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Coflat monomorphisms of coalgebras

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Abstract

We consider left coflat monomorphisms of coalgebras, and establish a 1–1 correspondence between the set of isomorphism classes of left coflat monomorphisms, the set of some coidempotent subcoalgebras and the set of equivalence classes of perfect localization bicomodules as well. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

In this paper, we consider coflat monomorphisms of coalgebras. These are dual to the perfect localizations of algebras (or the flat epimorphisms of algebras). If $\phi : C \longrightarrow D$ is a left coflat monomorphism of coalgebras, then ϕ determines a Morita–Takeuchi context

$$(C, D, {}_CU_D, {}_DC_C, f, g)$$

and the bicolinear map f is an isomorphism. It follows that C is quasi-finite as a left D-comodule, and the coendomorphism coalgebra of the left D-comodule ${}_DC$ is canonically isomorphic to C (cf. Theorem 3.4 and Corollary 3.5). It has been shown in [2] that a left coflat monomorphism $\phi : C \longrightarrow D$ of coalgebras determines a hereditary torsion theory $\mathrm{Ker}(-)^{\phi}$, $(-)^{\phi} = - \square_D C$, of the comodule category $\mathbf{M}^D$, and any hereditary torsion theory is uniquely determined by a coidempotent subcoalgebra A of D. It is natural to ask what conditions on A allow to reconstruct the coflat monomorphism ϕ. This leads us to define localization bicomodules of a coalgebra D in Section 2. We

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show that coidempotent subcoalgebras of D bijectively correspond to localization bicomodules. This correspondence yields a 1–1 correspondence between left coflat monomorphisms and the so-called left perfect localizations which precisely answers the question.

1. Preliminaries

Throughout, k is a fixed field. All coalgebras, algebras, vector spaces, and unadorned $\otimes$, Hom, etc. are over k. Throughout, Λ, Γ, C and D always stand for coalgebras. The character $\mathbf{M}$ indicates the category of k-modules. We refer to [4] for detail of coalgebras and comodules. If C is a coalgebra, we denote by $\mathbf{M}^C$ the category of right C-comodules. Similarly, we let ${}^C\mathbf{M}$ stand for the left C-comodule category. A right C-comodule X is injective (or C-injective) if the functor $\mathrm{Com}_{-C}(-,X)$ is exact.

A C–D-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_CX_D$, such that the C-comodule structure map $\rho_C : X \longrightarrow C\otimes X$ is D-colinear, or equivalently the D-comodule structure map $\rho_D : X \longrightarrow X \otimes D$ is C-colinear. In particular, C is a C–C-bicomodule through Δ.

Cotensor product. For a right C-comodule M and a left C-comodule N, the cotensor product $M \,\square_C\, N$ is the kernel of

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \rightrightarrows M \otimes C \otimes N.$$

The functors $M \,\square_C\, -$ and $- \,\square_C\, N$ are left exact and preserve direct sums. If ${}_\Lambda X_C$ and ${}_C Y_\Gamma$ are bicomodules, then $X \,\square_C\, Y$ is a $\Lambda - \Gamma$-bicomodule induced by the structure maps: $\rho_\Lambda : X \longrightarrow \Lambda \otimes X$ and $\rho_\Gamma : Y \longrightarrow Y \otimes \Gamma$. The cotensor product is associative. For comodules X_C and ${}_CY$ the structure maps ρ_X and ρ_Y induce C-colinear isomorphisms $X \simeq X \,\square_C\, C$ and $Y \simeq C \,\square_C Y$. If X is a right C-comodule which is finite dimensional as vector space, then the dual X^* is a left C-comodule with structure map

$$X^* \longrightarrow \mathrm{Com}_{-C}(X,C) \hookrightarrow \mathrm{Hom}(X,C) \simeq C \otimes X^*, \quad x^* \mapsto (x^* \otimes 1)\rho_X.$$

If Y is a right C-comodule, then we have the canonical isomorphism

$$Y \square_C X^* \simeq \mathrm{Com}_{-C}(X,Y). \tag{1}$$

If, moreover, Y is a D–C-bicomodule, then $\mathrm{Com}_{-C}(X,Y)$ is a left D-comodule induced by (1). A right C-comodule Y is called a coflat comodule if the functor $Y \square_C -$ is exact. Since every comodule is the union of its finite-dimensional subcomodules, it follows from (1) that Y_C is coflat if, and only if, $\mathrm{Com}_{-C}(-,Y)$ is exact if and only if Y is C-injective, cf. [6].

Co-hom functor. A comodule X_C is quasi-finite if $\mathrm{Com}_{-C}(Y,X)$ is finite-dimensional for every finite-dimensional comodule Y_C. We recall from [5] the definition of the co-hom functor and some of its basic properties.

