is quite suitable as was recently shown in the context of probability theory [2] and [3], hyperspaces [15], functional analysis [16] or complexity analysis [5] and [6]. A first lax-algebraic description of approach spaces was established by Clementino and Hofmann in [11]. The category there constructed and its isomorphic description App became an important example in the development of monoidal topology [12].

App contains Met, the category of extended metric spaces with non-expansive maps, as well as its non-symmetric counterpart, the category $q$Met of extended quasi-metric spaces with non-expansive maps, as fully embedded subcategories, where for a (quasi-)metric $d$ the associated approach structure is the natural distance $\delta_d(x, A) = \inf_{a \in A} d(x, a)$. Also the category Top of topological spaces with continuous maps is fully embedded in App where the distance $\delta_{\mathcal{T}}(x, A)$ associated to a topology $\mathcal{T}$ takes only two values 0 and $\infty$, depending on whether the point $x$ belongs to the closure of $A$ or not.

As such approach spaces are to be considered a simultaneous generalization of both quasi-metric and topological spaces. Especially the fundamental notion of distance is reminiscent of the closure operator in a topological space and of the point-to-set distance in a quasi-metric space. However, note that whereas in a quasi-metric space the distance from a point $x$ to a subset $A$ is calculated from the distances to the points of $A$, in an arbitrary approach space such a result does not hold.

The embedding of Top and $q$Met in App is moreover extremely nice. Every approach space $(X, \delta)$ has both a quasi-metric coreflection $(X, d_\delta)$ where $d_\delta(x, y) = \delta(x, \{y\})$, see (30), as well as a topological coreflection $(X, \mathcal{T}_\delta)$ with closures denoted by $\mathrm{cl}(A) = \overline{A} = \{x \mid \delta(x, A) = 0\}$ for $A \subseteq X$, see (32).

Different approach spaces though can have the same topological as well as the same quasi-metric coreflection. Hence these coreflections, in general, do not determine the approach space. However in [14] a result (3.2.15) is proved which says that if the approach space $(X, \delta)$ is weakly adjoint [14] and $A \subseteq X$ is relatively compact, then for any $x \in X$ we have $\delta(x, A) = \delta_{d_\delta}(x, \overline{A})$. As an immediate consequence, if $(X, \delta)$ is compact and weakly adjoint the formula $\delta(x, A) = \delta_{d_\delta}(x, \overline{A})$ holds for any $A \subseteq X$ and any $x \in X$. Hence in such a space, via the above formula, the topological and quasi-metric coreflections together do fully determine the approach space.

In this paper we investigate this property in detail and we elucidate which types of spaces satisfy it. We will call such a space $(X, \delta)$ *supported*, meaning that it is completely determined (supported) by its quasi-metric coreflection $(X, d_\delta)$ and its topological coreflection $(X, \mathcal{T}_\delta)$ via the above formula (see 3.1 for the formal definition). We give several characterizations of this property and show that quasi-metric approach spaces as well as topological approach spaces are both supported. We prove that in the setting of compact approach spaces many examples of supported approach spaces can be found. Thus, in the presence of base-regularity (see definition 4.3), which is a weakening of regularity [14], compact spaces are always supported which strengthens the result on weakly adjoint compact spaces from [14] mentioned higher up (see 4.9).

An important feature of supported approach spaces is their description of contractions. On a supported domain contractivity is characterized by the combination of continuity for the topological coreflections and non-expansiveness for the quasi-metric coreflections. We show that this result implies that a supported approach space is the infimum of its quasi-metric and its topological coreflection. We show that in a similar way closed-expansiveness and open-expansiveness are closely related to the respective properties in terms of the quasi-metric and topological coreflections.

We answer the question when a given topology and quasi-metric on the same underlying set, can generate a supported approach space. The result is applied to construct further examples of supported spaces.

Finally stability properties of supportedness are studied. We show that although arbitrary products of compact base-regular approach spaces are supported, supportedness is not stable under the taking of arbitrary products. Neither do arbitrary subspaces preserve supportedness. We show that closed subspaces as well as coproducts do preserve supportedness. With respect to maps we prove that closed-expansive or open-expansive surjective contractions also do preserve supportedness.

## 2. PRELIMINARIES

For more details on concepts and results on approach spaces we refer to [14] or [13]. We recall terminology and basic results that will be needed in this paper.

Usually an extended quasi-pseudometric on a set $X$ is a function $q : X \times X \to [0, \infty]$ which vanishes on the diagonal and satisfies the triangular inequality and if $q$ moreover satisfies symmetry then it is called an extended pseudometric. In this paper all such $q : X \times X \to [0, \infty]$ are allowed to take the value $\infty$ and both distances between two different points can be zero. *Hence, for simplicity in terminology we drop the words "extended" and "pseudo"*, so in this respect our terminology differs from what is commonly used. However, it conforms with the terminology in [14] and [12]. We denote by $\mathsf{qMet}$ the category of all *quasi-metric spaces with non-expansive maps* as morphisms and by $\mathsf{Met}$ the full subcategory of all *metric spaces.*

A *distance* on a set $X$ is a function

$$\delta : X \times 2^X \to [0, \infty] \tag{1}$$

with the following properties:

- (D1) $\delta(x, \{x\}) = 0, \ \forall x \in X,$
- (D2) $\delta(x, \emptyset) = \infty, \ \forall x \in X,$
- (D3) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}, \ \forall x \in X, \ \forall A, B \in 2^X,$
- (D4) $\delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon, \ \forall x \in X, \ \forall A \in 2^X, \ \forall \varepsilon \in [0, \infty],$
  with the enlargement

$$A^{(\varepsilon)} = \{x | \delta(x, A) \leq \varepsilon\}.$$

A pair $(X, \delta)$ consisting of a set $X$ endowed with a distance $\delta$ is called an *approach space.* From (D4) it follows that

$$\forall x \in X, \forall A, B \subseteq X : \delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A). \tag{2}$$

Morphisms between approach spaces are called contractions. A map $f : (X, \delta_X) \to (Y, \delta_Y)$ is a *contraction* if

$$\forall x \in X, \ \forall A \subseteq X, \ \delta_Y(f(x), f(A)) \leq \delta_X(x, A). \tag{3}$$

The category of approach spaces and contractions is denoted by $\mathsf{App}$.

An approach space $X$ has an *approach tower* $\mathsf{t} = (\mathsf{t}_\varepsilon)_{\varepsilon \in [0, \infty[}$ where

$$\mathsf{t}_\varepsilon : 2^X \to 2^X : A \mapsto \mathsf{t}_\varepsilon(A) = A^{(\varepsilon)} \tag{4}$$

is a pretopological closure operator. At level 0 we have a topology. The distance can be recovered from the approach tower by

$$\delta(x, A) = \inf\{\varepsilon \mid x \in A^{(\varepsilon)}\}, \tag{5}$$

for $x \in X$ and $A \subseteq X$. Using suitable axioms for the approach tower and the morphisms, the category $\mathsf{App}$ can be isomorphically described in terms of approach towers.

An approach space also has an *approach system* which is a collection $(\mathcal{A}(x))_{x \in X}$ of ideals where

$$\mathcal{A}(x) = \{\varphi \in [0, \infty]^X \mid \forall A \subseteq X : \inf_{y \in A} \varphi(y) \leq \delta(x, A)\}. \tag{6}$$

The distance can be recovered from this system by

$$\delta(x, A) = \sup_{\varphi \in \mathcal{A}(x)} \inf_{y \in A} \varphi(y) \tag{7}$$

and with the right axioms for the approach system and the morphisms, the category App can be isomorphically described in terms of approach systems.

An approach space $(X, \delta)$ has a *gauge*, namely the collection of quasi-metrics on $X$ given by

$$\mathcal{G} = \{q \mid \text{quasi-metric on } X, \delta_q \leq \delta\}, \tag{8}$$

with

$$\delta_q(x, A) = \inf_{z \in A} q(x, z), \tag{9}$$

whenever $A \subseteq X$ and $x \in X$. The distance can be recovered from the gauge by

$$\delta = \sup_{q \in \mathcal{G}} \delta_q. \tag{10}$$

A subcollection $\mathcal{D} \subseteq \mathcal{G}$ stable for finite suprema is called a *gauge basis* if $\delta = \sup_{q \in \mathcal{D}} \delta_q$. Here too with the right axioms for a gauge and the morphisms, the category App can be isomorphically described in terms of gauges.

Convergence in an approach space $(X, \delta)$ is described by means of a limit operator on filters. For a given filter $\mathcal{F}$ and a point $x \in X$ the value $\lambda\mathcal{F}(x)$ is interpreted as the distance that the point is away from being a limit point of the filter. If $\mathsf{F}X$ is the set of all filters on $X$ and $\beta X$ the set of all ultrafilters on $X$, the *limit operator* is a function

$$\lambda : \mathsf{F}X \to [0, \infty]^X.$$

The transition from the distance to the limit operator is described by

$$\lambda\mathcal{F}(y) = \sup_{U \in \mathcal{U} \in \beta X, \mathcal{F} \subseteq \mathcal{U}} \delta(y, U), \tag{11}$$

for $\mathcal{F} \in \mathsf{F}X$ and $y \in X$. Furthermore, the following formula holds

$$\lambda\mathcal{F} = \sup_{\mathcal{F} \subseteq \mathcal{U}, \mathcal{U} \in \beta X} \lambda\mathcal{U}. \tag{12}$$

Using the following characterization for a map $f : (X, \lambda_X) \to (Y, \lambda_Y)$ to be a contraction iff

$$\lambda_Y f(\mathcal{F})(f(x)) \leq \lambda_X \mathcal{F}(x), \tag{13}$$

for every $\mathcal{F} \in \mathsf{F}X$ and $x \in X$ and with $f(\mathcal{F})$ the filter generated by $\{f(F) \mid F \in \mathcal{F}\}$, with suitable axioms for the limit operator, the category App can be isomorphically described in terms of limit operators. The limit operator is very useful when describing the initial lift of a source $(f_i : X \to (X_i, \lambda_i))_{i \in I}$. The limit operator of the initial lift is given by

$$\lambda\mathcal{F} = \sup_{i \in I} \lambda_i(f_i(\mathcal{F})) \circ f_i. \tag{14}$$

The adherence operator for a filter $\mathcal{F}$ and $x \in X$ can be derived from the value of $\lambda$ on ultrafilters $\mathcal{U} \in \beta(X)$,

$$\alpha\mathcal{F}(x) = \inf_{\mathcal{F} \subseteq \mathcal{U}, \mathcal{U} \in \beta X} \lambda\mathcal{U}(x). \tag{15}$$

The adherence operator and the distance are related by

$$\alpha\mathcal{F}(x) = \sup_{F \in \mathcal{F}} \delta(x, F). \tag{16}$$

The tower and the limit operator are related by

$$\lambda\mathcal{F}(x) \leq \varepsilon \Leftrightarrow \mathcal{F} \to x \text{ in the pretopology } \mathsf{t}_\varepsilon, \tag{17}$$

for all $\mathcal{F} \in \mathsf{F}X$, $x \in X$ and $\varepsilon \in [0, \infty[$. The tower and the adherence operator are related by

$$\alpha\mathcal{F}(x) = \sup_{F \in \mathcal{F}} \inf\{\varepsilon \in \mathbb{R}^+ | x \in \mathfrak{t}_\varepsilon(F)\}. \tag{18}$$

The gauge and the limit operator are related by

$$\lambda\mathcal{F}(x) = \sup_{d \in \mathcal{G}} \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x, y). \tag{19}$$

The gauge and the approach tower are related by

$$\mathcal{G} = \{d \text{ quasi-metric} \mid \forall A \subseteq X, \forall \varepsilon \geq 0 : \mathfrak{t}_\varepsilon(A) \subseteq \{\delta_d(\cdot, A) \leq \varepsilon\}\}. \tag{20}$$

An approach space $X$ is called *uniform* if the gauge $\mathcal{G}$ has a basis consisting of metrics.

