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SINGULAR AND TOTALLY SINGULAR UNITARY SPACES

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ABSTRACT. In this paper we present a decomposition theorem for unitary spaces over a central simple division algebra with involution in characteristic 2. This is a generalisation of a decomposition result for quadratic forms in characteristic 2 from [4] and extends a generalisation of the Witt decomposition theorem for nonsingular spaces in this setting to cover spaces that may be singular.

1. INTRODUCTION

Generalised quadratic forms (also known as pseudo-quadratic forms) are an extension of the concept of quadratic forms over a field to the setting of central simple division algebras with involution, first introduced in [9]. They generalise quadratic forms to this setting in an analogous manner to the generalisation of bilinear forms to hermitian forms. Rather than being maps from a vector space over a field to that field, they are defined as being maps on a vector space over a division algebra to that division algebra modulo alternating elements with respect to an involution. This concept was further generalised in [1] (see also [10]), where the alternating elements are replaced with a subset of the skew–symmetric elements that contains the alternating elements (see Section 3). These objects are known as unitary spaces. In fact, [1] also considers unitary spaces on modules over general (not necessarily division) central simple algebras, but we will not work with this level of generality here.

The decomposition theorem of Witt (see [11]) states that any regular quadratic form over a field of characteristic different from 2 uniquely decomposes into an orthogonal sum of an anisotropic part and a hyperbolic part. Our interest is in this theorem's generalisation to the theory of unitary spaces over fields of arbitrary characteristic, where it says that every nonsingular unitary space over a finite dimensional division algebra with involution can be decomposed into an orthogonal sum of an anisotropic part and a hyperbolic part in a unique way (see (4.8)).

We consider decompositions of unitary spaces that may be singular. That is, we allow the hermitian space associated to the unitary space to be degenerate. If the characteristic of the underlying field is different from 2, or the involution is of the second kind, then the hermitian space associated to a unitary space completely determines the unitary space (and vice versa), so nonsingular unitary spaces are the only interesting spaces in these cases. Indeed, spaces that are totally singular, that is, whose associated hermitian form is the zero map, are trivial in characteristic different from 2 or in the case of an involution of the second kind.

However, over fields of characteristic 2 and for involutions of the first kind, singular unitary spaces have a great deal of structure. In particular, in characteristic 2 and when the involution is of the first kind, there can be many different totally singular spaces of the same dimension. Singular quadratic forms have been studied over fields, in, for example, [4]. Here it is shown that totally singular forms can be studied somewhat independently and with methods quite distinct from the usual theory of quadratic forms (see also, for example, [7]). Singular spaces in the wider setting of unitary spaces have remained largely uninvestigated.

In Section 9 we show a decomposition theorem for unitary spaces that are not assumed to be nonsingular. This generalises [4, (2.4)] to our wider setting. Towards this end, in Section 7 we also show some results on totally singular unitary spaces and on the elements they represent, which may be of independent interest. In particular, in (7.2) we show that, excluding the split case, totally singular unitary spaces can be distinguished from non–totally singular spaces by looking at the elements they represent.

2. Algebras with involution

We refer to [8] as a general reference on finite-dimensional algebras over fields, and for central simple algebras in particular, and to [6] for involutions. Throughout, let F be a field. We denote the characteristic of F by $\mathrm{char}(F)$ and the multiplicative group of F by $F^\times$.

Let A be a finite-dimensional F–algebra and let $Z(A)$ denote its centre. If A is simple (i.e. it has no non-trivial two sided ideals), we can view A as a $Z(A)$–algebra and by Wedderburn's Theorem (see [6, (1.1)]) we have that $A \simeq \mathrm{End}_D(V)$ for an F–division algebra D with centre $Z(A)$ and a right D–vector space V. In this case $\dim_{Z(A)}(A)$ is a square and the positive root of this integer is called the *degree of A* and is denoted $\deg(A)$. If $Z(A) = F$, then we call the F–algebra A *central simple*. An F–*quaternion algebra* is a central simple F–algebra of degree 2.

Let Ω be an algebraic closure of F. By Wedderburn's Theorem, under scalar extension to Ω every central simple F–algebra of degree n becomes isomorphic to $M_n(\Omega)$, the algebra of $n \times n$ matrices over Ω. Therefore if A is a central simple F–algebra we may fix an F–algebra embedding $A \to M_n(\Omega)$ and view every element $a \in A$ as a matrix in $M_n(\Omega)$. The characteristic polynomial of this matrix has coefficients in F and is independent of the embedding of A into $M_n(\Omega)$ (see [8, §16.1]). We call this polynomial the *reduced characteristic polynomial of A* and denote it by

$$\mathrm{Prd}_{A,a} = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} - \ldots + (-1)^n s_n(a).$$

We call $s_1(a)$ the *reduced trace of a* and $s_n(a)$ the *reduced norm of a* and denote them by $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$ respectively. We also denote $s_2(a)$ by $\mathrm{Srd}_A(a)$.

An F–*involution* on A is an F–linear map $\sigma : A \to A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$ and $\sigma^2 = \mathrm{id}_A$. An F–*algebra with involution* is a pair (A, σ) consisting of a finite-dimensional F–algebra A and an F–involution σ on A such that one has $F = \{x \in Z(A) \mid \sigma(x) = x\}$, and such that either A is simple or A is a product of two simple F–algebras that are mapped to one another by σ. In this situation, there are two possibilities: either $Z(A) = F$, so that A is a central simple F–algebra, or $Z(A)/F$ is a quadratic étale extension and $\sigma|_{Z(A)}$ is the nontrivial F–automorphism of $Z(A)$. To distinguish these two situations, we

speak of algebras with involution of the *first* and *second kind*: we say that the F–algebra with involution (A, σ) is of the *first kind* if $Z(A) = F$ and of the *second kind* otherwise. For more information on involutions of the second kind, also called *unitary involutions*, we refer to [6, Section 2.B].

Let (A, σ) be an F–algebra with involution. For $\lambda \in Z(A)$ such that $\lambda\sigma(\lambda) = 1$, let

$$\mathrm{Sym}_\lambda(A, \sigma) = \{a \in A \mid \lambda\sigma(a) = a\} \quad \text{and} \quad \mathrm{Alt}_\lambda(A, \sigma) = \{a - \lambda\sigma(a) \mid a \in A\}.$$

These are F–linear subspaces of A and we write $\mathrm{Sym}(A, \sigma) = \mathrm{Sym}_1(A, \sigma)$ and $\mathrm{Alt}(A, \sigma) = \mathrm{Alt}_1(A, \sigma)$. Note that if (A, σ) is of the first kind one must have that $\lambda = \pm 1$.

Lemma 2.1. *Let (A, σ) be an F–algebra with involution. If $\mathrm{char}(F) \neq 2$ or (A, σ) is of the second kind, then for all $\lambda \in Z(A)$ such that $\lambda\sigma(\lambda) = 1$ we have $\mathrm{Sym}_\lambda(A, \sigma) = \mathrm{Alt}_{-\lambda}(A, \sigma)$.*

Proof. If $\mathrm{char}(F) \neq 2$ or (A, σ) is of the second kind, then there exists $b \in Z(A)$ such that $b + \sigma(b) = 1$. Then for $a \in \mathrm{Sym}_\lambda(A, \sigma)$ we have $a = ba + \lambda\sigma(ba) \in \mathrm{Alt}_{-\lambda}(A, \sigma)$. $\square$

Example 2.2. Let $(1, u, v, w)$ be basis elements of a 4–dimensional F-vector space. Fix $a \in F$ such that $-4a \neq 1$ and $b \in F^\times$. Defining a multiplication on this F–vector space through the relations $u^2 = u + a$, $v^2 = b$ and $w = uv = v - vu$ gives the vector space the structure of an F–quaternion algebra, and we denote this F–quaternion algebra by $[a, b)_F$. We call a basis $(1, u, v, w)$ for $[a, b)_F$ as above a *quaternion basis*.

Let $Q = [a, b)_F$ for $a \in F$ and $b \in F^\times$ as above. Let γ be the F–involution on Q such that $\gamma(u) = 1 - u$ and $\gamma(v) = -v$. This is the unique symplectic involution on Q (see [6, (2.21)]) and for all $x \in Q$ we have $\gamma(x) = \mathrm{Trd}_Q(x) - x$ and $\gamma(x)x = \mathrm{Nrd}_Q(x)$. Direct computation then shows that $\mathrm{Alt}_{-1}(Q, \gamma) = F$.