Basic lemma. *Let ${}_CX_D$ be a bicomodule. Then X_D is quasi-finite if and only if the functor $-\,\square_C X : \mathbf{M}^C \longrightarrow \mathbf{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X,-)$. That is, for comodules Y_D and W_C,*

$$\mathrm{Com}_{-C}(h_{-D}(X,Y),W) \simeq \mathrm{Com}_{-D}(Y, W\,\square_C\, X). \tag{2}$$

Assume that X_D is a quasi-finite comodule, then $e_{-D}(X) = h_{-D}(X,X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1\otimes\theta)\theta : X \longrightarrow e_{-D}(X)\otimes e_{-D}(X)\otimes X$ in (2) when $C = k$, and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X. X is an $e_{-D}(X)-D$-bicomodule with the left comodule structure map θ given by the canonical map $X \longrightarrow h_{-D}(X,X) \otimes X$.

Morita–Takeuchi (M–T) context. An M–T context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ consists of coalgebras C, D, bicomodules ${}_CP_D, {}_DQ_C$, and bicolinear maps $f: C \longrightarrow P\,\square_D\, Q$ and $g: D \longrightarrow Q\,\square_C\, P$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\sim} & P\square_D D \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle 1\square g} \\ C\square_C P & \xrightarrow{f\square 1} & P\square_D Q\,\square_C P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\sim} & Q\square_C C \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle 1\square f} \\ D\square_D Q & \xrightarrow{g\square 1} & Q\square_C P\square_D Q \end{array}$$

The context is said to be *strict* if both f and g are injective (equivalently, isomorphic). In this case, we say that C is M–T equivalent to D. Let P_D be a quasi-finite comodule and $C = e_{-D}(P)$. Then ${}_CP_D$ is a bicomodule. Set ${}_DQ_C = h_{-D}(P,D)$, $g = \theta : D \longrightarrow Q\,\square_C\, P$, and $f: C \cong h_{-D}(P, P\,\square_D\, D) \xrightarrow{\partial} P\,\square_D\, h_{-D}(P,D) = P\,\square_D\, Q$. Then $(C, D, {}_CP_D, {}_DQ_C, f, g)$ is an M–T context, where f is injective if and only if P_D is injective, and g is injective if and only if P_D is a cogenerator in $\mathbf{M}^D$, cf. [5].

Let $\phi : C \longrightarrow D$ be a coalgebra map. Every right C-comodule X may be viewed as a right D-comodule with the structure map

$$(1\otimes\phi)\rho : X \longrightarrow X\otimes C \longrightarrow X \otimes D.$$

In this case, we will say that X_C restricts to the right D-comodule X_D. The map ϕ induces a left exact (restriction) functor:

$$(-)_\phi : \mathbf{M}^C \longrightarrow \mathbf{M}^D.$$

Let us recall from [2] the relation between monomorphisms of coalgebras and torsion theories in a comodule category. A coalgebra map $\phi: C \longrightarrow D$ is said to be a monomorphism if it is a monomorphism in the coalgebra category $\mathbf{Cog}_k$. Let $(-)^\phi$ be the cotensor functor

$$\mathbf{M}^D \longrightarrow \mathbf{M}^C, \quad M \mapsto M \,\square_D\, C.$$

Theorem 1.1 (Nastasescu and Torrecillas [2, Theorem 3.5]). *Let ϕ $C \longrightarrow D$ be a coalgebra map. The following are equivalent*:

(1) *ϕ is a monomorphism in* $\mathbf{Cog}_k$.

(2) $C \,\square_D \operatorname{Ker}\phi = 0$.

(3) *The canonical morphism $\overline{\Delta} : C \longrightarrow C \,\square_D\, C$ is an isomorphism.*

(4) *The restriction functor $(-)_\phi : \mathbf{M}^C \longrightarrow \mathbf{M}^D$ is full.*

(5) *The canonical functorial morphism $I_{\mathbf{M}^C} \longrightarrow (-)^\phi \circ (-)_\phi$ is an isomorphism.*

Note that conditions (4) and (5) in the above theorem may be replaced by the left comodule versions since condition (3) is symmetric. Let D be a coalgebra, $\mathbf{M}^D$ the comodule category. A subcategory $\mathscr{C}$ of $\mathbf{M}^D$ is a closed subcategory if $\mathscr{C}$ is closed under subobjects, quotient objects and direct sums. If, in addition, $\mathscr{C}$ is closed under extensions, then $\mathscr{C}$ is called a localizing subcategory. We refer to [3] for detail on (hereditary) torsion theories. A subcoalgebra A of D is said to be coidempotent if $A = A \wedge A = \operatorname{Ker}(D \xrightarrow{\Delta} D/A \otimes D/A)$.

Theorem 1.2 (Nastasescu and Torrecillas [2, Theorems 4.2, 4.5]). *Let D be a coalgebra and A be a subcoalgebra of D. We denote by $\mathscr{T}_A = \{M \in \mathbf{M}^D | \rho_M(M) \subseteq M \otimes A\}$. Then*

(1) *$\mathscr{T}_A$ is a closed subcategory of $\mathbf{M}^D$.*

(2) *The map $A \mapsto \mathscr{T}_A$ is a bijective map between the set of all subcoalgebras of D and the set of all closed subcategories of $\mathbf{M}^D$.*

(3) *$A \mapsto \mathscr{T}_A$ gives an one-to-one correspondence between the set of coidempotent subcoalgebras of D and the set of localizing subcategories of $\mathbf{M}^D$.*

(4) *All the localizing subcategories of $\mathbf{M}^D$ are hereditary torsion theories.*

Note that the theorem still holds if one considers the left comodule category ${}^D\mathbf{M}$.