The following approach space $\mathbb{P} = ([0, \infty], \delta_{\mathbb{P}})$ is an initially dense object in $\mathsf{App}$. It is defined by

$$\delta_{\mathbb{P}}(x, A) = \begin{cases} x \ominus \sup A & A \neq \emptyset \\ \infty & A = \emptyset, \end{cases} \tag{21}$$

where we use the notation $x \ominus y = (x - y) \vee 0$.

An alternative approach structure on $[0, \infty]$ gives us the space $\mathbb{P}_E = ([0, \infty], \delta_E)$ where

$$\delta_E(x, A) = \begin{cases} 0 & x = \infty, A \text{ unbounded} \\ \infty & x = \infty, A \text{ bounded} \\ \inf_{y \in A} |x - y| & x \in [0, \infty[. \end{cases} \tag{22}$$

For an approach space $(X, \delta)$ the class $\mathfrak{L}_\delta$ of *lower regular* functions, is defined by

$$\mathfrak{L}_\delta = \{f : (X, \delta) \to ([0, \infty], \delta_{\mathbb{P}}) \mid f \text{ contractive}\}. \tag{23}$$

The class is stable for taking suprema and finite infima. The function

$$\delta(\cdot, A) : X \to [0, \infty], \tag{24}$$

for $A \subseteq X$, is a lower regular function and with $\theta_A$ the function which takes the value 0 on $A$ and $\infty$ elsewhere, we have

$$\delta(\cdot, A) = \mathfrak{l}(\theta_A)(\cdot), \tag{25}$$

where $\mathfrak{l}(\theta_A)(\cdot)$ stands for the largest lower regular function below $\theta_A$, called the *lower regular hull*.

The distance can be recovered from the lower regular functions by

$$\delta(x, A) = \sup\{\rho(x) | \rho \in \mathfrak{L}, \rho | A = 0\}, \tag{26}$$

for $x \in X$ and $A \subseteq X$.

Using the following characterisation for a map $f : (X, \mathfrak{L}_X) \to (Y, \mathfrak{L}_Y)$ to be a contraction iff

$$\rho \circ f \in \mathfrak{L}_X, \tag{27}$$

whenever $\rho \in \mathfrak{L}_Y$, and suitable axioms for the lower regular function class, the category $\mathsf{App}$ can be isomorphically described in terms of lower regular functions. The lower regular functions are very useful when describing the final lift of a sink $(f_i : (X_i, \mathfrak{L}_i) \to X)_{i \in I}$. The final lift has lower regular functions

$$\mathfrak{L} = \{\mu \in \mathbb{P}^X \mid \forall i \in I, \mu \circ f_i \in \mathfrak{L}_i\}. \tag{28}$$

The category $\mathsf{App}$ constitutes a framework wherein other important categories can be fully embedded. The embedding of $q\mathsf{Met}$ is given in the usual way that one defines a distance $\delta_q$ between points and sets in a metric space as in (9).

The gauge of $(X, \delta_q)$ is given by

$$\mathcal{G}_{\delta_q} = q\downarrow. \tag{29}$$

qMet is embedded as a concretely coreflective subcategory. The concrete qMet coreflection of a given approach space $X$ with distance $\delta$ is given by the quasi-metric space $(X, d_\delta)$ where

$$d_\delta(x, y) = \delta(x, \{y\}), \tag{30}$$

for $x, y \in X$.

Top is embedded as a full concretely reflective and concretely coreflective subcategory. The embedding of topological spaces is determined by associating with every topological space $(X, \mathcal{T})$ (with closure of $A$ written as cl$A$ or as $\overline{A}$) the distance

$$\delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & x \in \mathrm{cl}A \\ \infty & x \notin \mathrm{cl}A. \end{cases} \tag{31}$$

Every approach space $(X, \delta)$ has two natural topological spaces associated with it, the topological coreflection, and the topological reflection. In this paper we will only deal with the coreflection which is the topological space $(X, \mathcal{T}_\delta)$ determined by the closure

$$x \in \mathrm{cl}A \Leftrightarrow \delta(x, A) = 0 \Leftrightarrow x \in A^{(0)}. \tag{32}$$

$\mathcal{T}_\delta$ coincides with the topology at level 0 of the approach tower. If $(\mathcal{A}(x))_{x \in X}$ is the approach system of the space then the neighborhood system $(\mathcal{V}(x))_{x \in X}$ of the topological coreflection is given by

$$\mathcal{V}(x) = \{V \subseteq X \mid \exists \varepsilon > 0 \text{ and } \varphi \in \mathcal{A}(x) \text{ such that } \{\varphi < \varepsilon\} \subseteq V\}. \tag{33}$$

When $(X, \delta)$ is an approach space notions such as closure, open and closed will always refer to the topological coreflection. For $f : (X, \delta) \to (Y, \delta')$, a map between approach spaces, continuity will always refer to the topological coreflections of $(X, \delta)$ and $(Y, \delta')$. The same holds for properties such as, compact, $T_1$ or $T_2$ when applied to an approach space $(X, \delta)$. What is meant is that the topological space $(X, \mathcal{T}_\delta)$ has the respective property. Other approach properties like regularity are not equivalent with the corresponding property of the topological coreflection, we will recall their definitions in the sequel, when and where they are used.

## 3. Supported approach spaces

In this section we formally introduce the notion of an approach space being supported. As already mentioned in the introduction, if $(X, \delta)$ is compact and weakly adjoint the formula $\delta(x, A) = \delta_{d_\delta}(x, \overline{A})$ holds for any $A \subseteq X$ and any $x \in X$ [14]. Hence in such a space the topological and quasi-metric coreflections together do determine the approach space via the above formula.

In an arbitrary approach space however such a result does not necessarily hold, i.e. the space need not be fully determined by its topological and quasi-metric coreflections.

For example, let $X$ be an infinite set equipped with the approach structure $\delta$ with tower

$$\mathfrak{t}_\varepsilon = \begin{cases} \text{discrete} & \varepsilon \in [0, 1[ \\ \mathcal{T} & \varepsilon \in [1, 2[ \\ \text{indiscrete} & \varepsilon \in [2, \infty[ \end{cases}$$

where $\mathcal{T}$ is any $T_1$ topology on $X$. Then $\mathcal{T}_\delta$ is discrete and $d_\delta$ is two-valued with $d_\delta(x, y) = 2$ if $x \neq y$ irrespective of which $T_1$ topology one considers.

Hence, this gives us an infinite collection of different approach spaces all with the same topological and (quasi-)metric coreflections.

A supported approach space however will be completely determined by its quasi-metric coreflection and its topological coreflection as in the following definition.

3.1. **Definition.** An approach space $(X, \delta)$ with quasi-metric coreflection $(X, d_\delta)$ and topological coreflection $(X, \mathcal{T}_\delta)$ for which for any $A \subseteq X$ and $x \in X$ we have

$$\delta(x, A) = \delta_{d_\delta}(x, \overline{A}),$$

with $\overline{A}$ the closure in $\mathcal{T}_\delta$, will be called *supported*.

Remark that always

$$\delta(x, A) = \delta(x, \overline{A}) \leq \inf_{y \in \overline{A}} \delta(x, \{y\}) = \delta_{d_\delta}(x, \overline{A}). \tag{34}$$

There are several ways in which a supported approach space can be characterized.

3.2. **Proposition.** *The following are equivalent:*

*(1) $(X, \delta)$ is supported.*

*(2) For any $A \subseteq X$ and $\varepsilon \geq 0$ we have $A_\delta^{(\varepsilon)} = \overline{A}_{d_\delta}^{(\varepsilon)}$.*

*(3) For any filter $\mathcal{F}$ we have $\alpha_\delta \mathcal{F} = \alpha_{d_\delta} \overline{\mathcal{F}}$.*

*(4) For any ultrafilter $\mathcal{F}$ we have $\alpha_\delta \mathcal{F} = \alpha_{d_\delta} \overline{\mathcal{F}}$.*

*Proof.* (1) ⇔ (2): This is clear. (1) ⇒ (3): Let $\mathcal{F}$ be a filter on $X$ and $x \in X$. By (18) we have

$$\begin{aligned} \alpha_{d_\delta} \overline{\mathcal{F}}(x) &= \sup_{F \in \mathcal{F}} \inf\{\varepsilon \in \mathbb{R}^+ | x \in \overline{F}_{d_\delta}^{(\varepsilon)}\} \\ &= \sup_{F \in \mathcal{F}} \inf\{\varepsilon \in \mathbb{R}^+ | x \in F_\delta^{(\varepsilon)}\} \\ &= \alpha_\delta \mathcal{F}(x). \end{aligned}$$

(3) ⇔ (4): We only have to prove that (4) implies (3). Let $\mathcal{F}$ be a filter on $X$, then for every ultrafilter $\mathcal{F} \subseteq \mathcal{U}$ we have $\overline{\mathcal{F}} \subseteq \overline{\mathcal{U}}$ and $\alpha_{d_\delta} \overline{\mathcal{F}} \leq \alpha_{d_\delta} \overline{\mathcal{U}}$ and by (15)

$$\alpha_{d_\delta} \overline{\mathcal{F}} \leq \inf_{\mathcal{F} \subseteq \mathcal{U}} \alpha_{d_\delta} \overline{\mathcal{U}} = \inf_{\mathcal{F} \subseteq \mathcal{U}} \alpha_\delta \mathcal{U} = \alpha_\delta \mathcal{F}.$$

(3) ⇒ (1): For any $A \subseteq X$ and $x \in X$ applying (16) we have

$$\delta_{d_\delta}(x, \overline{A}) = \alpha_{d_\delta}[\overline{A}](x) = \alpha_\delta[A](x) = \delta(x, A).$$

□

Next, as expected, we show that topological approach spaces and quasi-metric approach spaces both are supported. We also show that the initially dense object $\mathbb{P}$ in App, (21), is supported. Actually all these spaces are special cases of a particular type of space which we now define. We recall that $\theta_W$ stands for the function which takes the value 0 on $W$ and $\infty$ elsewhere.

In the next definition we use the description of an approach space in terms of the approach system (6).

