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Special functions in higher spin settings

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I don't know half of you half as well as I should like
and I like less than half of you half as well as you deserve.
Bilbo Baggins

You might be wondering: how did we end up here? Well, that is an interesting story which I choose to reserve for another time, but I can tell you that it involves ice cream, mathematics and a life dragon (which you should never laugh at). Nevertheless, I would like to take this opportunity to clarify one very important matter: even though it is my name on the title page, I could not have done this alone. No matter how small you deem your own contributions to this work, know that it has not gone unnoticed and I am eternally grateful! However, there are still some people that I would like to thank in particular.

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These past nine years at the university have been a great chapter in my life but it is time to turn the page. As the road goes ever on and on, it takes me on another adventure. To white shores and perhaps beyond, to a far green country under a swift sunrise.

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Introduction

Do. Or do not. There is no try.
Yoda

In physics and engineering, many phenomena are described by differential equations, i.e. equations written in terms of a differential operator. One example of this is the Laplace operator

$$\Delta_x = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2},$$

a second order differential operator that was first used by Pierre-Simon Laplace to study celestial mechanics. Since then, the Laplace operator has popped up in a variety of problems: from the diffusion of heat to the flow of fluids, from electric potentials to quantum mechanics. Therefore it should not be a surprise that this operator is widely studied in mathematics as well, where it even has a field of its own: harmonic analysis.
Another example of a differential operator which has caused a revolution in the world of physics is the Dirac operator

$$\partial_x = \sum_{i=1}^{m} e_i \frac{\partial}{\partial x_i},$$

a first order differential operator that factorises the Laplacian and acts on functions taking values in certain subspaces of Clifford algebras. This operator appears in particle physics, as it describes the behaviour of an electron (or, more generally, particles with spin $\frac{1}{2}$), see [19]. From a mathematical point of view, this operator is studied in e.g. differential geometry, abstract representation theory and Clifford analysis, the discipline which has inspired this thesis (see later). Note that it is also possible to describe particles with higher spin, but for this one needs so-called higher spin operators, an example of which is the Rarita-Schwinger operator, see [70].

An important concept when solving differential equations is symmetry, a notion which is (mathematically) described in terms of groups and algebras. If, for example, the differential operator governing the equation is invariant under the action of a Lie algebra $\mathfrak{g}$, then the solutions of said equation will span a module for $\mathfrak{g}$, allowing us to use representation theoretical methods to find more solutions. Another way to generate solutions for differential equations is to transform solutions in a space of lower dimension (which may be easier to describe) into solutions depending on more variables. Two crucial theorems that achieve this goal are the well-known CK-extension and the celebrated Fueter theorem (see later). Amongst (polynomial) solutions a special role is played by those that exhibit additional symmetry properties, e.g. solutions that are invariant with respect to the action of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. An example of this is the following well-known fact from classical harmonic analysis which serves as the main motivation for this thesis:

> *Up to a multiplicative constant, there exists a unique polynomial solution for the Laplace operator Δ_x on $\mathbb{R}^m$ which is homogeneous of degree k and depends on the inner product of the position vector $x \in \mathbb{R}^m$ with a fixed (unit) vector. From a geometrical point of view, this means that these solutions can actually be related to spherical functions on the sphere (considered as a homogeneous space).*

This particular solution for the Laplace equation is given by

$$H_k(x) := |x|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle x, e_1\rangle}{|x|}\right) \in \ker \Delta_x \ ,$$

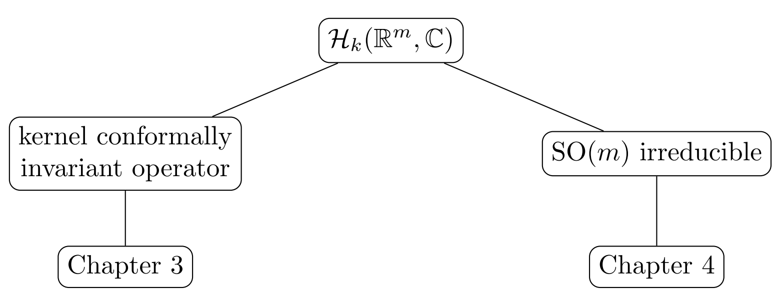
where $C_k^\alpha(t)$ denotes a classical Gegenbauer polynomial of order α and degree k, and where the fixed unit vector was chosen to be the basis vector e_1. This particular solution plays a crucial role in both harmonic analysis and its refinement Clifford analysis, for the following reasons:

(i) The sheer existence of this solution is dictated by an algebraic principle, known as the 'branching rules'. The general idea is the following: if one restricts the action of an algebra (or group) on a vector space $\mathbb{V}_\lambda$ to a subalgebra (or subgroup), then this space will in general decompose into subspaces. Given the (unique) label λ defining the initial vector space $\mathbb{V}_\lambda$, the branching rules automatically describe all the labels μ characterising the subspaces which will occur. In the setting from above, a crucial ingredient is the following: the space of k-homogeneous harmonic polynomials $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ describes such a vector space $\mathbb{V}_\lambda$ under the action of the orthogonal Lie algebra $\mathfrak{so}(m)$, for the specific choice $\lambda = (k, 0, ..., 0)$. This is essentially related to the fact that the Laplace operator is invariant under rotations, see also section 2.3. Applying the branching rules for the subalgebra $\mathfrak{so}(m-1)$ (i.e. the subalgebra leaving the direction e_1 invariant) to this label shows that also the trivial label $(0, ..., 0)$ will occur, which amounts to saying that there exists an invariant solution. In chapter 4, we will work with more complicated labels, but the branching rules will again dictate the existence of such an invariant polynomial solution.

(ii) These solutions are the basic building blocks in terms of which the Fueter images of holomorphic functions $f(z)$ will be expressed. This theorem, which allows the construction of monogenic functions (in the kernel of the Dirac operator ∂_x, which satisfies $\partial_x^2 = -\Delta_x$) starting from a complex function satisfying $\bar{\partial}_z f(z) = 0$, plays a crucial role in Clifford analysis (see later in the introduction for a more detailed account).

(iii) As the Gegenbauer polynomials are classical orthogonal polynomials, there exist plenty of special function identities for them. Amongst these, there are relations which can essentially be seen as raising and lowering relations (a concept borrowed from e.g. quantum mechanics). In particular, there exists an infinite family (the solutions mentioned above, for all possible degrees k) and two operators M and P which act between them (they raise or lower the index k, but they stay within the family). This embeds the theory of Gegenbauer harmonics into the framework of so-called Appell sequences. From a mathematical point of view, this typically indicates that there lies a small Lie algebra behind the family, for which the members of the family then become so-called weight spaces. In this particular case, this Lie algebra is a peculiar copy of $\mathfrak{sl}(2)$ which can be realised inside the conformal Lie algebra of generalised symmetries for the Laplace operator. In other words: the existence of a ladder operator formalism is hidden in the invariance properties of the equation. We have generalised this approach to our operator of interest, in chapter 3.

The goal of this thesis is to find a suitable generalisation of the Gegenbauer solutions, together with its properties mentioned above, to higher spin Clifford analysis and there are essentially two possible approaches based on how one interprets this notion of 'higher spin'. One can either focus on the fact that the operator Δ_x is conformally invariant and generalise the operator to the higher spin setting (the approach adopted in our paper [22] and chapter 3), or one can focus on the fact that they should for instance reproduce certain (higher spin) irreducible kernel spaces (which is the path we have pursued in the paper [23] and chapter 4).



Many of the results, if not all, found in this thesis can be generalised to the case of several vector variables. However, one is quickly faced with difficult computations which complicates things while the underlying idea is often the same as the case of two vector variables. Hence we have chosen to focus on the latter case throughout this thesis.

Chapter 1 and 2 serve a similar purpose as they both provide the reader with the necessary background required for this thesis. The first chapter focusses on Lie theory, which is crucial for our purposes as the existence of the generalised Gegenbauer solutions is guaranteed by the branching rules for the orthogonal Lie group (or algebra) $\mathrm{SO}(m)$ and also because our solutions for the partial differential equations under considerations will again be investigated as carrier spaces for Lie algebra actions. We start with some general definitions regarding Lie groups, Lie algebras and their representations and as an example we take a more in depth look at $\mathfrak{sl}(2)$ and $\mathfrak{so}(m)$, which were already mentioned above. Finally, we explore the branching rules that describe us how certain representations decompose when restricting to the action of subgroups, e.g. the orthogonal branching rules, as these play a crucial role in this thesis.
Classical harmonic and Clifford analysis are function theories in m dimensions in which conformally invariant operators are studied using a unified framework. In chapter 2 we will try to give the reader a brief introduction into the domain of Clifford analysis, a function theory centred around the Dirac operator on $\mathbb{R}^m$ (considered as a generalisation of the classical Cauchy-Riemann operator).

$$\begin{tikzcd}
\boxed{\text{Complex analysis}} \arrow[r, leftrightarrow, "\text{refinement}"] \arrow[d, "\text{higher dim.}"'] & \boxed{\text{Harmonic analysis on } \mathbb{R}^2} \arrow[d, "\text{higher dim.}"] \\
\boxed{\text{Clifford analysis}} \arrow[r, leftrightarrow, "\text{refinement}"'] & \boxed{\text{Harmonic analysis on } \mathbb{R}^m}
\end{tikzcd}$$

We refer to some of the standard textbooks, see [6,18,42,47]. While traditionally most of the attention was indeed aimed at the Laplace operator Δ_x and the Dirac operator ∂_x, it seems that various higher spin generalisations of both operators have recently gained their place in the aforementioned function theories. Far from claiming completeness, we refer to e.g. [4,5,8,17,26,27,30,31] for papers in the context of Clifford analysis. Moreover, one can construct special subgroups of the Clifford algebra and the Spin group $\mathrm{Spin}(m)$ will be such a subgroup of special interest to us as this group defines a double cover of $\mathrm{SO}(m)$. In e.g. [15] it was shown that for every finite-dimensional irreducible representation of $\mathrm{Spin}(m)$ one can find a polynomial model in several vector variables; the case of one variable is explored in section 2.3, whereas functions in two vector variables are studied in section 2.4. This natural connection between certain kernel spaces and irreducible representations of $\mathrm{Spin}(m)$ allows us to combine analysis and representation theory, which results in a synergy between the two disciplines.

In the third chapter, and in [22], we study special polynomial solutions to the higher spin Laplace operator

$$\mathcal{D}_k = \Delta_x - \frac{4}{2k+m-2}\left(\langle u, \partial_x \rangle - \frac{|u|^2}{2k+m-4}\langle \partial_u, \partial_x \rangle\right)\langle \partial_u, \partial_x \rangle ,$$

which is connected to both the Laplace operator (if $k = 0$ then $\mathcal{D}_0 = \Delta_x$) and the Rarita-Schwinger operators $\mathcal{R}_k$, see [8]. This corresponds to the first option mentioned earlier: focus on the conformal invariance (this has its advantages, as these operators are, up to a constant, uniquely defined, in sharp contrast to differential operators which are merely rotationally invariant). The second order operators $\mathcal{D}_k$ are conformally invariant and defined on functions taking values in the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of k-homogeneous harmonics in a dummy variable (our model for higher spin fields, hereby drawing inspiration from [15], which reduces to scalar values in $\mathbb{C}$ for $k = 0$). In particular, we will exploit the principle described in (iii): inside the full Lie algebra of generalised symmetries, which is (by choice) again $\mathfrak{so}(1, m+1)$, we will be able to isolate a peculiar subalgebra $\mathfrak{sl}(2)$ which serves as our algebra of raising and lowering operators. However, in sharp contrast to the classical case (for the Laplace operator), recognising the solutions that result from repeated action of the raising operator in terms of well-known special functions turns out to be a difficult problem; we believe that this stems from the fact that the solution spaces are not irreducible.

As mentioned before, for the case of one vector variable, the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ plays a double role: it is an irreducible representation for the Lie algebra $\mathfrak{so}(m)$ that is also defined by means of the conformally invariant equation $\Delta_x H_k(x) = 0$. In several variables, one still has conformally invariant equations (such as $\mathcal{D}_k f(x, u) = 0$, see chapter 3), but their (polynomial) solutions spaces are no longer irreducible (they decompose into a direct sum of several irreducible spaces, see e.g. [16]). Physicists are familiar with this behaviour, as they often impose so-called gauge conditions to make the solution spaces for certain invariant operators smaller. As mentioned earlier, this leaves a second option to generalise the classical situation: instead of focusing on the conformal invariance of the Laplace operator, one can also start from higher spin irreducible spaces of polynomials in several variables, which then generalise the harmonics. Note that these will still be solutions for a set of rotationally invariant operators, but no longer for a single conformally invariant operator. This means that, among the merely rotationally invariant systems, there is still a preferred choice (leading to irreducible solution spaces), and in chapter 4 (see also [23]) we look for our solutions in these representation spaces. Just like in the classical case, their existence will be dictated by the branching rules, which can now indeed be applied as we start from a label characterising an irreducible module for the orthogonal Lie algebra (in sharp contrast to the situation in chapter 3). Moreover, we will also be able to generate them using a ladder operator formalism and we will obtain an explicit expression (a polynomial in two scalar variables) and connect it with the existing results on hypergeometric functions in several variables. Moreover, we will construct an associated Appell sequence. These sequences are traditionally defined as a set of holomorphic polynomials $P_k(z)$ that satisfy

$$\frac{d}{dz}P_k(z) = kP_{k-1}(z),$$

where the derivation is often denoted as $P = \frac{d}{dz}$. If we can now find an operator M that satisfies $MP_{k-1}(z) = P_k(z)$ then the Appell sequence is a representation for the Heisenberg algebra. Indeed, it is easily seen that $[P, M] = PM - MP = 1$ and the operator M formally defines an integration within the Appell sequence. Notable examples of such sequences are the holomorphic powers, the Hermite polynomials, et cetera. Recently they have been generalised to higher dimensions and they include Clifford valuable polynomials, see e.g. [9, 21].

In chapter 5 we deal with the generalisation of the Cauchy-Kovaleskaya method for the solutions of a system of PDE's which has to be seen as the generalisation of the classical Dirac equation to the case of a matrix variable $(x, u) \in \mathbb{R}^{2m}$. In its most general form, the CK-method can be traced back to Cauchy (who proved a special case) and Kovaleskaya (who proved the general case); it is a theorem about the existence of solutions to a system of differential equations when the coefficients are analytic functions, given certain initial value conditions. For instance, if one chooses the differential operator to be the Cauchy-Riemann operator, then the theorem says that:

> *For a holomorphic function f, in a suitable region $\Omega \subset \mathbb{C}$, the function f is completely determined by its values on $\Omega \cap \mathbb{R}$.*

There even exists a further generalisation involving a cohomological formulation, due to Kashiwara [58]. The CK-extension has always played an important role in Clifford analysis where, for $x = x_1e_1 + \tilde{x}$ and $\partial_x = e_1\partial_{x_1} + \partial_{\tilde{x}}$, it can be explicitly written as

$$\begin{aligned} \mathrm{CK} : \mathcal{P}(\mathbb{R}^{m-1}, \mathbb{S}) &\rightarrow \mathcal{P}(\mathbb{R}^m, \mathbb{S}) \cap \ker \partial_x : \\ P(\tilde{x}) &\mapsto \mathrm{CK}[P(\tilde{x})] := \exp\big(x_1e_1\partial_{\tilde{x}}\big)P(\tilde{x}). \end{aligned} \tag{1}$$

It is crucial to point out here that every polynomial in a space of lower dimension can thus be used to construct a (unique) monogenic polynomial in one dimension higher. One of its key aspects is its connection with the construction of orthonormal bases for spaces of polynomial solutions which has been exploited in e.g [18]. This is then also connected to the theory of Gelfand-Tsetlin bases, see for instance [67]. In this thesis, and in [25], we focus on the CK-extension for the space of simplicial monogenics. This means that we will replace the space $\ker \partial_x$ to a higher spin equivalent, and investigate which polynomials in lower dimension can be used as a starting point to generate elements in $\mathcal{S}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^{\pm})$.

While generalising the CK extension to spaces of simplicial monogenics on $\mathbb{R}^{2m}$, we find that not every polynomial on $\mathbb{R}^{m-1}$ admits a CK extension. In sharp contrast to the case described above, only solutions to the so called skew-wedge system have to be considered, a fact that was also found in [78], and we thus take a closer look at this system in chapter 6. For instance, we obtain a Fischer decomposition for polynomials depending on wedge-variables. Roughly speaking, this means that one writes the full space of polynomials in terms of solutions for a certain operator, and the action of the dual operator. The original result, proven by Ernst Fischer in [35], states that:

> *Given a homogeneous polynomial $q(x)$ on $\mathbb{R}^m$ and an arbitrary homogeneous polynomial $P_k(x)$ of degree k, then we can decompose $P_k(x)$ as*
>
> $$P_k(x) = Q_k(x) + q(x)R(x),$$
>
> *where $R(x)$ is of the appropriate degree and $Q_k(x)$ belongs to the kernel of the differential operator $q(\partial_x)$.*

An example of this is the classical harmonic Fischer decomposition where one chooses $q(x) = |x|^2$. Indeed, one for instance has that

$$P_k(x) = H_k(x) + |x|^2 Q_{k-2}(x),$$

where $H_k(x)$ is a harmonic polynomial of degree k. Note that because of the $\mathrm{SO}(m)$-invariance of $q(x)$, the Fischer decomposition essentially tells us how polynomials will decompose into irreducible components under the $\mathrm{SO}(m)$-action. In order to arrive at a Fischer decomposition in which the spaces $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ we are interested in appear, one has to start from polynomials in two vector variables. However, seen our interest in the wedge-system (inspired by the CK-problem from above), it turns out that it suffices to start from a subspace, containing only polynomials depending on wedge-variables (in a sense, these are the duals of the derivatives appearing in the wedge system). This space of homogeneous scalar-valued polynomials depending on wedge-variables, i.e. $X_{ij} = x_i u_j - x_j u_i$, span, for any given degree of homogeneity, an irreducible $\mathrm{GL}(m)$-module. Using the branching rules from the second chapter will allow us to identify the irreducible $\mathrm{SO}(m)$-components that appear in said polynomial spaces. Just like in the classical case, we will consider the scalar situation first: once the analogue of harmonics are well understood, we then switch to the analogue of monogenic polynomials. Finding a suitable operator to fulfil the role of the Laplace operator prompts the appearance of the so-called Cayley-Laplace operator

$$\Delta_w = \Delta_x \Delta_u - \langle \partial_x, \partial_u \rangle^2 = |\partial_x \wedge \partial_u|^2,$$

an operator that was studied in e.g. [59, 71]. This operator is neither elliptic nor hyperbolic but it satisfies the strong Huygens principle, a principle that is connected to wave propagation (its physical interpretation is that an instantaneous light signal remains instantaneous for every observer at each later time). While the abstract decomposition was already discovered in [52], we obtain explicit embeddings for the irreducible components that appear. Moreover, in the spirit of

Clifford analysis, we find a refinement to functions taking values in spinor spaces, something that will prove to be crucial when considering a higher spin Fueter theorem. Finally, we generalise these results to the space of polynomials depending on so-called flag variables, i.e. polynomials $P(x, x \wedge u) \in \ker \langle x, \partial_u \rangle$, and obtain a similar decomposition to polynomials depending on wedge-variables.

In chapter 7 we introduce a higher spin version of Fueter's theorem, an elegant result in classical Clifford analysis. It originated in quaternionic analysis in 1935, see [36], where it was inspired by the following fact: it is well-known that complex powers z^n are holomorphic, but if one considers powers $q^n = (q_0+q_1i+q_2j+q_3k)^n$ of the quaternionic variable, then these powers are not null solutions for the generalized Cauchy Riemann operator $D = \partial_0 + i\partial_1 + j\partial_2 + k\partial_3$ (such null solutions are called regular). It was Fueter who pointed out that while q^n is not regular, the function $\Delta_4(q^n)$ does belong to the kernel of D (with Δ_4 the Laplace operator on $\mathbb{R}^4$). The Fueter theorem uses this insight to provide us with a way to construct quaternionic regular functions starting from holomorphic functions in the complex plane (which are much easier to describe).
Seen the fact that the algebra $\mathbb{H}$ of quaternions can be seen as a particular example of a (real) Clifford algebra, with $\mathbb{H} \cong \mathbb{R}_{0,2}$, it should come as no surprise that Fueter's classical result could be generalised to a more general framework. This was done by Sce in 1957, see [73], and his conclusion can be formulated as follows:

> *Let $m \in 2\mathbb{Z}^+$. Given a holomorphic function $f(z)$ on $\Omega \subset \mathbb{C}$, the function*
>
> $$F(x) := \Delta_x^{\mu}[f(\bar{e}_1 x)] \qquad \left(\mu = \frac{m}{2} - 1\right)$$
>
> *belongs the kernel of the Dirac operator (i.e. is monogenic).*

Note that the exponent of the Laplace operator becomes fractional for odd dimensions m, which means that one can no longer work with local (differential) operators. However, it is still possible to formulate a generalisation of the above result in the framework of non-local operators, which is precisely what Qian did in 1997: he was able to prove Fueter's theorem in the setting of Fourier multipliers, see [69]. Since then, this theorem has been generalised in a variety of directions by several other people, and the connection between Fueter's theorem and other branches of mathematics has been thoroughly explored. Without claiming completeness, we point out a few of these results here.
First of all, we mention the work of Sommen and Peña-Peña, see e.g. [61,68,74]. Over the years, they have refined Sce's result, for instance by proving that one can consider slightly deformed functions $f(\bar{e}_1 x)M_k(x)$ to start the Fueter procedure from, by multiplying the holomorphic function with a given monogenic polynomial $M_k(x)$ from the right. This was often done using the theory of Vekua systems, which appear naturally in the framework of axial monogenics (the term 'axial' refers to the preferred direction e_1 here).
The connection between the Fueter theorem and abstract representation theory

has been exploited in e.g. [28, 29]. First of all, one can formulate the Fueter theorem as an intertwining map between (infinite-dimensional) irreducible representations for the Lie algebra $\mathfrak{sl}(2)$, which appears in this framework as the algebra spanned by the ladder operators mentioned earlier. Not only does this approach connect Fueter's theorem with the theory of special functions (in casu Chebyshev and Gegenbauer polynomials), it also underlines the importance of the Lie algebra framework in which the theory is embedded. One can immediately observe from Sce's formulation that the direction e_1 plays a preferential role, which thus hints towards a connection with the branching rules. Indeed, the Fueter images are by construction invariant with respect to the subalgebra $\mathfrak{so}(m-1)$, which means that they can be expressed in terms of the Gegenbauer solutions mentioned earlier. As a result, this also means that the Fueter theorem is deeply connected with the Gelfand-Tsetlin construction, a method to construct orthonormal bases for irreducible representations, see e.g. [67].
A recent application of the Fueter theorem is rooted in the notion of so-called slice monogenic functions. As mentioned above, powers $(\bar{e}_1 x)^k$ are not monogenic, one needs the action of a power of the Laplace operator to arrive at a function in the kernel of the Dirac operator. However, these powers themselves do define solutions for a related system of equations, which has been dubbed the slice monogenic equations (for more information we e.g. refer to the monograph [12] and the references mentioned therein). This essentially means that the Fueter theorem can also be seen as a means to generate monogenic functions starting from slice monogenic functions (this has the advantage that this is a connection between two function theories in the same dimension). The function theory for the latter has given rise to a functional calculus for n-tuples of noncommuting operators, with a rich quaternionic functional calculus in particular. One of the problems that needed to be addressed in this framework is the surjectivity of the Fueter mapping and the inversion on a suitable function class, something that was for instance established in [12, 13, 20].
All in all, the Fueter theorem has proven to be a keystone result with many applications and connections to other results, and the fact that there exists an analogue in the framework of Cayley-Laplace and Cayley-Dirac operators could open up an interesting avenue for future research.