Let $\phi : C \longrightarrow D$ be a coalgebra map. ϕ is said to be a left coflat monomorphism if ϕ is a monomorphism and the comodule ${}_DC$ is coflat. Let ϕ be a left coflat monomorphism. The canonical functor:

$$(-)^\phi : \mathbf{M}^D \longrightarrow \mathbf{M}^C, \quad X \mapsto X \square_D C$$

is an exact functor that commutes with direct sums. It follows that the kernel $\operatorname{Ker}(-)^\phi = \mathscr{T}$ is a localizing subcategory of $\mathbf{M}^D$. By [2, Theorem 4.5] there exists a unique coidempotent subcoalgebra A of D such that

$$\mathscr{T} = \{M \in \mathbf{M}^D | \rho(M) \subseteq M \otimes A\} = \mathbf{M}^A.$$

Let us denote it by $\mathscr{T}_A$. Since $\mathscr{T}_A$ is closed under products $\mathscr{T}_A$ is a hereditary torsion theory and it is a TTF class. Note that A is a subcoalgebra of D. Hence, ${}_A\mathscr{T}$ is a hereditary torsion theory and a TTF class in ${}^D\mathbf{M}$.

2. Localization bicomodules

In this section, we define (perfect) localizations and show that any left coflat monomorphisms $\phi: C \longrightarrow D$ of coalgebras comes from some coidempotent subcoalgebra of D. There is ono-to-one correspondence between the set of left coflat monomorphisms to D and the set of equivalence classes of perfect localization bicomodules.

Let D be a coalgebra. By a *localization bicomodule*, we mean a pair (U,ψ) of a D-bicomodule U and a D-bicomodule map $\psi: D \longrightarrow U$ such that $U \square_D \psi$ and $\psi \square_D U$ are isomorphisms.

First, we establish a correspondence between localization bicomodules of a coalgebra D and coidempotent subcoalgebras of D.

Lemma 2.1. *Let (U,ψ) be a localization bicomodule of D. Then* $\operatorname{Ker}\psi$ *is a coidempotent subcoalgebra of D.*

Proof. $A = \operatorname{Ker}\psi$ is a subcoalgebra since ψ is D-bicolinear. Let $\mathscr{T}_A$ be the category $\mathbf{M}^A$ of right A-comodules. If $X \in \mathbf{M}^D$, then $X \in \mathscr{T}_A$ if and only if $X \square_D U = 0$. Indeed, if $X \in \mathscr{T}_A$, then $X \square_D U \simeq X \square_A A \square_D U = 0$ since $A \square_D U = 0$. Conversely, if $X \square_D U = 0$, then $X \square_D A \simeq X$, and $X \in \mathscr{T}_A$. Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence in $\mathbf{M}^D$ such that X, Z are in $\mathscr{T}_A$, then $X \square_D U = 0 = Z \square_D U$. This implies that $Y \square_D U = 0$, and Y is in $\mathscr{T}_A$. This means that $\mathscr{T}_A$ is closed under extension. By [2, Theorem 4.5] A is a coidempotent subcoalgebra of D. □

Lemma 2.2. *Let (U,ψ) be a localization D-bicomodule, A the coidempotent subcoalgebra* $\operatorname{Ker}\psi$. *If $X \xrightarrow{f} Y$ is a left (or right) D-comodule map, then* $\operatorname{Ker} f, \operatorname{Coker} f$ *are in ${}_A\mathscr{T}$ (or $\mathscr{T}_A$) if and only if $U \square_D f$ (or $f \square_D U$) is an isomorphism.*

Proof. By the argument in the proof of Lemma 2.1, a D-comodule is torsion (or in ${}_A\mathscr{T}$) iff $U \square_D X = 0$. Since $\operatorname{Ker} f$ is torsion, we have $U \square_D \operatorname{Ker} f = 0$ and hence, $U \square_D f$ is injective. To show that $U \square_D f$ is surjective, we may assume thet f is surjective since $U \square_D f(X) = U \square_D Y$ (because $U \square_D \operatorname{Coker} f = 0$). Now, the map

$$\psi \square_D X : X \longrightarrow U \square_D X$$

restricts to zero on $\operatorname{Ker} f$ since $\operatorname{Ker} f$ is torsion and $U \square_D X$ is torsion free as left D-comodules. Hence, it factors through f, and we may write

$$\psi \square_D X = h \circ f, \quad h: Y \longrightarrow U \square_D X.$$

Thus, we obtain

$$\psi \square_D Y = (U \square_D f) \circ h : Y \longrightarrow U \square_D Y.$$

Hence, $U \square_D \psi \square_D Y = (U \square_D U \square_D f) \circ (U \square_D h)$. But this is an isomorphism since $U \square_D \psi$ is. It follows that $U \square_D U \square_D f$ is surjective. Since $U \simeq U \square_D U$ as

D-bicomodule, we obtain that $U\Box_D f$ should be surjective, and we conclude that $U\Box_D f$ is an isomorphism.