3.3. **Definition.** We will call an approach space $(X, \delta)$ *split* if $X$ can be written as the disjoint union of two subsets $X_t$ and $X_m$ whereby

(1) for all $x \in X_t$ there exists a filterbasis $\mathcal{W}(x)$ on $X$ such that the collection $\{\theta_W \mid W \in \mathcal{W}(x)\}$ is a basis for the approach system $\mathcal{A}(x)$ in $x$.

(2) for all $x \in X_m$ there exists a function $\varphi_x : X \to [0, \infty]$ such that $\{\varphi_x\}$ is a basis for the approach system $\mathcal{A}(x)$ in $x$.

If necessary, to be precise we shall say that $(X,\delta)$ is split by $(\mathcal{W}(x))_{x\in X_t}$ and $(\varphi_x)_{x\in X_m}$.

The idea of this definition being that a split space consists exactly of a set of points where the approach structure is purely topological in nature (part (1) in 3.3) and a complementary set of points where it is purely metric in nature (part (2) in 3.3).

3.4. **Example.** *Any topological approach space, any quasi-metric approach space and the spaces* $(\mathbb{P},\delta_{\mathbb{P}})$, (21) *and* $(\mathbb{P},\delta_E)$, (22) *are examples of split spaces.*

*(1) In the case of a topological approach space we have that* $X_t = X$ *and* $X_m = \emptyset$.

*(2) In the case of a quasi-metric approach space we have that* $X_t = \emptyset$ *and* $X_m = X$.

*(3) In the cases of* $(\mathbb{P},\delta_{\mathbb{P}})$ *and* $(\mathbb{P},\delta_E)$ *the topological part reduces to* $\{\infty\}$ *and the quasi-metric part is* $[0,\infty[$.

3.5. **Proposition.** *If an approach space* $(X,\delta)$ *is split by* $(\mathcal{W}(x))_{x\in X_t}$ *and* $(\varphi_x)_{x\in X_m}$ *then it is supported. Moreover,* $\mathcal{W}(x)$ *is a basis for the neighborhoodsystem in* $x\in X_t$ *of the topological coreflection* $(X,\mathcal{T}_\delta)$, (31), *and* $\varphi_x = d_\delta(x,\cdot)$ *for* $x\in X_m$, *where* $(X,d_\delta)$ *is the quasi-metric coreflection,* (30).

*Proof.* Let $A\subseteq X$ and $x\in X$.

First suppose $x\in X_t$. The inequality $\delta(x,A)\le \delta_{d_\delta}(x,\overline{A})$ holds in any space (34), hence we only need to show the other inequality. By (7) have

$$\delta(x,A) = \sup_{W\in\mathcal{W}(x)} \inf_{z\in A} \theta_W(z) = \begin{cases} 0 & \forall W\in\mathcal{W}(x) : W\cap A\neq\emptyset \\ \infty & \exists W\in\mathcal{W}(x) : W\cap A=\emptyset \end{cases}$$

Hence it follows from (32) that

$$\delta(x,A)=0 \Rightarrow x\in\overline{A} \Rightarrow \delta_{d_\delta}(x,\overline{A})=0$$

which proves that $\delta_{d_\delta}(x,\overline{A})\le\delta(x,A)$. Furthermore, if $\mathcal{V}(x)$ is the neighborhoodsystem of the topological coreflection, then from (33) clearly $\mathcal{W}(x)\subseteq\mathcal{V}(x)$ and if $V\in\mathcal{V}(x)$ there exists $\varepsilon>0$ and $W\in\mathcal{W}(x)$ such that

$$W=\{\theta_W<\varepsilon\}\subseteq V$$

hence $\mathcal{W}(x)$ is indeed a basis for $\mathcal{V}(x)$.

Second, suppose that $x\in X_m$, then note by (30) that for any $y\in X$

$$d_\delta(x,y)=\delta(x,\{y\})=\varphi_x(y)$$

hence by (7)

$$\delta(x,A)=\delta(x,\overline{A})=\inf_{y\in\overline{A}}\varphi_x(y)=\inf_{y\in\overline{A}}d_\delta(x,y)=\delta_{d_\delta}(x,\overline{A}).$$

□

3.6. **Corollary.** *(1) For a topological space* $(X,\mathcal{T})$ *the approach space* $(X,\delta_{\mathcal{T}})$ *as in* (31) *is supported.*

*(2) For a quasi-metric space* $(X,d)$ *the approach space* $(X,\delta_d)$ *as in* (9) *is supported.*

*(3) The approach spaces* $(\mathbb{P},\delta_{\mathbb{P}})$ *and* $(\mathbb{P},\delta_E)$ *defined in* (21) *and* (22) *are supported.*

3.7. **Proposition.** *If* $(X,\delta)$ *is an approach space such that the topology* $\mathcal{T}_\delta$ *is discrete then the approach space is quasi-metric, meaning* $\delta=\delta_{d_\delta}$ *if and only if it is supported.*

*Proof.* One implication is clear from (2) in 3.6. For the other one observe that for $A \subseteq X$ and $x \in X$ we have

$$\delta(x, A) = \delta_{d_\delta}(x, \overline{A}) = \delta_{d_\delta}(x, A).$$

□

Remark that a characterization as in 3.2 by the limit operator does not hold.

3.8. **Example.** *On the real line $\mathbb{R}$ consider the topology $\mathcal{T} = \mathcal{T}_{d_E}$ for the Euclidean metric $d_E$, and the topological approach space $(\mathbb{R}, \delta_\mathcal{T})$. We know from 3.6 that $(\mathbb{R}, \delta_\mathcal{T})$ is supported. Consider the neighborhoodfilter $\mathcal{V}_\mathcal{T}(0)$ then*

$$\lambda_{\delta_\mathcal{T}} \mathcal{V}_\mathcal{T}(0)(0) = 0.$$

*However for the discrete quasi-metric coreflection $(X, d_{\delta_\mathcal{T}})$ applying (19) we have*

$$\lambda_{d_{\delta_\mathcal{T}}} \overline{\mathcal{V}_\mathcal{T}(0)}(0) = \lambda_{d_{\delta_\mathcal{T}}} \mathcal{V}_\mathcal{T}(0)(0) = \inf_{V \in \mathcal{V}_\mathcal{T}(0)} \sup_{y \in V} d_{\delta_\mathcal{T}}(0, y) = \infty.$$

## 4. Compact spaces

In this section we restrict ourselves to compact approach spaces, in the sense that the topological coreflection is compact. We show that in the presence of a weakening of regularity, namely base-regularity, compact spaces are supported. This also strengthens the result on weakly adjoint compact spaces from [14].

4.1. **Compactness and base-regularity.** Regularity of an approach space is an important notion in approach theory. Its role was demonstrated for instance in [1], [8] and [9]. Its meaning in a monoidal setting was treated in [4], [12] and in [7]. Note that regularity is not equivalent with the topological coreflection being regular.

4.1. **Definition.** A space $(X, \delta)$ is called *regular* if

$$\lambda \mathcal{F}^{(\varepsilon)} \leq \lambda \mathcal{F} + \varepsilon$$

for every $\varepsilon \geq 0$ and with $\mathcal{F}^{(\varepsilon)}$ the filter generated by $\{F^{(\varepsilon)} | F \in \mathcal{F}\}$.

Recall that a quasi-metric approach space $(X, \delta_d)$ is regular if and only if the quasi-metric $d$ is symmetric, i.e. is a metric. In our context in particular in 4.5, a weaker form of regularity will be sufficient.

4.2. **Proposition.** *Let $(X, \delta)$ be an approach space. The following are equivalent:*

*(1) For any filter $\mathcal{F}$ we have $\lambda \overline{\mathcal{F}} = \lambda \mathcal{F}$.*

*(2) For any ultrafilter $\mathcal{U}$ we have $\lambda \overline{\mathcal{U}} = \lambda \mathcal{U}$.*

*Proof.* We only have to prove that (2) implies (1). Let $\mathcal{F}$ be an arbitrary filter on $(X, \delta)$. By proposition 1.1.4 in [14] we have that for every ultrafilter $\overline{\mathcal{F}} \subseteq \mathcal{W}$ there exists an ultrafilter $\mathcal{F} \subseteq \mathcal{U}$ satisfying $\overline{\mathcal{U}} \subseteq \mathcal{W}$. Applying (12) it follows that

$$\lambda \overline{\mathcal{F}} = \sup_{\mathcal{W} \in \beta(X),\ \overline{\mathcal{F}} \subseteq \mathcal{W}} \lambda \mathcal{W} \leq \sup_{\mathcal{U} \in \beta(X),\ \mathcal{F} \subseteq \mathcal{U}} \lambda \overline{\mathcal{U}} = \sup_{\mathcal{U} \in \beta(X),\ \mathcal{F} \subseteq \mathcal{U}} \lambda \mathcal{U} = \lambda \mathcal{F}.$$

□

4.3. **Definition.** A space which satisfies either of the equivalent conditions of the previous proposition will be called *base-regular.*

It is well known that regularity is preserved by initial sources, and analogously we have

4.4. **Proposition.** *Base-regularity is preserved by initial sources.*

*Proof.* For the characterization of contractions with limit operators we refer to (14). Let $(f_i : (X,\lambda) \to (X_i,\lambda_i))_{i\in I}$ be an initial source with $(X_i,\lambda_i)$ base-regular for every $i \in I$ then we have

$$\begin{aligned} \lambda\overline{\mathcal{F}} &= \sup_{i\in I} \lambda_i f_i(\overline{\mathcal{F}}) \circ f_i \\ &\leq \sup_{i\in I} \lambda_i \overline{f_i(\mathcal{F})} \circ f_i \\ &= \sup_{i\in I} \lambda_i f_i(\mathcal{F}) \circ f_i \\ &= \lambda\mathcal{F}. \end{aligned}$$

So the initial lift is base-regular. □

4.5. **Theorem.** *A compact base-regular space is supported.*

*Proof.* Let $(X,\delta)$ be compact and base-regular with gauge $\mathcal{G}$ and let $x \in X$ and $A \subseteq X$. Applying 1.2.67 from [14] there exists an ultrafilter $\mathcal{U}$ on $X$ containing $\overline{A}$ such that

$$\delta(x,A) = \delta(x,\overline{A}) = \alpha_\delta[\overline{A}](x) = \lambda\mathcal{U}(x)$$

where we used (16) for the first equality. In view of the compactness of $(X,\delta)$ there exists $y \in \overline{A}$ such that $\mathcal{U}$ converges to $y$ in the topological coreflection $(X,\mathcal{T}_\delta)$. Then we have

$$\begin{aligned} \delta_{d_\delta}(x,\overline{A}) &\leq \delta_{d_\delta}(x,\{y\}) \\ &= \sup_{d\in\mathcal{G}} d(x,y) \\ &= \sup_{d\in\mathcal{G}} \inf_{U\in\mathcal{U}} d(x,y) \\ &\leq \sup_{d\in\mathcal{G}} \inf_{U\in\mathcal{U}} \sup_{z\in\overline{U}} d(x,z) \\ &= \lambda\overline{\mathcal{U}}(x) \\ &= \lambda\mathcal{U}(x) \\ &= \delta(x,A), \end{aligned}$$

where the equality on line 2 follows from (10) and the inequality on line 4 follows from the fact that $y \in \overline{U}$ for every $U \in \mathcal{U}$ and the equality on line 5 is (19). □

In contrast to the fact that arbitrary products of supported spaces need not be supported, as we will show in section 7, we have that products of compact base-regular spaces are supported, as both compactness and base-regularity are preserved under arbitrary products.