Remark 2.3. Let Q be an F–quaternion algebra with a quaternion basis $(1, u, v, w)$ as in (2.2). If $\mathrm{char}(F) \neq 2$, then setting $i = u - \frac{1}{2}$, $j = v$ and $k = ij$ gives the more commonly used basis $(1, i, j, k)$ for Q with $i^2, j^2 \in F^\times$ and $ij = -ji$ (see [6, p.25]).

3. Sesquilinear and Unitary spaces

In this section we recall the basic terminology and results we use from sesquilinear and hermitian form theory. We refer to [5, Chapter 1] as a general reference on sesquilinear and hermitian forms.

Let (D, θ) be an F–division algebra with involution and let V be a right D–vector space. A *sesquilinear form on V with respect to θ* is a bi-additive map $k : V \times V \to D$ such that $k(xd, yd') = \theta(d)k(x, y)d'$ for all $x, y \in V$ and $d, d' \in D$. The set $\mathrm{Sesq}_\theta(V)$ of sesquilinear forms on V with respect to θ is a $Z(D)$–module for the addition $k_1 + k_2$ of forms and the scalar multiplication $(ak)(x, y) = ak(x, y)$ for $a \in Z(D)$. For a given $k \in \mathrm{Sesq}_\theta(V)$, we define $k^* \in \mathrm{Sesq}_\theta(V)$ to be the form given by $k^*(x, y) = \theta(k(y, x))$ for all $x, y \in V$. For a subspace W of V, we denote the restriction of k to W in $\mathrm{Sesq}_\theta(W)$ by $k|_{W \times W}$ or simply by $k|_W$. A *sesquilinear space with respect to θ* is a pair (V, k) such that V is a right D–vector space and $k \in \mathrm{Sesq}_\theta(V)$.

Let $\varphi_1 = (V_1, k_1)$ and $\varphi_2 = (V_2, k_2)$ be sesquilinear spaces with respect to θ. We say that φ_1 and φ_2 are *isometric* if there exists a D–vector space isomorphism

$\alpha : V_1 \to V_2$ such that $k_1(x,y) = k_2(\alpha(x), \alpha(y))$ for all $x, y \in V_1$, and in this case we write $\varphi_1 \simeq \varphi_2$. For any homomorphism of D–vector spaces $\alpha : V_1 \to V_2$, we have a $Z(A)$–linear map

$$\mathrm{Sesq}(\alpha) : \mathrm{Sesq}_\theta(V_2) \to \mathrm{Sesq}_\theta(V_1)$$

given by $\mathrm{Sesq}(\alpha)(k)(x,y) = k(\alpha(x), \alpha(y))$ for all $x, y \in V_1$.

Fix $\lambda \in Z(A)$ such that $\lambda\theta(\lambda) = 1$. We say a form $h \in \mathrm{Sesq}_\theta(V)$ is λ–*hermitian* if $h = \lambda h^*$. That is, $h(y,x) = \lambda\theta(h(x,y))$ holds for all $x, y \in V$. We denote the set of λ–hermitian sesquilinear forms on a right D–vector space V with respect to θ by $\mathrm{Sesq}_\theta^\lambda(V)$. A λ–*hermitian space over* (D, θ) is a pair (V, h) such that V is a right D–vector space and $h \in \mathrm{Sesq}_\theta^\lambda(V)$. If $(D, \theta) = (F, \mathrm{id})$ and $\lambda = 1$, then a λ–hermitian space over (D, θ) is a symmetric bilinear space over F (see [2, (1.1)]).

Let $\varphi = (V, h)$ be a λ–hermitian space over (D, θ). We call the set

$$\mathrm{rad}(\varphi) = \{v \in V \mid h(v, w) = 0 \text{ for all } w \in V\}$$

the *radical of* φ. We say φ is *nondegenerate* if $\mathrm{rad}(V, h) = \{0\}$. We say that φ *represents an element* $a \in D$ if $h(x, x) = a$ for some $x \in V \backslash \{0\}$. We call φ *even* if $h(x, x) \in \mathrm{Alt}_{-\lambda}(D, \theta)$ for all $x \in V$. Note that by (2.1), if $\mathrm{char}(F) \neq 2$ or (D, θ) is of the second kind, then all λ–hermitian spaces over (D, θ) are even. We refer to $\dim_D(V)$ as *the dimension of* φ and write $\dim(\varphi)$ for this integer.

Let $\varphi_1 = (V, h_1)$ and $\varphi_2 = (W, h_2)$ be λ–hermitian spaces over (D, θ). The *orthogonal sum* of φ_1 and φ_2 is defined to be the λ–hermitian space $(V \times W, h)$ over (D, θ) where the F–linear map $h : (V \times W) \times (V \times W) \to D$ is such that $h((v_1, w_1), (v_2, w_2)) = h_1(v_1, v_2) + h_2(w_1, w_2)$ for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$; we also denote it by $\varphi_1 \perp \varphi_2$.

Lemma 3.1. *Let* $\varphi = (V, h)$ *be a* λ–*hermitian space over an* F–*division algebra with involution* (D, θ) *such that* h *is not identically zero on* $V \times V$. *Then* φ *represents a non–zero element in* D *if and only if* $(D, \theta) \neq (F, \mathrm{id})$ *or* $\lambda \neq -1$.

Proof. Since h is not identically zero on $V \times V$, there exists a subspace W of V such that $V = W \oplus \mathrm{rad}(\varphi)$. Then $\varphi = (W, h|_W) \perp (\mathrm{rad}(\varphi), h|_{\mathrm{rad}(\varphi)})$ as for all $x_1, x_2 \in W$ and $y_1, y_2 \in \mathrm{rad}(\varphi)$ we have $h(x_1 + y_1, x_2 + y_2) = h(x_1, x_2) + h(y_1, y_2) = h(x_1, x_2)$. Then by [5, Chapter 1, (6.2.3)], $(W, h|_W)$ represents a non-zero element in D if and only if $(D, \theta) \neq (F, \mathrm{id})$ or $\lambda \neq -1$. $\square$

For a fixed $\lambda \in Z(D)$ such that $\lambda\theta(\lambda) = 1$, let Λ be an additive subgroup of D such that

(1) $\mathrm{Alt}_\lambda(D, \theta) \subseteq \Lambda \subseteq \mathrm{Sym}_{-\lambda}(D, \theta)$.
(2) $\theta(d)xd \in \Lambda$ for all $x \in \Lambda$ and $d \in D$.

We call the pair (λ, Λ), with Λ as above, a *form parameter on* (D, θ). Note that by (2.1), Λ is uniquely determined by λ if $\mathrm{char}(F) \neq 2$ or (D, θ) is of the second kind. For a given form parameter (λ, Λ) on (D, θ) we define

$$\mathrm{Sesq}_\Lambda^{-\lambda}(V) = \{h \in \mathrm{Sesq}_\theta^{-\lambda}(V) \mid h(x, x) \in \Lambda \text{ for all } x \in V\}.$$

Note that if $\Lambda = \mathrm{Sym}_{-\lambda}(D, \theta)$, then $\mathrm{Sesq}_\Lambda^{-\lambda}(V) = \mathrm{Sesq}_\theta^{-\lambda}(V)$.

A (λ, Λ)–*unitary space over* (D, θ) is a pair $(V, [k])$ where V is a right D–vector space and $[k]$ is the class of

$$k \in \mathrm{Sesq}_\theta(V) \mod \mathrm{Sesq}_\Lambda^{-\lambda}(V).$$

Let $\rho = (V, [k])$ be a (λ, Λ)–unitary space over (D, θ). As $\mathrm{Sesq}_\Lambda^{-\lambda}(V) \subseteq \mathrm{Sesq}_\theta^{-\lambda}(V)$, there is a unique even λ–hermitian space associated to ρ given by $(V, k + \lambda k^*)$, which we call the *polar λ–hermitian space of ρ* and denote by ρ_{pol}. We say that ρ is *nonsingular* if ρ_{pol} is nondegenerate, and *singular* otherwise. We say that ρ is *totally singular* if $k + \lambda k^*$ is the zero map on $V \times V$. We refer to $\dim_D(V)$ as *the dimension of ρ* and write $\dim(\rho)$ for this integer.