Now, let $0 \longrightarrow \mathrm{Ker} f \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$ be an exact sequence of left D-comodules such that $U\Box_D f$ is an isomorphism. Since $U\Box_D -$ is a left exact functor $U\Box\, \mathrm{Ker} f = 0$, that is, $\mathrm{Ker} f$ is torsion. For any object $X \in^D \mathbf{M}$, we have a left D-colinear map $\psi_X = \psi\Box_D X : X \longrightarrow U\Box_D X$. Since $U\Box_D \psi_X$ is an isomorphism, $\mathrm{Ker}\, \psi_X$ is torsion. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 = & U\Box_D \mathrm{Ker} f & \longrightarrow & U\Box_D X & \xrightarrow[\sim]{U\Box f} & U\Box_D Y & \xrightarrow{U\Box p} & U\Box_D \mathrm{Coker}\, f & \\
& \uparrow \psi_{\mathrm{Ker} f} & & \uparrow \psi_X & & \uparrow \psi_Y & & \uparrow \psi_{\mathrm{Coker}\, f} & \\
0 \longrightarrow & \mathrm{Ker} f & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{p} & \mathrm{Coker}\, f & \longrightarrow 0
\end{array}$$

Since $U\Box_D f$ is an isomorphism, $U\Box_D p$ should be zero. It follows from the above diagram that $\psi_{\mathrm{Coker}\, f}$ is zero. But the kernel of $\psi_{\mathrm{Coker}\, f}$ is torsion, and hence, $\mathrm{Coker}\, f$ is torsion. $\Box$

For hereditary torsion theory $\mathscr{T}_A$, one may form a quotient category $\mathbf{M}^D/\mathscr{T}_A$, denoted by $\overline{\mathbf{M}^D}$. Let (T_A, S_A) be an adjoint pair of canonical functors. We have

Lemma 2.3. *The section functor $S_A : \mathbf{M}^D/\mathscr{T}_A \longrightarrow \mathbf{M}^D$ preserves direct sums and quasi-finiteness.*

Proof. Let $\{X_i\}$ be a family of objects in $\mathbf{M}^D/\mathscr{T}_A$. We have the canonical map

$$0 \longrightarrow \bigoplus_i S_A(X_i) \xrightarrow{\alpha} S_A(\bigoplus_i X_i),$$

where α is injective since $\mathrm{Ker}\,\alpha$ is torsion and $\bigoplus_i S_A(X_i)$ is torsion free. Suppose that α is not surjective. Since $\mathbf{M}^D$ is locally Noetherian, there exists a set of Noetherian generators $\{V_j\}$. If the monomorphism α is not epic, then there is some $V_l \in \{V_j\}$ and a non-zero morphism $f : V_l \longrightarrow S_A(\bigoplus_i X_i)$ such that f cannot factor through α. However, since V_l and $T_A(V_l)$ are Noetherian, we have

$$\begin{aligned}
\mathrm{Com}_{-D}(V_l, S_A(\bigoplus_i X_i)) &\simeq \mathrm{Hom}(T_A(V_l), \oplus_i X_i) \\
&\simeq \bigoplus_i \mathrm{Hom}(T_A(V_l), X_i) \\
&\simeq \bigoplus_i \mathrm{Com}_{-D}(V_l, S_A(X_i)) \\
&\simeq \mathrm{Com}_{-D}(V_l, \bigoplus S_A(X_i)),
\end{aligned}$$

where Hom means the Hom in $\mathbf{M}^D/\mathscr{T}_A$. It follows that f factors through α, a contradiction. So α is an isomorphism. Note that S_A is a right adjoint functor of T_A and T_A preserves objects of finite dimensions. These facts yield that S_A respects quasi-finiteness. $\square$

Lemma 2.4. *Let A be a coidempotent subcoalgebra of D. Then $S_AT_A(D)$ together with the canonical adjunction map $\psi: D \longrightarrow S_AT_A(D)$ is a localization bicomodule. Moreover, by symmetry, $({}_AS_AT(D), \psi')$ is a localization bicomodule and there is a bicolinear isomorphism $\theta: S_AT_A(D) \longrightarrow {}_AS_AT(D)$ such that $\psi' = \theta \circ \psi$.*

Proof. Let A be a coidempotent subcoalgebra of D. The localization functor S_AT_A: $\mathbf{M}^D \longrightarrow \mathbf{M}^D$ is a left exact functor and preserves direct sums by Lemma 2.3. So it is of form $- \square_D U$ for some D-bicomodule U by [5, 2.1]. In this case, the adjunction $\psi : I_D \longrightarrow S_AT_A$ is represented by a D-bicomodule map $\psi: D \longrightarrow U$. Since a comodule X is torsion if and only if $S_AT_A(X) = 0$, we obtain that X is torsion iff $X \square_D U = 0$. Now $\operatorname{Ker}\psi$ and $\operatorname{Coker}\psi$ are torsion. By Lemma 2.2, we obtain that $\psi \square_D U$ is an isomorphism. To show $U \square_D \psi$ is also an isomorphism, we consider the difference map

$$f = U \square_D \psi - \psi \square_D U : U \longrightarrow U \square_D U,$$

which is obviously right and left D-colinear. It is clear that $f \circ \psi = 0$. So f factors through $\operatorname{Coker}\psi$ which is torsion. Since $U \square_D U$ is torsion free as a right D-comodule, any right D-colinear map from $\operatorname{Coker}\psi$ to $U \square_D U$ should be zero. It follows that $f = 0$. Therefore (U, ψ) is a localization