Remark. At the start of my PhD, I spent two months in Milan collaborating with Colombo, F., Gantner, J. and Sabadini, I. which resulted in the papers [10,11,39] concerning quaternionic operator theory and slice hyperholomorphic functions, something we have thus far not been able to connect to special functions in higher spin setting. As for this thesis, we have chosen to present a self-contained and coherent story, which is why we did not include this material.

CHAPTER 1

Lie groups, Lie algebras and their representations

The game, Mrs. Hudson, is on!
Sherlock Holmes

Lie theory is of the utmost importance when studying symmetries, differential equations, physics, special functions,... and as we will see, in Clifford analysis. There are numerous connections and Clifford analysis provides practical toy-models for irreducible representations for certain Lie groups/algebras (in particular the Spin group and the orthogonal Lie algebra). In this chapter, we begin with some basic definitions and give a short overview of the representation theory of Lie algebras. Afterwards we take a look at the branching rules, an important concept in representation theory, that will be used throughout this thesis.

1.1 General definitions

Definition 1.1.1. A Lie group is a manifold with a compatible group structure, meaning that the inverse and the group action are smooth maps.

Example 1.1.2. The most basic example is the general linear group $\mathrm{GL}_{\mathbb{R}}(m)$ of invertible $m \times m$ matrices which we can also interpret as the group of automorphisms on an m-dimensional real vector space V, which we denote by $\mathrm{GL}(V)$ or $\mathrm{Aut}(V)$. We can also define the following subgroups:

1. The special linear group $\mathrm{SL}_{\mathbb{R}}(m)$ is the group of $m \times m$ matrices with determinant 1.

2. We define the orthogonal group $\mathrm{O}_{\mathbb{R}}(m)$ as the group of $m \times m$ matrices that satisfy $XX^T = I$ with $X \in M_m(\mathbb{R})$ and I the identity matrix. The

special orthogonal group $\mathrm{SO}_{\mathbb{R}}(m)$ is then defined as

$$\mathrm{SO}_{\mathbb{R}}(m) := \mathrm{O}_{\mathbb{R}}(m) \cap \mathrm{SL}_{\mathbb{R}}(m).$$

3. The symplectic group $\mathrm{Sp}_{\mathbb{R}}(2m)$ is defined as

$$\mathrm{Sp}_{\mathbb{R}}(2m) := \{X \in M_{2m}(\mathbb{R}) : X^T J X = J\} \qquad \text{with } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Note that one could also define the groups $\mathrm{O}_{\mathbb{R}}(m), \mathrm{SO}_{\mathbb{R}}(m)$ and $\mathrm{Sp}_{\mathbb{R}}(m)$ in terms of bilinear forms on m-dimensional vectorspaces, but these definitions will be sufficient for our purposes (see e.g. [45] for details). Completely similar one can define the complex version of these groups and since we will mostly be working with the complex realisations (and to lighten the notations) we will drop the field in the subscript.

Next, we include the definition of a representation of a group:

Definition 1.1.3. A representation ρ of a group G on a vector space $\mathbb{V}$ is a homomorphism from G to $\mathrm{Aut}(\mathbb{V})$. If $G \subset \mathrm{GL}(m)$ then we call the representation rational if $\dim V = n < \infty$ and if, after fixing a basis of $\mathbb{V}$, for all $g \in G$, the matrix entries of $\rho(g)$ are regular functions, i.e. a function generated by the matrix entry functions and $\det^{-1}$. The name *rational* stems from the fact that each entry $\rho_{ij}(g)$ is a rational function of the matrix entries of g and has powers of $\det g$ in its denominator. Throughout this thesis, we will only consider rational representations.

When studying representations of Lie groups there are two main approaches: one could keep working on the group level by considering the action of so-called maximal tori, or one could move the problem into the field of linear algebra by looking at Lie algebras. Note that these approaches are not always equivalent (an example of this is the orthogonal group $\mathrm{O}(m)$, see section 1.4.1), but for our intent and purposes the latter proves to be the more convenient one. The Lie algebras are defined as follows:

Definition 1.1.4. A Lie algebra $\mathfrak{g}$ is a vector space endowed with a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is anticommutative and satisfies the following identity, know as the Jacobi identity:

$$\forall X, Y, Z \in \mathfrak{g} : [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

A Lie algebra is called abelian if $\forall X, Y \in \mathfrak{g} : [X,Y] = 0$. Moreover, if $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ such that for all $X, Y \in \mathfrak{h} : [X,Y] \in \mathfrak{h}$, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and it inherits the Lie bracket of $\mathfrak{g}$, making it a Lie subalgebra of $\mathfrak{g}$. Similar to Lie groups, we will mostly work over the field of complex numbers $\mathbb{C}$ unless otherwise specified.

Definition 1.1.5. A non-abelian Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ does not have any non-trivial ideals (i.e. ideals different from 0 or $\mathfrak{g}$).

Definition 1.1.6. If $\mathfrak{g}$ is a Lie algebra then we define the derived sequence $\mathfrak{g}^{(n)}$ by:

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(n+1)} &= [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]. \end{aligned}$$

We call a Lie algebra solvable if there exists an $n \in \mathbb{Z}_0^+$ such that $\mathfrak{g}^{(n)} = 0$. One can show that every Lie algebra $\mathfrak{g}$ has a unique maximal solvable ideal which we call the radical of $\mathfrak{g}$ and denote it by $\mathrm{Rad}(\mathfrak{g})$. A Lie algebra $\mathfrak{g}$ is called semi-simple if $\mathrm{Rad}(\mathfrak{g})$ is trivial.

Using these definitions we can construct a short exact sequence:

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0,$$

in which $\mathrm{Rad}(\mathfrak{g})$ is solvable and $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semi-simple. This reduces the study of Lie algebras to the solvable ones, and the semi-simple ones. The former is not fully understood yet, in sharp contrast to the latter. As for the thesis, we only need the simple algebras, and these can be used to describe all the semi-simple ones:

Theorem 1.1.7. *Every semi-simple Lie algebra is the direct sum of simple Lie algebras.*

The simple Lie algebras have been classified by using Dynkin diagrams, and we have listed them in theorem 1.1.8.
It should come as no surprise that there is a natural connection between Lie groups and Lie algebras. If we define

$$\Psi_g : G \to G : h \mapsto ghg^{-1}$$

then we have a natural map

$$\Psi : G \to \mathrm{Aut}(G) : g \mapsto \Psi_g.$$

We look at the differential of the map Ψ_g at e and set:

$$\mathrm{Ad}(g) := (d\Psi_g)_e : T_eG \to T_eG$$

and this defines a representation of the group G on its own tangent space called the adjoint representation of the group. Taking the differential of the map Ad yields

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$$

and this can be seen as a bilinear map

$$[\cdot,\cdot] : T_eG \times T_eG \to T_eG : (X, Y) \mapsto \mathrm{ad}(X)(Y),$$

which turns T_eG in a Lie algebra. All the Lie groups we have defined thus far have been subgroups of $\mathrm{GL}(m)$ which makes it easier to describe the associated Lie algebras. Moreover, in this case the action $\mathrm{ad}(X)(Y)$ reduces to the commutator bracket. If we define, for $X \in M_m(\mathbb{R})$,

$$\exp(X) := \sum_{i=0}^{\infty} \frac{X^i}{i!},$$

then $\exp : M_m(\mathbb{R}) \to \mathrm{GL}(m)$. Now let G be any closed subgroup of $\mathrm{GL}(m)$ then

$$\mathrm{Lie}(G) = \{X \in M_m(\mathbb{R}) \text{ such that } \forall t \in \mathbb{R} : \exp(tX) \in G\}$$

and $\mathrm{Lie}(G)$ is a Lie subalgebra of $M_m(\mathbb{R})$ equipped with the commutator bracket. Note that there exist several other equivalent ways of describing the connection between Lie groups and Lie algebras, see e.g. [38, 45].
Let $m \in \mathbb{Z}_0^+$, then the classical Lie algebras are given by:

1. The Lie algebra $\mathfrak{gl}(m)$ is the vectorspace of all $m \times m$ matrices where the Lie bracket is given by the commutator, i.e. $[X, Y] = XY - YX$. We call this the general linear Lie algebra.

2. The special linear Lie algebra $\mathfrak{sl}(m)$ is defined as all traceless $m \times m$ matrices, i.e.

$$\mathfrak{sl}(m) := \{X \in \mathfrak{gl}(m) : \mathrm{tr}(X) = 0\},$$

   and because $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, it becomes a Lie subalgebra of $\mathfrak{gl}(m)$ with the commutator as bracket.

3. Another important Lie subalgebra of $\mathfrak{gl}(m)$ is the (special) orthogonal Lie algebra given by

$$\mathfrak{so}(m) := \{X \in \mathfrak{gl}(m) : X^T = -X\}.$$

4. Finally, the symplectic Lie algebra $\mathfrak{sp}(2m)$ is defined as the Lie algebra

$$\mathfrak{sp}(2m) := \{X \in M_{2m}(\mathbb{C}) : X^T J = -JX\},$$

   where J is the matrix from example 1.1.2 and the bracket is given by the commutator.

From the definition it is then clear that $\mathrm{Lie}(\mathrm{GL}(m)) = \mathfrak{gl}(m)$ and using the fact that

$$\det(\exp(X)) = e^{\mathrm{tr}(X)}$$

yields that $\mathrm{Lie}(\mathrm{SL}(m)) = \mathfrak{sl}(m)$. Using a similar reasoning one can also see that $\mathrm{Lie}(\mathrm{Sp}(2m)) = \mathfrak{sp}(2m)$. While the Lie algebra is uniquely determined by the Lie group, different Lie groups can still have the same Lie algebra. For example we have that $\mathrm{Lie}(\mathrm{O}(m)) = \mathrm{Lie}(\mathrm{SO}(m)) = \mathfrak{so}(m)$. Almost all the simple Lie algebras are isomorphic to these classical ones:

Theorem 1.1.8. *Every simple (complex) Lie algebra is isomorphic to either* $\mathfrak{sl}(m), \mathfrak{so}(m)$ *or* $\mathfrak{sp}(2m)$*, for some* $m \in \mathbb{Z}^+$*, or to one of the five exceptional ones, see e.g. [38].*

1.1.1 Representation theory

There is a natural connection between Lie groups and Lie algebras and the same can be said about their representations, but first we define representations of a Lie algebra;

Definition 1.1.9. A representation of a Lie algebra $\mathfrak{g}$ on a vectorspace $\mathbb{V}$ is a map

$$\rho_{\mathbb{V}} : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$$

that preserves the bracket:

$$\forall v \in \mathbb{V} : \rho_{\mathbb{V}}([X,Y])(v) = [\rho_{\mathbb{V}}(X), \rho_{\mathbb{V}}(Y)](v).$$

A representation is called irreducible if there is no (non-trivial) subspace $\mathbb{W} \subset \mathbb{V}$ for which $\mathbb{W}$ is invariant under the action of $\mathfrak{g}$.

One can always regard the Lie algebra $\mathfrak{g}$ as a representation of itself, which is precisely what we had before:

Example 1.1.10. The adjoint representation is defined as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) : X \mapsto \mathrm{ad}(X)$$

with action $\mathrm{ad}(X)[Y] := [X,Y]$ for all $X, Y \in \mathfrak{g}$.

Let $G \subset \mathrm{GL}(m)$ be an algebraic group, i.e. there exists a set of polynomial functions on $M_m(\mathbb{C})$ such that G is given by the elements of $\mathrm{GL}(m)$ that belong to the kernel of these polynomials. All the Lie groups we have defined thus far have been algebraic groups, consider for example the group $\mathrm{O}(m)$, which consists of the matrices X that satisfy $XX^T = I$. This condition can then be written as

$$\sum_{i=1}^{m} x_{li}x_{ki} = \delta_{lk} \qquad 1 \leq l,k \leq m.$$

The group $\mathrm{O}(m)$ is thus defined as the subgroup of $\mathrm{GL}(m)$, where the elements of $\mathrm{O}(m)$ satisfy these m^2 polynomial equations. Let the Lie algebra of G be given by $\mathfrak{g} \subset \mathfrak{gl}(m)$ and consider a rational representation $(\rho, \mathbb{V})$ of G. We can then look at the differential of the representation ρ, defined as:

$$d\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V}),$$

which is also called the derived action of $\mathfrak{g}$ on $\mathbb{V}$ and it is given by

$$d\rho(X) = \frac{d}{dt}\rho(\exp(tX))\bigg|_{t=0}.$$

Whether or not this derived action yields an irreducible representation for $\mathfrak{g}$ depends on the structure of the group G.

Theorem 1.1.11. *Let G be $\mathrm{SL}(m+1), \mathrm{SO}(2m+1)$ or $\mathrm{Sp}(m)$, with $m \geq 1$ or let G be $\mathrm{SO}(2m)$ with $m \geq 2$, then G is generated by its unipotent elements.*

The importance of these groups is provided in the following theorem:

Theorem 1.1.12. *Let $G \subset \mathrm{GL}(m)$ be an algebraic group that is generated by unipotent elements with corresponding Lie algebra $\mathfrak{g}$ and let $(\rho, \mathbb{V})$ be a rational representation of G. Then $\mathbb{V}$ is irreducible under the action ρ of G if and only if it is irreducible under the action $d\rho$ of $\mathfrak{g}$.*

In this thesis, we will mostly work with $\mathrm{SO}(m)$ (or a double cover thereof, the Spin group $\mathrm{Spin}(m)$, which we will introduce in the next chapter) and thus we will focus on the study of representations of Lie algebras. At the end of the chapter we will say something about the representation theory of $\mathrm{O}(m)$, as this will be briefly needed to formulate a branching problem. Let $\mathfrak{g}$ be a simple Lie algebra then we are going to look at the action of a particular subalgebra of $\mathfrak{g}$ on its irreducible representations:

Definition 1.1.13. A Cartan subalgebra $\mathfrak{h}$ of a simple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra in $\mathfrak{g}$. This subalgebra is unique up to automorphism and thus we can call the dimension of $\mathfrak{h}$ the rank of the Lie algebra $\mathfrak{g}$.

We are going to decompose irreducible representations $(\rho_{\mathbb{V}}, \mathbb{V})$ of $\mathfrak{g}$ in eigenspaces for the subalgebra $\mathfrak{h}$. Each of these eigenspaces is then characterized by an amount of eigenvalues equal to the rank of the Lie algebra. To describe these eigenvalues we use the dual algebra $\mathfrak{h}^*$ with dual basis $\{L_i : 1 \leq i \leq n\}$. This means that if $\{H_i : 1 \leq i \leq n\}$ is a basis for $\mathfrak{h}$ that $L_i(H_j) = \delta_{ij}$. Every eigenspace $V_\lambda \subset \mathbb{V}$ is determined by an element $\lambda \in \mathfrak{h}^*$ meaning that

$$\forall H \in \mathfrak{h} : \forall v \in V_\lambda : \rho_{\mathbb{V}}[H](v) = \lambda(H)v.$$

We call $\lambda \in \mathfrak{h}^*$ a weight and we can use these to decompose every irreducible representation for a simple Lie algebra $\mathfrak{g}$ in weight spaces . In other words, we can find a finite collection of weights such that

$$\mathbb{V} = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where we sum over the $\lambda \in \mathfrak{h}^*$ for which

$$V_\lambda := \{v \in \mathbb{V} \mid \forall H \in \mathfrak{h} : \ \rho_{\mathbb{V}}(H)[v] = \lambda(H)v\} \neq \{0\}.$$

This decomposition is called the weight space decomposition. If we order all the weights lexicographically then we find a unique highest weight. This highest weight, and its corresponding highest weight vector v_λ , uniquely determine a finite dimensional irreducible representation. Every element of $\mathbb{V}$ is generated by the action of $\mathfrak{g}$ on the highest weight vector, meaning that:

$$\forall v \in \mathbb{V} : \exists X_{i_1}, \dots, X_{i_k} \in \mathfrak{g} : v = \rho_{\mathbb{V}}(X_{i_1}) \cdots \rho_{\mathbb{V}}(X_{i_k})[v_\lambda].$$

We denote $\mathbb{V}_\lambda$ for the irreducible representation with highest weight λ.

Remark 1.1.14. If we use the adjoint representation of a Lie algebra then we speak of roots instead of weights and of the root space decomposition instead of the weight space decomposition. We denote the set of roots by R. Since the adjoint representation characterizes and determines the structure of the Lie algebra, it is obvious that this representation will be very significant.

Lemma 1.1.15. *Let $\mathfrak{g}$ be a simple Lie algebra with root space decomposition:*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \right)$$

then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

Therefore it follows that if $\alpha + \beta$ is not a root for the given Lie algebra that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ and hence the elements of these root spaces commute with each other and $\mathfrak{g}_0 = \mathfrak{h}$. We also have the following proposition:

Proposition 1.1.16. *All the root spaces are one dimensional and if $\alpha \in R$ then $-\alpha \in R$ and this is the only multiple of α for which this holds.*

The roots of a Lie algebra $\mathfrak{g}$ form a lattice and a basis for these roots is defined as:

Definition 1.1.17. The simple roots of a Lie algebra $\mathfrak{g}$ of rank n are roots $\{\alpha_1, \ldots, \alpha_n\}$ with the property that for every root α we have:

$$\exists a_i \in \mathbb{Z}^+ : \alpha = \sum_{i=1}^{n} a_i \alpha_i \quad \text{or} \quad \alpha = -\sum_{i=1}^{n} a_i \alpha_i.$$

We call a root that is a positive combination of simple roots a positive root, the negative roots are defined as the negative combinations of simple roots. We write

$$R = R^+ \cup R^-$$

to denote the negative and the positive roots.

The root space decomposition of $\mathfrak{g}$ is orthogonal w.r.t. a certain inner product that contains a lot of information about the structure of $\mathfrak{g}$:

Definition 1.1.18. If $\mathfrak{g}$ is a Lie algebra then we define the Killing form $B(X, Y)$ by means of:

$$\forall X, Y \in \mathfrak{g} : B(X, Y) = \operatorname{tr}\Big(\operatorname{ad}(X) \circ \operatorname{ad}(Y) : \mathfrak{g} \to \mathfrak{g} \Big).$$

This form will be non-degenerate if and only if $\mathfrak{g}$ is semi-simple, which we will assume from now on. Given a semi-simple Lie algebra $\mathfrak{g}$ with root decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \right),$$

one can then draw the following conclusions:

1. The given decomposition into $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is orthogonal with respect to the Killing form.
2. Restricting the Killing form to the Cartan algebra yields

$$B(X, Y) = \sum_{\alpha \in R} \alpha(X)\alpha(Y),$$

   for $X, Y \in \mathfrak{h}$.

Moreover, the Killing form satisfies:

$$\forall X, Y, Z \in \mathfrak{g} : B([X, Y], Z) = B(X, [Y, Z]).$$

If we take $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ then we define $H_\alpha = [X_\alpha, Y_\alpha]$. For any given $\alpha \in R^+$ we write

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

and this is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$. Therefore we can choose our elements X_α and Y_α in a way such that they satisfy the standard commutation relations for $\mathfrak{sl}(2)$ (see section 1.3.1), in particular we have for any positive root $\alpha \in R^+ : B(H_\alpha, H_\alpha) = 2B(X_\alpha, Y_\alpha)$. Now define the element $T_\alpha \in \mathfrak{h}$, for $\alpha \in R$, as

$$T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)} = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)}$$

then this is the unique element of $\mathfrak{h}$ that satisfies the condition:

$$\forall H \in \mathfrak{h} : B(T_\alpha, H) = \alpha(H).$$

This means that the Killing form induces an isomorphism:

$$\mathfrak{h}^* \to \mathfrak{h} : \alpha \mapsto T_\alpha.$$

and thus $B(\alpha, \beta) = B(T_\alpha, T_\beta)$. To illustrate the importance of the Killing form, we give just one of its numerous applications, the Weyl dimension formula, which tells us the dimension of (finite dimensional) irreducible representations of a Lie algebra $\mathfrak{g}$ (for the proof we refer you to [38]):

Theorem 1.1.19 (Weyl dimension formula). *Let $\mathfrak{g}$ be a Lie algebra, ρ half the sum of the positive roots and $\mathbb{V}_\lambda$ an irreducible representation with highest weight λ. We then have:*

$$\dim \mathbb{V}_\lambda = \prod_{\alpha \in R^+} \frac{B(\lambda + \rho, \alpha)}{B(\rho, \alpha)},$$

1.2 The universal enveloping algebra

We will introduce a very important operator associated to a simple Lie algebra $\mathfrak{g}$, the Casimir operator. While simple Lie algebras have no central elements, it is useful to have operators that commute with the action of $\mathfrak{g}$ and we have to look for them in a larger algebra associated to $\mathfrak{g}$, namely:

Definition 1.2.1. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the associative algebra with unity defined as:

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I,$$

with

$$T(\mathfrak{g}) = \bigoplus_{k \in \mathbb{Z}^+} \otimes^k \mathfrak{g}$$

the tensoralgebra of $\mathfrak{g}$ and $I \subset T(\mathfrak{g})$ the two-sided ideal in $\mathfrak{g}$ spanned by elements of the form

$$a \otimes b - b \otimes a - [a, b].$$

Theorem 1.2.2 (Poincaré-Birkhoff-Witt). *Let $\mathfrak{g}$ be a Lie algebra and take an ordered basis $\mathcal{B}(X_1, \ldots, X_n)$ for the vector space $\mathfrak{g}$. We then have that:*

$$\mathcal{B}_U := \{X_1^{i_1} \otimes X_2^{i_2} \otimes \cdots \otimes X_n^{i_n} \;\mid\; (i_1, \ldots, i_n) \in (\mathbb{Z}^+)^n\}$$

is a basis for the vector space $\mathcal{U}(\mathfrak{g})$.

From now on we will omit the tensor symbols to lighten the notation.

Definition 1.2.3. An element $\Omega \in \mathcal{U}(\mathfrak{g})$ is a Casimir operator for $\mathfrak{g}$ if

$$\forall X \in \mathfrak{g} : [\Omega, X] = 0.$$

In other words the Casimir operator is an element of the center of the universal enveloping algebra.

One can show that a simple Lie algebra of rank n has n independent Casimir operators and that they generate all possible Casimir operators. Given an irreducible representation $\mathbb{V}$ for $\mathfrak{g}$, we define the action of $\mathcal{U}(\mathfrak{g})$ on $\mathbb{V}$ canonically by:

$$\rho_{\mathbb{V}}(X_1^{i_1} \otimes X_2^{i_2} \otimes \cdots \otimes X_n^{i_n}) := \rho_{\mathbb{V}}(X_1)^{i_1} \otimes \rho_{\mathbb{V}}(X_2)^{i_2} \otimes \cdots \otimes \rho_{\mathbb{V}}(X_n)^{i_n}.$$

The following lemma shows how the Casimir operator acts on any irreducible representation of $\mathfrak{g}$:

Lemma 1.2.4 (Schur's lemma). *Let $(\rho_{\mathbb{V}}, \mathbb{V})$ be an irreducible finite dimensional representation for $\mathfrak{g}$ and $\varphi \in End(\mathbb{V})$ such that*

$$\forall X \in \mathfrak{g} : [\varphi, \rho_{\mathbb{V}}(X)] = 0$$

then there exists a constant $\lambda \in \mathbb{C}$ such that $\varphi = \lambda 1_{\mathbb{V}}$.