4.6. **Example.** *Let* $[0,1]$ *be endowed with the Euclidean metric* $d_E$. *The product approach space* $X = [0,1]^{[0,1]}$ *is compact and base-regular. Hence* $X$ *is supported. The topological coreflection is the product topology and the metric coreflection is the uniform metric.*

Clearly regularity of an approach space implies base-regularity, so we have

4.7. **Corollary.** *A compact regular approach space is supported.*

Recall from [14], that a quasi-metric space $(X,d)$ and its quasi-metric $d$ are called *weakly adjoint* if $\mathcal{T}_{d^-} \subseteq \mathcal{T}_d$ or equivalently if $\mathcal{T}_{d^*} = \mathcal{T}_d$ with $d^* = d \vee d^-$. This concept was shown to be equivalent to $d(\cdot,x)$ being upper semi-continuous for each $x \in X$ in [10]. An approach space is called *weakly adjoint* if the gauge has a base consisting of weakly adjoint quasi-metrics.

4.8. **Theorem.** *A compact weakly adjoint space is base-regular.*

*Proof.* Suppose $(X,\delta)$ is compact and weakly adjoint and let $\mathcal{U}$ be an ultrafilter on $X$ and $x \in X$. By compactness both filters $\mathcal{U}$ and $\overline{\mathcal{U}}$ are total in the sense of [17] or [18], so we can apply 3.2.14 in [14] for the calculation of their limitfunction in terms of the adherence set in the topological coreflection,

$$\lambda_\delta \overline{\mathcal{U}}(x) = \sup_{y \in adh_{\mathcal{T}_\delta} \overline{\mathcal{U}}} d_\delta(x,y) = \sup_{y \in adh_{\mathcal{T}_\delta} \mathcal{U}} d_\delta(x,y) = \lambda_\delta \mathcal{U}(x).$$

□

By 4.5 and 4.8 we now have

4.9. **Corollary.** *A compact weakly adjoint approach space is supported.*

4.10. **Corollary.** *A compact uniform approach space is supported.*

4.2. **Examples and counterexamples.** Every regular approach space is base-regular and a base-regular approach space has a regular topological coreflection. That the reverse implications do not hold follows from the following examples.

4.11. **Examples.** *(1) A weakly adjoint compact base-regular approach space that is not regular.*

*Let $[0,1]$ be endowed with the weakly adjoint quasi-metric*

$$d(x,y) = \begin{cases} |x-y| & x \le y \\ 2|x-y| & y \le x. \end{cases}$$

*As a quasi-metric space it is supported. By 4.8 the approach space $(X,\delta_d)$ is base-regular. Since the quasi-metric is not symmetric the approach space $(X,\delta_d)$ is not regular.*

*(2) A compact supported approach space with a regular topological coreflection that is neither base-regular nor weakly adjoint.*

*Consider $X = [0,1]$, let $\mathcal{T}$ be the usual topology on $X$ and $\mathcal{S}$ the topology defined by its neighborhoodfilters*

$$\mathcal{V}_\mathcal{S}(x) = \mathcal{V}_\mathcal{T}(x)$$

*if $x \neq 0$ and*

$$\mathcal{V}_\mathcal{S}(0) = \mathcal{V}_\mathcal{T}(0) \cap [G],$$

*with $X \setminus G = \{0\} \cup \{1/n | n \ge 1\}$ and with $[G]$ the filter generated by $G$. We define an approach space $(X,\delta)$ with tower*

$$\mathfrak{t}_\varepsilon = \begin{cases} \mathcal{T} & \varepsilon \in [0,1[ \\ \mathcal{S} & \varepsilon \in [1,\infty[. \end{cases}$$

*Clearly $(X,\delta)$ has a regular topological coreflection. For the filter $[G]$, applying (17) we have $\lambda[G](0) \le 1$ since $[G]$ converges to $0$ in $\mathcal{S}$. Its closure $\overline{[G]} = [X]$ does not converge to $0$ in $\mathcal{S}$ and therefore $\lambda\overline{[G]}(0) \not\le 1$. So $(X,\delta)$ is not base-regular. Since $(X,\delta)$ is compact, by 4.8 it is not weakly adjoint. The space is supported. To see this we only check the case where $x = 0$.*

$$d_\delta(0,y) = \begin{cases} 0 & y = 0 \\ \infty & y \neq 0, y \notin G \\ 1 & y \in G. \end{cases}$$

*For $A \subseteq X$ and $x \in X$ by (5) we now have*

$$\delta(0,A) = \inf\{\varepsilon | 0 \in \mathfrak{t}_\varepsilon(A)\} = \begin{cases} 0 & 0 \in \overline{A} \\ \infty & 0 \notin \overline{A} \ \&\ G \cap A = \emptyset \\ = 1 & 0 \notin \overline{A} \ \&\ G \cap A \neq \emptyset \end{cases}$$

*which clearly equals* $\delta_{d_\delta}(0,\overline{A})$.

In theorem 4.5 base-regularity cannot be replaced by the topological coreflection being regular.

4.12. **Example.** *A compact Hausdorff approach space with a regular topological coreflection that is not supported.*

*Let* $X=[0,1]$ *and let* $\mathcal{T}$ *be the usual topology on* $X$. *An approach structure* $\delta$ *is defined by its tower*

$$\mathfrak{t}_\varepsilon = \begin{cases} \mathcal{T} & \varepsilon\in[0,1[ \\ \textit{cofinite} & \varepsilon\in[1,2[ \\ \textit{indiscrete} & \varepsilon\in[2,\infty[. \end{cases}$$

*Then* $d_\delta$ *is only two-valued with* $d_\delta(x,y)=2$ *if* $x\neq y$. *Let* $A=[0,\frac{1}{2}]$ *then for* $x\notin A$ *we have* $\delta_{d_\delta}(x,A)=2$ *and* $\delta(x,A)=1$, *so* $(X,\delta)$ *is not supported.*

From corollary 4.10 we know that a compact uniform approach space is supported. In this respect compactness cannot be replaced by completeness.

4.13. **Example.** *A complete* $T_2$ *uniform approach space that is not supported.*

*Let* $X=[0,1]$ *be equipped with the approach structure* $\delta$ *with tower*

$$\mathfrak{t}_\varepsilon = \begin{cases} \textit{discrete} & \varepsilon\in[0,1[ \\ \textit{cofinite} & \varepsilon\in[1,2[ \\ \textit{indiscrete} & \varepsilon\in[2,\infty[. \end{cases}$$

*Then* $\mathcal{T}_\delta$ *is discrete and* $d_\delta$ *is only two-valued with* $d_\delta(x,y)=2$ *if* $x\neq y$. *So the approach space is not quasi-metric and by 3.6 (3) it is not supported.*

*Furthermore, an easy verification, making use of* (5), *shows that*

$$\delta(y,B)=\begin{cases} 2 & B \textit{ finite}, y\notin B \\ 1 & B \textit{ infinite}, y\notin B \\ 0 & y\in B. \end{cases}$$

*Next we show that* $(X,\delta)$ *is uniform. Let* $x\in X, A\subseteq X$, $\varepsilon>0$ *and* $\omega<\infty$. *Applying 3.1.7 in* [14] *we construct a contraction* $f:(X,\delta)\to([0,\infty],d_E)$ *satisfying* $f(x)=0$ *and*

$$f|A+\varepsilon\geq\delta(x,A)\wedge\omega.$$

*We may assume* $x\notin A$.

*For* $A$ *infinite we have* $\delta(x,A)=1$ *and we put*

$$f=\begin{cases} 0 & \textit{in } x \\ 1 & \textit{elsewhere.} \end{cases}$$

*To see that this function is a contraction consider* $y\in X$ *and* $B\subseteq X$ *arbitrary such that* $y\notin B$ *then on the one hand, obviously,* $\delta_{d_E}(f(y),f(B))\leq 1$ *and on the other hand* $\delta(y,B)\geq 1$. *Thus always,* $\delta_{d_E}(f(y),f(B))\leq\delta(y,B)$.

*For* $A$ *finite we have* $\delta(x,A)=2$ *and we put*

$$f=\begin{cases} 0 & \textit{in } x \\ 2 & \textit{on } A \\ 1 & \textit{elsewhere.} \end{cases}$$

*To see that this function is a contraction, again consider* $y\in X$ *and* $B\subseteq X$ *arbitrary such that* $y\notin B$ *then* $\delta_{d_E}(f(y),f(B))=2$ *only if* $B\subseteq A$ *and* $y=x$ *and in all other cases the value is less than or equal to* 1. *However, since* $A$ *is finite, also* $B$ *is finite and then* $\delta(y,B)=2$. *Hence again, in all cases* $\delta_{d_E}(f(y),f(B))\leq\delta(y,B)$.

*By construction, both in the case that $A$ is infinite and in the case that it is finite the function constructed satisfies the uniformity condition mentioned higher up.*

*Next we show that the space is complete. Since for a filter $\mathcal{F}$ on $(X, \delta)$ and $y \in X$, the limit function $\lambda\mathcal{F}(y)$ has values in $\{0, 1, 2\}$ the property of being a Cauchy filter, meaning $\inf_{y \in X} \lambda\mathcal{F}(y) = 0$, implies that the filter converges, in the sense that there exists $y \in X$ with $\lambda\mathcal{F}(y) = 0$.*

## 5. Contractions in supported approach spaces

In this section an important feature of supported approach spaces will be made clear. We show that on a supported domain contractivity is characterized by continuity for the topological coreflections and non-expansiveness of the quasi-metric coreflections. This result implies that a supported approach space is the infimum of its quasi-metric and its topological coreflection. From the characterization of contractivity on a supported space, new examples of contractive maps emerge, leading to a characterization of uniform approach spaces. We show that in a similar way closed-expansiveness and open-expansiveness are closely related to the respective property in terms of the quasi-metric and topological coreflections.