By symmetry, $({}_AS_AT(D), \psi)$ is a localization bicomodule. Let U' be ${}_AS_AT$. Since $\operatorname{Ker}\psi'$ and $\operatorname{Coker}\psi'$ are in ${}_A\mathscr{T}$, By Lemma 2.2, we obtain that $U \square_D \psi: U \longrightarrow U \square_D U'$ is a bicolinear isomorphism. By symmetry, $\psi \square U'$ is a bicolinear isomorphism too. Let θ be $(\psi \square U')^{-1} \circ (U \square \psi')$. Then $\theta: U \longrightarrow U'$ is a bicolinear isomorphism such that $\psi' = \theta \circ \psi$. $\square$

From the proof of Lemma 2.4, we obtain that $\psi \square U = U \square \psi$ if (U, ψ) is a localization bicomodule. Two localizations (U, ψ) and (U', ψ') are *equivalent* if there exists an D-bicolinear isomorphism $\mu : U \longrightarrow U'$ such that $\psi = \psi'\mu$. Let Ł be the set of equivalence classes of localization bicomodules of coalgebra D. Denote by $\mathscr{C}$ the set of coidempotent subcoalgebras of D. Now we are allowed to define two maps Φ and Ψ as follows:

- Φ: Ł $\longrightarrow \mathscr{C}$; $(U, \psi) \mapsto \operatorname{Ker}\psi$, and
- Ψ: $\mathscr{C} \longrightarrow$ Ł; $A \mapsto S_AT_A(D)$, where S_AT_A is the localizing functor associated to A.

Theorem 2.5. *Let D be a coalgebra. The maps Φ and Ψ defined as above are isomorphisms and inverse to each other.*

Proof. Given a coidempotent subcoalgebra A of D, we have to show that $\Psi\Phi(A) = A$. Let S_AT_A be the localizing functor with respect to the torsion theory $\mathscr{T}_A$, and let

$\psi : D \longrightarrow U = S_A T_A(D)$ be the representing bicolinear D-map. We have to show that $\mathrm{Ker}\,\psi = A$. We know that $\mathrm{Ker}\,\psi$ is a subcoalgebra of D which is torsion, i.e, a right A-comodule. This implies that $\mathrm{Ker}\,\psi \subseteq A$. On the other hand, $\mathrm{Ker}\,\psi$ is the maximal torsion subcomodule of D since ψ is the adjunction map. But A is obviously a torsion subcomodule of D. It follows that $A \subseteq \mathrm{Ker}\,\psi$. Therefore, $A = \Psi\Phi(A)$.

Conversely, suppose that (U, ϕ) is a localization bicomodule of D. $A = \mathrm{Ker}\phi$ is a coidempotent subcoalgebra. Let $S_A T_A$ be the localizing functor with respect to the torsion theory $\mathscr{T}_A$. Let $\psi : D \longrightarrow S_A T_A(D)$ be the adjunction map with which $\mathrm{Coker}\,\psi$ and $\mathrm{Ker}\,\psi$ are torsion. By Lemma 2.2, $\psi \,\square_D\, U$ is an isomorphism. On the other hand, $(S_A T_A(D), \psi)$ is a localization bicomodule, and $\mathrm{Ker}\,\phi$, $\mathrm{Coker}\,\phi$ are torsion by Lemma 2.2, we have that $S_A T_A(D) \,\square_D\, \phi$ is an isomorphism by Lemma 2.2. This gives bicolinear isomorphism from U to $S_A T_A(D)$, and hence $\Phi\Psi([U]) = [U]$, where $[U]$ represents the equivalence class of U. □

A localization bicomodule (U, ψ) is called a left *perfect localization* if U_D is quasi-finite and injective as a right D-comodule.

Let (U, ψ) be a left perfect localization. Since U_D is quasi-finite and injective, by [5, 2.5] we may associate an MT-context to U_D

$$(C, D, {}_C U_D, {}_D Q_C, f, g)$$

such that $f : C \xrightarrow{\simeq} U \,\square_D\, Q$, where $C = e_{-D}(U_D)$. The bicomodule structure of ${}_D U_D$ induces a coalgebra map $\phi : C \longrightarrow D$. It is easy to check that $\psi \,\square_D C : C \longrightarrow U \square_D C$ is the following composite D-bicolinear isomorphism:

$$C \xrightarrow{f} U \,\square_D Q \xrightarrow{\psi \square U \square Q} U \,\square_D\, U \,\square_D\, Q \xrightarrow{U \square f^{-1}} U \,\square_D C.$$

