

**This item is the archived peer-reviewed author-version of:**

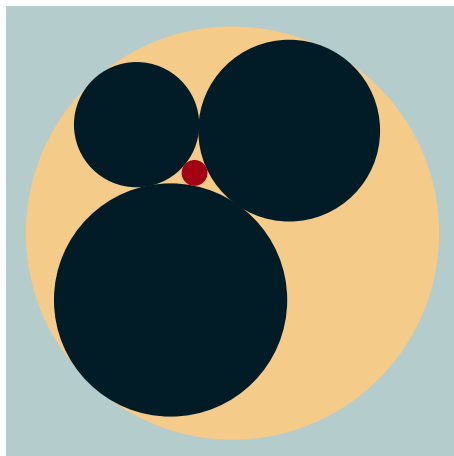
A straightforward proof of Descartes's circle theorem

**Reference:**

Levrie Paul.- A straightforward proof of Descartes's circle theorem  
The mathematical intelligencer - ISSN 0343-6993 - 41:3(2019), p. 24-27  
Full text (Publisher's DOI): <https://doi.org/10.1007/S00283-019-09883-X>  
To cite this reference: <https://hdl.handle.net/10067/1621880151162165141>

# A straightforward proof of Descartes' Circle Theorem

Paul Levrie



## Abstract

In this paper we give a new, straightforward and short proof of Descartes' Circle theorem, a proof that Descartes might have liked.

## Introduction

There are probably not many formulas in mathematics (a) where a princess was instrumental in finding them and (b) that are described in a poem written by a Nobel prize laureate (in Chemistry!). This is the case for what is now known as Descartes' Circle Formula.

Let  $C_1$ ,  $C_2$  and  $C_3$  be circles with radius  $r_1$ ,  $r_2$  and  $r_3$ , that are mutually tangent. Let us assume that the radius of a fourth circle tangent to the other three, the red circle in the figure above, is  $r_4$ . Descartes' Circle Theorem asserts the following:

**Theorem.** *The radii  $r_1$ ,  $r_2$ ,  $r_3$ ,  $r_4$  of four mutually tangent circles satisfy:*

$$2 \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \right) = \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)^2. \quad (1)$$

The problem of finding the radius of the fourth circle is a special case of a problem of Apollonius<sup>1</sup> [2]: given three circles, construct the circles tangent to these three circles.

We will give a straightforward proof of Descartes' theorem, using only elementary algebra and Heron's formula for the area of a triangle.

---

<sup>1</sup>A problem a countess tried her hand on, Countess Skorzewska [1, p. 308]. Lambert describes her in his correspondence as a learned Polish lady and a great lover ("Liebhaber") of the mathematical sciences.

## A short history of Descartes' Circle Theorem [3]

Descartes' Circle Theorem was first described by Descartes in 1643 in his correspondence with Princess Elisabeth of Bohemia, one of his pupils [4]. In a letter to her, Descartes posed the following problem:

*« A M. la Princesse Elizabeth, etc. Touchant le Probleme : trois cercles estant donnez, trouuer le quatriéme qui touche les trois »*

which is Apollonius' problem. Descartes soon realised that this might be too difficult, and reduced the problem to the case that the three given circles are mutually tangent. He also gave the following solution:

$$\begin{array}{rcl} ddeeff & \infty & 2deffxx + 2deeffx \\ + ddeexx & & + 2deefxx + 2ddeeffx \\ + ddffxx & & + 2ddefxx + 2ddeeffx \\ + eeffxx, & & \end{array}$$

Here  $e$ ,  $f$  and  $g$  are the given radii, and  $x$  is the radius to be found.

Exercise for the reader: Show that this solution is equivalent with (1).

The formula was rediscovered and proved by Jakob Steiner in 1826, and again by Philip Beecroft in 1842. In 1936 Frederick Soddy, who received the Nobel Prize in chemistry in 1921, rediscovered the result, and wrote a poem about it [5]:

### The Kiss Precise

For pairs of lips to kiss maybe  
Involves no trigonometry.  
'Tis not so when four circles kiss  
Each one the other three.  
To bring this off the four must be  
As three in one or one in three.  
If one in three, beyond a doubt  
Each gets three kisses from without.  
If three in one, then is that one  
Thrice kissed internally.

Four circles to the kissing come.  
The smaller are the benter.  
The bend is just the inverse of  
The distance from the center.  
Though their intrigue left Euclid dumb  
There's now no need for rule of thumb.