### 5.1. **A characterization of contractivity.**

5.1. **Theorem.** *Let $(X, \delta)$ be supported, then the following properties are equivalent:*

*(1) $f : (X, \delta) \to (Y, \delta')$ is a contraction.*

*(2) For the topological and quasi-metric coreflections*

*(i) $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ is continuous*

*(ii) $f : (X, d_\delta) \to (Y, d_{\delta'})$ is non-expansive.*

*Proof.* That (1) implies (2) always holds.
To prove that (2) implies (1), let $x \in X$ and $A \subseteq X$. We have

$$\begin{aligned} \delta'(f(x), f(A)) &= \delta'(f(x), \overline{f(A)}) \\ &\leq \delta'(f(x), f(\overline{A})) \\ &\leq \delta_{d_{\delta'}}(f(x), f(\overline{A})) \\ &\leq \delta_{d_\delta}(x, \overline{A}) \\ &= \delta(x, A), \end{aligned}$$

where the first inequality uses continuity, the second non-expansiveness and the last equality uses the fact that the domain is supported. □

That even in uniform approach spaces this characterisation of contractions is not satisfied follows from our example 4.13

5.2. **Example.** *Take the uniform approach space $(X, \delta)$ in 4.13 of which we already know that it is not supported. Let $f : (X, \delta) \to ([0, 2], \delta_{d_E})$ be the function defined by $f(x) = 2x$.*
*The function $f : (X, d_\delta) \to ([0, 2], d_E)$ is non-expansive and $f : (X, \mathcal{T}_\delta) \to ([0, 2], \mathcal{T}_{d_E})$ is continuous. However*

$$\delta_{d_E}(f(0), f([\frac{2}{3}, 1])) = \frac{4}{3} \not\leq 1 = \delta(0, [\frac{2}{3}, 1])$$

*So $f$ is not contractive.*

5.3. **Corollary.** *Using the notations in (23), when $(X, \delta)$ is supported we have*

$$\mathfrak{L}_\delta = \mathfrak{L}_{d_\delta} \cap \mathfrak{L}_{\mathcal{T}_\delta}$$

*and by (28) this means that $(X, \delta)$ is the infimum of the approach spaces $(X, \delta_{d_\delta})$ and $(X, \delta_{\mathcal{T}_\delta})$.*

*Proof.* By definition we have $\mu \in \mathfrak{L}_{d_\delta}$ if and only if $\mu : (X, d_\delta) \to (\mathbb{P}, d_{\mathbb{P}})$ is non-expansive and $\mu \in \mathfrak{L}_{\mathcal{T}_\delta}$ if and only if $\mu : (X, \mathcal{T}_\delta) \to (\mathbb{P}, \mathcal{T}_{\mathbb{P}})$ is continuous. The rest follows from (5.1). □

In order to obtain a characterization of supportedness in terms of the lower regular function class, we prove the following preliminary result.

5.4. **Proposition.** *Let $X$ be a set endowed with a topology $\mathcal{T}$ and a quasi-metric $d$. For a $\mathcal{T}$-closed subset $A$ the following are equivalent:*

*(1) $\delta_d(\cdot, A) \in \mathfrak{L}_{\mathcal{T}}$.*

*(2) $A^\varepsilon_d$ is $\mathcal{T}$-closed, for all $\varepsilon \geq 0$.*

*Proof.* It is sufficient to observe that for $A$ a $\mathcal{T}$-closed subset and $\varepsilon \geq 0$ we have $[0, \varepsilon]$ is closed in $\mathcal{T}_{\mathbb{P}}$ and

$$A^\varepsilon_d = (\delta_d(\cdot, A))^{-1}[0, \varepsilon].$$

□

Next we give a new characterization of supportedness.

5.5. **Theorem.** *For an approach space $(X, \delta)$ with lower regular function class $\mathfrak{L}_\delta$, topological coreflection $(X, \mathcal{T}_\delta)$ and quasi-metric coreflection $(X, d_\delta)$ the following are equivalent*

*(1) $(X, \delta)$ is supported.*

*(2) $\mathfrak{L}_\delta = \mathfrak{L}_{d_\delta} \cap \mathfrak{L}_{\mathcal{T}_\delta}$ and $\delta_{d_\delta}(\cdot, A) \in \mathfrak{L}_{\mathcal{T}_\delta}$ for every $\mathcal{T}_\delta$-closed subset $A$.*

*(3) $\mathfrak{L}_\delta = \mathfrak{L}_{d_\delta} \cap \mathfrak{L}_{\mathcal{T}_\delta}$ and $A^\varepsilon_{d_\delta}$ is $\mathcal{T}_\delta$-closed for every $\mathcal{T}_\delta$-closed subset $A$ and for every $\varepsilon \geq 0$.*

*Proof.* That (2) and (3) are equivalent follows from 5.4.

Assume $(X, \delta)$ is supported then by 5.3 we have $\mathfrak{L}_\delta = \mathfrak{L}_{d_\delta} \cap \mathfrak{L}_{\mathcal{T}_\delta}$.

Next let $A$ be a $\mathcal{T}_\delta$-closed subset. Since $\delta_{d_\delta}(\cdot, A) = \delta(\cdot, A)$ and $\delta(\cdot, A) \in \mathfrak{L}_\delta$ by (24), we have $\delta_{d_\delta}(\cdot, A) \in \mathfrak{L}_{\mathcal{T}_\delta}$.

Assume that $A$ is a $\mathcal{T}_\delta$-closed subset, $\mathfrak{L}_\delta = \mathfrak{L}_{d_\delta} \cap \mathfrak{L}_{\mathcal{T}_\delta}$ and $\delta_{d_\delta}(\cdot, A) \in \mathfrak{L}_{\mathcal{T}_\delta}$. Applying equation (2) to the quasi-metric $d_\delta$ we have that $\delta_{d_\delta}(\cdot, A) \in \mathfrak{L}_{d_\delta}$. It follows that $\delta_{d_\delta}(\cdot, A) \in \mathfrak{L}_\delta$. Clearly $\delta_{d_\delta}(\cdot, A) \leq \theta_A(\cdot)$. By (25) we have

$$\delta_{d_\delta}(\cdot, A) \leq \mathfrak{l}(\theta_A)(\cdot) = \delta(\cdot, A).$$

Since the other inequality is always true, we can conclude that

$$\delta_{d_\delta}(\cdot, A) = \delta(\cdot, A),$$

which in view of the $\mathcal{T}_\delta$-closedness of $A$ proves supportedness. □

5.2. **A characterization of uniform approach spaces.** We use the notation $\dot{x}$ for the ultrafilter generated by $\{x\}$.

5.6. **Proposition.** *For an approach space $(X, \delta)$ the following conditions are equivalent:*

*(1) The quasi-metric coreflection of $(X, \delta)$ is symmetric*

*(2) $\lambda \dot{x}^{(\varepsilon)} \leq \lambda \dot{x} + \varepsilon$ holds for all $x \in X$ and $\varepsilon \geq 0$.*

*Proof.* (1)$\Rightarrow$ (2) For all $x, y$ we have

$$\begin{aligned}
\lambda\dot{x}^{(\varepsilon)}(y) &= \sup_{d\in\mathcal{G}} \inf_{F\in\dot{x}^{(\varepsilon)}} \sup_{z\in F} d(y,z) \\
&= \sup_{z\in\{x\}^{(\varepsilon)}} \sup_{d\in\mathcal{G}} d(y,z) \\
&= \sup\{d_\delta(y,z) | d_\delta(z,x) \leq \varepsilon\} \\
&\leq \{d_\delta(y,x) + d_\delta(x,z) | d_\delta(z,x) \leq \varepsilon\} \\
&\leq d_\delta(y,x) + \varepsilon \\
&= \lambda\dot{x}(y) + \varepsilon,
\end{aligned}$$

where the first equality applies (19), the third equality applies (10) and the last equality applies (11).

(2) $\Rightarrow$ (1) Since $\lambda\dot{x}^{(\varepsilon)}(x) \leq \varepsilon$ holds for any $x \in X$ and $\varepsilon \geq 0$, we have

$$\sup_{z\in\{x\}^{(\varepsilon)}} d_\delta(x,z) \leq \varepsilon$$

and hence $d_\delta(z,x) \leq \varepsilon$ implies $d_\delta(x,z) \leq \varepsilon$ for any $\varepsilon, x$ and $z$ and thus $d_\delta(x,z) = d_\delta(z,x)$. □

As an application of Theorem 5.1 we have the following result.

5.7. **Proposition.** *Suppose $(X,\delta)$ is supported, weakly adjoint and assume $d_\delta$ is symmetric. If $\mathcal{H}$ is a base of the gauge consisting of weakly adjoint quasi-metrics and $A \subseteq X$ then the function*

$$\delta_d(\cdot, A) : (X,\delta) \to ([0,\infty], \delta_{d_E})$$

*is contractive for every $d \in \mathcal{H}$.*

*Proof.* Let $d \in \mathcal{H}$ and $d^* = d \vee d^{-1}$. For $x, y \in X$ and $A \subseteq X$ by (2) we have

$$\delta_d(x,A) - \delta_d(y,A) \leq d(x,y) \leq d^*(x,y) \leq d_\delta(x,y)$$

which implies

$$|\delta_d(x,A) - \delta_d(y,A)| \leq d^*(x,y) \leq d_\delta(x,y). \tag{35}$$

This means that $\delta_d(\cdot,A) : (X,d^*) \to ([0,\infty], d_E)$ is non-expansive and hence is continuous as a function $\delta_d(\cdot,A) : (X,\mathcal{T}_{d^*}) \to ([0,\infty], \mathcal{T}_{d_E})$. Since we have $\mathcal{T}_{d^*} = \mathcal{T}_d \leq \mathcal{T}_\delta$ we conclude that

$$\delta_d(\cdot,A) : (X,\mathcal{T}_\delta) \to ([0,\infty], \mathcal{T}_{d_E})$$

is continuous. By (35) we also have that

$$\delta_d(\cdot,A) : (X,d_\delta) \to ([0,\infty], d_E)$$

is non-expansive. From Theorem 5.1 the conclusion follows. □

5.8. **Corollary.** *Under the same assumptions as in 5.7, for a fixed point $x \in X$ we have that*

$$d(\cdot,x) : (X,\delta) \to ([0,\infty], \delta_{d_E})$$

*is contractive for every $d \in \mathcal{H}$.*

5.9. **Theorem.** *For supported approach spaces the following are equivalent:*

*(1) $(X,\delta)$ is a uniform approach space.*

*(2) $(X,\delta)$ is weakly adjoint and has a symmetric quasi-metric coreflection.*

*Proof.* That (1) implies (2) is clear. To show the other implication, let $(X,\delta)$ be supported weakly adjoint and assume $d_\delta$ is symmetric. Let $\mathcal{H}$ be a base of the gauge consisting of weakly adjoint quasi-metrics. To show that $(X,\delta)$ is uniform, fix $A \subseteq X$, $x \in X$, $\varepsilon > 0$ and $\omega < \infty$. Choose $d \in \mathcal{H}$ satisfying

$$\delta_d(x,A) + \varepsilon \geq \delta(x,A) \wedge \omega.$$

Now as in 3.1.7 in [14] we consider $f(\cdot) = d(\cdot,x) : (X,\delta) \to ([0,\infty], \delta_{d_E})$ which is contractive by 5.8. Then this function satisfies $f(x) = 0$ and the inequality

$$f|A + \varepsilon \geq \delta_d(x,A) + \varepsilon \geq \delta(x,A) \wedge \omega.$$