So for every irreducible representation $\mathbb{V}$ we have that

$$\rho_{\mathbb{V}}(\Omega) = c(\Omega, \mathbb{V}) 1_{\mathbb{V}}.$$

The Casimir operator can therefore be used to distinguish the irreducible components of a given representation. While determining all Casimir operators of a given Lie algebra $\mathfrak{g}$ is no trivial matter, by using the Killing form one can easily calculate one of these operators, namely (apart from the scalars) the one having the lowest order. Let $\mathfrak{g}$ be a semisimple Lie algebra and fix a basis $\{X_i\}$ for $\mathfrak{g}$, where we denote $\{X_i^*\}$ for the dual basis with respect to the Killing form, i.e. $B(X_i, X_j^*) = \delta_{ij}$. Then we define

$$\mathcal{C}_2 := \sum_i X_i X_i^*$$

and one can show that this operator is independent of the choice of basis and that it commutes with $\mathfrak{g}$.

1.3 Examples of simple Lie algebras

We take a closer look at two (semi-)simple Lie algebras that will be used in this thesis.

1.3.1 The Lie algebra $\mathfrak{sl}(2)$

Let us start with the (complex) Lie algebra $\mathfrak{sl}(2)$, which was the matrix algebra defined as

$$\mathfrak{sl}(2) := \{X \in \mathfrak{gl}(2) : \mathrm{tr}(X) = 0\}$$

with the commutator as Lie bracket. If we define the elements:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then these matrices form a basis for the vectorspace $\mathfrak{sl}(2)$ and they satisfy the relations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. This means that the spaces $\mathbb{C}X$ and $\mathbb{C}Y$ are eigenspaces for the action of H with eigenvalues ± 2.

Proposition 1.3.1. *If we take $v \in \mathbb{V}$ an eigenvector with eigenvalue $\alpha \in \mathbb{C}$ for the action of H on an irreducible finite dimensional $\mathfrak{sl}(2)$ representation then*

$$\rho_{\mathbb{V}}(X)[v] = X(v)$$

is an eigenvector with eigenvalue $\alpha + 2$. The element

$$\rho_{\mathbb{V}}(Y)[v] = Y(v)$$

is an eigenvector with eigenvalue $\alpha - 2$.

Because we can write every irreducible representation $\mathbb{V}$ as a direct sum of eigenspaces:

$$\mathbb{V} = \bigoplus_\lambda V_\lambda,$$

there must exist a $\lambda_0 \in \mathbb{C}$ such that

$$\mathbb{V} = \bigoplus_{n \in \mathbb{Z}} V_{\lambda_0 + 2n}.$$

One can then show that:

Theorem 1.3.2. *Every finite dimensional irreducible representation $\mathbb{V}$ for $\mathfrak{sl}(2)$ with $\dim \mathbb{V} = n + 1$ is of the form:*

$$\mathbb{V} = \bigoplus_{j=0}^{n} V_{-n+2j}.$$

The highest weight of this representation is n and thus we denote it by $\mathbb{V}_n$.

This can be illustrated by the following scheme:

$$\begin{tikzcd} 0 & V_{-n} \arrow[l, "Y"'] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & V_{-n+2} \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & \bullet \arrow[l, shift left, "Y"] \arrow[r, dotted, no head] & \bullet \arrow[r, shift left, "X"] & V_{n-2} \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & V_n \arrow[l, shift left, "Y"] \arrow[r, "X"] \arrow[loop below, "H"] & 0. \end{tikzcd}$$

We need to know how tensor products of two $\mathfrak{sl}(2)$- representations behave:

Theorem 1.3.3 (Clebsch-Gordan)**.** *Let $a, b \in \mathbb{Z}^+$ and let $\mathbb{V}_a$ and $\mathbb{V}_b$ be finite-dimensional irreducible $\mathfrak{sl}(2)$-representations. Then*

$$\mathbb{V}_a \otimes \mathbb{V}_b \cong \mathbb{V}_{a+b} \oplus \mathbb{V}_{a+b-2} \oplus \cdots \oplus \mathbb{V}_{|a-b|+2} \oplus \mathbb{V}_{|a-b|}.$$

Finally, let us calculate the Casimir operator $\mathcal{C}_2$ for $\mathfrak{g} = \mathfrak{sl}(2)$. It is not hard to see that $X^* = Y$, $Y^* = X$ and $2H^* = H$ which means that

$$\mathcal{C}_2 = XY + YX + \frac{1}{2}H^2$$

and thus, for an arbitrary finite-dimensional irreducible representation $(\rho, \mathbb{V}_n)$ for $\mathfrak{sl}(2)$, we have

$$\rho(\mathcal{C}_2)\Big|_{\mathbb{V}_n} = \frac{n(n+2)}{2} 1_{\mathbb{V}_n}.$$

1.3.2 The special orthogonal Lie algebra $\mathfrak{so}(m)$

Depending on the value of m, this algebra is not always simple:

Proposition 1.3.4. *Let $m = 3$, then:*

$$\mathfrak{so}(3) \cong \mathfrak{sl}(2).$$

If $m = 4$, then the Lie algebra $\mathfrak{so}(4)$ is semisimple and is given by:

$$\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2),$$

see example 2.1.14 for a more explicit description of this decomposition. Finally, for every $m \geq 5$ we have that $\mathfrak{so}(m)$ is simple.

To end this section we will study the adjoint representation of $\mathfrak{so}(m)$ and determine the root space decomposition. Here we use matrices but another convenient realisation using Clifford algebras will be discussed in example 2.1.14. To determine the root space decomposition we have to treat the cases $2n$ and $2n+1$ separately but in both cases $\mathfrak{so}(m)$ is a Lie algebra of rank n and thus $\dim \mathfrak{h} = n$. To simplify matters we will realise $\mathfrak{so}(m)$ in such a way that the Cartan subalgebra is given by diagonal matrices (which is slightly different from our realisation earlier). For this purpose, let us define

$$M_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \qquad \text{and} \qquad M_{2n+1} = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then realise the (complex) special orthogonal Lie algebra as

$$\mathfrak{so}(m) := \{X \in \mathbb{C}^{m\times m} : X^T M + MX\}.$$

The Cartan subalgebra $\mathfrak{h}$ is then given by

$$\mathfrak{h} = \text{span}\{H_i := E_{i,i} - E_{n+i,n+i} : 1 \leq i \leq n\},$$

where $E_{i,j}$ denotes the matrix with all zero element and a 1 as entry (i,j) and the dual basis will be denoted by $\{L_1, \ldots, L_n\}$. Let us then define the matrices:

$$\begin{aligned} X_{i,j} &= E_{i,j} - E_{n+j,n+i} \\ Y_{i,j} &= E_{i,n+j} - E_{j,n+i} \\ Z_{i,j} &= E_{n+i,j} - E_{n+j,i} \end{aligned}$$

where $1 \leq i,j \leq n$. If $m = 2n$ then, together with $\mathfrak{h}$, these form a basis for $\mathfrak{so}(m)$. The root vectors $X_{i,j}$ have as their root $L_i - L_j$, $Y_{i,j}$ corresponds to $L_i + L_j$ and $Z_{i,j}$ belong to the root $-L_i - L_j$.

Theorem 1.3.5. *The orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(2n)$ (with $n \geq 2$) has root space decomposition:*

$$\mathfrak{so}(2n) = \mathfrak{h} \ \oplus \ \bigoplus_{p<q} (\mathfrak{g}_{L_p+L_q} \oplus \mathfrak{g}_{L_p-L_q} \oplus \mathfrak{g}_{-L_p-L_q} \oplus \mathfrak{g}_{-L_p+L_q}).$$

The simple roots are given by:

$$\Delta = \{L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n\}.$$

If $m = 2n+1$ then we find a similar result, but here one needs to add the elements

$$\begin{aligned} U_i &= E_{i,2n+1} - E_{2n+1,n+i} \\ V_i &= E_{n+i,2n+1} - E_{2n+1,i} \end{aligned}$$

for $1 \leq i \leq n$ and these root vectors will belong to the roots L_i and $-L_i$:

Theorem 1.3.6. *The orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(2n+1)$ (with $n \geq 2$) has root space decomposition:*

$$\mathfrak{so}(2n+1) = \mathfrak{h} \oplus \bigoplus_{p<q} (\mathfrak{g}_{L_p+L_q} \oplus \mathfrak{g}_{L_p-L_q} \oplus \mathfrak{g}_{-L_p-L_q} \oplus \mathfrak{g}_{-L_p+L_q}) \\ \oplus \bigoplus_{p} (\mathfrak{g}_{L_p} \oplus \mathfrak{g}_{-L_p}).$$

Here we have that the simple roots are given by:

$$\Delta = \{L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n\}.$$

From this, by applying theorem 1.1.19, we can find the dimensions of irreducible $\mathfrak{so}(m)$ representations:

Theorem 1.3.7. *Consider an irreducible representation $\mathbb{V}_\lambda$ of $\mathfrak{so}(m)$ with highest weight $(\lambda_1, \ldots, \lambda_n)$ then we have the following expressions (in both cases $1 \leq i, j \leq n$):*
if $\underline{m = 2n+1}$:

$$\dim \mathbb{V}_\lambda = \prod_{i<j} \left(\frac{\lambda_i + \lambda_j + 2n - i - j + 1}{2n - i - j + 1} \; \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \prod_{i} \left(\frac{\lambda_i + n - i + \frac{1}{2}}{n - i + \frac{1}{2}} \right)$$

and if $\underline{m = 2n}$:

$$\dim \mathbb{V}_\lambda = \prod_{i<j} \left(\frac{\lambda_i + \lambda_j + 2n - i - j}{2n - i - j} \; \frac{\lambda_i - \lambda_j + j - i}{j - i} \right).$$

1.4 Branching rules

An important part of this thesis is devoted to studying solutions for differential equations which have certain invariance properties. The language we need to describe this comes from the so-called branching rules for Lie algebras, which we describe here. In general we formulate the branching problem as follows: let us consider a finite dimensional irreducible representation $\mathbb{V}_\lambda$ for a (simple) Lie algebra (resp. group) G with proper subalgebra (resp. subgroup) $H \subset G$. We ask ourselves how the module $\mathbb{V}_\lambda$ decomposes if we consider it as a representation for H and the answer is generally given by

$$\mathbb{V}_\lambda \Big|_H^G \cong \bigoplus_{\mu} n_\mu \mathbb{W}_\mu$$

with μ a highest weight characterising the irreducible H-representations $\mathbb{W}_\mu$ and n_μ the associated multiplicity. For a given Lie algebra (resp. group) G and given subalgebra (resp. subgroup) H there exist abstract rules which describe the $\mathbb{W}_\mu$ that one has to consider in the decomposition. We mention these rules for the orthogonal Lie algebra as we will need them in the following chapters:

Theorem 1.4.1 (Orthogonal Lie algebra Branching rules)**.**
Let $\mathbb{V}_\lambda$ be a finite-dimensional irreducible representation for $G = \mathfrak{so}(m)$ and let the considered subalgebra be $H = \mathfrak{so}(m-1)$. If we write $\lambda = (\lambda_1, \dots, \lambda_n)$ then if m is odd we have that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, if m is even then $\lambda_1 \geq \dots \geq |\lambda_n|$. We then have that:

$$\mathbb{V}_\lambda \Big|_H^G \cong \bigoplus_\mu \mathbb{V}_\mu$$

is multiplicity-free, where we take the sum over all the μ satisfying:

- $\underline{m = 2n+1}$*: Here we have that $\mu = (\mu_1, \dots, \mu_n)$ and*

  $$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n \geq |\mu_n|$$

  where the λ_i and μ_i are simultaneously all integers or all half integers.

- $\underline{m = 2n}$*: In this case $\mu = (\mu_1, \dots, \mu_{n-1})$ and*

  $$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq |\lambda_n|$$

  where the λ_i and μ_i are simultaneously all integers or all half integers.

Another branching problem that will be of particular interest to us is the one where $G = \mathrm{GL}(m)$ and $H = \mathrm{SO}(m)$ but this will be slightly more complicated than the branching of the orthogonal Lie algebra. We will take two steps: first we use a result from [53] to branch from $\mathrm{GL}(m)$ to $\mathrm{O}(m)$ and afterwards we will further restrict the irreducible $\mathrm{O}(m)$ representations to $\mathrm{SO}(m)$. This will require us to take a closer look at irreducible (finite dimensional) representations of $\mathrm{O}(m)$.

1.4.1 Representations of GL(m) and O(m)

Because theorem 1.1.12 only applies to Lie groups that are generated by unipotent elements, we need to use an alternate approach to prove that one can label the irreducible representations of $\mathrm{GL}(m)$ and $\mathrm{O}(m)$ by means of highest weights, as one cannot simply consider the derived representations of their Lie algebras. Therefore, we will classify their irreducible (rational) representations by looking at the restriction to $\mathrm{SL}(m)$ and $\mathrm{SO}(m)$ resp. since for these subgroups we can use the connection with the representations of the corresponding Lie algebras to simplify matters. Let us start with $\mathrm{GL}(m)$, the proof of which can be found in [45]:

Theorem 1.4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a weight with $\lambda_i \in \mathbb{Z}$ that satisfies $\lambda_1 \geq \dots \geq \lambda_m$, then there exists a unique irreducible rational representation $(\pi_\lambda, \mathbb{V}_\lambda)$ of $\mathrm{GL}(m)$ that is completely determined by its restriction to $\mathrm{SL}(m)$ and by the action of zI_m, with $z \in \mathbb{C}_0$ which is given by $z^{\lambda_1 + \dots + \lambda_m}$ (this follows from the fact that $\mathrm{GL}(m) = \mathbb{C}_0 \times \mathrm{SL}(m)$).*

Next, we will determine the irreducible (finite dimensional) representations of $\mathrm{O}(m)$ in terms of the irreducible representations of $\mathrm{SO}(m)$. First let us fix some notations, where we follow [45], which also contains the proofs. Let $G = \mathrm{O}(m)$, $G^0 = \mathrm{SO}(m)$ (i.e. the connected component of $\mathrm{O}(m)$ containing the unit element) and let H be a diagonal subgroup of G, i.e. the subgroup of diagonal matrices in $\mathrm{O}(m)$ (or a conjugate thereof). Furthermore, let $H^0 = H \cap G^0$, we denote $\mathfrak{n}^+$ for the positive roots in $\mathfrak{so}(m)$ and for a representation $(\rho, \mathbb{W})$ of G we define:

$$\mathbb{W}^{\mathfrak{n}^+} := \{w \in \mathbb{W} : \mathfrak{n}^+.w = 0\}$$

i.e. all the possible highest weight vectors.

Let $m = 2n+1$ be odd, then $\det(-I) = -1$ and $-I$ belongs to the centre of G. If we let $(\rho, \mathbb{W})$ be an irreducible representation of $\mathrm{O}(m)$ then, by Schur's lemma, we know that $\rho(-I) = \pm I$ and the restriction of ρ to G^0 is still irreducible. This means that we can find a weight to represent the action of $\mathfrak{h}$ on $\mathbb{W}^{\mathfrak{n}^+}$. Conversely, one could also start from a highest weight representation of G^0 and lift it to an irreducible representation in two different ways, by defining the sign that determines the action of $-I$. All in all we have the following:

Theorem 1.4.3. *The irreducible (regular) representations of* $\mathrm{O}(m)$*, with* m *odd, are of the form* $(\rho_{\lambda,\pm}, \mathbb{V}_{\lambda,\pm})$ *where* λ *is the highest weight for the action of* $\mathrm{SO}(m)$ *and* $-I \in \mathrm{O}(m)$ *acts by* $\pm I$*.*

The case $m = 2n$ is significantly more complicated and would require some more theoretical background to fully explain, for which we refer the reader to [45], but we shall include the main idea here (which will be more than sufficient for our purposes). We start by defining an element $g_0 \in \mathrm{O}(m)$ by: $g_0 e_{n+1} = e_n$, $g_0 e_n = e_{n+1}$ and $g_0 e_i = e_i$ if $i \notin \{n, n+1\}$. This means that the action of g_0 on the weight lattice is given by:

$$g_0.\lambda = g_0.(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n).$$

If we now start with an irreducible representation $(\pi_\lambda, \mathbb{V}_\lambda)$ for $\mathrm{SO}(m)$ then we can define

$$I(\mathbb{V}_\lambda) := \{f : G \to \mathbb{V}_\lambda \text{ regular} : f(xg) = \pi_\lambda(x) f(g),\ \forall x \in G^0 \text{ and } \forall g \in G\}$$

and let ρ_λ, the induced representation, be the right translation action of $\mathrm{O}(m)$ on $I(\mathbb{V}_\lambda)$. One can then show that $\rho_\lambda\big|_{\mathrm{SO}(m)} \cong \pi_\lambda \oplus \pi_{g_0.\lambda}$, i.e. as a $\mathrm{SO}(m)$ representation it decomposes into two irreducible components that have highest weight λ and $g_0.\lambda$. There are two possibilities:

(i) $\underline{g_0.\lambda \neq \lambda}$ (i.e. $\lambda_n \neq 0$), in this case $(\rho_\lambda, I(\mathbb{V}_\lambda))$ is an irreducible $\mathrm{O}(m)$ representation.

(ii) $\underline{g_0.\lambda = \lambda}$ (i.e. $\lambda_n = 0$), here one has that $I(\mathbb{V}_\lambda) = \mathbb{V}_{\lambda,+} \oplus \mathbb{V}_{\lambda,+}$ is a decomposition into irreducible $\mathrm{O}(m)$-representations where the sign is determined by $\rho(g_0)$. Moreover, each of these components is also $\mathrm{SO}(m)$-irreducible and if we denote $\pi_{\lambda,\pm}$ for the restriction of ρ_λ to $\mathbb{V}_{\lambda,\pm}$ then $\pi_{\lambda,\pm}\big|_{\mathrm{SO}(m)} = \pi_\lambda$.

Conversely, if $(\sigma, \mathbb{W})$ is an irreducible representation of G then one can always find a weight λ and a subspace $\mathbb{V}_\lambda$ such that $\rho\big|_{\mathbb{V}_\lambda} = \pi_\lambda$. Then,

$$\sigma \cong \begin{cases} \rho_\lambda & \lambda_n \neq 0 \\ \pi_{\lambda,\pm} & \lambda_n = 0. \end{cases}$$

Moreover, if $\lambda_n \neq 0$ then $\dim \mathbb{W}^{\mathfrak{n}^+} = 2$, and it is equal to 1 otherwise. Combining everything we have the following theorem:

Theorem 1.4.4. *Let $m \geq 4$ be even, then the irreducible (regular) representations $(\sigma, \mathbb{W})$ of $\mathrm{O}(m)$ are of the following type:*

1. *If $\dim \mathbb{W}^{\mathfrak{n}^+} = 1$ and $\mathfrak{h}$ acts by the weight λ on $\mathbb{W}^{\mathfrak{n}^+}$, then g_0 acts by $\pm I$ and*
$$(\sigma, \mathbb{W}) \cong (\pi_{\lambda,\pm}, \mathbb{V}_{\lambda,\pm}).$$
2. *If $\dim \mathbb{W}^{\mathfrak{n}^+} = 2$ then $\mathfrak{h}$ has two distinct weights on $\mathbb{W}^{\mathfrak{n}^+}$, namely λ and $g_0.\lambda$, and*
$$(\sigma, \mathbb{W}) \cong (\rho_\lambda, I(\mathbb{V}_\lambda)).$$

In this thesis, we will always start with a given weight λ and then associate to it a $\mathrm{O}(m)$ representation, after which we will restrict it to $\mathrm{SO}(m)$ (see e.g. the next section). But before we continue we will explain how we will relate $\mathrm{O}(m)$-representations to certain Young diagrams (or partitions) λ, as this will be of particular interest to us.

Definition 1.4.5. A non-negative integer partition λ with k parts is defined as a sequence

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0.$$

We define the length $l(\lambda)$ of such a partition to be k (i.e. the amount of non-zero entries). Its dual, λ', is given by:

$$\lambda_i' := \left|\{\lambda_j : \lambda_j \geq i\}\right|.$$

Let us now consider a non-negative integer partition μ such that $\mu_1' + \mu_2' \leq m$, i.e. the sum of the first two columns of the corresponding Young diagram is no greater than m. We will now associate irreducible $\mathrm{O}(m)$-representations to such partitions. Because $\mu_1' + \mu_2' \leq m$ we can write:

$$\mu = (\mu_1, \ldots, \mu_m), \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0$$

and we can thus look at it as a $\mathrm{GL}(m)$ weight. Let $\tilde{\mathfrak{h}}$ denote the $m \times m$ diagonal matrices and let $\mathfrak{h} = \mathfrak{so}(m) \cap \tilde{\mathfrak{h}}$. Seeing as $\mu \in \tilde{\mathfrak{h}}^*$, one can consider $\overline{\mu} := \mu\big|_{\mathfrak{h}^*}$, and for $n = \left\lfloor \frac{m}{2} \right\rfloor$, it is given by:

$$\overline{\mu} = (\mu_1 - \mu_m, \mu_2 - \mu_{(m-1)}, \ldots, \mu_n - \mu_{m-n+1}).$$

Let m be odd, then we make the following distinction between possible values for μ : either $\mu_i = 0$ for all $i > n$ (for these partition we write $\epsilon(\mu) = +1$) or there exists a $p \leq n$ such that $\mu_i = 1$ if $p+1 \leq i \leq m-p$ and $\mu_i = 0$ if $i > m-p$ (here we write $\epsilon(\mu) = -1$). We then define:

$$(\rho_\mu, \mathbb{W}_\mu) := (\pi_{\overline{\mu},\epsilon(\mu)}, \mathbb{V}_{\overline{\mu},\epsilon(\mu)}).$$

Note that for us the sign does not matter as we will afterwards restrict these to $\mathrm{SO}(m)$. The case $m = 2n$ even is similar but here we must consider three types of partitions:

1. If $\mu_i = 0$ for $i < n$ then we write $\epsilon(\mu) = +1$.
2. If there exists a $p < n$ such that $\mu_i = 1$ for $p+1 \leq i \leq m-p$ and $\mu_i = 0$ if $i > m-p$, then we write $\epsilon(\mu) = -1$.
3. The last possibility is the one where $\mu_n > 0$ and $\mu_i = 0$ if $i > n$.

In the first two cases, $\overline{\mu}_n = 0$ and thus we know that there exist irreducible representations $(\pi_{\overline{\mu},\epsilon(\mu)}, \mathbb{V}_{\overline{\mu},\epsilon(\mu)})$ that remain irreducible when restricting to $\mathrm{SO}(m)$. In the last case we will associate to μ the representation $(\rho_\lambda, I(\mathbb{V}_\lambda))$.
We will use this in the next section when branching from $\mathrm{GL}(m)$ to $\mathrm{O}(m)$, where we will consider an example.

1.4.2 Branching from GL(m) to O(m)

Definition 1.4.6. We define the Littlewood-Richardson coefficients in terms of a product on Young tableaux (or partitions), namely:

$$c^\lambda_{\mu,\nu} = \langle c_\mu \cdot c_\nu, c_\lambda \rangle \ .$$

This means that one has to look at the multiplicity of the diagram c_λ in the decomposition of the product $c_\mu \cdot c_\nu$. We quickly recall how the product of Young tableaux is defined: we need to add to the boxes of the partition μ, the boxes of the partition $\nu = (\nu_1, \ldots, \nu_n)$ and we need to label these boxes:

- We label the first ν_1 boxes with a
- The next ν_2 boxes are labeled with b
- And so on ....

We need to add these labelled boxes alphabetically to μ, one at a time, and at each step the following rules must hold:

1. The diagram must be regular, which means that if one considers the length of the rows then this has to be a decreasing sequence.
2. No two identical letters ever appear in the same column.