Since zero bend's a dead straight line  
And concave bends have minus sign,  
*The sum of the squares of all four bends  
Is half the square of their sum.*

To spy out spherical affairs  
An oscular surveyor  
Might find the task laborious,  
The sphere is much the gayer,  
And now besides the pair of pairs  
A fifth sphere in the kissing shares.  
Yet, signs and zero as before,  
For each to kiss the other four  
*The square of the sum of all five bends  
Is thrice the sum of their squares.*

F. SODDY

(After the publication of the poem another verse was added to it by Thorold Gosset [6], the generalization of the result to  $n$  dimensions.)

More recently, Lagarias a.o. [7] proved that a relation very similar to (1) relates the centers of the tangent circles in the complex plane.

The proof of (1) by Steiner [8] uses a result about Pappus chains and a generalization of Viviani's theorem to general triangles. Beecroft's proof makes use of four other mutually tangent circles through the points where the four circles meet. It was later simplified by Coxeter [2]. Coxeter himself gave a new proof, based on inversion with respect to a circle [3]. Pedoe [9] lists some other proofs, one based on a symmetry argument, another using Grassmann calculus. None of these proofs is really straightforward. It's not known which path Descartes and Elisabeth followed to derive their result.

## The proof (that Descartes missed?)

The proof given here is based on Heron's formula [10] for the area of a triangle with sides  $a$ ,  $b$  and  $c$ :

$$\text{Area} = \sqrt{o(o-a)(o-b)(o-c)} \quad \text{with} \quad o = \frac{a+b+c}{2}.$$

This formula was known to Descartes and Princess Elisabeth, and they both probably used it in trying to solve the problem of the touching circles.

If we connect the centers of these four circles (see Figure 1), four triangles are formed, the area of the biggest triangle being the sum of the areas of the other three. We can write this out using Heron's formula:

$$\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)} = \sqrt{r_1 r_2 r_4 (r_1 + r_2 + r_4)} + \sqrt{r_1 r_3 r_4 (r_1 + r_3 + r_4)} + \sqrt{r_2 r_3 r_4 (r_2 + r_3 + r_4)}. \quad (2)$$

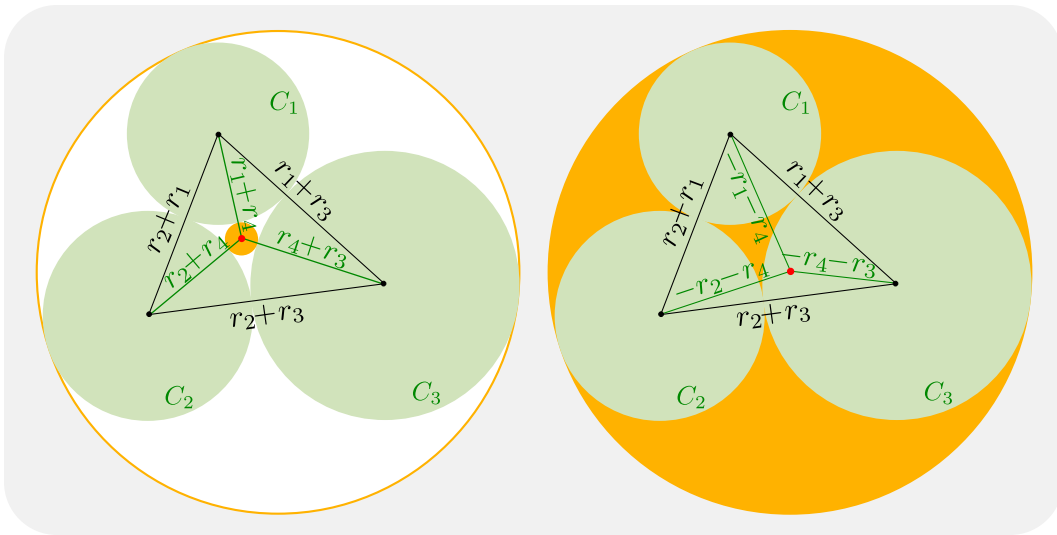


Figure 1: Three mutually touching circles  $C_1$ ,  $C_2$  and  $C_3$  and the two solutions of Apollonius' problem

Note that there's also a circle touching the three given circles externally. If we assume that the radius of this circle is taken to be a negative number, then in the configuration of Figure 1 the same equation (2) is satisfied, as can be seen in the right-hand part.

Solving the equation (2) in the traditional way by repeatedly squaring leads to enormous calculations.<sup>2</sup> However, by carefully simplifying in each step it is possible to get the result in one page.

In the sequel we will use the following notations:

$$s = r_1 + r_2 + r_3 + r_4, p = r_1 r_2 r_3 r_4, t = \frac{p}{s}, u = \frac{1}{s}.$$

Furthermore let

$$\alpha = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}, \beta = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}.$$

Using these we can rewrite (2) as:

$$\sqrt{r_1 r_2 r_3 s - p} = \sqrt{r_1 r_2 r_4 s - p} + \sqrt{r_1 r_3 r_4 s - p} + \sqrt{r_2 r_3 r_4 s - p}.$$

<sup>2</sup>As Descartes writes in his letter to Elisabeth dated November 29, 1643 [4]: 'But this route seems to me to lead to so many superfluous multiplications that I would not want to undertake to solve them in three months.'