Let $\rho_1 = (V_1, [k_1])$ and $\rho_2 = (V_2, [k_2])$ be (λ, Λ)–unitary spaces over (D, θ). We say that ρ_1 and ρ_2 are *isometric* if there exists a D–linear map $\alpha : V_1 \to V_2$ such that $[\mathrm{Sesq}(\alpha)(k_2)] = [k_1]$, and in this case we write $\rho_1 \simeq \rho_2$. The *orthogonal sum of ρ_1 and ρ_2* is the (λ, Λ)–unitary space given by $(V_1 \oplus V_2, [k_1 \oplus k_2])$, which we denote by $\rho_1 \perp \rho_2$. Since $\mathrm{Sesq}_\Lambda^{-\lambda}(V_1) \oplus \mathrm{Sesq}_\Lambda^{-\lambda}(V_2) \subseteq \mathrm{Sesq}_\Lambda^{-\lambda}(V_1 \oplus V_2)$, the class $[k_1 \oplus k_2]$ only depends on the classes $[k_1]$ and $[k_2]$. We say ρ' is a *subform of ρ* if there exists a (λ, Λ)–unitary space ρ'' over (D, θ) such that $\rho \simeq \rho' \perp \rho''$. For $n \in \mathbb{N}$ we denote the orthogonal sum of n copies of ρ by $n \times \rho$. For $c \in F^\times$ we denote by $c\rho$ the (λ, Λ)–unitary space $(V, [ck])$.

Lemma 3.2. *Let ρ, ρ_1 and ρ_2 be (λ, Λ)–unitary spaces over (D, θ). Then $\rho = \rho_1 \perp \rho_2$ if and only if $\rho_{\mathrm{pol}} = (\rho_1)_{\mathrm{pol}} \perp (\rho_2)_{\mathrm{pol}}$.*

Proof. See [5, Chapter 1, (5.4)]. $\qquad\square$

Corollary 3.3. *Let $\rho = (V, [k])$ be a (λ, Λ)–unitary space over (D, θ). Then for a D–vector subspace U of V such that $V = U \oplus \mathrm{rad}(\rho_{\mathrm{pol}})$ we have*

$$\rho \simeq (U, [k|_U]) \perp (\mathrm{rad}(\rho_{\mathrm{pol}}), [k|_{\mathrm{rad}(\rho_{\mathrm{pol}})}]).$$

Proof. Let $\rho_{\mathrm{pol}} = (V, h)$. For all $x \in \mathrm{rad}(\rho_{\mathrm{pol}})$ and $y \in U$ we have $h(x, y) = 0$ and hence $\rho_{\mathrm{pol}} \simeq (U, h|_U) \perp (\mathrm{rad}(\rho_{\mathrm{pol}}), h|_{\mathrm{rad}(\rho_{\mathrm{pol}})})$. The result then follows from (3.2). $\qquad\square$

Lemma 3.4. *Let $(V, [k])$ be a (λ, Λ)–unitary space over (D, θ). If U is a D–vector subspace of V such that $(U, [k|_U])$ is nonsingular, then $(U, [k|_U])$ is a subform of $(V, [k])$.*

Proof. See [5, Chapter 1 (5.4.1)]. $\qquad\square$

4. The Quadratic map associated to a unitary space

Let (D, θ) be an F–division algebra with involution and let $\rho = (V, [k])$ be a (λ, Λ)–unitary space over (D, θ). We define a map $q_\rho : V \to D/\Lambda$ by

$$q_\rho(x) = k(x, x) \mod \Lambda.$$

We call this map the *quadratic map associated to ρ*. For all $d \in D$ and $x \in V$ we have $q_\rho(xd) = \theta(d)q_\rho(x)d$. Moreover by [5, Chapter 1, (5.3)], the map q_ρ is related to $\rho_{\mathrm{pol}} = (V, h)$ through the formula

(4.1) $\qquad q_\rho(x + y) - q_\rho(x) - q_\rho(y) = h(x, y) \mod \Lambda, \quad$ for all $x, y \in V$.

The following is a straightforward generalisation of [3, (1.1)], where the result is proven only in the case of $\mathrm{char}(F) = 2$ and $\Lambda = \mathrm{Alt}(D, \theta)$.

Proposition 4.2. *Let $\rho = (V, [k])$ be a (λ, Λ)–unitary space over (D, θ) with associated quadratic map q_ρ and polar λ–hermitian space $\rho_{\mathrm{pol}} = (V, h)$. Excluding the case where $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ and $\Lambda = \mathrm{Sym}(D, \theta)$, we have that*

(i) the map h is uniquely determined by q_ρ and condition (4.1),

(ii) *for all* $x \in V$, $h(x, x) = \kappa + \lambda\theta(\kappa)$, *where* $\kappa \in D$ *is a representative of* $q(x) \in D/\Lambda$.

Proof. (i) Note first that as we are not in the case of $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ and $\Lambda = \mathrm{Sym}(D, \theta)$, we have that $\Lambda \neq D$. Suppose (V, h') is another λ–hermitian space satisfying (4.1). Then $h - h'$ is a λ–hermitian form with values in Λ. Since the set of values of a sesquilinear form is either $\{0\}$ or D, and since $\Lambda \neq D$, we must have that $h - h' = 0$.

(ii) Firstly we have that $q_\rho(x) = k(x, x) \bmod \Lambda$. Using (4.1) we have for all $x, y \in V$,

$$h(x, y) = k(x, y) + k(y, x) = k(x, y) + \lambda\theta(k(y, x)) \mod \Lambda.$$

By the uniqueness of h from part (i), it follows that

$$h(x, y) = k(x, y) + \lambda\theta(k(y, x)) \quad \text{for all } x, y \in V.$$

The result then easily follows. □

Corollary 4.3. *Assume that we are not in the case of* $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ *and* $\Lambda = \mathrm{Sym}(D, \theta)$. *Let* $\rho = (V, [k])$ *and* $\rho' = (V', [k'])$ *be* (λ, Λ)–*unitary spaces over* (D, θ). *Then* $\rho \simeq \rho'$ *if and only if there exists a* D–*vector space isomorphism* $\alpha : V \to V'$ *such that* $q_\rho(x) = q_{\rho'}(\alpha(x))$ *for all* $x \in V$.

Proof. By [5, Chapter 1, (5.3.4)], a D–vector space isomorphism $\alpha : V \to V'$ is an isometry $\rho \to \rho'$ if and only if it is an isometry of $\rho_{\mathrm{pol}} \to \rho'_{\mathrm{pol}}$ and $q_\rho(x) = q_{\rho'}(\alpha(x))$ for all $x \in V$. The 'only if' part of the statement follows immediately.

Conversely, assume that there exists a D–vector space isomorphism $\alpha : V \to V'$ such that $q_\rho(x) = q_{\rho'}(\alpha(x))$ for all $x \in V$. Let $\rho_{\mathrm{pol}} = (V, h)$ and $\rho'_{\mathrm{pol}} = (V', h')$. Then using (4.1) we have for all $x, y \in V$

$$h(x, y) = h'(\alpha(x), \alpha(y)) \mod \Lambda.$$

That α is an isometry of $\rho_{\mathrm{pol}} \to \rho'_{\mathrm{pol}}$ now follows by (4.2, (i)), and the result follows from [5, Chapter 1, (5.3.4)]. □

Note that if $(D, \theta) = (F, \mathrm{id})$, $\lambda = 1$ and $\Lambda = \mathrm{Alt}(D, \theta)$, then the quadratic map associated to a (λ, Λ)–unitary space over (D, θ) is a quadratic form over F (see [2, (7.1)]). In particular, (4.3) shows that in this case a unitary space can be considered equivalent to the concept of a quadratic form on an F–vector space.

Let ρ be a (λ, Λ)–unitary space over (D, θ). We call the set

$$\mathrm{rad}(\rho) = \{x \in \mathrm{rad}(\rho_{\mathrm{pol}}) \mid q_\rho(x) = 0\}$$

the *radical of* ρ. We say that ρ is *regular* if $\mathrm{rad}(\rho) = \{0\}$.

Remark 4.4. (4.3) shows that if we exclude the case of $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ and $\Lambda = \mathrm{Sym}(D, \theta)$, then when we consider unitary spaces up to isometry, we only need to consider the quadratic map associated to the space.