3. If we read the sequence of added letters from right to left across each row in turn (from top to bottom) then, at any point, the number of letters a one has encountered is bigger than the encountered number of letters b and so on.

As an example we calculate some of the Littlewood-Richardson coefficients if the Young tableaux are given by:

$$\mu = (3,2) = \ydiagram{3,2} \qquad \text{and} \qquad \nu = (2,1) = \ydiagram{2,1}.$$

For example, we have that $c_{\mu,\nu}^{(4,3,1)} = 2$, because there are two ways we can add ν to μ that yield $(4,3,1)$ namely:

$$\begin{ytableau} \, & \, & \, & a \\ \, & \, & a \\ b \end{ytableau} \qquad \text{and} \qquad \begin{ytableau} \, & \, & \, & a \\ \, & \, & b \\ a \end{ytableau}.$$

The full product is given by:

$$\begin{aligned}(3,2)\cdot(2,1) = (5,3) \oplus (5,2,1) \oplus (4,4) \oplus 2(4,3,1) \oplus (4,2,2) \oplus (4,2,1,1)\\ \oplus(3,3,2) \oplus (3,3,1,1) \oplus (3,2,2,1).\end{aligned}$$

In [53] these coefficients were used to decompose irreducible $\mathrm{GL}(m)$ representations into irreducible $\mathrm{O}(m)$ components:

Theorem 1.4.7 (Branching from $\mathrm{GL}(m)$ to $\mathrm{O}(m)$)**.** *Let λ be a non-negative integer partition with $l(\lambda) \leq \frac{m}{2}$ and let $\mathbb{F}_\lambda$ be the irreducible $\mathrm{GL}(m)$-representation with weight λ. Then:*

$$\mathbb{F}_\lambda \Big|_{\mathrm{O}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{\mu} m_{\lambda,\mu} \mathbb{E}_\mu$$

where we sum over all partitions μ such that $\mu_1' + \mu_2' \leq m$, and the multiplicity of the irreducible $\mathrm{O}(m)$-module $\mathbb{E}_\mu$ (which is the $\mathrm{O}(m)$-representation from the previous section) is given by

$$m_{\lambda,\mu} = \sum_{2\delta} c_{\mu,2\delta}^{\lambda},$$

which is a sum of Littlewood-Richardson coefficients over all Young tableaux with even parts (hence the 2δ in the summation).

In the future we will simply denote the representations by their highest weight to lighten the notations and if the need arises we will add a subscript to denote the group/algebra for which we are considering the representations.

Example 1.4.8. Let $m \geq 3$ and consider $\lambda = (k, 0, \ldots, 0) = (k)$ (we drop the zeroes to reduce the notation). We then know from theorem 1.4.7 that:

$$(k)\Big|_{\mathrm{O}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{\mu} \left(\sum_{2\delta} c_{\mu,2\delta}^{(k)} \right) \mu_{\mathrm{O}(m)}.$$

From the definition of the Littlewood-Richardson coefficients we know that the weights occurring in the sum on the right hand side are of the form $\mu = (\mu_1)$ and $2\delta = (2\delta_1)$, because if any of them had a second entry different from zero then we would be unable to obtain λ as part of their product. Therefore we have that:

$$\mu \cdot 2\delta = \bigoplus_{i=0}^{\min(\mu_1, 2\delta_1)} (\mu_1 + 2\delta_1 - i, i).$$

And thus the only component that could be equal to λ is the one where $i = 0$ and $\mu_1 + 2\delta_1 = k$. From this we can conclude that:

$$(k)\Big|_{\mathrm{O}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} (k - 2i)_{\mathrm{O}(m)}.$$

Finally, because $m \geq 3$ we have that the representation $(k - 2i)_{\mathrm{O}(m)}$ is also irreducible under the action of $\mathrm{SO}(m)$, which leads to the result:

$$(k)\Big|_{\mathrm{SO}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} (k - 2i)_{\mathrm{SO}(m)},$$

which, as we will see in the next chapter, is nothing more then the Fischer decomposition of the space of k-homogeneous polynomials in terms of spaces of harmonic polynomials. This example truly captures the spirit of this thesis: in what follows we will introduce polynomial models for these abstract branching rules, as this will allow us to recognise them as a decomposition for certain invariant differential operators in several vector variables.

CHAPTER 2

A brief introduction to Clifford analysis

You haven't lived until you've seen Bonanza dubbed in French-Canadian!
Cliff Clavin

We start this chapter by introducing Clifford algebras and take a closer look at some important subspaces, for example one can find a copy of the groups $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ inside these algebras. We will also construct double covers for these groups, the so-called $\mathrm{Pin}(m)$ and $\mathrm{Spin}(m)$ group, and the latter will play a particularly important role in this thesis. We then turn our attention towards certain $\mathrm{Spin}(m)$-representations, namely the spinor spaces, which can be realised inside the Clifford algebra, that are crucial in Clifford analysis. Afterwards, we will study (conformally or $\mathrm{Spin}(m)$) invariant operators on function spaces $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$, where $\mathbb{V}_\lambda$ is an irreducible $\mathrm{Spin}(m)$-representation. If $\lambda = (0)$ (resp. $\lambda = (0)'$) then the operator of interest is given by the Laplace (resp. Dirac) operator and we will give a short overview of some of their properties. To conclude this chapter, we will explain what we mean by higher spin Clifford analysis (basically we take a different weight λ) as well as define the irreducible $\mathrm{Spin}(m)$-spaces that we will be working with for the remainder of the thesis.

2.1 Clifford algebras and the Spin group

Clifford algebras can be seen as a generalisation of complex numbers, considered as a real algebra of dimension 2, where, instead of one, we have several complex units that anti-commute:

Definition 2.1.1. Let $m \in \mathbb{Z}^+$, then the algebra $\mathbb{R}_m$ which is defined as the associative algebra generated by the orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$ which satisfies:

$$\begin{cases} e_i^2 = -1 & 1 \leq i \leq m \\ e_i e_j + e_j e_i = 0 & i \neq j \end{cases}$$

is called the (real) universal Clifford algebra.

We then have that $\dim_{\mathbb{R}}(\mathbb{R}_m) = 2^m$ and

$$\mathbb{R}_m = \mathrm{span}_{\mathbb{R}}\{e_{i_1} e_{i_2} \cdots e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq m\}.$$

Let $A \subset \{1, \ldots, m\}$ with $A = \{i_1, \ldots, i_l\}$ then we define

$$e_A := e_{i_1 \cdots i_l} := e_{i_1} \cdots e_{i_l},$$

which means that

$$\mathbb{R}_m = \mathrm{span}_{\mathbb{R}}\{e_A : A \subset \{1, 2, \ldots, m\}\}.$$

Moreover, we can consider $\mathbb{R}^m$ as a subspace of this algebra:

$$(x_1, \ldots, x_m) \mapsto \sum_{i=1}^{m} x_i e_i$$

and this yields an interesting connection between the product in our Clifford algebra, the Euclidean inner product and the wedge product by means of:

$$xy = -\langle x, y \rangle + x \wedge y = -\sum_{i=1}^{m} x_i y_i + \sum_{i<j} e_i e_j (x_i y_j - x_j y_i).$$

This justifies the common saying that the Clifford algebra unifies two geometric structures, the Euclidean and the Grassmannian structure.

Definition 2.1.2. For every $0 \leq k \leq m$ we define the space $\mathbb{R}_m^{(k)}$ of k-vectors as:

$$\mathbb{R}_m^{(k)} := \mathrm{span}_{\mathbb{R}}\{e_A : A \subset \{1, 2, \ldots, m\} \text{ with } |A| = k\}.$$

In particular, we have that $\mathbb{R}^m \cong \mathbb{R}_m^{(1)}$ and we call the space $\mathbb{R}_m^{(2)}$ the space of bivectors. The latter will be used to realise the (special) orthogonal Lie algebra $\mathfrak{so}(m)$ inside the Clifford algebra.

We can easily describe the even and odd part of the Clifford algebra $\mathbb{R}_m$, the even subalgebra is defined as:

$$\mathbb{R}_m^+ := \bigoplus_{j=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{R}_m^{(2j)}$$

whereas the odd subspace is given by:

$$\mathbb{R}_m^- := \bigoplus_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \mathbb{R}_m^{(2j+1)}.$$

Definition 2.1.3. We can define the following (anti-)automorphisms on $\mathbb{R}_m$:

1. The inversion $a \mapsto \hat{a}$ on $\mathbb{R}_m$ is defined on the basis elements as

$$e_{i_1\cdots i_k} \mapsto \hat{e}_{i_1\cdots i_k} := (-1)^k e_{i_1\cdots i_k},$$

   and then linearly extended to the entire algebra. It is then obvious that:

$$\hat{\hat{a}} = a \quad \text{and} \quad \widehat{ab} = \hat{a}\hat{b}.$$

2. The conjugation $a \mapsto \overline{a}$ on $\mathbb{R}_m$ is defined as:

$$e_{i_1\cdots i_k} \mapsto \overline{e_{i_1\cdots i_k}} := (-1)^k e_{i_k\cdots i_1} = (-1)^{\frac{k(k+1)}{2}} e_{i_1\cdots i_k},$$

   and once again we linearly extend this to the entire algebra. We then note that:

$$\overline{\overline{a}} = a \quad \text{and} \quad \overline{ab} = \overline{b}\overline{a}.$$

Using these maps we can describe special subgroups of $\mathbb{R}_m$, starting with the Clifford group:

Definition 2.1.4. The Clifford group is given by

$$\Gamma(m) := \{s \in \mathbb{R}_m \text{ invertible such that } \forall x \in \mathbb{R}^m : sx\hat{s}^{-1} \in \mathbb{R}^m\}.$$

Let $s \in \Gamma(m)$ then we can define the linear transformation $\chi(s)$ as

$$\chi(s) : \mathbb{R}^m \to \mathbb{R}^m : x \mapsto \chi(s)x := sx\hat{s}^{-1}$$

which has the following property:

Lemma 2.1.5. *Let $s \in \Gamma(m)$ then $\chi(s) \in \mathrm{O}(m)$.*

Let $v = (v_1, \ldots, v_m) \in \mathbb{R}_m^{(1)}$ be nonzero then v is invertible where, using the fact that $v^2 = -\sum_{i=1}^{m} v_i^2 = -|v|^2$, the inverse v^{-1} is given by:

$$v^{-1} = -\frac{v}{|v|^2}.$$

Combined with the following lemma, this shows that the Clifford group is not trivial.

Lemma 2.1.6. *Every invertible vector $v \in \mathbb{R}^m$ belongs to the group $\Gamma(m)$ and defines a reflection w.r.t. the hyperplane perpendicular to v.*

Recall the theorem of Cartan-Dieudonné, which states that each orthogonal transformation in $\mathrm{O}(m)$ is the composition of at most m reflections. This means that every orthogonal transformation T in $\mathrm{O}(m)$ can be written as

$$T = \prod_{i=1}^{k} \chi(v_i)$$

with $1 \leq k \leq m$ and $v_i \in \mathbb{R}^m$. Note that $\chi(v_1)\chi(v_2) = \chi(v_1v_2)$ for every $v_1, v_2 \in \Gamma(m)$ so that we can conclude that

$$T = \chi\left(\prod_{i=1}^{k} v_i\right)$$

and thus we have a surjective group homomorphism:

$$\chi : \Gamma(m) \to \mathrm{O}(m).$$

The kernel of χ is given by $\mathbb{R}_0$ which allows us to find a realisation of $\mathrm{O}(m)$ within our Clifford algebra:

$$\mathrm{O}(m) \cong \Gamma(m)/\mathbb{R}_0.$$

From this we can also derive an equivalent definition for the Clifford group:

$$\Gamma(m) := \left\{\prod_{i=1}^{k} v_i \mid k \in \mathbb{Z}^+, \forall i : 1 \leq i \leq k : v_i \in \mathbb{R}^m \text{ such that } |v_i| \neq 0\right\}.$$

We consider the following two cases:

1. $s \in \Gamma(m)$ induces a rotation if $s \in \Gamma^+(m) := \Gamma(m) \cap \mathbb{R}_m^+$. Since for these s we have that $s = \hat{s}$ we can write this rotation as $\chi(s)x = sxs^{-1}$.
2. $s \in \Gamma(m)$ induces an anti-rotation if $s \in \Gamma^-(m) := \Gamma(m) \cap \mathbb{R}_m^-$. For these elements we have $-s = \hat{s}$ and therefore we can write the anti-rotation as $\chi(s)x = -sxs^{-1}$.

We thus have that

$$\Gamma(m) = \Gamma^+(m) \cup \Gamma^-(m)$$

and we call $\Gamma^+(m)$ the even Clifford group. If we now consider

$$\chi : \Gamma^+(m) \to \mathrm{SO}(m)$$

then we can write

$$\mathrm{SO}(m) \cong \Gamma^+(m)/\mathbb{R}_0.$$

We can turn the Clifford group into a normed space in the following way:

Definition 2.1.7. We define the spinor norm $\mathcal{N}$ for $s \in \Gamma(m)$ as

$$\mathcal{N}(s) := s\bar{s}.$$

This norm defines a homomorphism from $\Gamma(m)$ to $\mathbb{R}_0^+$ which leads to the following subgroups of $\Gamma(m)$:

$$\begin{array}{lcl} \mathrm{Pin}(m) & := & \{s \in \Gamma(m) : |\mathcal{N}(s)| = 1\} \\ \mathrm{Spin}(m) & := & \{s \in \Gamma^+(m) : |\mathcal{N}(s)| = 1\}. \end{array}$$

Both of these subgroups are normal subgroups of $\Gamma(m)$ and:

$$\mathrm{Pin}(m) \cong \Gamma(m)/\mathbb{R}_0^+ \quad \text{and} \quad \mathrm{Spin}(m) \cong \Gamma^+(m)/\mathbb{R}_0^+.$$

And thus we can finally conclude that the Pin (resp. Spin) group is a double cover of $\mathrm{O}(m)$ (resp. $\mathrm{SO}(m)$):

$$\mathrm{O}(m) \cong \mathrm{Pin}(m)/\mathbb{Z}_2 \quad \text{and} \quad \mathrm{SO}(m) \cong \mathrm{Spin}(m)/\mathbb{Z}_2.$$

We can identify the elements of $x \in \mathbb{R}^m$ for which $\mathcal{N}(x) = 1$ with the sphere $S^{m-1} \subset \mathbb{R}^m$ to find that

$$\begin{aligned} \mathrm{Pin}(m) &= \{\prod_{j=1}^{k} \omega_j : \omega_j \in S^{m-1}\} \\ \mathrm{Spin}(m) &= \{\prod_{j=1}^{2k} \omega_j : \omega_j \in S^{m-1}\}. \end{aligned}$$

Both of these groups are Lie groups and because $\mathrm{Spin}(m)$ is of particular interest to us, we need its Lie algebra, see e.g. [18]:

Theorem 2.1.8. *The Lie algebra associated to Spin(m) is the Lie algebra $(\mathbb{R}_m^{(2)}, [\cdot,\cdot])$ of the bivectors, which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_{\mathbb{R}}(m)$.*

Until now we have been working over $\mathbb{R}$ but we can also consider a complexification. This is for instance useful in the light of the previous theorem, as it will provide us with a model for the simple complex Lie algebra $\mathfrak{so}(m)$ in terms of bivectors.

Definition 2.1.9. The complex Clifford algebra $\mathbb{C}_m$ is defined as $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$.

These algebras satisfy the following, rather useful, relation:

Theorem 2.1.10. *Let $m \in \mathbb{Z}_0^+$ then $\mathbb{C}_m^+ \cong \mathbb{C}_{m-1}$.*

Apart from the automorphisms of $\mathbb{R}_m$ we had before, we can define yet another anti-involution here:

Definition 2.1.11. We define the Hermitian conjugate $a \mapsto a^\dagger$ on $\mathbb{C}_m$ as the composition of the Clifford conjugation and the complex conjugation. Therefore we have for every $a \in \mathbb{C}_m$ that:

$$\left(\sum_A a_A e_A\right)^\dagger := \sum_A \overline{a}_A \overline{e_A}.$$

Using this conjugation, we can define a very important set of vectors:

Definition 2.1.12. If we write $n = \lfloor \frac{m}{2} \rfloor$ then we define for $j = 1, \ldots, 2n$ the Witt-vectors as:

$$\mathfrak{f}_j := \frac{e_{2j-1} - ie_{2j}}{2} \quad \text{and} \quad \mathfrak{f}_j^\dagger := -\frac{e_{2j-1} + ie_{2j}}{2}.$$

We can use these vectors to span the space $\mathbb{C}^m$ if m is even, but if m is odd we have to add the vector e_m to form a basis. These vectors also satisfy the following relation:

$$\mathfrak{f}_j^2 = 0 \quad \text{and for } j \neq k: \ \mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0,$$

which also holds for $\mathfrak{f}_j^\dagger$. One can use these vectors to generate a rather familiar algebra:

Proposition 2.1.13. *Each of the sets $\{\mathfrak{f}_i : 1 \leq i \leq n\}$ and $\{\mathfrak{f}_i^\dagger : 1 \leq i \leq n\}$ generates the Grassmann algebra.*

Moreover, we have that

$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}.$$

In the next section, we will introduce the spinor spaces but to do this one needs the elements $I_j := \mathfrak{f}_j\mathfrak{f}_j^\dagger$ which satisfy:

$$I_j^2 = I_j \quad \text{and} \quad I_jI_k = I_kI_j \quad \text{for} \quad j \neq k.$$

This leads us to the definition of the primitive idempotent

$$I := \prod_{j=1}^{n} I_j.$$

Using the Witt-vectors we can also find an elegant realisation of the Lie algebra $\mathfrak{so}(m)$ inside the Clifford algebra $\mathbb{C}_m$:

Example 2.1.14. The Cartan algebra $\mathfrak{h}$ of $\mathfrak{so}(m)$ is given by:

$$\mathfrak{h} := \left\{ H_k := \mathfrak{f}_k\mathfrak{f}_k^\dagger - \frac{1}{2} \mid 1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor \right\}$$

and again we denote $\{L_1, \ldots, L_n\}$ for the dual basis of $\mathfrak{h}^*$. If $m = 2n$ is even then for each $p \neq q$ we have the root spaces:

$$\mathbb{C}\mathfrak{f}_p\mathfrak{f}_q \ , \ \mathbb{C}\mathfrak{f}_p\mathfrak{f}_q^\dagger \text{ and } \mathbb{C}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger,$$

and their corresponding roots $\pm L_p \pm L_q \in \mathfrak{h}^*$ (where the dagger determines the minus sign) with $1 \leq p \neq q \leq n$. If $m = 2n + 1$ is odd then we have to add the root spaces

$$\mathfrak{g}_{L_p} := \mathbb{C}\mathfrak{f}_pe_{2n+1} \quad \text{and} \quad \mathfrak{g}_{-L_p} := \mathbb{C}\mathfrak{f}_p^\dagger e_{2n+1}.$$

The copies of $\mathfrak{sl}(2)$ that can be found, for each $\alpha \in R^+$, are then given by:

- The copy $s_{L_p-L_q}$ of $\mathfrak{sl}(2)$ associated to the root $L_p - L_q$ is spanned by the root spaces $\mathbb{C}\mathfrak{f}_p\mathfrak{f}_q^\dagger$ and $\mathbb{C}\mathfrak{f}_p^\dagger\mathfrak{f}_q$ with $H_{L_p-L_q} = H_p - H_q$.
- The copy $s_{L_p+L_q}$ of $\mathfrak{sl}(2)$ associated to the root $L_p + L_q$ is spanned by the root spaces $\mathbb{C}\mathfrak{f}_p\mathfrak{f}_q$ and $\mathbb{C}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger$ with $H_{L_p+L_q} = H_p + H_q$.
- Both of the above cases are applicable to both the even and odd case. In the odd case $\mathfrak{so}(2n+1)$ we also have a copy s_{L_p} of $\mathfrak{sl}(2)$ associated to the root L_p and this one is spanned by the root spaces $\mathbb{C}\mathfrak{f}_p e_{2n+1}$ and $\mathbb{C}\mathfrak{f}_p^\dagger e_{2n+1}$ with $H_{L_p} = 2H_p$.

In particular, if $m = 4$, there are two copies of $\mathfrak{sl}(2)$ to be found in $\mathfrak{so}(4)$, namely $s_{L_1+L_2}$ and $s_{L_1-L_2}$. Moreover, from lemma 1.1.15, we can conclude that these are commuting copies, proving that

$$\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

2.2 Spinor spaces

As we have mentioned before, see theorem 1.4.1, the possible highest weights of irreducible $\mathfrak{so}(m)$ representations are given by n-tuples $(\lambda_1, \ldots, \lambda_n)$, with $n = \lfloor \frac{m}{2} \rfloor$, that satisfy:

$$\begin{array}{ll} \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n| & \text{for } m \in 2\mathbb{Z}^+ \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 & \text{for } m \in 2\mathbb{Z}^+ + 1 \end{array}$$

and where all λ_i are either all simultaneously integers or all simultaneously half-integers. If all $\lambda_i \in \mathbb{Z}^+$ then this is also the highest weight of the corresponding $\mathrm{SO}(m)$ representation but this is no longer the case for the half-integers. It is here that the Spin group proves its use: each of these weights is the highest weight of precisely one irreducible $\mathrm{Spin}(m)$ representation (whereas the special orthogonal group was insufficient). The highest weights of the form

$$\left(\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\right)$$

are of particular interest to us because, by considering the Cartan product of these representations with representations of $\mathrm{SO}(m)$, one can use them to construct all possible highest weight representations. This also justifies the following notation: let $\lambda_1 \geq \cdots \geq \lambda_k > 0$ be strictly positive integers with $k \leq n$, then

$$\begin{array}{ll} (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) & := (\lambda_1, \ldots, \lambda_k) \\ (\lambda_1 + \frac{1}{2}, \ldots, \lambda_k + \frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2}) & := (\lambda_1, \ldots, \lambda_k)'_{\pm}. \end{array}$$

In other words, we omit redundant zeros and the prime denotes the Cartan product with the spinor space $\mathbb{S}_{2n}^{\pm}$. We will construct an explicit realisation within the Clifford algebra $\mathbb{C}_m$ for the representations $(0)'_{\pm}$, which are the so-called spinor spaces. To do this we consider the following action of the Spin group $\mathrm{Spin}(m)$ on the Clifford algebra $\mathbb{C}_m$:

$$l : \mathrm{Spin}(m) \to \mathbb{C}_m$$

which is given by the left multiplication

$$l(s)a = sa.$$

The irreducible $\mathrm{Spin}(m)$ representations can be labelled by considering the action of the maximal torus, given by:

$$T = \left\{ \exp\left(\frac{1}{2}e_{12}t_1\right) \cdots \exp\left(\frac{1}{2}e_{(2n-1)(2n)}t_n\right) \mid \forall j = 1, \ldots, n : t_j \in \mathbb{R} \right\}$$

or one could look at the action of $\mathfrak{so}(m)$ (see example 2.1.14). If $(\rho, \mathbb{V})$ is an irreducible $\mathrm{Spin}(m)$ representation, then the action of T on $\mathbb{V}$ reduces to:

$$\rho(s_t)v = \exp\left(i\left(\lambda_1 t_1 + \cdots + \lambda_n t_n\right)\right),$$

with $s_t \in T$. Let $m = 2n+1$ be odd, then we define $\mathbb{S}_{2n} = \mathbb{C}_m^+ I$. It is clear that this space is $l(s)$-invariant and the action of the maximal torus T on the idempotent I yields:

$$l(s)I = \exp\left(\frac{i}{2}\left(t_1 + \cdots + t_n\right)\right) I$$

and thus $\mathbb{S}_{2n}$ provides a model for the $\mathrm{Spin}(m)$ representation with highest weight $(0)'$. If $m = 2n$ be even then there exist two inequivalent spinor spaces, namely

$$\mathbb{S}_{2n}^+ = \mathbb{C}_m^+ I \qquad \text{and} \qquad \mathbb{S}_{2n}^- = \mathbb{C}_m^+ \mathfrak{f}_n^\dagger I.$$

Letting the maximal torus T act on the highest weight vectors I and $\mathfrak{f}_n^\dagger I$ gives us the weights $(0)'_+$ and $(0)'_-$.