We now divide by  $\sqrt{s}$  and rearrange:

$$\sqrt{r_1 r_2 r_3 - t} - \sqrt{r_1 r_2 r_4 - t} = \sqrt{r_1 r_3 r_4 - t} + \sqrt{r_2 r_3 r_4 - t}.$$

Squaring both sides and rearranging leads to:

$$r_1 r_2 r_3 + r_1 r_2 r_4 - r_1 r_3 r_4 - r_2 r_3 r_4 = 2(\sqrt{r_1 r_2 r_3 - t} \sqrt{r_1 r_2 r_4 - t} + \sqrt{r_1 r_3 r_4 - t} \sqrt{r_2 r_3 r_4 - t}).$$

We divide this result by  $p$ :

$$\frac{1}{r_4} + \frac{1}{r_3} - \frac{1}{r_2} - \frac{1}{r_1} = 2 \left( \sqrt{\frac{1}{r_4} - u} \cdot \sqrt{\frac{1}{r_3} - u} + \sqrt{\frac{1}{r_2} - u} \cdot \sqrt{\frac{1}{r_1} - u} \right).$$

Again we square both sides:

$$\begin{aligned} \beta + \frac{2}{r_3 r_4} - \frac{2}{r_2 r_4} - \frac{2}{r_1 r_4} - \frac{2}{r_2 r_3} - \frac{2}{r_1 r_3} + \frac{2}{r_1 r_2} \\ = \frac{4}{r_3 r_4} + \frac{4}{r_1 r_2} - 4\alpha u + 8u^2 + 8\sqrt{\frac{1}{r_4} - u} \cdot \sqrt{\frac{1}{r_3} - u} \cdot \sqrt{\frac{1}{r_2} - u} \cdot \sqrt{\frac{1}{r_1} - u} \end{aligned}$$

or, after rearranging:

$$\beta - \frac{2}{r_3 r_4} - \frac{2}{r_2 r_4} - \frac{2}{r_1 r_4} - \frac{2}{r_2 r_3} - \frac{2}{r_1 r_3} - \frac{2}{r_1 r_2} + 4\alpha u - 8u^2 = 8\sqrt{\frac{1}{r_4} - u} \cdot \sqrt{\frac{1}{r_3} - u} \cdot \sqrt{\frac{1}{r_2} - u} \cdot \sqrt{\frac{1}{r_1} - u}.$$

Note that since  $\alpha^2 - \beta = \frac{2}{r_3 r_4} + \frac{2}{r_2 r_4} + \frac{2}{r_1 r_4} + \frac{2}{r_2 r_3} + \frac{2}{r_1 r_3} + \frac{2}{r_1 r_2}$  we can rewrite this equation as:

$$(2\beta - \alpha^2) + 4\alpha u - 8u^2 = 8\sqrt{\frac{1}{r_4} - u} \cdot \sqrt{\frac{1}{r_3} - u} \cdot \sqrt{\frac{1}{r_2} - u} \cdot \sqrt{\frac{1}{r_1} - u}. \quad (3)$$

Squaring both sides (again), we find for the left-hand side:

$$(2\beta - \alpha^2)^2 + 8(2\beta - \alpha^2)\alpha u - 16(2\beta - \alpha^2)u^2 + 16\alpha^2 u^2 - 64\alpha u^3 + 64u^4.$$

The right-hand side is given by:

$$\frac{64}{r_1 r_2 r_3 r_4} - 64 \left( \frac{1}{r_1 r_2 r_3} + \frac{1}{r_1 r_2 r_4} + \frac{1}{r_1 r_3 r_4} + \frac{1}{r_2 r_3 r_4} \right) u + 32(\alpha^2 - \beta)u^2 - 64\alpha u^3 + 64u^4.$$

The first two terms on the right-hand side cancel out, and a number of terms of left and right-hand side are equal. Hence we get after rearranging:

$$(2\beta - \alpha^2)^2 + 8(2\beta - \alpha^2)\alpha u = 0 \quad \text{or} \quad (2\beta - \alpha^2) \cdot (2\beta - \alpha^2 + 8\alpha u) = 0.$$

The second factor cannot be zero, since in that case  $2\beta - \alpha^2 = -8\alpha u$  which results in a negative left-hand side in (3). Hence we have that  $2\beta = \alpha^2$ , or

$$2 \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \right) = \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)^2.$$

Note that given  $r_1$ ,  $r_2$  and  $r_3$  this is a quadratic equation for  $r_4$  with two solutions: the radii of the two tangent circles in Figure 1, called the inner and the outer Soddy circle.