Note that in the case of $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ and $\Lambda = \mathrm{Sym}(D, \theta)$ we have that $\Lambda = \mathrm{Sym}(F, \mathrm{id}) = F$. Hence in this case the quadratic map associated to any (λ, Λ)–unitary space is trivial. In particular, any (-1)–hermitian form over (F, id) will satisfy (4.1). Therefore, by [5, Chapter 1, (5.3.4)], in this case any (λ, Λ)–unitary space ρ over (D, θ) is uniquely determined by ρ_{pol}. Further, in this case all regular (-1)–hermitian spaces are uniquely determined up to isometry by their dimension, as they are simply the hyperbolic skew–symmetric bilinear spaces (V, b)

such that $b(x,x) = 0$ for all $x \in V$ (see [2, (1.8)]). More explicitly, in this case any regular (λ, Λ)–unitary space is isometric to an orthogonal sum of hyperbolic planes (see below).

In the sequel, we will be mainly interested in unitary spaces only up to isometry, hence this case is rather trivial from our point of view. We will refer to it as the *negligible case*.

Remark 4.5. If char$(F) \neq 2$ or (D, θ) is of the second kind, then any (λ, Λ)–unitary space is uniquely determined by its polar λ–hermitian space (see [5, Chapter 1, (6.6.1)]), and in these cases we can consider the concepts of a (λ, Λ)–unitary space and of a λ–hermitian space as coinciding. In particular, in these cases a (λ, Λ)–unitary space is regular if and only if it is nonsingular.

Indeed, whenever $\Lambda = \mathrm{Sym}_{-\lambda}(D, \theta)$ any (λ, Λ)–unitary space is uniquely determined by its polar hermitian space (see [5, Chapter 1, (5.3.4)]). As this polar λ–hermitian space must be even, we can consider the concepts of a (λ, Λ)–unitary space and of an even λ–hermitian space as coinciding when $\Lambda = \mathrm{Sym}_{-\lambda}(D, \theta)$.

We say that ρ *represents an element* $a \in D$ if $q_\rho(x) = a + \Lambda$ for some $x \in V \setminus \{0\}$. We call ρ *isotropic* if there exists a vector $x \in V \setminus \{0\}$ such that $q_\rho(x) \in \Lambda$, and *anisotropic* otherwise. Assume that ρ is nonsingular and let $\rho_{\mathrm{pol}} = (V, h)$. Then we call a subspace $W \subset V$ *totally isotropic* (with respect to ρ) if $q_\rho|_W = 0$ and $h|_{W \times W} = 0$. We call ρ *hyperbolic* if there exists a totally isotropic subspace $W \subset V$ with $\dim_D(W) = \frac{1}{2}\dim(\rho)$. Note that for any nonsingular (λ, Λ)–unitary space ρ' over (D, θ), we have that $\rho' \perp (-\rho')$ is hyperbolic.

We denote by $\mathbb{H}_{(\lambda, \Lambda)}$ the (λ, Λ)–unitary space $(D^2, [k])$ over (D, θ) where $k : D^2 \to D$ is given by $(x_1, x_2) \times (y_1, y_2) \mapsto \theta(x_1)y_2$. We call a (λ, Λ)–unitary space isometric to $\mathbb{H}_{(\lambda, \Lambda)}$ a *hyperbolic plane*.

Lemma 4.6. $\mathbb{H}_{(\lambda, \Lambda)}$ *is nonsingular and hyperbolic.*

Proof. Let $\rho = \mathbb{H}_{(\lambda, \Lambda)}$. Then ρ_{pol} is given by the map $(x_1, x_2) \times (y_1, y_2) \mapsto \theta(x_1)y_2 + \lambda\theta(x_2)y_1$. Hence ρ_{pol} is nondegenerate and ρ is nonsingular. The quadratic map q_ρ is given by $(x_1, x_2) \to \theta(x_1)x_2$, hence the subspace of D^2 generated by $(1, 0)$ is totally isotropic and has dimension equal to $\frac{1}{2}\dim(\rho)$. $\square$

Proposition 4.7. *Let ρ be a nonsingular hyperbolic (λ, Λ)–unitary space over (D, θ). Then $\rho \simeq n \times \mathbb{H}_{(\lambda, \Lambda)}$, where $n = \frac{1}{2}\dim_D(\rho)$.*

Proof. It follows from [5, (5.6.1)] and induction on the dimension that a nonsingular hyperbolic (λ, Λ)–unitary space over (D, θ) is uniquely determined up to isometry by its dimension. The result then follows immediately from (4.6). $\square$

Proposition 4.8. *Let ρ be a nonsingular (λ, Λ)–unitary space over (D, θ). Then there exist an anisotropic nonsingular (λ, Λ)–unitary space ρ' over (D, θ) and a nonnegative integer n such that $\rho \simeq \rho' \perp n \times \mathbb{H}_{(\lambda, \Lambda)}$. Moreover, n is uniquely determined and ρ' is determined up to isometry by ρ.*

Proof. This follows immediately from [5, Chapter 1, (6.5.3)] and (4.7). $\square$

5. Witt Cancellation

Throughout this section, let (D, θ) be an F–division algebra with involution. The following results taken together are an extension of the 'Witt Cancellation'

result for nonsingular unitary spaces from [5, Chapter 1, (6.5.2)] to cover the case where the unitary spaces are potentially singular.

The proof of [5, Chapter 1, (6.5.2)] is not explicitly given, but it is noted that the proof of an analogous result for even hermitian forms, [5, Chapter 1, (6.4.2)], is straightforward to adapt to this case. In fact, such an adaptation does not require the assumption that the (λ, Λ)–unitary space is nonsingular.

Let $\rho = \mathbb{H}_{(\lambda,\Lambda)}$ and let $\rho_{\mathrm{pol}} = (D^2, h)$. By [5, Chapter 1, (5.6.2)], we can find vectors $x, y \in D^2$ such that $q_\rho(x), q_\rho(y) \in \Lambda$ and $h(x, y) = 1$. For a (λ, Λ)–unitary space ρ' over (D, θ) with $\rho'_{\mathrm{pol}} = (V, h')$, we call a pair $x, y \in V$ such that $q_{\rho'}(x), q_{\rho'}(y) \in \Lambda$ and $h'(x, y) = 1$ a *hyperbolic pair in ρ'*. Let H and H' be hyperbolic planes that are subforms of a (λ, Λ)–unitary space over (D, θ). We say H and H' are *adjacent* if there is a hyperbolic pair $\{x, y\}$ in H and a hyperbolic pair $\{x', y'\}$ in H' with a common element. We say that H and H' are *related* if there is a finite chain of adjacent hyperbolic planes connecting H and H'.

Lemma 5.1. *Let ρ be a (λ, Λ)–unitary space over (D, θ). Two hyperbolic planes H and H' that are subforms of ρ are always related.*

Proof. The proof follows similarly to the analogous hermitian space result in [5, Chapter 1, (6.4.3)]. □

For a (λ, Λ)–unitary space $(V, [k])$ over (D, θ) with polar λ–hermitian space (V, h) and a nonsingular subform $(U, [k'])$ of $(V, [k])$, we denote the set $\{x \in V \mid h(x, y) = 0$ for all $y \in U\}$ by $U^\perp$ and the (λ, Λ)–unitary space $(U^\perp, [k|_{U^\perp}])$ by $(U, [k'])^\perp$. This space exists and is a subform of $(V, [k])$ by (3.4).

Lemma 5.2. *Let ρ be a (λ, Λ)–unitary space over (D, θ). If H and H' are two adjacent hyperbolic planes that are subforms of ρ, then the (λ, Λ)–unitary spaces $H^\perp$ and $H'^\perp$ are isometric.*

Proof. The proof follows similarly to the analogous hermitian space result in [5, Chapter 1, (6.4.4)]. □

Corollary 5.3. *Let ρ, ρ_1 and ρ_2 be (λ, Λ)–unitary spaces over (D, θ) such that ρ is nonsingular. If $\rho_1 \perp \rho \simeq \rho_2 \perp \rho$ then $\rho_1 \simeq \rho_2$.*

Proof. As $\rho \perp (-\rho)$ is hyperbolic and $\rho_1 \perp \rho \perp (-\rho) \simeq \rho_2 \perp \rho \perp (-\rho)$, the result follows from (5.1) and (5.2). □

6. Diagonalisability of unitary spaces

Throughout this section, let (D, θ) be an F–division algebra with involution. For $a_1, \ldots, a_n \in D$ we denote the (λ, Λ)–unitary space $(D^n, [k])$ over (D, θ) with $[k]$ the class of $k : D^n \times D^n \to D$ given by

$$(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \mapsto \sum_{i=1}^n \theta(x_i) a_i y_i$$

by $\langle a_1, \ldots, a_n \rangle_{(\lambda,\Lambda)}$. We call such a space a *diagonal space*, and we call a (λ, Λ)–unitary form over (D, θ) *diagonalisable* if it is isometric to a diagonal space and we are not in the negligible case. Similarly, for $b_1, \ldots, b_n \in D \cap \mathrm{Sym}_\lambda(D, \theta)$ we denote the λ–hermitian space (D^n, h) over (D, θ) with $h : D^n \times D^n \to D$ given by

$$(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \mapsto \sum_{i=1}^n \theta(x_i) b_i y_i$$

by $\langle b_1, \ldots, b_n \rangle^{her}_{(\lambda,\theta)}$, and say that a λ–hermitian space is *diagonalisable* if it is isometric to such a space.