□

5.3. **Closed and open expansiveness.** Next we study closed-expansive maps in terms of the associated maps between the topological and quasi-metric coreflections. Recall from [14] that a function $f : (X,\delta) \to (Y,\delta')$ is *closed expansive* if for all $A \subseteq X$ and $y \in Y$

$$\inf_{x \in f^{-1}(y)} \delta(x,A) \leq \delta'(y, f(A)). \tag{36}$$

Equivalently this means that for all $A \subseteq X$ and for all $\alpha \in \mathbb{R}^+$

$$(f(A))^{(\alpha)}_{\delta'} \subseteq \bigcap_{\varepsilon > 0} f(A^{(\alpha+\varepsilon)}_{\delta}). \tag{37}$$

5.10. **Proposition.** *Let $(X,\delta)$ and $(Y,\delta')$ be supported approach spaces, then the condition (i) + (ii)*

*(i) $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ is closed*

*(ii) $f : (X, d_\delta) \to (Y, d_{\delta'})$ is closed expansive*

*implies that $f : (X,\delta) \to (Y,\delta')$ is closed expansive*

*Proof.* Let $A \subseteq X$ and $\alpha \geq 0$.

$$\begin{aligned} (f(A))^{(\alpha)}_{\delta'} &= (\overline{f(A)}^{\mathcal{T}'})^{(\alpha)}_{d_{\delta'}} \\ &\subseteq (f(\overline{A}^{\mathcal{T}}))^{(\alpha)}_{d_{\delta'}} \\ &\subseteq \bigcap_{\varepsilon>0} f((\overline{A}^{\mathcal{T}})^{(\alpha+\varepsilon)}_{d_\delta}) \\ &\subseteq \bigcap_{\varepsilon>0} f((A)^{(\alpha+\varepsilon)}_{\delta}) \end{aligned}$$

where the first line uses supportedness of $(X,\delta')$, the second line uses (i), the third line (ii) and the last line supportedness of $(X,\delta)$. □

5.11. **Proposition.** *Let $(X,\delta)$ and $(Y,\delta')$ be supported approach spaces and assume $(X,\mathcal{T}_\delta)$ is $T_1$, if $f : (X,\delta) \to (Y,\delta')$ is closed expansive then for the quasi-metric coreflections $f : (X, d_\delta) \to (Y, d_{\delta'})$ is closed expansive.*

*Proof.* Let $x, y \in X$ then for $A = \{x\}$, by supportedness of $(X,\delta)$ and $(Y,\delta')$ we have

$$\inf_{z \in f^{-1}(y)} \delta_{d_\delta}(z, \overline{\{x\}}) \leq \delta_{d_{\delta'}}(y, \overline{\{f(x)\}}).$$

Since $(X,\mathcal{T})$ is $T_1$ it follows that

$$\inf_{z \in f^{-1}(y)} d_\delta(z,x) \leq \delta_{d_{\delta'}}(y, \{f(x)\}) = d_{\delta'}(y, f(x)),$$

and by (9) (ii) follows. □

That closed expansiveness of $f : (X, \delta) \to (Y, \delta')$ does not imply closedness of $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ follows from the example 2.4.11 in [14]

5.12. **Example.** *Consider the projection $\mathbb{R}^2 \to \mathbb{R}$, with domain and codomain endowed with the Euclidean metrics. Then the projection is closed expansive, but for the Euclidean topologies it is not a closed map.*

Next we study open-expansive maps in terms of the associated maps between the topological and quasi-metric coreflections. Recall from [14] that a function $f : (X, \delta) \to (Y, \delta')$ is *open expansive* if for all $B \subseteq Y$ and $x \in X$

$$\delta(x, f^{-1}(B)) \leq \delta'(f(x), B). \tag{38}$$

Equivalently this means that for all $B \subseteq Y$ and for all $\varepsilon \in \mathbb{R}^+$

$$f^{-1}(B^{(\varepsilon)}_{\delta'}) \subseteq (f^{-1}(B))^{(\varepsilon)}_{\delta}. \tag{39}$$

5.13. **Proposition.** *Let $(X, \delta)$ and $(Y, \delta')$ be supported approach spaces and assume $(Y, \mathcal{T}_{\delta'})$ is $T_1$ and $f^{-1}(y)$ is closed in $(X, \mathcal{T}_\delta)$ for every $y \in Y$ then the following conditions are equivalent:*

*(1) $f : (X, \delta) \to (Y, \delta')$ is open expansive.*

*(2) For the topological and quasi-metric coreflections*

*(i) $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ is open*

*(ii) $f : (X, d_\delta) \to (Y, d_{\delta'})$ is open expansive.*

*Proof.* (2) $\Rightarrow$ (1): Let $B \subseteq Y$ and $\varepsilon \geq 0$, then we have

$$\begin{aligned} f^{-1}(B^{(\varepsilon)}_{\delta'}) &= f^{-1}((\overline{B})^{(\varepsilon)}_{d_{\delta'}}) \\ &\subseteq (f^{-1}(\overline{B}))^{(\varepsilon)}_{d_\delta} \\ &\subseteq (\overline{f^{-1}(B)})^{(\varepsilon)}_{d_\delta} \\ &= (f^{-1}(B))^{(\varepsilon)}_{\delta} \end{aligned}$$

where on the first line we use supportedness of $(X, \delta')$, on the second line the open expansiveness of $f : (X, d_\delta) \to (Y, d_{\delta'})$, on the third line the openness of $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ and on the last line the supportedness of $(X, \delta)$.

(1) $\Rightarrow$ (2): We first prove (i). Let $B \subseteq Y$ arbitrary. We apply $f^{-1}(B^{(\varepsilon)}_{\delta'}) \subseteq (f^{-1}(B))^{(\varepsilon)}_{\delta}$ to the case $\varepsilon = 0$ and obtain

$$f^{-1}(\overline{B}^{\mathcal{T}_{\delta'}}) \subseteq \overline{f^{-1}(B)}^{\mathcal{T}_\delta}.$$

This means $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ is open.

To prove (ii), let $y \in Y$ then for $B = \{y\}$ we have

$$\begin{aligned} \inf_{z \in f^{-1}(y)} d_\delta(x, z) &= \delta_{d_\delta}(x, f^{-1}(y)) \\ &= \delta_{d_\delta}(x, \overline{f^{-1}(y)}) \\ &= \delta(x, f^{-1}(y)) \\ &\leq \delta'(f(x), \{y\}) \\ &= \delta_{d_{\delta'}}(f(x), \overline{\{y\}}) \\ &= \delta_{d_{\delta'}}(f(x), \{y\}) \\ &= d_{\delta'}(f(x), y) \end{aligned}$$

where on the third line we use supportedness of $(X, \delta)$, on the fourth line the open expansiveness of $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{T}_{\delta'})$ and on the fifth line the supportedness of $(X, \delta')$. Finally by (9) (ii) follows. $\square$

## 6. CONSTRUCTION OF SUPPORTED APPROACH SPACES

An obvious question is whether, given a topology and a quasi-metric on the same set, they can generate a supported approach space. In this section, under a certain condition on the topology and the quasi-metric, we answer this question positively. The supported approach space we obtain is the infimum in App of the given topology and quasi-metric. We apply the results to construct several supported approach spaces.

6.1. **Theorem.** *Let $X$ be a set with topology $\mathcal{T}$ and quasi-metric $d$. Define*

$$\delta : X \times 2^X \to \mathbb{P} : (x, A) \mapsto \delta_d(x, \overline{A}) \tag{40}$$

*where the overline stands for closure in $\mathcal{T}$. Then the following are equivalent:*

*(1) $(X, \delta)$ is a supported approach space with $d_\delta \leq d$ and $\mathcal{T}_\delta \subseteq \mathcal{T}$.*

*(2) For any $\mathcal{T}$-closed set $A$ and any $\varepsilon \geq 0$ the set $A_d^{(\varepsilon)}$ is $\mathcal{T}$-closed.*

*Proof.* (2) $\Rightarrow$ (1) It follows at once from the definition that (D1), (D2) and (D3) from (1) are fulfilled. In order to verify (D4), first note that for any $A$ and $\varepsilon \geq 0$ we have $A_\delta^{(\varepsilon)} = \overline{A}_d^{(\varepsilon)}$. It then follows that for any $A$ and $\varepsilon \geq 0$

$$\begin{aligned}\delta(x, A) &= \delta_d(x, \overline{A}) \\ &\leq \delta_d(x, \overline{A}_d^{(\varepsilon)}) + \varepsilon \\ &= \delta_d(x, \overline{\overline{A}_d^{(\varepsilon)}}) + \varepsilon \\ &= \delta(x, \overline{A}_d^{(\varepsilon)}) + \varepsilon \\ &= \delta(x, A_\delta^{(\varepsilon)}) + \varepsilon\end{aligned}$$

where the second line applies (D4) to $(X, \delta_d)$, the third line uses the assumption (2) and the fourth line is the definition of $\delta$. It follows that $\delta$ is indeed a distance. To see that $d_\delta \leq d$, for any $x, y$ apply (30), then

$$d_\delta(x, y) = \delta(x, \{y\}) = \delta_d(x, \overline{\{y\}}) \leq d(x, y).$$

In order to show that $\mathcal{T}_\delta \subseteq \mathcal{T}$ first note that

$$\delta(x, \overline{A}) = \delta_d(x, \overline{\overline{A}}) = \delta_d(x, \overline{A}) = \delta(x, A)$$

and hence

$$x \in \overline{A} \Rightarrow \delta(x, \overline{A}) = 0 \Rightarrow \delta(x, A) = 0 \Leftrightarrow x \in \mathrm{cl}_{\mathcal{T}_\delta} A.$$

Finally we show that $(X, \delta)$ is supported. This follows from

$$\delta_{d_\delta}(x, \mathrm{cl}_{\mathcal{T}_\delta} A) \leq \delta_{d_\delta}(x, \overline{A}) \leq \delta_d(x, \overline{A}) = \delta(x, A).$$

Since the other inequality always holds we are done.

(1) $\Rightarrow$ (2) Assume $(X, \delta)$ defined by (40) is a supported approach space with $d_\delta \leq d$ and $\mathcal{T}_\delta \subseteq \mathcal{T}$. For any $A$ and $\varepsilon \geq 0$, $A_\delta^{(\varepsilon)}$ is $\mathcal{T}$-closed as

$$A_\delta^{(\varepsilon)} = \delta^{-1}(\cdot, A)[0, \varepsilon]$$

and $\delta(\cdot, A) \in \mathfrak{L}_\delta$ by (24), which implies $\delta(\cdot, A) \in \mathfrak{L}_{\mathcal{T}_\delta} \subseteq \mathfrak{L}_\mathcal{T}$. From (40) it follows that $A_\delta^{(\varepsilon)} = \overline{A}_d^{(\varepsilon)}$ and consequently if $A$ is $\mathcal{T}$-closed also $A_d^{(\varepsilon)}$ is $\mathcal{T}$-closed.