## Final remark

The problem of the ‘kissing circles’ and Descartes’ circle theorem are now as actual as some 400 years ago. To give but one example: Descartes’ formula plays an important role in the theory of circle packings in the plane. A circle packing is an arrangement of circles that all touch each other. A special case are the Apollonian circle packings, which are constructed by starting with three mutually touching circles and adding the two circles tangent to the first three. Taking one of those two and combining it with two of the original circles leads to a similar situation in which we can find two new circles tangent to them. And we can continue in the same way. The first steps in such a process can be seen in Figure 2.

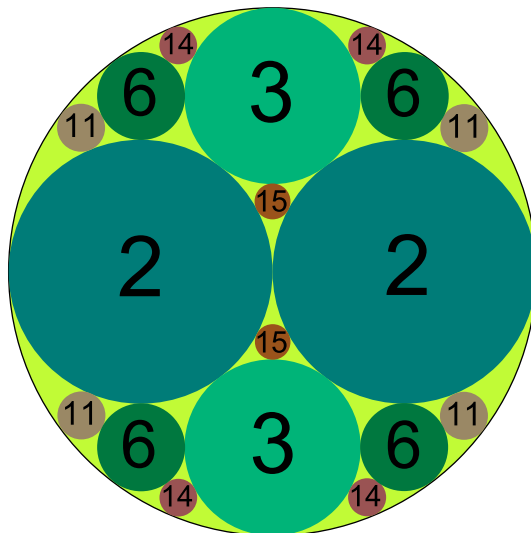


Figure 2: The biggest circles in an integral Apollonian circle packing

The numbers in the circles denote their curvature ( $1/\text{radius}$ ). It is a nice consequence of the form of Descartes’ circle formula that if we start with four circles with integral radius, one of them being negative, all the other curvatures will be integers too. Such a packing is called an integral Apollonian circle packing [11].

**Acknowledgment** I want to thank Damaris Schindler from Utrecht University for bringing me back to this old favourite of mine.

## References

- [1] Lambert, J.H., *Deutscher gelehrter Briefwechsel*, Band I, Berlin, 1781.  
<https://www.e-rara.ch/zut/content/structure/2334211>
- [2] Coxeter, H.S.M., The Problem of Apollonius, *Amer. Math. Monthly* 75 (1968), 5–15.
- [3] Coxeter, H.S.M., Loxodromic Sequences of Tangent Spheres. *Aequationes Math.* 1 (1968), 104–121.  
[http://www.digizeitschriften.de/download/PPN356261603\\_0001/PPN356261603\\_0001\\_\\_\\_log17.pdf](http://www.digizeitschriften.de/download/PPN356261603_0001/PPN356261603_0001___log17.pdf)
- [4] Shapiro, L., *The correspondence between Princess Elisabeth of Bohemia and René Descartes*, University of Chicago Press, Chicago, 2007.

- [5] Soddy, F., The Kiss Precise, *Nature* 137 (1936), 1021.  
[http://maths.ac-noumea.nc/polyhedr/packing\\_kiss/kiss.htm](http://maths.ac-noumea.nc/polyhedr/packing_kiss/kiss.htm)
- [6] *Nature* 139 (1937), 62.
- [7] Lagarias, J.C., Mallows, C.L., and Wilks A.R., Beyond the Descartes circle theorem, *Amer. Math. Monthly* 109 (2002), no. 4, 338–361.
- [8] Steiner, J., Fortsetzung der geometrische Betrachtungen, *J. reine Angew. Math.* 1 (1826), 252–288.  
[https://gdz.sub.uni-goettingen.de/id/PPN243919689\\_0001](https://gdz.sub.uni-goettingen.de/id/PPN243919689_0001)
- [9] Pedoe, D., On a theorem in geometry, *Amer. Math. Monthly* 74 (1967), 627–640.
- [10] Nelsen, R.B., Heron’s Formula via Proofs Without Words, *College Math. J.* 32, 4 (2001), 290–292.  
<https://www.maa.org/sites/default/files/0746834212944.di020798.02p0691h.pdf>
- [11] Sarnak, P., Integral Apollonian Packings. *Amer. Math. Monthly* 118 (2011), 291–306.

*Faculty of Applied Engineering, UAntwerpen, Groenenborgerlaan 171, B-2020 Antwerp, Belgium*

*Department of Computer Science, KU Leuven, Celestijnenlaan 200A, P.O. Box 2402, B-3001 Heverlee (Leuven), Belgium*  
*paul.levrie@cs.kuleuven.be*