Example 2.2.1. Let $m = 5$, then using example 2.1.14 we can write

$$\mathfrak{so}(5) = \mathrm{Lie}\left(\mathfrak{so}(4), \mathfrak{f}_1 e_5, \mathfrak{f}_1^\dagger e_5, \mathfrak{f}_2 e_5, \mathfrak{f}_2^\dagger e_5\right).$$

From theorem 2.1.10 we can conclude that

$$\mathbb{S}_4 \cong \mathbb{S}_4^+ \oplus \mathbb{S}_4^- = \mathrm{Span}(I, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger I) \oplus \mathrm{Span}(\mathfrak{f}_1^\dagger I, \mathfrak{f}_2^\dagger I),$$

which means that there exists an action of $\mathfrak{so}(5)$ on $\mathbb{S}_4^\pm$. This action is given by

$$X \cdot \psi := \varphi^{-1}(X)\psi,$$

where the action on the right hand side is the Clifford multiplication, $\psi \in \mathbb{S}_4^\pm$ and φ is the algebra isomorphism

$$\varphi : \mathbb{C}_4 \to \mathbb{C}_5^+.$$

As an example, we have that

$$(\mathfrak{f}_1 e_5) \cdot \psi = \mathfrak{f}_1 \psi.$$

Representing the action of $\mathfrak{so}(5)$ on a diagram illustrates that $\mathbb{S}_4$ decomposes into two irreducible parts under the action of $\mathfrak{so}(4)$ and the action of $\mathfrak{so}(5)$ allows us to move from one of said components to the other one:

$$\begin{tikzcd} I \arrow[r, leftrightarrow, "\mathfrak{so}(4)"] \arrow[d, bend left, "\mathfrak{f}_1^\dagger e_5"] & \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger I \arrow[d, bend left, "\mathfrak{f}_1 e_5"] \\ \mathfrak{f}_1^\dagger I \arrow[u, bend left, "\mathfrak{f}_1 e_5"] \arrow[r, leftrightarrow, "\mathfrak{so}(4)"'] & \mathfrak{f}_2^\dagger I \arrow[u, bend left, "\mathfrak{f}_1^\dagger e_5"] \end{tikzcd}$$

2.3 Polynomial representations of Spin(m)

As mentioned earlier, we will study certain differential operators in this thesis. These will act on functions taking values in certain representations for the Spin group. As such, having a convenient model at our disposal will be crucial. In this thesis we work with polynomial models, a nice alternative for the abstract index notation mostly used in theoretical physics. Let $(\rho, \mathbb{V})$ and $(\phi, \mathbb{W})$ be representations of a group G then we can define an action of G on the mappings f from $\mathbb{V}$ to $\mathbb{W}$ by means of:

$$(g \cdot f)(v) := \phi(g) f(\rho(g^{-1})v),$$

i.e. it is the action that makes the following scheme commute:

$$\begin{tikzcd} \mathbb{V} \arrow[r, "f"] \arrow[d, "\rho(g)"'] & \mathbb{W} \arrow[d, "\phi(g)"] \\ \mathbb{V} \arrow[r, "g \cdot f"] & \mathbb{W} \end{tikzcd}$$

for all $g \in G$. Let $\mathbb{V} = \mathbb{R}^m$ and $\mathbb{W} = \mathbb{C}$, then we consider:

Definition 2.3.1. The space of k-homogeneous scalar-valued polynomials is denoted by $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$ and we write:

$$\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}).$$

We have a natural action of the Spin group on $\mathbb{R}^m$ given by:

$$H : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{R}^m) : s \mapsto H(s),$$

with

$$H(s)[x] = sx\overline{s}.$$

This action then induces one on the space of scalar valued functions by means of

$$H(s)[f](x) := f(\overline{s}xs).$$

Note that one could also consider the action of $\mathrm{SO}(m)$, which is given by:

$$\forall A \in \mathrm{SO}(m) : A \cdot P(x) = P(A^{-1}x),$$

but because that will no longer be sufficient when $\mathbb{W} = \mathbb{S}^{\pm}_{2n}$, we prefer to work with the $\mathrm{Spin}(m)$ group for the sake of continuity. The Lie algebra associated to $\mathrm{Spin}(m)$ is the orthogonal Lie algebra and thus the H-action induces the following action of $\mathfrak{so}(m)$:

Definition 2.3.2. The derived action dH of $\mathfrak{so}(m)$ on $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$ is given by:

$$dH : \mathfrak{so}_{\mathbb{R}}(m) \to \mathrm{End}(\mathcal{P}(\mathbb{R}^m, \mathbb{C})),$$

with

$$dH(e_{ij})[P](x) := \frac{d}{dt} H(e^{te_{ij}})[P](x)\Big|_{t=0} = 2\left(x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\right) P(x).$$

The operators

$$L^x_{ij} := x_i \partial_{x_j} - x_j \partial_{x_i}$$

with $1 \leq i < j \leq n$ are called the angular momentum operators and they define a model for the Lie algebra $\mathfrak{so}(m)$.
There are numerous operators that act on $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$ but the ones that are of particular interest to us, are the ones that are invariant to the H-action of the Spin group $\mathrm{Spin}(m)$.

Definition 2.3.3. The following operators are H-invariant:

1. For every $x \in \mathbb{R}^m$ we define the norm squared as:

$$|x|^2 := r^2 = \sum_{j=1}^{m} x_j^2$$

   and it is often seen as a multiplication-operator.

2. We define the Euler operator on $\mathbb{R}^m$ as:

$$\mathbb{E}_x := \sum_{j=1}^{m} x_j \partial_{x_j} = r\partial_r.$$

   Note that for a k-homogeneous polynomial $P_k(x) \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$ we have that

$$\mathbb{E}_x P_k(x) = k P_k(x).$$

3. The Laplace operator on $\mathbb{R}^m$ is defined as:

$$\Delta_x := \sum_{j=1}^{m} \partial^2_{x_j}.$$

We know that these generate the algebra of all the possible operators that are invariant under the H-action of $\mathrm{Spin}(m)$ because of the Howe duality, see e.g. [44], which states that the $\mathrm{SO}(m)$-invariant operators on $\mathbb{C}^\infty(\mathbb{R}^m, \mathbb{C})$ span a Lie algebra isomorphic to $\mathfrak{sl}(2)$:

Lemma 2.3.4. *We have that*

$$\mathfrak{sl}(2) \cong \mathrm{Alg}\left(\frac{1}{2}|x|^2, -\frac{1}{2}\Delta_x, \mathbb{E}_x + \frac{m}{2}\right) \subset \mathrm{End}(\mathcal{P}(\mathbb{R}^m, \mathbb{C})).$$

We then want to rewrite the Laplace operator in polar coordinates, and for this we need:

Definition 2.3.5. The following operators will act solely on the angular part of a vector x and thus they can be regarded as operators on the sphere S^{m-1}.

1. The Gamma operator is defined as

$$\Gamma_x := -\sum_{i<j} e_{ij} L_{ij}^x.$$

2. The Laplace-Beltrami operator is defined as:

$$\Delta_{LB}^x := \Gamma_x(m - 2 - \Gamma_x).$$

   Note that this is a scalar operator, despite the Gamma operators in its definition. Moreover, one could also write the Laplace-Beltrami in terms of the angular momentum operators as

$$\Delta_{LB}^x = \sum_{i<j} \left(L_{ij}^x\right)^2,$$

   from which one can conclude that, for the Casimir operator $\mathcal{C}_2$ of $\mathfrak{so}(m)$ belonging to the dH-action defined above, the following holds:

$$\frac{1}{4}\mathcal{C}_2 = \Delta_{LB}^x.$$

These two operators can be used to write:

$$\Delta_x = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta_{LB}^x$$

from which it follows that, where $\mathbb{E}_x = r\partial_r$:

$$|x|^2\Delta_x = \mathbb{E}_x(\mathbb{E}_x + m - 2) + \Delta_{LB}^x.$$

To show the importance of the Laplace-Beltrami operator we compute the Casimir operator for the realisation for $\mathfrak{sl}(2)$ given in lemma 2.3.4:

$$2\mathcal{C}_2 = H^2 + 2\{X, Y\} = -\Delta_{LB}^x + \frac{m}{2}\left(\frac{m}{2} - 2\right).$$

We find that the Casimir operator for $\mathfrak{sl}(2)$ is, up to a constant and a shift, given by the Laplace-Beltrami operator.
While it can be shown that $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$ is an irreducible $\mathrm{GL}(m)$-representation with weight (k), it decomposes under the H-action of $\mathrm{Spin}(m)$ (or $\mathrm{SO}(m)$). There are two ways to see this: either one uses the abstract branching rules, see example 1.4.8 and below, or one employs the power of the polynomial model, as this allows us to realise invariant subspaces in terms of a differential operator which is rotationally invariant. In this case, the irreducible components are solutions for the Laplace operator Δ_x:

Definition 2.3.6. A function $f(x)$ on $\mathbb{R}^m$ is called harmonic if and only if $f(x) \in \ker \Delta_x$. If we focus on the polynomial solutions we write:

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) := \{H_k(x) \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) : \Delta_x H_k(x) = 0\}$$

and

$$\mathcal{H}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}).$$

One can then prove the following result:

Theorem 2.3.7. *Let $m \geq 3$. The space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ defines an irreducible module for the Spin group $\mathrm{Spin}(m)$ (or $\mathrm{SO}(m)$ or the orthogonal Lie algebra) with highest weight (k) under the $H(s)$-action defined by $H(s)[P(x)] = P(\overline{s}xs)$. The highest weight vector is then given by*

$$w_k = (x_1 - i x_2)^k.$$

As we have seen in example 1.4.8, we know that

$$(k)\Big|^{\mathrm{GL}(m)}_{\mathrm{SO}(m)} \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} (k - 2i)_{\mathrm{SO}(m)}.$$

A polynomial version for this branching rule can be described as follows:

Theorem 2.3.8 (Fischer decomposition)**.** *For every $k \in \mathbb{Z}^+$ we have:*

$$\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} |x|^{2j} \mathcal{H}_{k-2j}(\mathbb{R}^m, \mathbb{C}).$$

Another way to arrive at this result, and to show that the Laplace operator is surjective, is by using the following lemma:

Lemma 2.3.9. *For every $k \in \mathbb{Z}^+$ and $P_q(x) \in \mathcal{P}_q(\mathbb{R}^m, \mathbb{C})$ we have that:*

$$\Delta_x |x|^{2k} P_q(x) = 4k \left(k + q - 1 + \frac{m}{2}\right) |x|^{2k-2} P_q(x) + |x|^{2k} \Delta_x P_q(x).$$

This can be seen as an operator identity for $[Y, X^k] = -kX^{k-1}(H + k - 1)$ in the Lie algebra $\mathfrak{sl}(2)$ as realised in lemma 2.3.4.

This identity can be used to show that:

Corollary 2.3.10. *The Laplace operator*

$$\Delta_x : \mathcal{P}_{k+2}(\mathbb{R}^m, \mathbb{C}) \to \mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$$

is surjective.

The Fischer decomposition can also be represented graphically, by decomposing the space of polynomials $\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$ into irreducible modules under the action of the Spin group, which yields the following diagram:

$$\begin{array}{cccccccc}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \mathcal{P}_6 & \cdots \\
|| & || & || & || & || & || & || & \\
 & & & & & & & \\
\mathcal{H}_0 & & |x|^2\mathcal{H}_0 & & |x|^4\mathcal{H}_0 & & |x|^6\mathcal{H}_0 & \cdots \\
 & \mathcal{H}_1 & \oplus & |x|^2\mathcal{H}_1 & \oplus & |x|^4\mathcal{H}_1 & \oplus & \cdots \\
 & & \mathcal{H}_2 & \oplus & |x|^2\mathcal{H}_2 & \oplus & |x|^4\mathcal{H}_2 & \cdots \\
 & & & \mathcal{H}_3 & \oplus & |x|^2\mathcal{H}_3 & \oplus & \cdots \\
 & & & & \mathcal{H}_4 & \oplus & |x|^2\mathcal{H}_4 & \cdots \\
 & & & & & \mathcal{H}_5 & \oplus & \cdots \\
 & & & & & & \mathcal{H}_6 & \cdots \\
 & & & & & & & \ddots
\end{array}$$

Note that the total decomposition is not multiplicity free but using lemma 2.3.4 one can group these multiplicities row-wise to find an irreducible infinite dimensional $\mathfrak{sl}(2)$-module:

$$\begin{tikzcd}
0 & \mathcal{H}_k \arrow[l, "\Delta_m"'] \arrow[r, shift left, "|x|^2"] \arrow[loop below, "\mathbb{E}_x"] & {|x|^2\mathcal{H}_k} \arrow[l, shift left, "\Delta_m"] \arrow[r, shift left, "|x|^2"] \arrow[loop below, "\mathbb{E}_x"] & {|x|^4\mathcal{H}_k} \arrow[l, shift left, "\Delta_m"] \arrow[r, shift left, "|x|^2"] \arrow[loop below, "\mathbb{E}_x"] & {|x|^6\mathcal{H}_k} \arrow[l, shift left, "\Delta_m"] \arrow[r, shift left, "|x|^2"] \arrow[loop below, "\mathbb{E}_x"] & \cdots \arrow[l, shift left, "\Delta_m"]
\end{tikzcd}$$

We denote this module by $\mathbb{V}_k^\infty$ and these modules are called Verma modules and are generated by the action of $X \in \mathfrak{sl}(2)$. If we look again at the decomposition of the space of polynomials then we can write this as:

$$\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{V}_k^\infty.$$

If we consider this as a decomposition for the action of $\mathfrak{so}(m) \times \mathfrak{sl}(2)$ then this decomposition is multiplicity free. We call $\mathfrak{so}(m) \times \mathfrak{sl}(2)$ a Howe dual pair. Moreover, when one looks intuitively to functions on the sphere S^{m-1}, then each row in the diagram will "roll up" since for $|x|^2 = 1$ each row reduces to $\mathcal{H}_k$. This is confirmed by the following well-known proposition:

Proposition 2.3.11. *Let $m \in \mathbb{Z}^+$ then*

$$L^2(S^{m-1}) \cong \bigoplus_{k \in \mathbb{Z}^+} \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}).$$

Proof. This follows either from the Peter-Weyl theorem for homogeneous spaces or one can use the Fischer decomposition found above combined with the Stone-Weierstrass approximation theorem, see e.g. [63]. □

Note that the results found above can be "refined" by factorising the Laplace operator. In view of the fact that $\Delta_2 = 4\partial_z\overline{\partial_z}$ in 2 dimensions, the following operator can thus be seen as a generalisation of the Cauchy-Riemann operator in complex analysis, which we mentioned earlier in the introduction of this chapter:

Definition 2.3.12. The space of (smooth) Clifford-valued functions (resp. polynomials) on $\mathbb{R}^m$ is denoted by $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C}_m)$ (resp. $\mathcal{P}(\mathbb{R}^m, \mathbb{C}_m)$). We have that

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C}_m) := \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{C}_m.$$

The Dirac operator ∂_x on $\mathbb{C}^\infty(\mathbb{R}^m, \mathbb{C}_m)$ is defined as:

$$\partial_x := \sum_{j=1}^m e_j \partial_{x_j}.$$

Note that $\partial_x^2 = -\Delta_x$, which justifies the common saying that Clifford analysis is a refinement of harmonic analysis.

Definition 2.3.13. A Clifford-valued function $f(x)$ is called monogenic if and only if $\partial_x f(x) = 0$. We then denote the monogenic polynomials of degree k by

$$\mathcal{M}_k(\mathbb{R}^m, \mathbb{C}_m) = \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m) \cap \ker \partial_x.$$

The action of the Spin group on the space of Clifford algebra-valued polynomials is given by

$$L(s)[P](x) = (H(s) \otimes l(s))[P](x) = sP(\bar{s}xs)$$

with $l(s)$ the left multiplication in $\mathbb{C}_m$. We can then consider the derived action dL of $\mathfrak{so}(m)$ on $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m)$ by:

$$dL(e_{ij})[P](x) := \frac{d}{dt} L(e^{te_{ij}})[P](x)\bigg|_{t=0} = (-2L_{ij} + e_{ij})P(x).$$

Definition 2.3.14. We define the operators $M_{ij}^x \in \mathrm{End}(\mathcal{P}(\mathbb{R}^m, \mathbb{C}_m))$ as

$$M_{ij}^x = L_{ij}^x - \frac{1}{2} e_{ij}.$$

The following proposition is then obvious:

Proposition 2.3.15. *The operators M_{ij}^x with $1 \leq i < j \leq m$ define a model for the Lie algebra $\mathfrak{so}(m)$ with the commutator as Lie bracket.*

Similarly to the Laplace operator, the Dirac operator is invariant with respect to the $L-$action of the group $\mathrm{Spin}(m)$:

Corollary 2.3.16. *For all $s \in Spin(m)$ we have that*

$$\partial_x \circ L(s) = L(s) \circ \partial_x.$$

From this it follows that:

Theorem 2.3.17. *Let $k \in \mathbb{Z}^+$, then $\mathcal{M}_k(\mathbb{R}^m, \mathbb{C}_m)$ is an invariant subspace under the action of the Spin group.*

However, this space is not irreducible as for this one needs to consider spinor-valued functions instead because the Clifford algebra as a whole can be seen as a direct sum of many isomorphic copies of a spinor space. When working with spinor spaces we need to take the parity of m into account. If $m = 2n+1$ is odd then $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n})$ defines an irreducible module for the Spin group. If $m = 2n$ is even then both $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}^{\pm}_{2n})$ will be irreducible and here one has to take into account that the action of the Dirac operator changes the sign of the spinor spaces, i.e.

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\pm}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\mp}).$$

Proposition 2.3.18. *Let $m \geq 3$. The space $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}^{\pm})$ (where we drop the sign if m is odd) defines an irreducible module for the Spin group (or the orthogonal Lie algebra $\mathfrak{so}(m)$) under the $L(s)$-action with highest weight $(k)'_{\pm}$. The corresponding highest weight vectors are then given by:*

$$(k)'_+ \sim w_k I \qquad (k)'_- \sim w_k \mathfrak{f}_n^\dagger I$$

Given a Clifford algebra-valued harmonic polynomial, we can then construct a monogenic polynomial using the following theorem.

Theorem 2.3.19. *Take $H_k(x) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}_m)$ a Clifford-valued k-homogeneous harmonic polynomial then*

$$H_k(x) = M_k(x) + x M_{k-1}(x),$$

in which $M_j(x) \in \mathcal{M}_j(\mathbb{R}^m, \mathbb{C}_m)$. Both polynomials are unique and defined as:

$$\begin{aligned} M_{k-1}(x) &= -\frac{1}{2k+m-2}\partial_x H_k(x) \\ M_k(x) &= \left(1 + \frac{1}{2k+m-2} x\partial_x\right) H_k(x). \end{aligned}$$

In other words,

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S}^{\pm}_{2n} = \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}^{\pm}_{2n}) + x\mathcal{M}_{k-1}(\mathbb{R}^m, \mathbb{S}^{\mp}_{2n})$$

or equivalently

$$(k) \otimes (0)'_{\pm} \cong (k)'_{\pm} \oplus (k-1)'_{\mp}$$

Combining both theorem 2.3.8 and theorem 2.3.19 yields the monogenic Fisher decomposition:

Theorem 2.3.20 (Monogenic Fischer decomposition)**.** *For every $k \in \mathbb{Z}^+$ we have the decomposition:*

$$\mathcal{P}_k(\mathbb{R}^m, \mathbb{S}^{\pm}_{2n}) = \bigoplus_{j=0}^{k} x^j \mathcal{M}_{k-j}(\mathbb{R}^m, \mathbb{S}^{\sigma_j}_{2n})$$

where σ_j is the sign that makes the parities on both sides of the equality match, i.e. if j is even then $\sigma_j = \pm$ and $\sigma_j = \mp$ otherwise.

As a by-product, we obtain a refinement of corollary 2.3.10, which says that

$$\partial_x : \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m) \to \mathcal{P}_{k-1}(\mathbb{R}^m, \mathbb{C}_m)$$

is surjective.

2.3.1 Conformal invariance

Both the Laplace and the Dirac operator are not just $\mathrm{Spin}(m)$-invariant, they are also conformally invariant. In other words, their first order generalised symmetries span the conformal Lie algebra $\mathfrak{so}(1, m+1)$. To clarify this we look at the structure of the conformal Lie algebra $\mathfrak{so}_{\mathbb{R}}(1, m+1)$ which can be seen as the bivectors of the real Clifford algebra $\mathbb{R}_{1,m+1}$. If we denote the basis of $\mathbb{R}^{m+2}$ by $e_0, e_1, \dots, e_{m+1}$ then we have that

$$\mathfrak{so}(1, m+1) \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} = \mathbb{R}^m \oplus (\mathbb{R} \oplus \mathfrak{so}_{\mathbb{R}}(m)) \oplus \mathbb{R}^m$$

as a graded Lie algebra. This means that there exists a grading element $E \in \mathfrak{g}_0$ such that $[E, X_\alpha] = \alpha X_\alpha$ and $[X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}$.
We can then make the following identifications:

- $\mathfrak{g}_{-1}$ with $\mathrm{span}_{\mathbb{R}}\{e_j(e_0 - e_{m+1}) | 1 \leq j \leq m\}$,
- $\mathfrak{g}_{+1}$ with $\mathrm{span}_{\mathbb{R}}\{e_j(e_0 + e_{m+1}) | 1 \leq j \leq m\}$,
- the grading element with $e_{m+1}e_0$,
- and $\mathfrak{so}_{\mathbb{R}}(m)$ is the Lie algebra spanned by e_{ij} with $1 \leq i < j \leq m$.

Definition 2.3.21. Given a differential operator $\mathcal{D}$ then we say that an operator $\mathcal{D}_1$ is a generalised symmetry of $\mathcal{D}$ if and only if there exists another operator $\mathcal{D}_2$ such that $\mathcal{D}\mathcal{D}_1 = \mathcal{D}_2\mathcal{D}$.

In particular we have that a generalised symmetry preserves the kernel of the operator. If we consider the first order generalised symmetries of the Laplace (resp. Dirac) operator then these span a Lie algebra isomorphic to the conformal Lie algebra. To determine such generalised symmetries we will need both the monogenic and the harmonic inversion, who will return later.

Definition 2.3.22. The harmonic inversion (or Kelvin inversion) for functions on $\mathbb{R}^m$ is defined as:

$$\mathcal{J}_\alpha[f](x) = |x|^{2-m+\alpha} f\left(\frac{x}{|x|^2}\right).$$

Then, for all α, $\mathcal{J}_\alpha^2 = \mathrm{Id}$ and for $\alpha = 0$

$$\Delta_m f(x) = 0 \iff \Delta_m \mathcal{J}_0[f](x) = 0.$$

The monogenic inversion for $\mathbb{C}_m$-valued functions on $\mathbb{R}^m$ is defined as:

$$\mathcal{I}_\alpha[f](x) = \frac{x}{|x|^{\alpha+m}} f\left(\frac{x}{|x|^2}\right).$$

Then $\mathcal{I}_\alpha^2 = -\mathrm{Id}$ for all α and $\mathcal{I}_0$ preserves the solutions of ∂_x.