We show that ϕ is a left coflat monomorphism

Lemma 2.6. *Let (U, ψ) be a left perfect localization and let $C = e_D(U), \phi : C \longrightarrow D$ the induced coalgebra map. Then $(C, D, {}_C U_D, {}_D C_C, F, G)$ is an MT-context and F is an isomorphism, where $F = \psi \,\square_D\, C$, $G = \psi$.*

Proof. It is enough to show that F, G are compatible. One may see that $U \,\square_D G = F \square_C U$ follows from the fact that $U \square_D \psi = \psi \square_D U$. To check that $C \square_C F = G \square_D C$, we compute that, $c \in C$,

$$\begin{aligned}(C \,\square_C F)(c) &= \sum c_{(1)} \,\square_C\, F(c_{(2)}) \\ &= \sum c_{(1)} \,\square_C\, \psi\phi(c_{(2)}) \,\square_D\, c_{(3)} \\ &= \sum \psi\phi(c_{(1)}) \,\square_D\, c_{(2)} \\ &= (G \,\square_D\, C)(c). \qquad \square\end{aligned}$$

Proposition 2.7. *Let $\phi : C \longrightarrow D$ be the coalgebra map as above. Then ϕ is a left coflat monomorphism.*

Proof. That ϕ is a left coflat map follows from Lemma 2.6 and [5, Theorem 2.5]. Let X be in ${}^C\mathbf{M}$, we have an isomorphism in ${}^C\mathbf{M}$

$$\psi \,\square_D X : X \longrightarrow U \,\square_D X,$$

since $(\psi \,\square_D C) \,\square_C X$ is a left C-isomorphism. If $X, Y \in {}^C\mathbf{M}$ and $h : X \longrightarrow Y$ is a D-comodule map, then it is a C-comodule map since the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi \square_D X} & U \square_D X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle U \square_D h} \\ Y & \xrightarrow{\psi \square_D Y} & U \square_D Y \end{array}$$

Hence the functor $(-)_\phi : {}^C\mathbf{M} \longrightarrow {}^D\mathbf{M}$ is full so that ϕ is a monomorphism by [2, Theorem 3.5]. $\square$

3. Coflat monomorphisms

In this section, we investigate left coflat monomorphisms of coalgebras, and establish a bijective correspondence between left coflat monomorphisms and left perfect localization bicomodules. First, we have an easy observation: the functor of direct sum preserves coflat monomorphisms. That is,

Proposition 3.1. *If $F_i : C_i \longrightarrow D_i$ are left coflat monomorphisms of coalgebras, then $\bigoplus_i : \bigoplus_i C_i \longrightarrow \bigoplus_i D_i$ is a left coflat monomorphism.*

Proof. Straightforward. $\square$

Lemma 3.2. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism of coalgebras, and ${}_A\mathscr{T}$ the corresponding torsion theory.*

(1) *Any left C-comodule X is torsion free as a left D-comodule.*

(2) *A comodule ${}_CX$ is C-injective if and only if ${}_DX$ is injective as a D-comodule.*

(3) *The torsion theory ${}_A\mathscr{T}$ is cogenerated by ${}_DC$.*

Proof. It is enough to show that ${}_DC$ as a left D-comodule is torsion free since any left C-comodule as a left D-comodule is still cogenerated by ${}_DC$. To show that ${}_DC$ is torsion free, it is sufficient to check that $\mathrm{Com}_{D-}(F, C) = 0$ for any finite dimensional object $F \in {}_A\mathscr{T}$ because ${}_A\mathscr{T}$ is hereditary and any comodule is locally finite. But we have

$$\mathrm{Com}_{D-}(F,C) \simeq F^* \,\square_D\, C = 0,$$

where F^* is a right A-comodule in $\mathcal{T}_A$.

(2) Suppose that ${}_CX$ is an injective comodule. Then there exists some set I such that ${}_CX \oplus_C Y = C^{(I)}$. Observe that the restriction functor $(-)_\phi : {}^C\mathbf{M} \longrightarrow {}^D\mathbf{M}$ is exactly the cotensor functor $C \,\square_C\, -$ which is exact. It follows that ${}_DX \oplus_D Y = {}_D\, C^{(I)}$. Now since ${}_DC^{(I)}$ is injective ${}_DX$ is injective. Conversely, if a left C-comodule X as a left D-comodule is injective, then there are some set J and D-comodule Z such that ${}_DX \oplus Z = D^{(J)}$. Now after cotensoring by C_D we obtain

$$C \square_D X \oplus C \square_D Z \simeq C^{(J)}$$

as left C-comodules. But ϕ is a monomorphism. By [2, Theorem 3.5] ${}_CX \simeq C \,\square_D X$ as left C-comodules. It follows that ${}_CX$ is injective.

(3) Follows from the proof of (1). □

Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. Let $\mathcal{T}_A$ be the kernel of $- \,\square_D C$. By Theorem 2.5 we may identify A with (U, ψ), where $U = S_A T_A(D), \psi = \psi_D$. In this case, the D-bicomodule C plays the role of U in Lemma 2.2. That is, if $f : X \longrightarrow Y$ is a map in $\mathbf{M}^D$, then $\mathrm{Ker} f, \mathrm{Coker}\, f$ are in $\mathcal{T}_A$ iff $f \,\square_D C$ is an isomorphism. It follows from $C \square_D C \cong C$ that $\mathrm{Ker}\phi$ and $\mathrm{Coker}\,\phi$ are torsion relative to $\mathcal{T}_A$.