□

6.2. **Proposition.** *Let $X$ be a set with topology $\mathcal{T}$ and quasi-metric $d$ and assume that for any $\mathcal{T}$-closed set $A$ and any $\varepsilon \geq 0$ the set $A_d^{(\varepsilon)}$ is $\mathcal{T}$-closed. Let $(X, \delta)$ be defined as in 6.1. Then we have:*

*(1) If $\mathcal{T}$ is a $T_1$ topological space, then $d_\delta = d$.*

*(2) If $\mathcal{T} \subseteq \mathcal{T}_d$, then $\mathcal{T}_\delta = \mathcal{T}$.*

*Proof.* First assume $\mathcal{T}$ is $T_1$. For any $x, y$ we have

$$d_\delta(x,y) = \delta(x,\{y\}) = \delta_d(x,\overline{\{y\}}) = \delta_d(x,\{y\}) = d(x,y),$$

which proves (1). Next assume $\mathcal{T} \subseteq \mathcal{T}_d$. For any $x$ and $A$ we apply (31) and have

$$\begin{aligned} x \in \mathrm{cl}_{\mathcal{T}_\delta} A &\Leftrightarrow \delta(x,A) = 0 \\ &\Leftrightarrow \delta_d(x,\overline{A}) = 0 \\ &\Leftrightarrow x \in \mathrm{cl}_{\mathcal{T}_d} \overline{A} \\ &\Rightarrow x \in \mathrm{cl}_{\mathcal{T}} \overline{A} \\ &\Rightarrow x \in \overline{A}, \end{aligned}$$

which proves $\mathcal{T} \subseteq \mathcal{T}_\delta$. Since the other inclusion was shown in 6.1 the equality holds. □

6.3. **Theorem.** *Let $X$ be a set with topology $\mathcal{T}$ and quasi-metric $d$ and assume that for any $\mathcal{T}$-closed set $A$ and any $\varepsilon \geq 0$ the set $A_d^{(\varepsilon)}$ is $\mathcal{T}$-closed. Then the approach space defined in 6.1 by*

$$\delta(x,A) = \delta_d(x,\overline{A})$$

*for $x \in X$, $A \subseteq X$ and $\overline{A}$ the closure in $\mathcal{T}$, is the infimum of $(X,\delta_d)$ and $(X,\delta_{\mathcal{T}})$, which by* (28) *means that*

$$\mathfrak{L}_\delta = \mathfrak{L}_d \cap \mathfrak{L}_{\mathcal{T}}.$$

*Proof.* First observe that $\mathfrak{L} = \mathfrak{L}_d \cap \mathfrak{L}_{\mathcal{T}}$ is stable for taking arbitrary suprema, finite infima and translations since both $\mathfrak{L}_d$ and $\mathfrak{L}_{\mathcal{T}}$ are so. Hence $\mathfrak{L}$ satisfies the axioms of a lower regular function class [14]. Let $(X,\delta')$ be the associated approach space described by (26).

We claim that

$$\mathcal{T}_{\delta'} \subseteq \mathcal{T}. \tag{41}$$

In order to see this observe that by (26)

$$\begin{aligned} x \in \overline{A} \quad &\Rightarrow \quad \mu(x) = 0, \forall \mu \in \mathfrak{L}_{\mathcal{T}} \text{ with } \mu|A = 0 \\ &\Rightarrow \quad \mu(x) = 0, \forall \mu \in \mathfrak{L} \text{ with } \mu|A = 0 \\ &\Leftrightarrow \quad \delta'(x,A) = 0 \\ &\Leftrightarrow \quad x \in \mathrm{cl}_{\mathcal{T}_{\delta'}} A. \end{aligned}$$

Next we prove that $(X,\delta)$ and $(X,\delta')$ coincide. Let $A \subseteq X$ and consider the function $\delta_d(\cdot,\overline{A})$. By 5.4 we have $\delta_d(\cdot,\overline{A}) \in \mathfrak{L}_{\mathcal{T}}$. Moreover by (2) we also have $\delta_d(\cdot,\overline{A}) \in \mathfrak{L}_d$, and hence $\delta_d(\cdot,\overline{A}) \in \mathfrak{L} = \mathfrak{L}_{\delta'}$. Since the function $\delta_d(\cdot,\overline{A})$ is 0 on A, for $x \in X$ we have

$$\delta'(x,A) = \sup\{\mu(x) | \mu \in \mathfrak{L}_{\delta'}, \mu|A = 0\} \geq \delta_d(x,\overline{A}).$$

For the other inequality observe that by (41) and applying $\mathfrak{L} \subseteq \mathfrak{L}_d$

$$\begin{aligned} \delta'(x,A) &= \delta'(x,\mathrm{cl}_{\mathcal{T}_{\delta'}} A) \\ &\leq \delta'(x,\overline{A}) \\ &= \sup\{\mu(x) | \mu \in \mathfrak{L}, \mu|\overline{A} = 0\} \\ &\leq \sup\{\mu(x) | \mu \in \mathfrak{L}_d, \mu|\overline{A} = 0\} \\ &= \delta_d(x,\overline{A}). \end{aligned}$$

Finally we can conclude that $\delta'(x,A) = \delta(x,A)$. □

Next we construct several supported approach spaces on the basis of Theorem 6.1 and Proposition 6.2.

6.4. **Examples.** *(1) With regard to proposition 6.2, we give an example showing that without the $T_1$ property, $d_\delta$ need not be equal to $d$.*
*Let $X = [0,\infty], d = d_E$ and let $\mathcal{T}$ be the right order topology. Clearly $\mathcal{T} \subseteq \mathcal{T}_d$. For $A$ a $\mathcal{T}$-closed set, i.e. $[0,a], \{0\}$ or $[0,\infty]$ the set $A^{(\varepsilon)}_{d_E}$ has the form $[0,a+\varepsilon], [0,\varepsilon], [0,\infty]$ respectively and hence is $\mathcal{T}$-closed.*
*Consequently $(X,\delta)$ defined by*

$$\delta(x,A) = \delta_{d_E}(x,\overline{A})$$

*is a supported approach space with $\mathcal{T}_\delta = \mathcal{T}$ and $d_\delta \leq d$. However since*

$$d_\delta(x,y) = \begin{cases} 0 & x \leq y \\ d_E(x,y) & y < x \end{cases}$$
$$= x \ominus y$$

*we clearly have $d_\delta \neq d_E$.*

*(2) Let $X = \mathbb{R}, d = d_E$ and let $\mathcal{T}$ be the topology of compact complements. Clearly $\mathcal{T} \subseteq \mathcal{T}_d$. For $A$ a $\mathcal{T}$-closed set, i.e. a compact set, the set $A^{(\varepsilon)}_{d_E}$ is $\mathcal{T}$-closed. Consequently $(X,\delta)$ defined by*

$$\delta(x,A) := \delta_{d_E}(x,\overline{A}) = \begin{cases} \delta_{d_E}(x,\mathbb{R}) = 0 & A \text{ unbounded} \\ \delta_{d_E}(x,\mathrm{cl}_{\mathcal{T}_{d_E}} A) & A \text{ bounded} \end{cases}$$

*is a supported approach space with $\mathcal{T}_\delta = \mathcal{T}$ and $d_\delta = d_E$.*

*(3) Let $X = [0,\infty[$, with the usual topology $\mathcal{T}$ and $d = q$ with*

$$q(x,y) = \begin{cases} y - x & x \leq y \\ \infty & \text{elsewhere} \end{cases}$$

*inducing the Sorgenfrey topology $\mathcal{T}_q$. Clearly $\mathcal{T} \subseteq \mathcal{T}_q$. Let $A$ be $\mathcal{T}$-closed. In order to see that $A^{(\varepsilon)}_q$ is $\mathcal{T}$-closed, first observe that the set $A^{(\frac{\varepsilon}{2})}_{d_E}$ is $\mathcal{T}$-closed. Then apply a translation over $\frac{\varepsilon}{2}$ to obtain the set $A^{(\varepsilon)}_q$ which hence is $\mathcal{T}$-closed. Consequently $(X,\delta)$ defined by*

$$\delta(x,A) = \delta_q(x,\overline{A}) = \begin{cases} \inf_{a\in\overline{A}, a\leq x} x - a & \exists a \in \overline{A}, a \leq x \\ \infty & \forall a \in \overline{A}, x < a, \end{cases}$$

*is a supported approach space with $\mathcal{T}_\delta = \mathcal{T}$ and $d_\delta = q$.*

## 7. Stability properties

In this section we study stability properties of supportedness. We show that although arbitrary products of compact base-regular approach spaces are supported, supportedness is not stable under taking arbitrary products. Neither do arbitrary subspaces preserve supportedness. We show that closed subspaces as well as coproducts do preserve supportedness. With respect to maps we prove that closed-expansive or open-expansive surjective contractions are preserving supportedness.

7.1. **Proposition.** *Supportedness is preserved by taking closed subspaces.*

*Proof.* Let $(X,\delta)$ be supported, $Y \subseteq X$ with the subspace distance $\delta_Y$. Then $\mathcal{T}_{\delta_Y}$ is the trace of $\mathcal{T}_\delta$ and $d_{\delta_Y}$ is the trace of $d_\delta$. Assume $Y$ is closed in the topological coreflection $(X,\mathcal{T}_\delta)$. For a subset $A \subseteq Y$ and $y \in Y$ we have

$$\delta_Y(y,A) = \delta(y,A) = \delta_{d_\delta}(y,\overline{A}^X) = \delta_{d_{\delta_Y}}(y,\overline{A}^Y),$$

with $\overline{A}^X$ the closure in $\mathcal{T}_\delta$ and $\overline{A}^Y$ the closure in $\mathcal{T}_{\delta_Y}$ which are equal in view of the closedness of $Y$. □

Remark that an arbitrary subspace of a supported approach space need not be supported. This follows from the fact that an arbitrary $T_2$ uniform approach space, which by 4.2 need not be supported, can be embedded into its Čech-Stone compactification, which is uniform and compact and hence supported.