These inversions can be used to construct the following symmetries:

Lemma 2.3.23. *Let $1 \leq j \leq m$, then we have the following operator identities:*

$$\begin{aligned} \mathcal{J}_0 \partial_{x_j} \mathcal{J}_0 &= |x|^2 \partial_{x_j} - x_j(2\mathbb{E}_x + m - 2) \\ \mathcal{I}_0 \partial_{x_j} \mathcal{I}_0 &= x e_j + x_j(2\mathbb{E}_x + m) - |x|^2 \partial_{x_j}. \end{aligned}$$

These operators are called the special conformal transformations.

This leads to the following theorems, which express the conformal invariance of the Laplace and the Dirac operator:

Theorem 2.3.24 (Conformal Invariance of the Laplace operator)**.** *The first order generalised symmetries of the Laplace operator are:*

1. *The (shifted) Euler operator:* $\mathbb{E}_x + \frac{m-2}{2}$.
2. *The infinitesimal rotations:* L_{ij}^x.
3. *The infinitesimal translations:* ∂_{x_j}.
4. *The special conformal transformations:* $\mathcal{J}_0 \partial_{x_j} \mathcal{J}_0$.

These operators span a Lie algebra, with the commutator as bracket, which is isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m+1)$. In other words we have that:

$$\mathfrak{so}(1, m+1) \cong \bigoplus_{j=1}^{m} \mathbb{R}\partial_{x_j} \oplus \left(\mathbb{R}\left(\mathbb{E}_x + \frac{m-2}{2}\right) \oplus \bigoplus_{i<j} \mathbb{R} L_{ij}^x\right) \oplus \bigoplus_{j=1}^{m} \mathbb{R}\mathcal{J}_0 \partial_{x_j} \mathcal{J}_0.$$

Theorem 2.3.25 (Conformal Invariance of the Dirac operator)**.** *The first order generalised symmetries of the Dirac operator are:*

1. *The (shifted) Euler operator:* $\mathbb{E}_x + \frac{m-1}{2}$.

2. *The infinitesimal rotations:* M_{ij}^x.

3. *The infinitesimal translations:* ∂_{x_j}.

4. *The special conformal transformations:* $\mathcal{I}_0 \partial_{x_j} \mathcal{I}_0$.

Analogue to the harmonic case we have that these operators span a Lie algebra isomorphic to the conformal Lie algebra and

$$\mathfrak{so}(1, m+1) \cong \bigoplus_{j=1}^{m} \mathbb{R}\partial_{x_j} \oplus \left(\mathbb{R}\left(\mathbb{E}_x + \frac{m-1}{2}\right) \oplus \bigoplus_{i<j} \mathbb{R}M_{ij}^x \right) \oplus \bigoplus_{j=1}^{m} \mathbb{R}\mathcal{I}_0 \partial_{x_j} \mathcal{I}_0.$$

2.4 Higher spin Clifford analysis

As we mentioned in at the beginning of this chapter, we are interested in the function spaces $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$ and the invariant operators that act on them. In the previous section we had chosen $\lambda = (0)$ (resp. $\lambda = (0)'$) and looked at the irreducible Spin(m) components which lead to the harmonic (resp. monogenic) polynomials of a given degree. In this section, we consider different values for λ, which will be equivalent to functions of several variables (also known as a matrix variable) that belong to the kernel of certain Spin(m) invariant operators (the analogue of harmonic polynomials as the kernel of the Laplace operator).

2.4.1 Two vector formalism

Let $\lambda = (k)$ then we are considering the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}))$, which consists of functions $f(x, u)$ such that

$$f(x, u) := f_x(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}), \quad \forall x \in \mathbb{R}^m.$$

This means that

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})) = \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker(\Delta_u, \mathbb{E}_u - k),$$

where we identify the space $\mathbb{R}^{2m}$ with $\mathbb{R}^{m\times 2}$, the space containing vectors (x, u) with $x, u \in \mathbb{R}^m$. Completely similar, if $\lambda = (k)'$ then

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^\pm)) = \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}_{2n}^\pm) \cap \ker(\partial_u, \mathbb{E}_u - k).$$

Note that this can be done for any highest weight λ describing a Spin(m) representation, as we can always consider functions in $l(\lambda) + 1$ variables that satisfy certain conditions. In this thesis we will work with functions of two vector variables and there is a natural action of the group Spin(m) on these functions. If $f(x, u)$ is scalar valued then, for $s \in \text{Spin}(m)$:

$$H(s)[f](x, u) = f(\overline{s}xs, \overline{s}us).$$

The derived action of the Lie algebra $\mathfrak{so}(m)$ is then given by

$$dH(e_{ij})[f](x, u) = -2(L_{ij}^x + L_{ij}^u)f(x, u)$$

and the Casimir operator $\mathcal{C}_2$ w.r.t. this action can be written as, see e.g. [15]:

$$\frac{1}{4}\mathcal{C}_2 = \Delta^x_{LB} + \Delta^u_{LB} + 2\left(\langle x,u\rangle\langle\partial_x,\partial_u\rangle - \langle u,\partial_x\rangle\langle x,\partial_u\rangle + \mathbb{E}_u\right).$$

We note $\mathcal{P}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ for the space of scalar-valued polynomials in two variables x and u with degree of homogeneity in x (resp. u) equal to ℓ (resp. k) then we are interested in the $\mathrm{SO}(m)$-invariant operators that act on them. In the one variable case, such operators spanned the Lie algebra $\mathfrak{sl}(2)$ which was expressed in the Howe duality $\mathfrak{sl}(2)\times\mathrm{SO}(m)$. Here, the Weyl-algebra $\mathcal{W}(\mathbb{R}^{m\times 2})$ is given by

$$\mathcal{W}(\mathbb{R}^{m\times 2}) = \mathrm{Alg}\left(x_a, u_b, \partial_{x_a}, \partial_{x_b} : 1\leq a,b\leq m\right)$$

and we are thus interested in $\mathcal{W}(\mathbb{R}^{m\times 2})^{\mathrm{SO}(m)}$. From [79] we can conclude that

$$\begin{aligned}\mathcal{W}(\mathbb{R}^{m\times 2})^{\mathrm{SO}(m)} =&\mathrm{Alg}\big(\Delta_x, \Delta_u, |x|^2, |u|^2, \mathbb{E}_x + \frac{m}{2}, \mathbb{E}_u + \frac{m}{2},\\ &\qquad\qquad \langle x,\partial_u\rangle, \langle u,\partial_x\rangle, \langle\partial_x,\partial_u\rangle, \langle x,u\rangle\big)\end{aligned}$$

and this algebra is a model for the symplectic Lie algebra $\mathfrak{sp}(4)$, as predicted by the Howe dual pair $\mathfrak{sp}(4)\times\mathrm{SO}(m)$, see e.g. [50, 51].
We now want to decompose $\mathcal{P}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ into irreducible components w.r.t. the H-action and the irreducible components that appear generalise the space of harmonic polynomials in one variable and are described using the invariant differential operators:

Definition 2.4.1. For $\ell \geq k \in \mathbb{Z}^+$, the space of simplicial harmonics is defined as:

$$\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C}) := \mathcal{P}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C}) \cap \ker(\Delta_x, \Delta_u, \langle x,\partial_u\rangle).$$

These spaces satisfy the following properties:

Proposition 2.4.2.

1. *For $m > 4$, the space $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ is an irreducible* $\mathrm{Spin}(m)$ *(or* $\mathrm{SO}(m)$ *or $\mathfrak{so}(m)$) module with highest weight (ℓ,k) and the highest weight vector is given by:*

$$w_{\ell,k} = (x_1 - ix_2)^{\ell-k}\left((x_1 - ix_2)(u_3 - iu_4) - (x_3 - ix_4)(u_1 - iu_2)\right)^k.$$

   A proof for this can be found in e.g. [15, 42].

2. *If $\ell = k$, we also have that $\mathcal{H}_{k,k}(\mathbb{R}^{2m},\mathbb{C}) \subset \ker\langle u,\partial_x\rangle$ due to the symmetry in x and u.*

Remark 2.4.3. In the case $m = 4$ it is slightly more complicated, as the space of simplicial harmonics is no longer irreducible; it can then be decomposed in two irreducible components. A well-known example of this is the splitting of $\Lambda^2\mathbb{C}^4$ into self-dual and anti-self-dual forms in $\Lambda^2_\pm\mathbb{C}^4$. We will come back to this later in section 4.1.6.

Let us now consider functions $f(x,u)$ that take values in a spinor space $\mathbb{S}^{\pm}_{2n}$, then we can let $\mathrm{Spin}(m)$ act on these functions by means of:

$$L(s)[f](x,u) = sf\left(\bar{s}xs, \bar{s}us\right).$$

The corresponding action of the orthogonal Lie algebra is given by:

$$\mathrm{d}L(e_{ij})[f](x,u) = -2\left(L^x_{ij} + L^u_{ij} - \frac{1}{2}e_{ij}\right)f(x,u) = \left(\mathrm{d}H(e_{ij}) + e_{ij}\right)[f](x,u).$$

Now let $\mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}_{2n}) = \mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^{\pm}_{2n}$ be the space of spinor-valued polynomials in two variables x and u with degree of homogeneity in x (resp. u) equal to ℓ (resp. k). This space will also decompose into irreducible summands, which brings us to the following spaces:

Definition 2.4.4. For $\ell \geq k \in \mathbb{Z}^+$ we define the space of simplicial monogenics by means of

$$\mathcal{S}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}_{2n}) := \mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}_{2n}) \cap \ker(\partial_x, \partial_u, \langle x, \partial_u\rangle).$$

If $m > 4$, these spaces define irreducible representations for the Spin group $\mathrm{Spin}(m)$ (or the orthogonal Lie algebra $\mathfrak{so}(m)$) with highest weight $(\ell,k)'_{\pm}$ and corresponding highest weight vector (see [15])

$$v^+_{\ell,k} := w_{\ell,k}I \qquad \text{or} \qquad v^-_{\ell,k} := w_{\ell,k}\mathfrak{f}^\dagger_n I,$$

depending on the parity of m and the sign of the spinor space, with $w_{\ell,k}$ as in proposition 2.4.2. Furthermore, there exists a generalisation for theorem 2.3.19, which said how the space of harmonic polynomials decomposes when considering the tensor product with $(0)'_{\pm}$, see e.g. [7, 60]:

Proposition 2.4.5. *Let $\ell \geq k$ and $m > 4$ then*

$$(\ell,k) \otimes (0)'_{\pm} = (\ell,k)'_{\pm} \oplus (\ell-1,k)'_{\mp} \oplus (\ell,k-1)'_{\mp} \oplus (\ell-1,k-1)'_{\pm}$$

where the space $(\ell-1,k)'$ only occurs when $\ell > k$. Note that the parity changes according to the total number of entries differing from the highest weight (ℓ,k) appearing at the left-hand side.

If $m = 4$ then a similar result holds, see theorem 6.2.12.

2.4.2 The higher spin Laplace operator $\mathcal{D}_k$

The operators Δ_x and ∂_x were not just $\mathrm{Spin}(m)$ invariant, they are also conformally invariant (see above) and one could ask what the higher spin versions of these operators are. The Dirac operator generalises as follows:

Definition 2.4.6. Consider $f(x,u) \in \mathbb{C}^\infty(\mathbb{R}^m, \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}^{\pm}_{2n}))$, then the Rarita-Schwinger operator is given by the first-order differential operator

$$\mathcal{R}_k f(x,u) := \left(1 + \frac{u\partial_u}{m+2k-2}\right)\partial_x f(x,u).$$

This operator is the unique conformally invariant first-order differential operator that acts on $\mathbb{C}^\infty(\mathbb{R}^m, \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}))$. For further details we refer the reader to [8, 30, 70], where this operator was constructed and studied in depth.

In this thesis we are interested in the higher spin Laplace operator which was constructed and studied in [16].

Definition 2.4.7. The generalisation of the Laplace operator to the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}))$ is defined as the operator

$$\mathcal{D}_k : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}))$$

with

$$\mathcal{D}_k = \Delta_x - \frac{4}{2k+m-2}\left(\langle u, \partial_x\rangle - \frac{|u|^2}{2k+m-4}\langle \partial_u, \partial_x\rangle\right)\langle \partial_u, \partial_x\rangle .$$

This operator is conformally invariant with respect to the following inversion:

$$f(x,u) \mapsto \mathcal{J}_R[f](x,u) := |x|^{2-m} f\left(\frac{x}{|x|^2}, \frac{xux}{|x|^2}\right) .$$

In terms of this conformal symmetry operator, one can define an operator which will be crucial for our purposes, as it will play the role of the ladder operator that generates Gegenbauer type solutions (see proposition 2.5.5), which will be explored in the next chapter.

Lemma 2.4.8. *One has that*

$$\mathcal{J}_R \partial_{x_i} \mathcal{J}_R = |x|^2 \partial_{x_i} + 2\langle x, u\rangle \partial_{u_i} - 2u_i \langle x, \partial_u\rangle - x_i(2\mathbb{E}_x + m - 2).$$

This operator raises the degree of homogeneity in x by 1, while preserving both $\ker \mathcal{D}_k$ and $\ker \Delta_u$. It should be seen as the higher spin analogue of the special conformal transformations from lemma 2.3.23.

Using this operator we can realise a copy of the Lie algebra $\mathfrak{sl}(2)$ inside the full conformal Lie algebra $\mathfrak{so}(1, m+1)$ of (generalised) symmetries for the operators $\mathcal{D}_k$ (see [16] for more details).

Proposition 2.4.9. *We have the following realisation of $\mathfrak{sl}(2)$:*

$$\mathfrak{sl}(2) \cong \mathrm{Alg}(\mathcal{J}_R \partial_{x_j} \mathcal{J}_R, \partial_{x_j}, 2\mathbb{E}_x + m - 2).$$

One of the main differences between the Laplace operator Δ_x and the operators $\mathcal{D}_k$ for $k \neq 0$ is the fact that the polynomial kernel for the latter operator is not irreducible under the action of $\mathfrak{so}(m)$. As a matter of fact, in [16] is was shown that one can decompose the ℓ-homogeneous kernel of $\mathcal{D}_k$ as follows:

$$\ker_\ell \mathcal{D}_k = \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-i} (\mathcal{J}_R \Delta_x \mathcal{J}_R)^i \langle u, \partial_x\rangle^{i+j} \mathcal{H}_{\ell-i+j, k-i-j}.$$

Note that these embedding operators are explicitly given as follows:

Lemma 2.4.10. *The operator* $\mathcal{J}_R \Delta_x \mathcal{J}_R$ *is given by:*

$$\mathcal{J}_R \Delta_x \mathcal{J}_R = |x|^4 \Delta_x + 4\left((2\mathbb{E}_u + m - 4)\langle u, x\rangle + |u|^2 \langle x, \partial_u\rangle\right)\langle x, \partial_u\rangle$$
$$+4|x|^2\left(\langle u, x\rangle\langle \partial_u, \partial_x\rangle - \langle u, \partial_x\rangle\langle x, \partial_u\rangle\right).$$

2.5 Gegenbauer polynomials

As the title of the thesis suggested, we are interested in special functions in higher spin Clifford analysis and we are particularly interested in the Gegenbauer polynomials. They are related to the reproducing kernel for the spaces $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ and they are crucial in the construction of Gelfand-Tsetlin bases, see e.g. [67], due to their connection with the branching problem from $\mathfrak{so}(m)$ to $\mathfrak{so}(m-1)$.

2.5.1 Basic properties

Definition 2.5.1. If we consider the differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + k(k+2\alpha)y = 0$$

with $k \in \mathbb{Z}^+$ and $\alpha > -\frac{1}{2}$, then the Gegenbauer polynomial of degree k and order α is a polynomial solution to this equation.

An explicit form is given by:

Theorem 2.5.2. *For* $\alpha \neq 0$ *the Gegenbauer polynomials satisfy:*

$$C_k^\alpha(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{\Gamma(k-j+\alpha)}{\Gamma(\alpha) j!(k-2j)!}(2x)^{k-2j}.$$

If $\alpha = 0$ *then*

$$C_k^0(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{\Gamma(k-j)}{j!(k-2j)!}(2x)^{k-2j}.$$

These polynomials satisfy some recurrence relations which we include for the sake of completeness, more details can be found in e.g. [1, 33, 34, 76].

Proposition 2.5.3. *For* $\alpha \neq 0$ *the Gegenbauer polynomials satisfy the following identities:*

$$\begin{aligned}
\frac{d}{dx}C_k^\alpha(x) &= 2\alpha C_{k-1}^{\alpha+1}(x)\\
(k+2\alpha)C_k^\alpha(x) &= 2\alpha(C_k^{\alpha+1}(x) - xC_{k-1}^{\alpha+1}(x))\\
kC_k^\alpha(x) &= 2\alpha(xC_{k-1}^{\alpha+1}(x) - C_{k-2}^{\alpha+1}(x))\\
(k+\alpha)C_k^\alpha(x) &= \alpha(C_k^{\alpha+1}(x) - C_{k-2}^{\alpha+1}(x)).
\end{aligned}$$

We point out that for all $\alpha > -\frac{1}{2}$ the Gegenbauer polynomials are a family of orthogonal polynomials with

$$\int_{-1}^{1} C_k^\alpha(x) C_l^\alpha(x)(1-x^2)^{\alpha-\frac{1}{2}} dx = \frac{\pi 2^{1-2\alpha}\Gamma(k+2\alpha)}{(k+\alpha)\Gamma(\alpha)^2\Gamma(k+1)}\delta_{kl}.$$

Moreover, they can be rewritten in terms of a hypergeometric function, using a suitable change of variables:

$$C_k^\mu(x) = \frac{\Gamma(2\mu+k)}{\Gamma(2\mu)k!}\,{}_2F_1\left(-k, k+2\mu, \mu+\frac{1}{2}; \frac{1-x}{2}\right).$$

2.5.2 Gegenbauer polynomials in harmonic analysis

The Gegenbauer polynomials occur naturally when looking for harmonic polynomials on $\mathbb{R}^m$ of a given degree that only depend on the norm of the vector and its inner product with a fixed unit vector. This leads to harmonic polynomials which are invariant under a large subgroup of the full rotation group, see below. If we consider the splitting

$$\mathbb{R}^m = \mathbb{R}e_1 \oplus \mathbb{R}^{m-1},$$

then we can always perform a change of basis to make sure that the inner product is taken with the vector e_1. Let us adopt the notation $x = x_1 e_1 + \tilde{x}$, with $x \in \mathbb{R}^m$ and $\tilde{x} \in \mathbb{R}^{m-1}$. We are thus looking for a harmonic polynomial of the form:

$$H_k(x) = |x|^k \varphi_k(t) \qquad \text{with } t = \left\langle \frac{x}{|x|}, e_1 \right\rangle.$$

Then, one has that:

$$\Delta_x H_k(x) = 0 \iff (1-t^2)\varphi_k''(t) - (m-1)t\varphi_k'(t) + k(m-2+k)\varphi_k(t) = 0$$

This is the differential equation for the Gegenbauer polynomials mentioned earlier and this shows us that the solution to our problem is given by:

$$H_k(x) = |x|^k C_k^{\frac{m}{2}-1}(t).$$

If one considers the Lie algebra $\mathfrak{so}(m)$ with subalgebra $\mathfrak{so}(m-1) \subset \mathfrak{so}(m)$, then $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ is a $\mathfrak{so}(m-1)$ representation but it is not irreducible. The abstract branching rules, see theorem 1.4.1, tell us that

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})\Big|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong \bigoplus_{j=0}^{k} \mathcal{H}_j(\mathbb{R}^{m-1}, \mathbb{C}).$$

In particular there must exist a $\mathfrak{so}(m-1)$-invariant polynomial in $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ which corresponds to the highest weight (0) for the action of $\mathfrak{so}(m-1)$. If we see $\mathfrak{so}(m-1)$ as the subgroup that fixes the unit vector e_1 and acts on $\mathbb{R}^{m-1}$, then it is clear that the Gegenbauer polynomial $H(x) = |x|^k C_k^{\frac{m}{2}-1}(t)$ is indeed such an invariant polynomial. Therefore we can see the trivial part of the sum as constant multiples of these Gegenbauer polynomials. There is a way to construct these special solutions for Δ_x using a raising operator, which is given by a composition of an inversion operator that preserves the kernel of Δ_x (see definition 2.3.22), with a $\mathfrak{so}(m-1)$-invariant differential operator. This raising operator will be one of the special conformal transformations of the Laplace operator and can be used to construct a realisation of $\mathfrak{sl}(2)$:

Proposition 2.5.4. *We have that:*

$$\mathfrak{sl}(2) = Alg(X, Y, H) \cong Alg(\partial_{x_1}, \mathcal{J}_0\partial_{x_1}\mathcal{J}_0, -2\mathbb{E}_x - (m-2)).$$

Since $\mathcal{J}_0$ is an endomorphism on the space of harmonic functions, and from the explicit form of the raising operator in lemma 2.3.23, we can conclude that

$$\forall k \in \mathbb{Z}^+ : (\mathcal{J}_0\partial_{x_1}\mathcal{J}_0)^k[1] \in \mathcal{H}_k(\mathbb{R}_0^m, \mathbb{C}).$$

In other words we can create an infinite dimensional lowest weight module for $\mathfrak{sl}(2)$ containing harmonic polynomials by letting the operator $\mathcal{J}_0\partial_{x_1}\mathcal{J}_0$ act repeatedly on 1 (or another harmonic). Because this operator is invariant for the action of $\mathfrak{so}(m-1)$ we know that the polynomials $(\mathcal{J}_0\partial_{x_1}\mathcal{J}_0)^k[1]$ have to be $\mathfrak{so}(m-1)$-invariant as well. However, from the branching rules we can infer that there exists a unique $\mathfrak{so}(m-1)$-invariant element and thus there must exist a constant $c_k \in \mathbb{C}$ such that:

$$(\mathcal{J}_0\partial_{x_1}\mathcal{J}_0)^k[1] = \left(|x|^2\partial_{x_1} - x_1(2\mathbb{E}_x + m - 2)\right)^k [1] = c_k|x|^k C_k^{\frac{m}{2}-1}(t),$$

with t as above. An easy computation leads to the explicit form for the constant c_k, which yields:

Proposition 2.5.5. *The following formula holds for all $k \in \mathbb{Z}^+$:*

$$(\mathcal{J}_0\partial_{x_1}\mathcal{J}_0)^k[1] = (-1)^k k!|x|^k C_k^{\frac{m}{2}-1}\left(\frac{x_1}{|x|}\right).$$

The Gegenbauer polynomials can also be used to turn the isomorphism from the branching problem

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})\Big|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong \bigoplus_{j=0}^{k} \mathcal{H}_j(\mathbb{R}^{m-1}, \mathbb{C})$$

into an equality:

Theorem 2.5.6. *Let $k \in \mathbb{Z}^+$, then*

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{j=0}^{k} (\mathcal{J}_0\partial_{x_1}\mathcal{J}_0)^{k-j}\mathcal{H}_j(\mathbb{R}^{m-1}, \mathbb{C}).$$

An equivalent way to realise these embeddings is by left multiplication with a suitable Gegenbauer polynomial:

$$\mathcal{H}_j(\mathbb{R}^{m-1}, \mathbb{C}) \hookrightarrow \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) : H_j(\tilde{x}) \mapsto |x|^{k-j} C_{k-j}^{\frac{m}{2}-1+j}\left(\frac{x_1}{|x|}\right) H_j(\tilde{x}).$$

2.5.3 Gegenbauer polynomials in Clifford analysis

Because Clifford analysis is a refinement of harmonic analysis, we can also introduce a refinement of the harmonic Gegenbauer solutions. From theorem 2.3.20 we know that:

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S}_{2n}^{\pm} = \mathcal{H}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}) = \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}) \oplus x\mathcal{M}_{k-1}(\mathbb{R}^m, \mathbb{S}_{2n}^{\mp})$$

with projection operator:

$$\pi_k = \left(1 + \frac{x\partial_x}{2k+m-2}\right) : \mathcal{H}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}) \to \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}).$$

This means that the action of π_k (which is an $\mathfrak{so}(m)$-invariant operator) on an $\mathfrak{so}(m-1)$-invariant harmonic will lead to an $\mathfrak{so}(m-1)$-invariant monogenic.