Lemma 3.3. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism.*

(1) *U may be furnished into a (C,D)-bicomodule.*

(2) *$f_A = \psi \,\square_D C : C \longrightarrow U \square_D C$ is a C-bicolinear isomorphism.*

Proof. By forgoing remark, $\mathrm{Ker}\phi$ and $\mathrm{Coker}\,\phi$ are torsion. It follows from Lemma 2.2 that $\phi \,\square_D U : C \,\square_D U \longrightarrow U$ is a D-bicolinear isomorphism, which gives U a left C-comodule structure. The second statement follows immediately from the fact that $\psi \,\square_D C$ factors as

$$\psi \,\square_D C : C \xrightarrow{\Delta} C \square_D C \xrightarrow{C\square_D\psi\square_D C} C \,\square_D U \,\square_D C \xrightarrow{\phi\square_D U\square_D C} U \square_D C. \qquad \square$$

Now let $f_A = \psi \square_D C$ and $g_A = \psi$. It is easy to see that $f_A \square_C U = U \square_D g_A$ follows from $\psi \,\square_D U = U \,\square_D \psi$ (because (U, ψ) is a localization). $C \,\square_C f_A = g \,\square_D C$ is trivial. Thus we have proved the following:

Theorem 3.4. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. Then $(C, D, {}_CU_D, {}_DC_C, f_A, g_A)$ forms a Morita–Takeuchi context. Moreover, f_A is an isomorphism and* $\mathrm{Ker} g_A = A$.

Corollary 3.5. *Let $\phi : C \longrightarrow D$ be a left coflat morphism, U as above. Then*

(1) *${}_DC$ is a quasi-finite (injective) comodule.*

(2) $C \cong e_{D-}(C) \cong e_{-D}(U)$ *as coalgebras.*
(3) (U, ψ) *is a left perfect localization bicomodule.*

Proof. It follows from [5, Theorem 2.5]. □

Now we are able to establish a correspondence between left perfect localizations and left coflat monomorphisms. Two coalgebra maps $u: C \longrightarrow D$ and $v: E \longrightarrow D$ are *isomorphic* if there is a coalgebra isomorphism $h: C \longrightarrow E$ such that $u = vh$.

Theorem 3.6. *Let D be a coalgebra. There is one-to-one correspondence between the set of isomorphism classes of left coflat monomorphisms* $\phi: C \longrightarrow D$ *and the set of equivalence classes of left perfect localizations* (U, ψ) *in* ${}^D\mathbf{M}$.

Proof. Let $\phi: C \longrightarrow D$ be a left coflat monomorphism. By Corollary 3.5, $(U = S_A T_A(D), \psi_D)$ is a left perfect localization bicomodule, and $C' = e_{-D}(U) \cong C$. Let $\phi': C' \longrightarrow D$ be the induced coalgebra map by bicomodule ${}_D U_D$. It is clear that ϕ' is isomorphic to ϕ.

Conversely, let (U, ψ) be a left perfect localization. Let $\phi: C \longrightarrow D$ be the resulted left coflat monomorphism in Proposition 2.7, where $C = e_{-D}(U)$. Let $A = \operatorname{Ker}\psi$ and A' be the coidempotent subcoalgebra corresponding to the torsion theory $\operatorname{Ker}(-\Box_D C)$. To show that (U, ψ) is equivalent to $(S_A T_A, \psi_D)$, it is equivalent to show $A = A'$ by Theorem 2.5. Since coidempotent subcoalgebras bijectively correspond to hereditary torsion theories in $\mathbf{M}^D$, it is enough to show $\mathcal{T}_A = \mathcal{T}_{A'}$. In fact, if $X \in \mathbf{M}^D$, $X \Box_D C = 0$ implies $X \Box_D U = (X \Box_D C) \Box_C U = 0$. Conversely, if $X \Box_D U = 0$, then $X \Box_D C \simeq X \Box_D (U \Box Q) = 0$, where $Q = h_{-D}(U, D)$. Thus we obtain that $X \Box_D U = 0$ iff $X \Box_D C = 0$. That is, $\mathcal{T}_A = T_{A'}$. □

Now we are able to show which coidempotent subcoalgebras correspond to left coflat monomorphisms.