In order to deal with arbitrary products, from the proof of 3.1.5 in [14] we recall that for an arbitrary uniform approach space $(X, \delta)$ with symmetric base $\mathcal{H}$ for the gauge, the diagonal map is an embedding

$$\Psi : (X, \delta) \to (X^{\mathcal{H}}, \prod_{d \in \mathcal{H}} \delta_d) : x \mapsto (x_d = x)_{d \in \mathcal{H}}.$$

7.2. **Proposition.** *For the $T_2$ uniform approach space $(X, \delta)$ of example 4.13 the embedding $\Psi$ is closed.*

*Proof.* Let $\mathcal{G}$ be the gauge of $(X, \delta)$ and $\mathcal{D}$ the gauge basis consisting of all symmetric $d \in \mathcal{G}$. We show that the Euclidean metric $d_E$ belongs to $\mathcal{D}$ by applying (20).
Let $A \subseteq X$ and $\varepsilon \geq 0$. Since $\{\delta_{d_E}(\cdot, A) \leq 1\} = X$ and $\mathfrak{t}_0(A) = A$, in all cases of $A$ and $\varepsilon$ we have

$$\mathfrak{t}_\varepsilon(A) \subseteq \{\delta_{d_E}(\cdot, A) \leq \varepsilon\}.$$

Let

$$\mathcal{H} = \{d \vee d_E \mid d \in \mathcal{D}\}.$$

Since $\mathcal{H}$ is closed under finite infima and satisfies $\delta = \sup_{d \in \mathcal{D}} \delta_{d \vee d_E}$, the collection $\mathcal{H}$ is a gauge basis for $\mathcal{G}$ consisting of separated metrics.
Next we consider the embedding

$$\Psi : (X, \delta) \to (X^{\mathcal{H}}, \prod_{d \in \mathcal{D}} \delta_{d \vee d_E}) : x \mapsto (x_d = x)_{d \in \mathcal{D}}$$

onto the diagonal $\Delta$ and show that it is closed. Let $z = (z_{d \vee d_E})_{d \in \mathcal{D}}$ and assume $z \notin \Delta$. Let $d', d'' \in \mathcal{D}$ such that $z_{d' \vee d_E} \neq z_{d'' \vee d_E}$. In $(X, d_E)$ we choose $\varepsilon > 0$ such that

$$B_{d_E}(z_{d' \vee d_E}, \varepsilon) \cap B_{d_E}(z_{d'' \vee d_E}, \varepsilon) = \emptyset.$$

Then also

$$B_{d' \vee d_E}(z_{d' \vee d_E}, \varepsilon) \cap B_{d'' \vee d_E}(z_{d'' \vee d_E}, \varepsilon) = \emptyset.$$

Finally the open neighborhood $\prod_{d \in \mathcal{D}} V_{d \vee d_E}$ of $z$ with

$$V_{d \vee d_E} = \begin{cases} B_{d' \vee d_E}(z_{d' \vee d_E}, \varepsilon) & d = d' \\ B_{d'' \vee d_E}(z_{d'' \vee d_E}, \varepsilon) & d = d'' \\ X & d \neq d', d \neq d'' \end{cases}$$

is disjoint from $\Delta$.

□

7.3. **Proposition.** *Supportedness is not preserved by arbitrary products.*

*Proof.* Take the $T_2$ uniform approach space $(X, \delta)$ of example 4.13. By 7.2 it is a closed subspace of the product

$$(X^{\mathcal{H}}, \prod_{d \in \mathcal{D}} \delta_{d \vee d_E})$$

with all factors $(X, \delta_{d \vee d_E})$ metric and hence supported. If the product would be supported then by 7.1 also $(X, \delta)$ would be supported. This contradicts the observation in 4.13.

□

7.4. **Proposition.** *Coproducts preserve supportedness.*

*Proof.* Let $(X_i, \delta_i)$ be approach spaces indexed by $i \in I$. Let

$$X = \coprod_{i \in I} X_i \times \{i\}$$

endowed with the coproduct approach structure given by

$$\delta((x,i), A) = \delta_i(x, A \cap (X_i \times \{i\})),$$

for $A \subseteq X$ where we use the same notation for $\delta_i$ considered on $X_i$ and on $X_i \times \{i\}$. For the quasi-metric coreflection we have with $(x,i) \in X$ and $(y,j) \in X$

$$d_\delta((x,i),(y,j)) = \delta((x,i), \{(y,j)\}) = \begin{cases} \infty & i \neq j \\ d_{\delta_i}(x,y) & i = j. \end{cases}$$

For the topological coreflection $(X, \mathcal{T}_\delta)$ and $A \subseteq X$ we have

$$(x,i) \in \mathrm{cl}_{\mathcal{T}_\delta}(A) \Leftrightarrow x \in \mathrm{cl}_{\mathcal{T}_{\delta_i}}(A \cap (X_i \times \{i\})).$$

Now suppose every $(X_i, \delta_i)$ is supported. Then for $(x,i) \in X$ we have

$$\begin{aligned} \delta((x,i), A) &= \delta_i(x, A \cap (X_i \times \{i\})) \\ &= \delta_{d_{\delta_i}}(x, \mathrm{cl}_{\mathcal{T}_{\delta_i}}(A \cap (X_i \times \{i\})) \\ &= \delta_{d_\delta}((x,i), \mathrm{cl}_{\mathcal{T}_\delta}(A)). \end{aligned}$$

□

7.5. **Proposition.** *If $f : (X, \delta) \to (Y, \delta')$ is a closed expansive, contractive surjection, then it is a quotient.*

*Proof.* Suppose $h : (Y, \delta') \to (Z, \sigma)$ is a function and $h \circ f$ is contractive. Let $y \in Y$ and $B \subseteq Y$. With $A = f^{-1}(B)$ and $x \in f^{-1}(y)$ we have

$$\sigma(h(y), h(B)) = \sigma(h(f(x)), h(f(A))) \leq \delta(x, A),$$

hence using (36)

$$\sigma(h(y), h(B)) \leq \inf_{x \in f^{-1}(y)} \delta(x, A) \leq \delta'(y, f(A)) = \delta'(y, B).$$

□

7.6. **Proposition.** *If $f : (X, \delta) \to (Y, \delta')$ is a closed expansive, contractive surjection and $(X, \delta)$ is supported then also $(Y, \delta')$ is supported.*

*Proof.* Let $y \in Y$ and $B \subseteq Y$ a closed set in $(Y, \mathcal{T}_{\delta'})$, then for $x \in f^{-1}(y)$ and the closed set $A = f^{-1}(B)$, applying non-expansiveness of $f : (X, d_\delta) \to (Y, d_{\delta'})$ and supportedness of $(X, \delta)$, see 5.11, we have

$$\delta_{d_{\delta'}}(y, B) = \delta_{d_{\delta'}}(f(x), f(A)) \leq \delta_{d_\delta}(x, A) = \delta(x, A),$$

and hence using (36)

$$\delta_{d_{\delta'}}(y, B) \leq \inf_{x \in f^{-1}(y)} \delta_{d_\delta}(x, A) = \inf_{x \in f^{-1}(y)} \delta(x, A) \leq \delta'(y, B).$$

Since the other inequality is always valid we are done. □

7.7. **Proposition.** *If $f : (X, \delta) \to (Y, \delta')$ is an open expansive, contractive surjection, then it is a quotient.*

*Proof.* Suppose $h : (Y, \delta') \to (Z, \sigma)$ is a function and $h \circ f$ is contractive. For $y = f(x)$ and $B \subseteq Y$ applying (38) we have

$$\begin{aligned}\sigma(h(y), h(B)) = \sigma(h(f(x)), hf(f^{-1}(B))) &\leq \delta(x, f^{-1}(B)) \\ &\leq \delta'(f(x), B) = \delta'(y, B)\end{aligned}$$

□

7.8. **Proposition.** *If $f : (X, \delta) \to (Y, \delta')$ is a open expansive, contractive surjection and $(X, \delta)$ is supported then also $(Y, \delta')$ is supported.*

*Proof.* Let $y \in Y$ and $B \subseteq Y$ a closed set in $(Y, \mathcal{T}_{\delta'})$, then for $x \in f^{-1}(y)$ we have

$$\delta_{d_{\delta'}}(f(x), B) \leq \delta_{d_\delta}(x, f^{-1}(B)) = \delta(x, f^{-1}(B)) \leq \delta'(f(x), B) \leq \delta_{d_{\delta'}}(f(x), B),$$

where the first inequality applies non-expansiveness of $f : (X, d_\delta) \to (Y, d_{\delta'})$, the next equality follows from the supportedness of $(X, \delta)$, the next inequality from the open expansiveness of $f : (X, \delta) \to (Y, \delta')$ (38) and the last inequality is always true. Hence we have

$$\delta_{d_{\delta'}}(y, B) = \delta'(y, B)$$

for all $B$ closed and hence $(Y, \delta')$ is supported. □

## References


[1] B. Banaschewski, R. Lowen, C. Van Olmen, Regularity in approach theory, *Acta Math. Hungar.*,115, (2007), 183-196.

[2] B. Berckmoes, R. Lowen, J.A. van Casteren, Distances on probability measures and random variables, *J. Math. Anal. and Appl.*, 374(2), (2011), 412-428.

[3] B. Berckmoes, T. Hellemans, M. Sioen, J.A. van Casteren, An application of approach theory to the relative Hausdorff measure of non-compactness for the Wasserstein metric, *J. Math. Anal. and Appl.*, 449(20), (2017), 1770-1789.

[4] M.M. Clementino, E. Colebunders, W. Tholen, Lax algebras as spaces, *Monoidal Topology, A categorical approach to order, metric and topology*, Cambridge University Press, (2014), 375-465.

[5] E. Colebunders, S. De Wachter, R. Lowen, Fixed points of contractive maps between DCPO's, *Math. Struct. in Comp. Sci.*, 24(1), (2014), e240102, 18 pp.

[6] E. Colebunders, S. De Wachter, M. Schellekens, Complexity analysis via approach theory, *Appl. Categ. Struct.*, 22 (1), (2014), 119-136.

[7] E. Colebunders, R. Lowen, K. Van Opdenbosch, Regularity for relational algebras and approach spaces, *Topol. Appl.* 200, (2016), 79-100.

[8] E. Colebunders, F. Mynard, W. Trott, Function spaces and contractive extensions in approach theory: the role of regularity, *Appl. Categ. Struct.*, 22 (3), (2014), 551-563.

[9] E. Colebunders, M. Sioen, W. Van Den Haute, Normality, regularity and contractive realvalued maps, *Appl. Categ. Struct.*, 26, (2018), 909-930.

[10] P. Fletcher, W. F. Lindgren, Quasi Uniform Spaces, Marcel Dekker, (1982).

[11] M. M. Clementino and D. Hofmann, Topological features of lax algebras, *Appl. Categ. Struct.*, 11, (2003) 267-286.

[12] D. Hofmann, G. J. Seal, W. Tholen eds, *Monoidal Topology, A categorical approach to order, metric and topology*, Cambridge University Press, (2014).

[13] R. Lowen, *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, Oxford University Press, (1997).

[14] R. Lowen, *Index Analysis: Approach Theory at Work*, Springer Monographs in Mathematics, Springer Verlag, (2015).

[15] R. Lowen, M. Sioen, The Wijsman and Attouch-Wets topologies on hyperspaces revisited, *Topol. Appl.*, 70, (1996), 179-197.
[16] M. Sioen, S. Verwulgen, Quantified functional analysis and seminormed spaces: a dual adjunction, *J. Pure Appl. Alg.*, 207, (2006), 675-686.
[17] J.E. Vaughan, Products of topological spaces, *Gen. Topol. Appl* 8, 207-217, (1976).
[18] J.E. Vaughan, Total nets and filters, *Lecture notes in Pure and Applied Mathematics*, 24, 259-265, (1976).

E. Colebunders
Department of Mathematics and Data Science, Vrije Universiteit Brussel, Pleinlaan 2,
1050 Brussel, België
*evacoleb@vub.be*

R. Lowen
Department of Mathematics, Universiteit Antwerpen, Middelheimlaan 1, 2020 Antwerpen, België
*bob.lowen@uantwerpen.be*