Lemma 2.5.7. *For all $k \in \mathbb{Z}^+$ and $\psi \in \mathbb{S}_{2n}^{\pm}$ the following formula holds:*

$$\pi_k\left(|x|^k C_k^{\frac{m}{2}-1}(t)\psi\right) = \frac{m-2}{2k+m-2}|x|^k\left(C_k^{\frac{m}{2}}(t) + \frac{x}{|x|}e_1 C_{k-1}^{\frac{m}{2}}(t)\right)\psi.$$

We define

$$\mathcal{G}_k^m(x) := r^k\left(C_k^{\frac{m}{2}}(t) + \frac{x}{|x|}e_1 C_{k-1}^{\frac{m}{2}}(t)\right)$$

and name these the monogenic Gegenbauer polynomials. Similar to the harmonic case, we can relate these solutions to a branching problem for the Spin group and use the conformal symmetries for the Dirac operator to generate these solutions. This yields a realisation for $\mathfrak{sl}(2)$ as a subalgebra of the full conformal algebra:

Lemma 2.5.8. *Let $\alpha \in \mathbb{R}$ then:*

$$\mathfrak{sl}(2) = Alg(X, Y, H) \cong Alg(\mathcal{I}_\alpha \partial_{x_1} \mathcal{I}_\alpha, -\partial_{x_1}, 2\mathbb{E}_x + \alpha + m - 1).$$

Moreover, for all $k \in \mathbb{Z}^+$ and $\psi \in \mathbb{S}_{2n}^{\pm}$, we have that:

$$(\mathcal{I}_0 \partial_{x_1} \mathcal{I}_0)^k[\psi] \in \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm}).$$

Let us consider the aforementioned branching problem for the Spin group:

$$\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}_{2n}^{\pm})\Big|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong \begin{cases} \bigoplus_{j=0}^{k} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathbb{S}_{2n-2}) & m \text{ even} \\ \bigoplus_{j=0}^{k} \mathcal{M}_j(\mathbb{R}^{m-1}, \mathbb{S}_{2n}^{+}) \oplus \mathcal{M}_j(\mathbb{R}^{m-1}, \mathbb{S}_{2n}^{-}) & m \text{ odd.} \end{cases}$$

Then the trivial components in the branching problems will be given by $\mathcal{G}_k^m(x)\psi$, because for all M_{ij}^x, with $1 < i < j \leq m$, the angular operator only acts on the spinor space, i.e.

$$M_{ij}^x \mathcal{G}_k^m(x)\psi = \mathcal{G}_k^m(x) M_{ij}^x \psi.$$

Because the $\mathfrak{so}(m-1)$-transformational behaviour of $(\mathcal{I}_0 \partial_{x_1} \mathcal{I}_0)^k[\psi]$ is also completely determined by the spinor element, we know that there has to exist an element $c_k \in \mathbb{C}_m^+$ that commutes with the action of $\mathfrak{so}(m-1)$ such that $(\mathcal{I}_0 \partial_{x_1} \mathcal{I}_0)^k[\psi] = \mathcal{G}_k^m(x) c_k \psi$, for all spinors ψ. Depending on the parity of m this means that either $c_k \in \mathbb{C}$ or c_k is the so-called pseudoscalar $e_2 \cdots e_m$ but because the latter reduces to a complex number when acting on ψ, we can assume that $c_k \in \mathbb{C}$. This is confirmed by the following proposition:

Proposition 2.5.9. *For all $k \in \mathbb{Z}^+$ and $\psi \in \mathbb{S}_{2n}^{\pm}$ we have the following equality:*

$$(\mathcal{I}_0 \partial_{x_1} \mathcal{I}_0)^k[\psi] = k! \mathcal{G}_k^m(x)\psi.$$

CHAPTER 3

Invariant polynomial solutions for $\mathcal{D}_k$

Do not worry about your difficulties in mathematics
I can assure you that mine are still greater!
Albert Einstein

In this chapter we study special polynomial solutions for the higher spin Laplace operator, where we will use a ladder formalism to generate solutions that are invariant under the action of a particular subalgebra of the orthogonal Lie algebra, which follows the lines of our paper [22]. For the normal Laplace operator this leads to harmonic polynomials expressed in terms of Gegenbauer polynomials (as we have seen in the previous chapter), in the higher spin case the resulting solutions are more complicated. We have seen that the space of ℓ-homogeneous solutions for $\mathcal{D}_k$ is no longer irreducible which will lead to a certain ambiguity in the search for a higher spin version of the Gegenbauer type solutions: one can either generalise them using the conformal inversion (the approach adopted in this chapter), or one can focus on the fact that they should for instance reproduce certain solution spaces (which we will do in chapter 4). We will start by introducing the ladder formalism that generates our solutions but recognising these solutions in terms of well-known special functions turns out to be a difficult problem. This stems from the fact that the solution spaces are not irreducible which will be made clear using the abstract branching rules, see theorem 1.4.1. We finish the chapter by calculating an explicit example.

3.1 Raising operator

Similar to the construction of the harmonic Gegenbauer polynomials, we will construct special solutions for $\mathcal{D}_k$ by repeated action of the operator from lemma 2.4.8. In the harmonic case ($k = 0$), this was done by letting it act on the constant 1, see proposition 2.5.5, but here we need the raising operator to act on a k-homogeneous polynomial in the dummy variable $u \in \mathbb{R}^m$. This polynomial must belong to the kernel of the operator Δ_u and should be $\mathfrak{so}(m-1)$-invariant, so as not to violate the invariance built into the raising operator. This is necessary, because the H-action affects both the dummy variable u representing the values, and the variable x (see section 2.4.1). Therefore, the only possibility is the harmonic Gegenbauer polynomial in the variable u. We will implement the following notation: for $j \leq k$ we put

$$P_k^j(u) = |u|^{k-j} C_{k-j}^{\frac{m}{2}-1+j} \left(\frac{u_1}{|u|} \right) .$$

For $j > k$ we adopt the convention that $P_k^j(u) = 0$. Note that for $j = 0$, this is precisely the harmonic polynomial in $u \in \mathbb{R}^m$ we use as a starting point. Although for $j > 0$ the resulting Gegenbauer polynomials are no longer harmonic, they still have a special meaning: they occur as embedding factors for the branching problem for harmonic polynomials in $u \in \mathbb{R}^m$, see theorem 2.5.6. In other words: $P_k^j(u)$ can be interpreted as a multiplication operator which gives harmonics on $\mathbb{R}^m$ when acting on a harmonic of a certain degree in a space of one dimension lower.

Let $\mathcal{J}_R$ again denote the inversion on the kernel of the higher spin Laplace operator as defined in definition 2.4.7, then we denote the raising operator by means of

$$X := \mathcal{J}_R \partial_{x_1} \mathcal{J}_R \ .$$

We are interested in finding an expression for $X[P_k^j(u)]$, as this often occurs in what follows. We first prove a Leibniz type rule:

Lemma 3.1.1. *For $f(x,u)$ and $g(x,u)$ in $\mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{C})$, one has:*

$$X[fg] = X[f]g + fX[g] + (m-2)x_1 fg \ .$$

Proof. Each term in the formula for X, see lemma 2.4.8, satisfies the Leibniz rule, except for the multiplication with $(2-m)x_1$. To compensate this we have to add the additional last term in the formula above. □

Lemma 3.1.2. *Write $x = \tilde{x} + x_1 e_1$, $u = \tilde{u} + u_1 e_1$ then, for every $j \in \mathbb{Z}^+$, we have that:*

$$X[P_k^j(u)] = 2(m-2+2j) \langle \tilde{x}, \tilde{u} \rangle P_k^{j+1}(u) - (m-2)x_1 P_k^j(u).$$

Proof. As we are acting on polynomials that are independent of x, our raising operator reduces to $X = 2\langle x, u \rangle \partial_{u_1} - 2u_1 \langle x, \partial_u \rangle - x_1(m-2)$. Using the fact that

$$\frac{d}{dt} C_n^\mu(t) = 2\mu C_{n-1}^{\mu+1}(t),$$

the desired result follows from straightforward calculations. □

Next, we show that the action of the raising operator yields polynomial solutions for $\mathcal{D}_k$ of a special form (linear combinations of Gegenbauer polynomials in u):

Theorem 3.1.3. *For each $k, \ell \in \mathbb{Z}^+$ we have that:*

$$X^\ell[P_k^0(u)] = \sum_{i=0}^{\min(k,\ell)} f_i^{(\ell)}(x_1, |x|, u_1, \langle x, u\rangle) P_k^i(u).$$

Proof. We take an arbitrary k fixed and proceed by induction on ℓ. If $\ell = 0$ the result is trivial since $f_0^{(0)} = 1$. Assume it holds for all values up to and including $(\ell - 1)$. This means that:

$$\begin{aligned} X^\ell[P_k^0(u)] &= X X^{\ell-1}[P_k^0(u)] \\ &= \sum_{i=0}^{\min(k,\ell-1)} X[f_i^{(\ell-1)}(x_1, |x|, u_1, \langle x, u\rangle) P_k^i(u)]. \end{aligned}$$

For each $0 \leq i \leq \min(k, \ell - 1)$ we have that

$$\begin{aligned} X[f_i^{(\ell-1)}(x_1, |x|, u_1, \langle x, u\rangle) P_k^i(u)] &= X[f_i^{(\ell-1)}(x_1, |x|, u_1, \langle x, u\rangle)] P_k^i(u) \\ &\quad + f_i^{(\ell-1)}(x_1, |x|, u_1, \langle x, u\rangle) X[P_k^i(u)] \\ &\quad + (m-2) x_1 f_i^{(\ell-1)}(x_1, |x|, u_1, \langle x, u\rangle) P_k^i(u). \end{aligned}$$

We know from the previous lemma that $X[P_k^i(u)]$ looks as follows:

$$X[P_k^j(u)] = 2(m-2+2i)\left(\langle x, u\rangle - u_1 x_1\right) P_k^{i+1}(u) - (m-2) x_1 P_k^i(u)$$

and thus this is of the correct form. All that we have to check is that $X[f_i^{(l-1)}]$ only depends on the given parameters to complete the proof which follows from the chain rule and straightforward calculations. □

From this proof we can extract a recursive relation for the functions $f_i^{(l)}$:

Proposition 3.1.4. *For $0 \leq i \leq \min(k, l)$ we have that:*

$$\begin{aligned} f_i^{(\ell)}\big(x_1, |x|, u_1, \langle x, u\rangle\big) &= X\left[f_i^{(\ell-1)}\big(x_1, |x|, u_1, \langle x, u\rangle\big)\right] \\ &\quad + 2(m-4+2i)\, \langle \tilde{x}, \tilde{u}\rangle\, f_{i-1}^{(\ell-1)}\big(x_1, |x|, u_1, \langle x, u\rangle\big). \end{aligned}$$

As an example, we obtain the explicit formula for $f_0^{(\ell)}$ in the following lemma:

Lemma 3.1.5. *For each $\ell \in \mathbb{Z}^+$ we have that:*

$$f_0^{(\ell)}(x_1, |x|, u_1, \langle x, u\rangle) = (-1)^\ell \ell! |x|^\ell C_\ell^{\frac{m}{2}-1}\left(\frac{x_1}{|x|}\right).$$

Proof. As the recursive relation reduces to $f_0^{(\ell)} = X[f_0^{(\ell-1)}]$ for all $\ell \in \mathbb{Z}^+$, and the fact that

$$X^\ell[1] = \left(|x|^2\partial_{x_1} - x_1(2\mathbb{E}_x + m - 2)\right)^\ell [1] \ ,$$

the result follows from the harmonic case ($k = 0$). $\square$

Note that this shows that $X^\ell[P_k^0(u)]$ can in fact be seen as 'a projection' of a harmonic polynomial in both x and $u \in \mathbb{R}^m$ (i.e. up to a remaining polynomial to make it a solution for $\mathcal{D}_k$):

$$X^\ell[P_k^0(u)] = (-1)^\ell \ell! |x|^\ell |u|^k C_\ell^{\frac{m}{2}-1}\left(\frac{x_1}{|x|}\right) C_k^{\frac{m}{2}-1}\left(\frac{u_1}{|u|}\right) + \mathrm{Rest}_{\ell,k}(x,u) \ .$$

We can illustrate the previous results with a scheme, where the arrows show which coefficients contribute to a specific term:

$$\begin{tikzcd}
P_k^0 \arrow[r] \arrow[dr] & P_k^0 \arrow[r] \arrow[dr] & P_k^0 \arrow[r] \arrow[dr] & P_k^0 \arrow[r] \arrow[dr] & P_k^0 \arrow[r] \arrow[dr] & P_k^0 \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & {} \\
& P_k^1 \arrow[r] \arrow[dr] & P_k^1 \arrow[r] \arrow[dr] & P_k^1 \arrow[r] \arrow[dr] & P_k^1 \arrow[r] \arrow[dr] & P_k^1 \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & {} \\
& & P_k^2 \arrow[r] \arrow[dr, dotted, no head] & P_k^2 \arrow[r] \arrow[dr, dotted, no head] & P_k^2 \arrow[r] \arrow[dr, dotted, no head] & P_k^2 \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & {} \\
& & & P_k^{k-1} \arrow[r] \arrow[dr] & P_k^{k-1} \arrow[r] \arrow[dr] & P_k^{k-1} \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & {} \\
& & & & P_k^k \arrow[r] & P_k^k \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

In the harmonic case, the repeated action of the raising operator leads to a polynomial depending on the squared norm of the vector variable (an invariant) and the inner product with a fixed unit vector (reducing the invariance by one dimension). This also leads to differential equations that corresponded to the defining differential equation of the Gegenbauer polynomials. In the present setting, the previous results suggest a result depending on the variables

$$(r, s, t) := \left(\frac{x_1}{|x|}, \frac{u_1}{|u|}, \left\langle \frac{x}{|x|}, \frac{u}{|u|} \right\rangle\right)$$

which is shown in the next theorem.

Theorem 3.1.6. *For all $\ell, k \in \mathbb{Z}^+$, there exists a scalar function $f_{\ell,k}(A, B, C)$ depending on three variables such that:*

$$X^\ell[P_k^0(u)] = |x|^\ell |u|^k f_{\ell,k}\left(r, s, t\right).$$

Proof. Take arbitrary k and ℓ. Using theorem 3.1.3 we know that:

$$\begin{aligned}
X^{\ell}[P_k^0(u)] &= \sum_{i=0}^{\min(k,\ell)} f_i^{(\ell)}(x_1, |x|, u_1, \langle x, u\rangle) P_k^i(u) \\
&= \sum_{i=0}^{\min(k,\ell)} |u|^{k-i} f_i^{(\ell)}(x_1, |x|, u_1, \langle x, u\rangle) C_{k-i}^{\frac{m}{2}-1+i}(s)\,.
\end{aligned}$$

Using proposition 3.1.4 we can conclude that

$$\begin{aligned}
\mathbb{E}_x f_i^{(\ell)} &= \ell f_i^{(\ell)} \\
\mathbb{E}_u f_i^{(\ell)} &= i f_i^{(\ell)}
\end{aligned}$$

which means that $f_i^{(\ell)}(x_1, |x|, u_1, \langle x, u\rangle) = |x|^{\ell}|u|^i g_i^{(\ell)}(r, s, t)$ for some function $g_i^{(\ell)}$. This finishes the proof. $\square$

One can rewrite the action of the raising operator X, using the variables (r, s, t). We want to find an operator Q_ℓ such that:

$$X[|x|^{\ell}|u|^k f_{\ell,k}(r, s, t)] = |x|^{\ell+1}|u|^k Q_\ell f_{\ell,k}(r, s, t).$$

Using lemma 3.1.1 and the chain rule, straightforward calculations give us that:

$$Q_\ell := (1 - r^2)\partial_r + 2(t - sr)\partial_s - (s - rt)\partial_t - (\ell + m - 2).$$

We can also use our $\mathfrak{sl}(2)$-realisation to find an inverse. Recall that we have that

$$\begin{tikzcd}
0 & P_k^0(u) \arrow[l, "\partial_{x_1}"'] \arrow[r, shift left, "\mathcal{J}_R\partial_{x_1}\mathcal{J}_R"] \arrow[loop below, "2\mathbb{E}_x+m-2"'] & (\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)[P_k^0(u)] \arrow[l, shift left, "\partial_{x_1}"] \arrow[r, shift left] \arrow[loop below, "2\mathbb{E}_x+m-2"'] & (\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^2[P_k^0(u)] \arrow[l, shift left, "\partial_{x_1}"] \arrow[r, shift left, "\mathcal{J}_R\partial_{x_1}\mathcal{J}_R"] \arrow[loop below, "2\mathbb{E}_x+m-2"'] & \bullet \cdots \arrow[l, shift left, "\partial_{x_1}"]
\end{tikzcd}$$

It is also well known that for $\mathfrak{sl}(2) \cong \mathrm{Alg}(X, Y, H)$ the following commutation relation holds (with $a \in \mathbb{Z}^+$):

$$[Y, X^{a+1}] = -(a + 1)X^a(H + a).$$

If we apply this to our situation we find that:

$$\begin{aligned}
\partial_{x_1}(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell+1}[P_k^0(u)] &= [\partial_{x_1}, (\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell+1}][P_k^0(u)] \\
&= -(\ell + 1)(m - 2 + \ell)(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell}[P_k^0(u)].
\end{aligned}$$

Moreover it is easy to see that $\partial_{x_1}|x|^{\ell+1}|u|^k f_{\ell+1,k}(r, s, t)$ can be written as:

$$|x|^{\ell}|u|^k \left((\ell + 1)r + (1 - r^2)\partial_r + (s - rt)\partial_t\right) f_{\ell+1,k}(r, s, t).$$

Defining the operator L_ℓ by means of

$$L_\ell = -\frac{1}{(\ell + 1)(m - 2 + \ell)} \left((\ell + 1)r + (1 - r^2)\partial_r + (s - rt)\partial_t\right),$$

we can now say that

$$\partial_{x_1}|x|^{\ell+1}|u|^k f_{\ell+1,k}(r,s,t) = -(\ell+1)(m-2+\ell)|x|^\ell|u|^k L_\ell(\partial_r,0,\partial_t)f_{\ell+1}(r,s,t).$$

This operator serves as the inverse, as can be observed from direct calculations. Indeed, we have that both

$$\begin{aligned}
&|x|^\ell|u|^k L_\ell(\partial_r,0,\partial_t)Q_\ell(\partial_r,\partial_s,\partial_t)f_{\ell,k}(r,s,t)\\
=&-\frac{1}{(\ell+1)(m-2+\ell)}\partial_{x_1}|x|^{\ell+1}|u|^k Q_\ell(\partial_r,\partial_s,\partial_t)f_{\ell,k}(r,s,t)\\
=&-\frac{1}{(\ell+1)(m-2+\ell)}\partial_{x_1}(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell+1}[P_k^0(u)]\\
=&(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^\ell[P_k^0(u)]\\
=&|x|^\ell|u|^k f_{\ell,k}(r,s,t)
\end{aligned}$$

and

$$\begin{aligned}
&|x|^{\ell+1}|u|^k Q_\ell(\partial_r,\partial_s,\partial_t)L_\ell(\partial_r,0,\partial_t)f_{\ell+1,k}(r,s,t)\\
=&(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)|x|^\ell|u|^k L_\ell(\partial_r,0,\partial_t)f_{\ell+1,k}(r,s,t)\\
=&-\frac{1}{(\ell+1)(m-2+\ell)}(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)\partial_{x_1}|x|^{\ell+1}|u|^k f_{\ell+1,k}(r,s,t)\\
=&-\frac{1}{(\ell+1)(m-2+\ell)}(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)\partial_{x_1}(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell+1}[P_k^0(u)]\\
=&(\mathcal{J}_R\partial_{x_1}\mathcal{J}_R)^{\ell+1}[P_k^0(u)]\\
=&|x|^{\ell+1}|u|^k f_{\ell+1,k}(r,s,t)\ .
\end{aligned}$$

The motivation for introducing this inverse is encoded in the following scheme, where we have also defined the polynomial:

$$S_k = -\frac{1}{k+1}\big((1-s^2)\partial_s-(k+m-2)s\big)$$

.

$$\begin{tikzcd}
f_{0,0} \arrow[r, shift left, "Q_0"] \arrow[d, "S_0"] & f_{1,0} \arrow[l, shift left, "L_0"] \arrow[r, shift left, "Q_1"] & f_{2,0} \arrow[l, shift left, "L_1"] \arrow[r, dotted, no head] & f_{\ell,0} \arrow[r, shift left, "Q_\ell"] & f_{\ell+1,0} \arrow[l, shift left, "L_\ell"] \arrow[r, dotted, no head] & {} \\
f_{0,1} \arrow[r, shift left, "Q_0"] \arrow[d, "S_1"] & f_{1,1} \arrow[l, shift left, "L_0"] \arrow[r, shift left, "Q_1"] & f_{2,1} \arrow[l, shift left, "L_1"] \arrow[r, dotted, no head] & f_{\ell,1} \arrow[r, shift left, "Q_\ell"] & f_{\ell+1,1} \arrow[l, shift left, "L_\ell"] \arrow[r, dotted, no head] & {} \\
f_{0,2} \arrow[r, shift left, "Q_0"] \arrow[d, dotted, no head] & f_{1,2} \arrow[l, shift left, "L_0"] \arrow[r, shift left, "Q_1"] \arrow[d, dotted, no head] & f_{2,2} \arrow[l, shift left, "L_1"] \arrow[r, dotted, no head] \arrow[d, dotted, no head] & f_{\ell,2} \arrow[r, shift left, "Q_\ell"] \arrow[d, dotted, no head] & f_{\ell+1,2} \arrow[l, shift left, "L_\ell"] \arrow[r, dotted, no head] \arrow[d, dotted, no head] & {} \\
f_{0,k} \arrow[r, shift left, "Q_0"] \arrow[d, "S_k"] & f_{1,k} \arrow[l, shift left, "L_0"] \arrow[r, shift left, "Q_1"] & f_{2,k} \arrow[l, shift left, "L_1"] \arrow[r, dotted, no head] & f_{\ell,k} \arrow[r, shift left, "Q_\ell"] & f_{\ell+1,k} \arrow[l, shift left, "L_\ell"] \arrow[r, dotted, no head] & {} \\
f_{0,k+1} \arrow[r, shift left, "Q_0"] & f_{1,k+1} \arrow[l, shift left, "L_0"] \arrow[r, shift left, "Q_1"] & f_{2,k+1} \arrow[l, shift left, "L_1"] \arrow[r, dotted, no head] & f_{\ell,k+1} \arrow[r, shift left, "Q_\ell"] & f_{\ell+1,k+1} \arrow[l, shift left, "L_\ell"] \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

Given any $f_{\ell,k}$, we can thus complete the scheme.