Remark 6.1. A nondegenerate even skew–symmetric bilinear form (V, b) over a field is never diagonalisable as $b(x, x) = 0$ for all $x \in V$ (see [2, (1.8)]). By (4.4), in the negligible case (λ, Λ)–unitary spaces are equivalent to even skew–symmetric bilinear spaces and hence only diagonalisable in the completely trivial case where the radical of the polar space is the whole vector space. This justifies our defining (λ, Λ)–unitary spaces as never diagonalisable in the negligible case.

Lemma 6.2. *Take $a_1, \ldots, a_n \in D$. Then the polar space of $\langle a_1, \ldots, a_n \rangle_{(\lambda,\Lambda)}$ is $\langle a_1 + \lambda\theta(a_1), \ldots, a_n + \lambda\theta(a_n) \rangle^{her}_{(\lambda,\theta)}$.*

Proof. It suffices to prove the result for $n = 1$. Let $a = a_1$ and $\rho = \langle a \rangle_{(\lambda,\Lambda)}$. Then for all $x, y \in D$ we have

$$
\begin{aligned}
q_\rho(x+y) - q_\rho(x) - q_\rho(y) &= \theta(x)ay + \theta(y)ax + \Lambda \\
&= \theta(x)ay + \theta(y)ax - (\theta(y)ax - \lambda\theta(x)\theta(a)y) + \Lambda \\
&= \theta(x)(a + \lambda\theta(a))y + \Lambda,
\end{aligned}
$$

as required. $\qquad\square$

Proposition 6.3. *Let ρ be a (λ, Λ)–unitary form over (D, θ). Then ρ is diagonalisable except if $(D, \theta) = (F, \mathrm{id})$, $\lambda = -1$ and ρ is not totally singular if $\mathrm{char}(F) = 2$.*

Proof. Every λ–hermitian space is isometric to the orthogonal sum of a totally degenerate space and a nonsingular space, and is therefore diagonalisable if the restriction to this nonsingular subspace is diagonalisable. As every nonsingular even λ–hermitian form is diagonalisable except if $(D, \theta) = (F, \mathrm{id})$ and $\lambda = -1$ by [5, Chapter 1 (6.2.4)], the result then follows from (3.2). $\qquad\square$

We finish this section with some remarks on the case of $\mathrm{char}(F) = 2$, $(D, \theta) = (F, \mathrm{id})$, $\Lambda \neq \mathrm{Sym}(D, \theta)$ and ρ nonsingular. That is, the non-negligible case in which we may not diagonalise our unitary spaces. Assume we are in this case. In particular, $\lambda = 1$, so we drop it from the notation. For $a, b \in F$, we denote the Λ–unitary space $(D^2, [k])$ over (D, θ) where k is given by $(x_1, x_2) \times (y_1, y_2) \to ax_1y_1 + x_1y_2 + bx_2y_2$ by $[a, b]_\Lambda$. It is easily checked that $[a, b]_\Lambda$ is nonsingular.

Lemma 6.4. *Assume that $\mathrm{char}(F) = 2$, $(D, \theta) = (F, \mathrm{id})$ and $\Lambda \neq \mathrm{Sym}(D, \theta)$. Let ρ be a nonsingular 2–dimensional Λ–unitary space over (D, θ). Then there exist $a, b \in F$ such that $\rho \simeq [a, b]_\Lambda$.*

Proof. Let $\rho_{\mathrm{pol}} = (V, h)$. Then as ρ is nonsingular, there exist some $v, w \in V \backslash \{0\}$ such that $h(v, w) = c \in F^\times$. By replacing v with vc^{-1}, we may assume that $c = 1$. Since ρ_{pol} is an even 1–hermitian space and $\mathrm{Alt}(F, \mathrm{id}) = \{0\}$, $h(v, w) \neq 0$ implies that $v \neq wd$ for any $d \in D$ and hence v and w are linearly independent over D. Let a and $b \in F$ be representatives of $q_\rho(x)$ and $q_\rho(y)$ respectively. Then we see immediately that ρ has the desired form. $\qquad\square$

Corollary 6.5. *Assume that $\mathrm{char}(F) = 2$, $(D, \theta) = (F, \mathrm{id})$ and $\Lambda \neq \mathrm{Sym}(D, \theta)$. Let ρ be a nonsingular Λ–unitary space over (D, θ). Then there exist an $n \in \mathbb{N}$ and elements $a_1 \ldots, a_n, b_1, \ldots, b_n \in F$ such that $\rho \simeq [a_1, b_1]_\Lambda \perp \ldots \perp [a_n, b_n]_\Lambda$.*

Proof. First note that as ρ_{pol} is a nondegenerate even hermitian space over (F, id), it must be even dimensional and is isometric to an orthogonal sum of 2–dimensional spaces by [2, (1.8)]. The result then follows by (3.2) and (6.4). $\qquad\square$

Lemma 6.6. *Assume that* $\mathrm{char}(F) = 2$, $(D, \theta) = (F, \mathrm{id})$ *and* $\Lambda \neq \mathrm{Sym}(D, \theta)$. *For all* $a, b \in F$

$$[a, b]_\Lambda \perp \langle a \rangle_\Lambda \simeq \mathbb{H}_\Lambda \perp \langle a \rangle_\Lambda \,.$$

Proof. Let $\rho = [a, b]_\Lambda \perp \langle a \rangle_\Lambda$ and let $\rho_{\mathrm{pol}} = (F^3, h)$. Then there exist vectors $v, w \in F^3 \backslash \{0\}$ such that $q_\rho(v) \in \Lambda$ and $h(v, w) = c \in F^\times$. That v and w are linearly independent follows as in (6.4). Replacing v by vc^{-1}, we may assume that $c = 1$. Let $\kappa \in F$ be a representative of $q_\rho(w) \in F/\Lambda$. Then we have $q_\rho(v), q_\rho(w - \kappa v) \in \Lambda$ and, as ρ_{pol} is even,

$$h(v, w - \kappa v) = h(v, w) + \kappa h(v, v) = 1 \,.$$

That is, $\{v, w - \kappa v\}$ is a hyperbolic pair in ρ. Hence $\rho \simeq \mathbb{H}_\Lambda \perp \langle d \rangle_\Lambda$ for some $d \in F$. It is clear that any isometry of ρ to another (λ, Λ)–unitary space ρ' over (F, id) maps $\mathrm{rad}(\rho_{\mathrm{pol}})$ to $\mathrm{rad}(\rho'_{\mathrm{pol}})$, hence we must have that $\langle a \rangle_\Lambda \simeq \langle d \rangle_\Lambda$. $\qquad\square$

7. Totally Singular unitary spaces

In this section we consider totally singular unitary spaces. Since totally singular spaces over (D, θ) are only of interest if $\mathrm{char}(F) = 2$ and (D, θ) is an F–division algebra with involution of the first kind, we assume we are in this case throughout the section. In particular, since we always have $\lambda = 1$, we drop it from our notation. Similarly, as these spaces are trivial in the negligible case by (4.4), we also assume throughout this section that we are not in this case.

Proposition 7.1. *Let* ρ *be a totally singular* Λ–*unitary space over* (D, θ). *Then there exists an anisotropic totally singular* Λ–*unitary space* ρ' *and a nonnegative integer* j *such that* $\rho \simeq \rho' \perp (j \times \langle 0 \rangle_\Lambda)$.