Corollary 3.7. *Let D be a coalgebra. There is a one-to-one correspondence between the set of isomorphism classes of left coflat monomorphisms* $\phi: C \longrightarrow D$ *and the set of coidempotent subcoalgebras A such that D/A is quasi-finite as a right D-comodule and the localizing functor* ${}_A S_A T$ *(equivalently* ${}_A S$*) is exact.*

Proof. It is enough to show that a localization bicomodule (U, ψ) is perfect if and only if D/A is quasi-finite and ${}_A S_A T$ is exact for coidempotent subcoalgebra $A = \operatorname{Ker}\psi$. By Lemma 2.2 $({}_A S_A T(D), \psi')$ is equivalent to $(\simeq S_A T_A(D), \psi)$. Since ${}_A S_A T$ is isomorphic to the cotensor functor ${}_A S_A T(D) \Box_D -$, ${}_A S_A T$ is exact if and only if ${}_A S_A T(D)$ is injective as a right D-comodule. In this case $S_A T_A(D) \simeq {}_A S_A T(D)$ is a injective hull of D/A. So $S_A T_A(D)$ is quasi-finite if and only if D/A is quasi-finite. □

Remark 3.8. The categorical translation of quasi-finiteness of D/A is that the canonical functor T_A has a left adjoint functor. Indeed, T_A is isomorphic to $T_A(D) \Box_D -$. So T_A

has a left adjoint functor if and only if $T_A(D)$ is quasi-finite in $\overline{\mathbf{M}^D}$ which is abelian category of finite type. For a torsion-free comodule $X \in \mathbf{M}^D$, $T_A(X)$ is quasi-finite iff X_D is quasi-finite. Since $T_A(D) \simeq T_A(D/A)$ and D/A is torsion free. It follows that T_A has a left adjoint functor iff D/A is quasi-finite as a right D-comodule.

Finally, we will see a left coflat monomorphism $\phi : C \longrightarrow D$ determines a categorical equivalence between ${}^C\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$ as well as it results in an equivalence between $\mathbf{M}^C$ and $\overline{\mathbf{M}^D}$ [2]. Consider the hereditary torsion theory ${}_A\mathcal{T}$ in ${}^D\mathbf{M}$. We may form the quotient category ${}^D\mathbf{M}/{}_A\mathcal{T}$, denoted by $\overline{{}^D\mathbf{M}}$. Let $({}_AT,{}_AS)$ be the canonical adjoint pair of functors between ${}^D\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$. The following proposition is the left version of the equivalence in [2, Theorem 6.1].

Proposition 3.9. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. The following composite functors define an equivalence between ${}^C\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$:*

$$ {}^C\mathbf{M} \underset{U\Box_D -}{\overset{(-)_\phi}{\rightleftarrows}} {}^D\mathbf{M} \underset{{}_AS}{\overset{{}_AT}{\rightleftarrows}} \overline{{}^D\mathbf{M}}. $$

Proof. Let $F = {}_AT \circ (-)_\phi$. It is clear that F is a left exact functor and preserves direct sums. So $F \simeq F(C) \Box_C -$. Since ${}_DC$ is a torsion free and injective object we have that ${}_DC \simeq {}_AS_AT(C)$. Let $V = {}_AT({}_DC)$. We claim that V is a cogenerator in $\overline{{}^D\mathbf{M}}$. In fact, $\forall X \in \overline{{}^D\mathbf{M}}$, there is a non-torsion C-comodule X' such that ${}_AT(X') = X$. Since ${}_DC$ cogenerated the torsion theory ${}_AT$, we have

$$ \mathrm{Hom}(X,V) = \mathrm{Hom}({}_AT(X'),V) \simeq \mathrm{Com}_{D-}(X',\ {}_AS(V)) \simeq \mathrm{Com}_{D-}(X',C) \neq 0, $$

where Hom are taken in $\overline{{}^D\mathbf{M}}$. Since ${}_DC$ is torsion free and injective, V is an injective object in $\overline{{}^D\mathbf{M}}$. A similar argument shows that V is a quasi-finite object in $\overline{{}^D\mathbf{M}}$. Now we may copy the argument of [5, 5.1–5.11] to get an equivalence between $\overline{{}^D\mathbf{M}}$ and the comodule category ${}^E\mathbf{M}$, where $E = h_{\overline{{}^D\mathbf{M}}}(V,V)$ is a coalgebra and the cohom functor $h_{\overline{{}^D\mathbf{M}}}(V,-)$ induces the equivalence. By the adjoint isomorphism and Corollary 3.5 we have

$$ h_{\overline{{}^D\mathbf{M}}}(V,V) \cong h_{D-}(C,C) \cong C. $$

It is clear that the inverse functor of the cohom functor is exactly the cotensor functor $V \Box_C -$. But $V = F(C)$ and hence $F = V \Box_C -$. It remains to show that the cohom functor $h_{\overline{{}^D\mathbf{M}}}(V,-)$ is isomorphic to the composite functor $G = (U \Box_D -) \circ {}_AS$. It is enough to show that G is a left inverse of functor F. Indeed, for any $M \in {}^C\mathbf{M}$,

$$ {}_CM \simeq C \Box_C M \simeq U \Box_D C \Box_C M \simeq U \Box_D M. $$

View M as a left D-comodule. We have an exact sequence

$$ 0 \longrightarrow \mathrm{Ker}\psi_M \longrightarrow M \xrightarrow{\psi_M} {}_AS_AT(M) \longrightarrow \mathrm{Coker}\,\psi_M \longrightarrow 0. $$

Note that a left D-comodule X is torsion if and only if $U \square_D X = 0$ Applying the exact functor $U \square_D -$ to the above exact sequence, we arrive at the isomorphisms:

$$_C M \simeq U \square_D M \simeq U \square_D, \qquad {}_A S_A T({}_D M) = GF(M),$$

and the proof is complete. □

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