Proof. If $\rho = (V, [k])$ is anisotropic, then we are done. Otherwise, take $x \in V \backslash \{0\}$ such that $q_\rho(x) \in \Lambda$. Then $V \simeq U \oplus xD$ for a D–vector subspace U of V. As ρ_{pol} is trivial, it follows that $q_\rho(y + xd) = q_\rho(y) + \Lambda$ for all $y \in U$ and $d \in D$. Hence $\rho \simeq (U, [k|_U]) \perp \langle 0 \rangle_\Lambda$. The result follows by induction on the dimension of V. $\qquad\square$

Proposition 7.2. *Assume that* $(D, \theta) \neq (F, \mathrm{id})$. *Let* ρ *be a* Λ–*unitary space over* (D, θ). *Then* ρ *is totally singular if and only if every element represented by* ρ *is in* $\mathrm{Sym}(D, \theta)$.

Proof. Let $\rho = (V, [k])$ and $\rho_{\mathrm{pol}} = (V, h)$. Suppose ρ only represents elements in $\mathrm{Sym}(D, \theta)$. By (4.2, (ii)) this implies that $h(x, x) = 0$ for all $x \in V$. Therefore h is the zero map on $V \times V$ by (3.1). Conversely, if there exists a vector $x \in V \backslash \{0\}$ such that $q_\rho(x) \notin \mathrm{Sym}(D, \theta)/\Lambda$ then by (4.2, (ii)), $h(x, x) \neq 0$ and hence ρ is not totally singular. $\qquad\square$

Remark 7.3. (7.2) can also be shown using (6.3) and (6.2).

Corollary 7.4. *Let* ρ *be an anisotropic* n–*dimensional* Λ–*unitary space over* (D, θ). *Then* ρ *is totally singular if and only if there exist* $b_1, \ldots, b_n \in \mathrm{Sym}(D, \theta)$ *such that* $\rho \simeq \langle b_1, \ldots, b_n \rangle_\Lambda$.

Proof. If $(D, \theta) \neq (F, \mathrm{id})$, the result follows directly from (6.3) and (7.2).

If $(D, \theta) = (F, \mathrm{id})$, then $\mathrm{Sym}(D, \theta) = F$ and the result says that ρ is totally singular if and only if it is diagonalisable. That totally singular forms are diagonalisable is clear from (3.2). That diagonal forms are totally singular in this case is clear from (6.2). $\qquad\square$

Corollary 7.5. *Assume that $(D, \theta) \neq (F, \mathrm{id})$. Take $a \in D$. Then $\langle a \rangle_\Lambda$ is totally singular if and only if $a \in \mathrm{Sym}(D, \theta)$. Otherwise $\langle a \rangle_\Lambda$ is nonsingular and does not represent any elements in $\mathrm{Sym}(D, \theta)$.*

Proof. All elements represented by $\langle a \rangle_\Lambda$ are of the form $\theta(x)ax + b$ for $x \in D$ and $b \in \Lambda$. If $x \neq 0$, then $\theta(x)ax + b \in \mathrm{Sym}(D, \theta)$ if and only if $a \in \mathrm{Sym}(D, \theta)$. Hence $\langle a \rangle_\Lambda$ represents an element in $\mathrm{Sym}(D, \theta)$ if and only if all elements represented by $\langle a \rangle_\Lambda$ are in $\mathrm{Sym}(D, \theta)$. That is, if and only if $\langle a \rangle_\Lambda$ is totally singular by (7.2). That $\langle a \rangle_\Lambda$ is nonsingular if $\langle a \rangle_\Lambda$ is not totally singular is clear as $\dim(\langle a \rangle_\Lambda) = 1$. $\qquad\square$

8. Nonsingular spaces representing symmetric elements

In (7.2) we showed that, in characteristic 2, totally singular unitary spaces only represent symmetric elements. This means that the isotropy of an orthogonal sum of an anisotropic nonsingular unitary space and an anisotropic totally singular unitary space depends on the symmetric elements represented by the nonsingular space. Therefore, in this section, we investigate nonsingular spaces that represent symmetric elements.

Again, throughout we assume that $\mathrm{char}(F) = 2$, that (D, θ) is an F–division algebra with involution of the first kind and that we are not in the negligible case. As we always have $\lambda = 1$ we drop it from our notation.

Lemma 8.1. *For all elements $a \in D$ and $b \in \mathrm{Sym}(D, \theta)$ we have $\langle a, b \rangle_\Lambda \simeq \langle a + b, b \rangle_\Lambda$.*

Proof. Let $\rho = \langle a, b \rangle_\Lambda$. Let α be the map $\alpha : D^2 \to D^2$ given by left multiplication by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then for $(x, y) \in D^2$ we have

$$
\begin{aligned}
q_\rho(\alpha(x, y)) &= q_\rho((x, x + y)) + \Lambda \\
&= \theta(x)ax + \theta(x)bx + \theta(y)by + \theta(x)by + \theta(y)bx + \Lambda \\
&= \theta(x)(a + b)x + \theta(y)by + \Lambda.
\end{aligned}
$$

Hence α gives an isometry $\langle a, b \rangle_\Lambda \simeq \langle a + b, b \rangle_\Lambda$ by (4.3). $\qquad\square$

Lemma 8.2. *Assume $(D, \theta) \neq (F, \mathrm{id})$. Let ρ be a nonsingular anisotropic Λ–unitary space over (D, θ). Take $b \in \mathrm{Sym}(D, \theta)$. Then ρ represents b if and only if there exists an element $a \in D \setminus \mathrm{Sym}(D, \theta)$ and an anisotropic Λ–unitary space ρ' such that $\rho \simeq \langle a, a + b \rangle_\Lambda \perp \rho'$.*

Proof. That $\langle a, a + b \rangle_\Lambda$ represents b, and hence the 'if' implication in the statement, is clear. Assume now that ρ represents b. If $\dim(\rho) = 1$ then ρ representing an element in $\mathrm{Sym}(D, \theta)$ would contradict the non-singularity of ρ by (7.5). Therefore we must have that $\dim(\rho) > 1$.

If ρ only represents elements in $\mathrm{Sym}(D, \theta)$, then ρ is totally singular by (7.2). Hence there exists an element $a \in D \setminus \mathrm{Sym}(D, \theta)$ represented by ρ. Since $\langle a \rangle_\Lambda$ is

nonsingular by (7.5), it follows from (3.4) that $\rho \simeq \langle a \rangle_\Lambda \perp \rho'$ for some anisotropic Λ–unitary space $\rho' = (V, [k])$ over (D, θ). By our hypothesis, we have

$$0 \neq b = \theta(d)ad + q_{\rho'}(y) + \Lambda$$

for some $y \in V$ and $d \in D$. If $d = 0$, then ρ' represents b and the result follows from induction on $\dim(\rho)$.

If $d \neq 0$, then we may scale b and y in order to assume that $d = 1$. We must have that $y \neq 0$ as otherwise $\langle a \rangle_\Lambda$ represents $b \in \text{Sym}(D, \theta)$, which cannot occur by (7.5). Hence ρ' represents $a + b$. Therefore, again by (3.4), we have that $\rho' \simeq \langle a + b \rangle_\Lambda \perp \rho''$ for an anisotropic Λ–unitary space ρ'' over (D, θ). Hence $\rho \simeq \langle a, a + b \rangle_\Lambda \perp \rho''$. $\qquad \square$

Proposition 8.3. *Let ρ_1 and ρ_2 be anisotropic Λ–unitary spaces over (D, θ) such that ρ_1 is nonsingular and ρ_2 is totally singular. If $\rho \simeq \rho_1 \perp \rho_2$ is isotropic, then there exists a nonsingular anisotropic Λ–unitary space ρ_1' such that $\rho \simeq \mathbb{H}_\Lambda \perp \rho_1' \perp \rho_2$.*

Proof. First assume that $(D, \theta) = (F, \text{id})$. Then as $\rho_1 \perp \rho_2$ is isotropic, but ρ_1 and ρ_2 are anisotropic, they represent a common element $a \in F \backslash \Lambda$. Therefore $\rho_1 \simeq [a, b]_\Lambda \perp \rho_1'$ and $\rho_2 \simeq \langle a \rangle_\Lambda \perp \rho_2'$ for some $b \in F$ and Λ–unitary spaces ρ_1' and ρ_2' over (F, id). The result then follows from (6.6).

Now assume that $(D, \theta) \neq (F, \text{id})$. The isotropy of ρ and the anisotropy of $\rho_1 = (W_1, [k])$ and $\rho_2 = (W_2, [k'])$ imply that there exist vectors $x \in W_1 \backslash \{0\}$ and $x' \in W_2 \backslash \{0\}$ such that $q_{\rho_1}(x), q_{\rho_2}(x') \notin \Lambda$ and $q_{\rho_1}(x) + q_{\rho_2}(x') \in \Lambda$. By (7.2) we have that $q_{\rho_1}(x) = q_{\rho_2}(x') \in \text{Sym}(D, \theta)/\Lambda$.

Let $b \in \text{Sym}(D, \theta)$ be a representative of the class of $q_{\rho_1}(x)$ in D/Λ. Then by (8.2) there exists an element $a \in D \backslash \text{Sym}(D, \theta)$ and anisotropic Λ–unitary spaces ρ_1' and ρ_2' over (D, θ) with ρ_1' nonsingular and ρ_2' totally singular such that $\rho \simeq \langle a, a + b \rangle_\Lambda \perp \rho_1' \perp \langle b \rangle_\Lambda \perp \rho_2'$. Finally, (8.1) and (4.7) give

$$\langle a, a + b, b \rangle_\Lambda \simeq \langle a, a, b \rangle_\Lambda \simeq \mathbb{H}_\Lambda \perp \langle b \rangle_\Lambda$$

and hence

$$\rho \simeq \langle a, a \rangle_\Lambda \perp \rho_1' \perp \langle b \rangle_\Lambda \perp \rho_2' \simeq \mathbb{H}_\Lambda \perp \rho_1' \perp \rho_2 \,.$$

$\qquad \square$

9. Witt Decomposition of singular unitary spaces

In this section we give our generalisation of [4, (2.4)] to the setting of unitary spaces. Throughout this section, let (D, θ) be an F–division algebra with involution. The following is a generalisation of [4, (2.6)]. The proof of [4, (2.6)] is easily adapted to our setting, but we include it for convenience.

Lemma 9.1. *Let ρ_1 and ρ_2 be regular (λ, Λ)–unitary spaces over (D, θ). If $\rho_1 \perp (j \times \langle 0 \rangle_{(\lambda, \Lambda)}) \simeq \rho_2 \perp (j \times \langle 0 \rangle_{(\lambda, \Lambda)})$ for a nonnegative integer j, then $\rho_1 \simeq \rho_2$.*

Proof. By (3.3), (4.8) and (8.3) we can write $\rho_1 \simeq m \times \mathbb{H}_{(\lambda, \Lambda)} \perp \rho_1'$ and $\rho_2 \simeq n \times \mathbb{H}_{(\lambda, \Lambda)} \perp \rho_2'$ where m, n are nonnegative integers and ρ_1' and ρ_2' are anisotropic. We may assume that $m \geqslant n$. By (5.3) we have $(m - n) \times \mathbb{H}_{(\lambda, \Lambda)} \perp \rho_1' \simeq \rho_2'$, and hence we must have that $m = n$. Therefore it suffices to prove that for $\pi_1 = \rho_1' \perp (j \times \langle 0 \rangle_{(\lambda, \Lambda)})$ and $\pi_2 = \rho_2' \perp (j \times \langle 0 \rangle_{(\lambda, \Lambda)})$, we have that $\pi_1 \simeq \pi_2$ implies $\rho_1' \simeq \rho_2'$.

Let q_i be the quadratic map associated to $\rho_i' = (W_i, [k_i])$ and p_i be the quadratic map associated to $\pi_i = (V_i, [k_i])$ for $i = 1, 2$. Further, for $i = 1, 2$, let U_i be a D–vector space such that $V_i = W_i \oplus U_i$ and $(U_i, [k_i|_{U_i}]) = j \times \langle 0 \rangle_{(\lambda, \Lambda)}$. Now let

$\phi : V_1 \to V_2$ be a D–vector space isomorphism that is an isometry of π_1 and π_2. Let $\sigma : V_2 = W_2 \oplus U_2 \to W_2$ be a projection onto W_2 and define $\tau : W_1 \to W_2$ by $\tau = \sigma \circ \phi|_{W_1}$.

If $w \in W_1$ and $\phi(w) = w' + u'$, for some $w' \in W_2$ and $u' \in U_2$, then $\tau(w) = w'$ and thus

$$q_2(\tau(w)) = p_2(\tau(w)) = p_2(w') = p_2(w' + u') = p_2(\phi(w)) = p_1(w) = q_1(w),$$

where the third equality holds as $u' \in U_2 \subset V_2^\perp$ and $p_2(u') = 0$. To show that τ is an isometry, it suffices to show that τ is bijective. If $0 \neq w \in W_1$ then $q_1(w) \notin \Lambda$ as ρ_1' is anisotropic. Hence $0 \neq q_2(\tau(w)) = q_1(w) \notin \Lambda$ and in particular $\tau(w) \neq 0$. Therefore τ is injective, and hence bijective as $\dim_D(W_1) = \dim_D(W_2)$. $\square$

Theorem 9.2. *Let ρ be a (λ, Λ)–unitary space over (D, θ). Then there exists a nonsingular (λ, Λ)–unitary space ρ_1, a totally singular (λ, Λ)–unitary space ρ_2 and nonnegative integers i and j such that $\rho_1 \perp \rho_2$ is anisotropic and*

$$\rho \simeq \rho_1 \perp (i \times \mathbb{H}_{(\lambda, \Lambda)}) \perp \rho_2 \perp (j \times \langle 0 \rangle_{(\lambda, \Lambda)}).$$

The integers i and j are uniquely determined, and the (λ, Λ)–unitary spaces $\rho_1 \perp \rho_2$ and ρ_2 are uniquely determined up to isometry.

Proof. If $\mathrm{char}(F) \neq 2$ or (D, θ) is of the second kind, then the result follows immediately from (9.1) and (4.8). In particular, in this case ρ_2 is trivial. If we are in the negligible case, then we have that $\rho \simeq i \times \mathbb{H}_{(-1, F)} \perp j \times \langle 0 \rangle_{(-1, F)}$ by [2, (1.8)] and the result follows from (9.1).

Now assume that $\mathrm{char}(F) = 2$, (D, θ) is of the first kind and that we are not in the negligible case. Since $\lambda = 1$ in this case, we drop it from the notation. By (3.3), (4.8), (7.1) and (8.3) we need only prove the uniqueness of the decomposition.

Suppose

$$\rho \simeq \rho_1 \perp (i \times \mathbb{H}_\Lambda) \perp \rho_2 \perp (j \times \langle 0 \rangle_\Lambda) \simeq \rho_1' \perp (i' \times \mathbb{H}_\Lambda) \perp \rho_2' \perp (j' \times \langle 0 \rangle_\Lambda)$$

for nonnegative integers i, i', j and j', nonsingular Λ–unitary spaces ρ_1 and ρ_1' over (D, θ) and totally singular Λ–unitary spaces ρ_2 and ρ_2' over (D, θ). We must have that $j = j'$ as j and j' are the dimensions of the respective radicals, and any isometry must send radical to radical. It then follows from (9.1) that

$$\rho_1 \perp (i \times \mathbb{H}_\Lambda) \perp \rho_2 \simeq \rho_1' \perp (i' \times \mathbb{H}_\Lambda) \perp \rho_2'.$$

That $i = i'$ and $\rho_1 \perp \rho_2 \simeq \rho_1' \perp \rho_2'$ and then follows from (5.3). Finally, we must have that $\rho_2 \simeq \rho_2'$ as any isometry of (λ, Λ)–unitary spaces must map the radical of one polar hermitian space to the radical of the other polar hermitian space. $\square$

Note that, in the situation of (9.2), ρ_1 is generally not uniquely determined up to isometry, as we show in (10.6).

10. EXPLICIT EXAMPLES

Throughout this section we assume that $\mathrm{char}(F) = 2$ and that (D, θ) is an F–division algebra with involution of the first kind. As we again have that $\lambda = 1$ throughout, we drop it from the notation. The importance of (8.2) in the proof of (9.2) suggests the following question.

Question 10.1. Assume $(D, \theta) \neq (F, \mathrm{id})$. For which $a \in D \setminus \mathrm{Sym}(D, \theta)$ and $b \in \mathrm{Sym}(D, \theta)$ is $\langle a, a + b \rangle_\Lambda$ isotropic?

We now give an example of an F–division algebra with involution over which there exist both anisotropic and isotropic unitary spaces of the type in Question 10.1. This also provides an example showing that totally singular Λ–unitary spaces cannot be cancelled in general.

In the following proof we consider quadratic forms over a field. These are Λ–unitary spaces over (F, id) with $\Lambda = \{0\}$. In this case, for $a, b \in F$ we write $[a, b]$ for $[a, b]_{\{0\}}$ and $\langle a \rangle$ for $\langle a \rangle_{\{0\}}$.

Lemma 10.2. *Let* $\mathbb{F}_2$ *be the field of two elements and let* $F = \mathbb{F}_2(X)$, *where* X *is an indeterminate. Let* $Q = [X, 1 + X)_F$. *Then* Q *is an* F–*division algebra.*

Proof. By [2, (12.5)], Q is division if and only if $\pi = [1, X] \perp (1 + X) \cdot [1, X]$ is anisotropic. By [2, (23.11)], π is anisotropic if and only if $\rho = [1, X] \perp \langle 1 + X \rangle$ is anisotropic. It is clear that ρ is isotropic if and only if either $[1, X]$ represents $1 + X$ or $[1, X]$ is isotropic. First we show that $[1, X]$ does not represent $1 + X$.

By [2, (17.3)], $[1, X]$ represents $1 + X$ if and only if there exist elements $a, b \in \mathbb{F}_2[X]$, not both zero, such that $a^2 + ab + b^2 X = 1 + X$. Assume such elements exist and write $a = \sum_{i=0}^{n} a_i X^i$ and $b = \sum_{i=0}^{m} b_i X^i$ for some $n, m \in \mathbb{N}$ and $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathbb{F}_2$ and such that $a_n \neq 0$ and $b_m \neq 0$.

Assume first that $m \geqslant n > 0$. Then we get that $1 + X = b_m^2 X^{2m+1} + c$ where $\deg_X(c) < 2m + 1$. This contradicts $b_m \neq 0$. Now assume that $n > m > 0$. Then we have that $1 + X = a_n X^{2n} + c'$ where $\deg_X(c') < 2n$, contradicting $a_n \neq 0$. Therefore we must have that $a, b \in \mathbb{F}_2$. In particular, we must have that $b = 1$ and $a^2 + a = 1$. However, $x^2 + x = 1$ has no solution in $\mathbb{F}_2$, therefore $[1, X]$ does not represent $1 + X$. That $[1, X]$ is anisotropic can be shown in a similar way. $\square$

We denote the central simple F–algebra of $n \times n$ matrices over D by $M_n(D)$ and the $n \times n$ identity matrix by I_n. For a matrix $M \in M_n(D)$, let M^t denote the transpose of M and let M^* denote the image of M under the F–involution on $M_n(D)$ given by

$$((a_{ij})_{1 \leqslant i,j \leqslant n})^* = (\theta(a_{ij})_{1 \leqslant i,j \leqslant n})^t.$$

Let $\rho = (V, [k])$ be an n–dimensional Λ–unitary space over (D, θ). By [5, Chapter 1, (5.1.1)] one can find a matrix $M \in M_n(D)$ such that $q_\rho : V \to D/\Lambda$ is given by

$$(x_1, \ldots, x_n) \mapsto (\theta(x_1), \ldots, \theta(x_n)) M (x_1, \ldots, x_n)^t + \Lambda.$$

We call M a *matrix associated to* $(V, [k])$. Then if $\rho_{\mathrm{pol}} = (V, h)$, the map $h : V \times V \to D$ is given by

$$(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \mapsto (\theta(x_1), \ldots, \theta(x_n))(M + M^*)(y_1, \ldots, y_n)^t,$$

(see [5, Chapter 1, (5.3)]), and we say $M + M^*$ is a *matrix associated to the polar of* $(V, [k])$.

We now fix $\Lambda = \mathrm{Alt}(D, \theta)$. Let ρ be an n–dimensional nonsingular Λ–unitary space over (D, θ). Let $M \in M_n(D)$ be a matrix associated to ρ and let N be a matrix associated to the polar of ρ. As ρ is nonsingular we have that N is invertible and that at least one of $\deg(D)$ or $\dim(\rho)$ is even. Let $2m = \deg(D) \cdot \dim(\rho)$. We denote the set $\{a^2 + a \mid a \in F\}$ by $\wp(F)$. The *Arf invariant of* ρ is then defined as the class in $F/\wp F$ given by

$$\mathrm{Srd}_{M_n(D)}(N^{-1} \cdot M) + \frac{m(m-1)}{2} + \wp(F).$$

We denote this class by $\Delta(\rho)$. By [9, Corollaire 4], $\Delta(\rho)$ depends only on the isometry class of ρ and not on the choice of M.

Lemma 10.3. *Fix $\Lambda = \mathrm{Alt}(D,\theta)$ and let ρ_1 and ρ_2 be nonsingular Λ–unitary spaces over (D,θ). Then $\Delta(\rho_1 \perp \rho_2) = \Delta(\rho_1) + \Delta(\rho_2)$.*

Proof. This follows from [3, (2.1)] and [6, (7.14)]. □

Lemma 10.4. *Fix $\Lambda = \mathrm{Alt}(D,\theta)$ and let ρ be a hyperbolic nonsingular Λ–unitary space over (D,θ). Then $\Delta(\rho) \in \wp(F)$.*

Proof. This follows from [3, (1.9) and (2.1)] and [6, (7.9)]. □

The following gives an example of an anisotropic and a hyperbolic Λ–unitary space of the type considered in Question 10.1.

Example 10.5. Let $\mathbb{F}_2$ be the field of two elements and let $F = \mathbb{F}_2(X)$, where X is an indeterminate. Let $Q = [X, 1 + X)_F$. Then Q is an F–division algebra by (10.2). Let γ be the unique symplectic involution on Q. Then there exist elements $u, v \in Q$ such that $u^2 = u + X$, $\gamma(u) = 1 + u$, $v^2 = 1 + X$ and $\gamma(v) = v$. Recall that $\mathrm{Alt}(Q, \gamma) = F$ and note that $u \notin \mathrm{Sym}(Q, \gamma)$ and $v, v + uv \in \mathrm{Sym}(Q, \gamma) \backslash \mathrm{Alt}(Q, \gamma)$.
Fix $\Lambda = \mathrm{Alt}(Q, \gamma)$ and let

$$\rho = \langle u \rangle_\Lambda, \rho_1 = \langle u + v \rangle_\Lambda \text{ and } \rho_2 = \langle u + v + uv \rangle_\Lambda.$$

By (6.2), $\langle 1 \rangle_\Lambda^{her}$ is the polar form of each of ρ, ρ_1 and ρ_2 and therefore ρ, ρ_1 and ρ_2 are all nonsingular. It is also clear that ρ, ρ_1 and ρ_2 are all anisotropic as they are 1-dimensional and $u, u + v$ and $u + v + uv \notin \Lambda$. Note that as Q is a quaternion algebra, we have that $\mathrm{Srd}_Q = \mathrm{Nrd}_Q$. Hence $\Delta(\rho) = \mathrm{Nrd}_Q(1 \cdot u) = X$. Similarly, $\Delta(\rho_1) = 1$ and $\Delta(\rho_2) = X^2 = X \mod \wp(F)$.

It follows from an argument similar to that in the proof of (10.2) that $[1, 1 + X]$ is anisotropic, and hence $1 + X \notin \wp(F)$. Therefore it follows from (10.3) that $\Delta(\rho \perp \rho_1) = 1 + X \notin \wp(F)$. Hence $\rho \perp \rho_1$ is not hyperbolic by (10.4) and is therefore anisotropic as $\dim(\rho \perp \rho_1) = 2$.

Finally as

$$\gamma(u + v)u(u + v) = u + v + uv + (1 + X)$$

and $1 + X \in \mathrm{Alt}(Q, \gamma)$, we have that $\rho \perp \rho_2$ is isotropic. That $\rho \perp \rho_2$ is hyperbolic follows as $\dim(\rho \perp \rho_2) = 2$.

The following example shows that totally singular unitary spaces cannot be cancelled in general.

Example 10.6. Let $F, (Q, \gamma), \Lambda, \rho$ and ρ_1 all be as in (10.5). Then $\langle v \rangle_\Lambda$ is totally singular by (7.5). Further, $\rho \perp \langle v \rangle_\Lambda \simeq \rho_1 \perp \langle v \rangle_\Lambda$ by (8.1). If $\rho \simeq \rho_1$, then $\rho \perp \rho_1$ is hyperbolic, but $\rho \perp \rho_1$ is anisotropic by (10.5). Hence $\rho \not\simeq \rho_1$.

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