Proposition 3.1.7. *The functions $f_{\ell,k}(r,s,t)$, defined by*

$$X^\ell\left[|u|^k C_k^{\frac{m}{2}-1}\left(\frac{u_1}{|u|}\right)\right] = |x|^\ell |u|^k f_{\ell,k}(r,s,t)$$

can be written as:

$$f_{\ell,k}(r,s,t) = \sum_{a=0}^{\ell}\sum_{b=0}^{k}\sum_{c=0}^{\min\{\ell-a,k-b\}} \alpha_{\ell,k}(a,b,c) r^a s^b t^c$$

where the $\alpha_{\ell,k}(a,b,c)$ satisfy the following recursive relation:

$$\begin{aligned}\alpha_{\ell,k}(a,b,c) = (a+1)\alpha_{\ell-1,k}(a+1,b,c) &- 2(b+1)\alpha_{\ell-1,k}(a,b+1,c-1)\\ &+ (c-a-2b-\ell-m+4)\alpha_{\ell-1,k}(a-1,b,c)\\ &- (c+1)\alpha_{\ell-1,k}(a,b-1,c+1).\end{aligned}$$

We have adopted the convention that $\alpha_{\ell,k}(a,b,c)=0$ if any of the indices are out of bounds. Moreover, because $|x|^\ell|u|^k f_{\ell,k}(r,s,t) \in \mathcal{P}_{\ell,k}$, if

$$\ell - a - c \not\equiv 0 \ mod\ 2 \ \text{or}\ k - b - c \not\equiv 0 \ mod\ 2$$

then $\alpha_{\ell,k}(a,b,c) = 0$.

Proof. This follows from the fact that $f_{\ell,k}(r,s,t) = Q_{\ell-1} f_{\ell-1,k}(r,s,t)$ and direct calculations. □

Despite the existence of this recursive relation, it proves difficult to find a general expression for the coefficients. We expected the coefficients to be a rational function involving polynomials in the dimension m of degree one. However, at some point in the calculation an irreducible (over $\mathbb{Q}$) second degree polynomial in m appears (even when restricting to low k values) which makes it impossible to recognise a product of Gamma functions (something which could lead to hypergeometric coefficients). For instance, let $k=2$ and look at the coefficient of the term r^2s^2 for the first values for ℓ:

ℓ-values	$\alpha_{\ell,2}(2,2,0)$
$\ell=2$	$\frac{1}{2}(m-2)m(m+2)(m+4)$
$\ell=4$	$-3(m-2)m(m+2)(m+4)(m+10)$
$\ell=6$	$\frac{45}{2}(m-2)m(m+2)(m+4)(m^2+22m+104)$
$\ell=8$	$-210(m-2)m(m+2)(m+4)(m+6)(m+10)(m+20)$
$\ell=10$	$\frac{4725}{2}(m-2)m(m+2)(m+4)(m+6)(m+8)(m^2+38m+328)$

The reason for this unexpected term could be the following: in the classical case we found a unique invariant when looking at the repeated action of the raising operator. In our current setting this is no longer the case: the space of the $\mathfrak{so}(m-1)$-invariant polynomial solutions to $\mathcal{D}_k$ is $(k+1)$-dimensional (provided that $\ell \geq k$), which means that we are dealing with a certain linear combination. Fortunately we can find a suitable basis for this space.

3.2 Representation theoretical arguments

In the previous section, we have found special solutions for $\mathcal{D}_k$ which can be written as $|x|^\ell |u|^k f_{\ell,k}(r,s,t)$. As they are polynomials on $\mathbb{R}^{2m}$, this implies that

$$f_{\ell,k}(r,s,t) = \sum_{a=0}^{\ell} \sum_{b=0}^{k} \sum_{c=0}^{\min\{\ell-a,k-b\}} \alpha_{\ell,k}(a,b,c) r^a s^b t^c.$$

Since multiplying with $|x|^\ell |u|^k$ has to give a polynomial, we can conclude that

$$\begin{array}{ll} \ell \text{ even (resp. odd)} & \Longrightarrow \ \alpha_{\ell,k}(a,b,c) = 0 \text{ if } a+c \text{ is odd (resp. even)} \\ k \text{ even (resp. odd)} & \Longrightarrow \ \alpha_{\ell,k}(a,b,c) = 0 \text{ if } b+c \text{ is odd (resp. even)} \end{array}$$

These polynomials belong to the $(k+1)$-dimensional space of $\mathfrak{so}(m-1)$-invariant polynomials in $\ker_\ell \mathcal{D}_k$ and the following theorem provides us with a suitable basis:

Theorem 3.2.1. *Let $m > 4$, and let $P_{\ell,k}(x,u) \in \ker_\ell \mathcal{D}_k$ be an $\mathfrak{so}(m-1)$-invariant solution for $\mathcal{D}_k$ with $\ell \geq k$. In that case there exist constants $c_i \in \mathbb{C}$ $(i = 0, \ldots, k)$ such that:*

$$P_{\ell,k}(x,u) = \sum_{i=0}^{k} c_i (\mathcal{J}_R \Delta_x \mathcal{J}_R)^i \langle u, \partial_x \rangle^k |x|^{\ell+k-2i} C^{\frac{m}{2}-1}_{\ell+k-2i}(r)$$

Proof. Recall that $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ is an irreducible $\mathfrak{so}(m)$-representation with highest weight (ℓ,k). This means that:

$$\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})\Big|^{\mathfrak{so}(m)}_{\mathfrak{so}(m-1)} \cong \bigoplus_{i=k}^{\ell} \bigoplus_{j=0}^{k} (i,j)$$

and thus there is no scalar component to be found unless $k = 0$. From the classical harmonic result we know that the $\mathfrak{so}(m-1)$-invariant subspace in $\mathcal{H}_{\ell+k-2i}(\mathbb{R}^m,\mathbb{C})$ is generated by the harmonic Gegenbauer polynomial of degree $\ell+k-2i$. Using the embedding factors for the simplicial harmonics in $\ker_\ell \mathcal{D}_k$ (see section 2.4.2 and [16]), we arrive at the proof. □

From this theorem we can also conclude that each $\mathfrak{so}(m-1)$-invariant polynomial in $\ker_\ell \mathcal{D}_k$ has to be of the form $|x|^\ell |u|^k g(r,s,t)$ since it can be shown that both the operators $\langle u, \partial_x \rangle$ and $\mathcal{J}_R \Delta_x \mathcal{J}_R$ preserve this form.

3.3 Explicit example

We will find an explicit formula for one of the $\mathfrak{so}(m-1)-$invariants in $\ker_\ell \mathcal{D}_k$ namely:

$$\langle u, \partial_x \rangle^k |x|^{\ell+k} C_{l+k}^{\frac{m}{2}-1}\left(\frac{x_1}{|x|}\right).$$

This is a rather special solution, as it is *not* induced by the solutions for $\ker \mathcal{D}_{k-1}$. By this we mean the following: a special class of solutions for $\mathcal{D}_k$ contains polynomials in (x, u) which belong to the kernel of both Δ_x and $\langle \partial_x, \partial_u \rangle$ (see the definition of $\mathcal{D}_k$ in section 2.4.2). In physics, these solutions are important as they satisfy certain gauge conditions (they are harmonic and satisfy the condition $\langle \partial_x, \partial_u \rangle f(x,u) = 0$). As was shown in [16], this operator $\langle \partial_x, \partial_u \rangle$ also maps solutions for $\mathcal{D}_k$ surjectively to solutions for $\mathcal{D}_{k-1}$, although the inversion is a non-trivial operator. The component we are about to describe does not come from such an inversion procedure, as it is killed by the operator $\langle \partial_x, \partial_u \rangle$. To obtain an explicit expression we calculate the repeated action of $\langle u, \partial_x \rangle$ on

$$\begin{aligned} |x|^n C_n^\mu(r) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j 2^{n-2j} \frac{\Gamma(n-j+\mu)}{\Gamma(\mu) j! (n-2j)!} x_1^{n-2j} |x|^{2j} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n(j,\mu) x_1^{n-2j} |x|^{2j}. \end{aligned}$$

We need to following lemmas, all of which are easily proven by straightforward calculations and induction.

Lemma 3.3.1. *Let $k \in \mathbb{Z}^+$ and consider k-times differentiable functions f, g on $\mathbb{R}^m$ then*

$$\langle u, \partial_x \rangle^k (fg) = \sum_{i=0}^{k} \binom{k}{i} \left(\langle u, \partial_x \rangle^{k-i} f\right) \left(\langle u, \partial_x \rangle^i g\right).$$

Lemma 3.3.2. *Let $a, b \in \mathbb{Z}^+$ then*

$$\langle u, \partial_x \rangle^a x_1^b = (b)_a u_1^a x_1^{b-a}$$

where $(b)_a = b(b-1)\cdots(b-a+1)$ is the lowering factorial.

Lemma 3.3.3. *Let $a, b \in \mathbb{Z}^+$ then*

$$\langle u, \partial_x \rangle^a |x|^{2b} = \sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} \frac{2^{a-2i} (a)_{2i} (b)_{a-i}}{i!} \langle u, x \rangle^{a-2i} |u|^{2i} |x|^{2b-2a+2i}.$$

This means that $\langle u, \partial_x \rangle^k |x|^n C_n^\mu(r)$ is given by:

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n(j,\mu) \sum_{i=0}^{k} \binom{k}{i} \left(\langle u, \partial_x \rangle^{k-i} x_1^{n-2j}\right) \left(\langle u, \partial_x \rangle^i |x|^{2j}\right)$$

$$=\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n(j,\mu) \sum_{i=0}^{k} \binom{k}{i} \left((n-2j)_{k-i} u_1^{k-i} x_1^{n-2j-k+i}\right) \times$$

$$\sum_{h=0}^{\lfloor \frac{i}{2} \rfloor} \frac{2^{i-2h}(i)_{2h}(j)_{i-h}}{h!} \langle u, x\rangle^{i-2h} |u|^{2h}|x|^{2j-2i+2h}$$

$$=\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{k} \sum_{h=0}^{\lfloor \frac{i}{2} \rfloor} \gamma_{n,k,\mu}(i,j,h) u_1^{k-i} x_1^{n-2j-k+i} \langle u, x\rangle^{i-2h} |u|^{2h}|x|^{2j-2i+2h}$$

$$=|x|^{n-k}|u|^k \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{k} \sum_{h=0}^{\lfloor \frac{i}{2} \rfloor} \gamma_{n,k,\mu}(i,j,h) s^{k-i} r^{n-2j-k+i} t^{i-2h}$$

with

$$\gamma_{n,k,\mu}(i,j,h) := \alpha_n(j,\mu) \binom{k}{i} (n-2j)_{k-i} \frac{2^{i-2h}(i)_{2h}(j)_{i-h}}{h!}.$$

If we want to write this into our chosen standard form then we would have to do the following substitutions:

$$\begin{aligned} a &:= n-2j-k+i \\ b &:= k-i \\ c &:= i-2h. \end{aligned}$$

It is here that the parity conditions on our coefficients will appear. Since $c = k-b-2h$ we know that $c+b \equiv k \mod 2$. Completely analogue one can use the fact that $a = n-2j-k+c+2h$ to conclude that $a+c \equiv n-k \mod 2$. Also we can see that $c \leq k-b$ and $c \leq n-k-a$ to end up with:

$$\sum_{a=0}^{n-k} \sum_{b=0}^{k} \sum_{c=0}^{\min(n-k-a,k-b)} \epsilon_{n,k}(a,b,c) \gamma_{n,k,\mu} \left(k-b, \frac{n-a-b}{2}, \frac{k-b-c}{2}\right) r^a s^b t^c$$

where $\epsilon_{n,k}(a,b,c) = 0$ if the parity conditions are not met, and equal to 1 otherwise. It are also these conditions that guarantee that the arguments of the $\gamma_{n,k,\mu}$ are positive integers. There is however a way to get rid of the factor $\epsilon_{n,k}(a,b,c)$ in the summation if we slightly change our summation indices. If we use the fact that, for each a,b,c, there have to exist i,j such that $a = n-k-c-2i$ and $b = k-c-2j$ we can write our expression as:

$$\sum_{c=0}^{\min(n-k,k)} \sum_{i=0}^{\lfloor \frac{n-k-c}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k-c}{2} \rfloor} \gamma_{n,k,\mu} (c+2j, c+i+j, j)\, r^{n-k-c-2i} s^{k-c-2j} t^c.$$

Combining all of the above leads to the following theorem:

Theorem 3.3.4. *For each $\ell, k \in \mathbb{Z}^+$ the following $\mathfrak{so}(m-1)$-invariant polynomial belongs to $\ker_\ell \mathcal{D}_k$:*

$$|x|^\ell |u|^k \sum_{a=0}^{\ell} \sum_{b=0}^{k} \sum_{c=0}^{\min(\ell-a,k-b)} \gamma_{\ell,k} \left(k-b, \frac{\ell+k-a-b}{2}, \frac{k-b-c}{2} \right) r^a s^b t^c$$

where

$$\gamma_{\ell,k}(i,j,h) = \epsilon_{\ell,k}(a,b,c) 2^{i-2j-2h} \binom{k}{i} \frac{(-1)^j (i)_{2h} (j)_{i-h}}{h! j! (\ell+i-2j)!} \frac{\Gamma(\ell+k-j+\frac{m}{2}-1)}{\Gamma(\frac{m}{2}-1)}$$

and $\epsilon_{\ell,k}(a,b,c)$ is equal to 1 when both $a+c \equiv \ell$ mod 2 and $b+c \equiv k$ mod 2, and equal to zero otherwise.

CHAPTER 4

Simplicial Gegenbauer polynomials

If it wasn't the way it is,
it wouldn't be the way it is.
Bernard Black

4.1 Special simplicial harmonic solutions

In this chapter we generalise the harmonic Gegenbauer solutions to higher spin Clifford analysis by focusing on the fact that they should reproduce certain solution spaces (see section 4.1.7), something we explored in [23]. This generalises the property that the reproducing kernel for $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$, w.r.t. the Fischer inner product, is given by a suitable normalisation of

$$Z_k(x, y) = |x|^k |y|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle x, y\rangle}{|x||y|}\right).$$

The existence of these solutions follows from the branching rules for the Lie algebra $\mathfrak{so}(m)$, where the special solutions realise the trivial representation for $\mathfrak{so}(m-2)$ when restricting the action of $\mathfrak{so}(m)$ to the subalgebra leaving the direction e_1 and e_2 invariant. One of the applications of these solutions is the generalisation of Fueter's theorem to a higher spin setting (see chapter 7).
We start by gathering the necessary (algebraic) tools which lead to the existence and uniqueness of our generalised Gegenbauer solutions in terms of a ladder operator. Afterwards, we find an explicit expression (a polynomial in 2 scalar variables) and examine its relation with the existing results on hypergeometric functions in several variables. Finally, we give an application of our newly derived special functions, where we will construct an associated Appell sequence which yields a representation for the Heisenberg algebra.

4.1.1 Representation theoretical arguments

As we have already mentioned, in this chapter we will single out two preferred directions. We will choose these to be the unit vectors e_1 and e_2 and we will write: $x = \underline{x}_2 + \underline{x}$ with $\underline{x}_2 = x_1e_1 + x_2e_2 \in \mathbb{R}^2$ and $\underline{x} \in \mathbb{R}^{m-2}$. While we have the following identity for the norm of a wedge at our disposal (Lagrange's identity)

$$|x \wedge u|^2 = |x|^2|u|^2 - \langle x, u\rangle^2 \ ,$$

we also need an expression that connects $|x \wedge u|$ to $|\underline{x} \wedge \underline{u}|$ and $|\underline{x}_2 \wedge \underline{u}_2|$:

Lemma 4.1.1. *If we put $x = \underline{x}_2 + \underline{x}$ and $u = \underline{u}_2 + \underline{u}$ as above, then:*

$$\begin{aligned} |x \wedge u|^2 &= |\underline{x} \wedge \underline{u}|^2 + |\underline{x}_2 \wedge \underline{u}_2|^2 + |\underline{x}|^2|\underline{u}_2|^2 + |\underline{x}_2|^2|\underline{u}|^2 - 2\langle \underline{x}, \underline{u}\rangle \langle \underline{x}_2, \underline{u}_2\rangle \\ &= |\underline{x} \wedge \underline{u}|^2 - |\underline{x}_2 \wedge \underline{u}_2|^2 + |x|^2|\underline{u}_2|^2 + |u|^2|\underline{x}_2|^2 - 2\langle x, u\rangle \langle \underline{x}_2, \underline{u}_2\rangle \,. \end{aligned}$$

Using a similar approach to the harmonic case, in which special functions were obtained as harmonic polynomials transforming as a scalar under a (maximal) subalgebra, we will try to find possible invariants in $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})$. To do so, we can use the abstract branching rules for irreducible representations for $\mathfrak{so}(m)$, see [38].

Theorem 4.1.2. *If we identify irreducible representations with their highest weights, we have the following decompositions:*

$$(\ell, k)\Big|^{\mathfrak{so}(m)}_{\mathfrak{so}(m-1)} \cong \bigoplus_{i=k}^{\ell} \bigoplus_{j=0}^{k} (i, j)_{\mathfrak{so}(m-1)}$$

for $\mathfrak{so}(m-1)$, and

$$(\ell, k)\Big|^{\mathfrak{so}(m)}_{\mathfrak{so}(m-2)} \cong \bigoplus_{i=0}^{\ell} \bigoplus_{j=0}^{\min(i,k)} \eta_{i,j}(i, j)_{\mathfrak{so}(m-2)}$$

for $\mathfrak{so}(m-2)$. We hereby used the notation

$$\eta_{i,j} = \begin{cases} (k-j+1)(\ell-i+1) & i \geq k \\ (i-j+1)(\ell-k+1) & i < k \end{cases}$$

to denote the multiplicities of the representation.

Remark 4.1.3. In what follows we will look at $\mathfrak{so}(m-2)$ as the subalgebra of $\mathfrak{so}(m)$ fixing (e_1, e_2).

From theorem 4.1.2 we can now immediately conclude that there does not exist an $\mathfrak{so}(m-1)$-invariant polynomial in $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})$, as the trivial representation has highest weight (0), unless $k = 0$ in which our space reduces to the space of harmonic polynomials in one variable. The corresponding unique $\mathfrak{so}(m-1)$-invariant polynomial is then given by the classic Gegenbauer harmonic, see proposition 2.5.5. If we branch further then we can see that there have to exist $(\ell-k+1)$ linearly independent $\mathfrak{so}(m-2)$-invariant polynomials, and thus if $\ell = k$ there is a unique one. We will therefore mostly focus on to the case where the degree in x and in u is equal, which turns out to be closely related to functions on Grassmannians, for which we refer to section 4.2.2. In section 4.1.5 we will take a look at the more general case.

In order to explicitly construct this unique polynomial (up to a multiplicative constant), we will introduce a raising operator

$$\mathrm{X} : \mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}) \to \mathcal{H}_{k+1,k+1}(\mathbb{R}^{2m}, \mathbb{C})$$

which preserves $\mathfrak{so}(m-2)$-invariance, on the analogy of the raising operator generating the Gegenbauer harmonics. This operator will serve as a ladder operator, creating the polynomials lying at the core of this chapter.

Definition 4.1.4. Define the following inversion on functions $f \in \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{C})$:

$$\mathcal{J}_{x,u} f(x,u) := |x \wedge u|^{3-m} f\left(\frac{x}{|x\wedge u|}, \frac{u}{|x \wedge u|}\right).$$

Note that the action of this inversion on a function can lead to non-isolated singularities on the surface $|x \wedge u|^2 = 0$. In particular, when acting on polynomials $H_{\ell,k}(x,u)$ this inversion can be written as:

$$\mathcal{J}_{x,u} H_{\ell,k}(x,u) = |x \wedge u|^{3-m-\ell-k} H_{\ell,k}(x,u).$$

It is obvious that $\mathcal{J}_{x,u}^2 = \mathrm{Id}$ but more importantly we have the following result. It can be proved for general functions, but we will stick to polynomials here (as this is our case of interest):

Theorem 4.1.5. *The inversion $\mathcal{J}_{x,u}$ preserves* $\ker(\Delta_x, \Delta_u, \langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$.

Proof. First of all we point out that the operators $\langle x, \partial_u\rangle$ and $\langle u, \partial_x\rangle$ satisfy the Leibniz rule. In view of the fact that $|x \wedge u|^a$ itself belongs to the kernel of these operators for all $a \in \mathbb{Z}$, it is clear that the inversion preserves the space $\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$. As for the Laplace operators, we note that as soon as we have treated one of these (e.g. Δ_x), then the conclusion also holds for the other one (i.e. Δ_u), because of symmetry considerations. To see why this is true, note that for functions $f(x,u)$ belonging to $\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$ one also has that $f(x,u)$ belongs to the kernel of $[\langle x, \partial_u\rangle, \langle u, \partial_x\rangle] = \mathbb{E}_x - \mathbb{E}_u$, which means that the degree in x and u must be equal. When acting on polynomials, one can thus conclude that $\ell = k$ and therefore we can use a single index to denote the degree of homogeneity of these polynomials. For each $k \in \mathbb{Z}^+$ we are now looking for a suitable choice of $j \in \mathbb{Z}$ such that we have the following operator identity when acting on k-homogeneous polynomials in $\ker(\Delta_x, \Delta_u, \langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$:

$$\Delta_x |x \wedge u|^j P_k(x,u) = 0. \tag{4.1}$$

In view of the fact that

$$
\begin{aligned}
\partial_{x_i}|x\wedge u|^j &= j|x\wedge u|^{j-2}\left(x_i|u|^2 - u_i\langle x,u\rangle\right)\\
\partial_{u_i}|x\wedge u|^j &= j|x\wedge u|^{j-2}\left(u_i|x|^2 - x_i\langle x,u\rangle\right)\ ,
\end{aligned}
$$

one has the following operator identity on $\ker(\Delta_x,\Delta_u,\langle x,\partial_u\rangle,\langle u,\partial_x\rangle)$:

$$
\Delta_x|x\wedge u|^j = |x\wedge u|^j\Delta_x + 2j|x\wedge u|^{j-2}\left(|u|^2\mathbb{E}_x - \langle x,u\rangle\langle u,\partial_x\rangle\right) + \left(\Delta_x|x\wedge u|^j\right).
$$

Note that in the last term of this expression, the Laplace operator acts on the wedge power only. Since $\Delta_x(|x\wedge u|^j) = j(m+j-3)|x\wedge u|^{j-2}|u|^2$, we get

$$
\Delta_x|x\wedge u|^j = j|x\wedge u|^{j-2}|u|^2(2\mathbb{E}_x+m+j-3) - 2j|x\wedge u|^{j-2}\langle x,u\rangle\langle u,\partial_x\rangle
$$

and the second term vanishes on $\ker(\Delta_x,\Delta_u,\langle x,\partial_u\rangle,\langle u,\partial_x\rangle)$. Equation (4.1) thus reduces to $(2\mathbb{E}_x+m+j-3)P_k(x)=0$, which indeed holds if one defines the inversion as above. This finishes the proof. □

It is important to note that this inversion only preserves the kernel of both Laplace operators if it also acts on the kernel of both $\langle x,\partial_u\rangle$ and $\langle u,\partial_x\rangle$, i.e. one cannot remove any of the operators in theorem 4.1.5. This means that in particular, the inversion only behaves well when acting on simplicial harmonics with equal degrees of homogeneity because this is the only case in which they are solutions for both $\langle x,\partial_u\rangle$ and $\langle u,\partial_x\rangle$.

Lemma 4.1.6. *If for $i\neq j$ we define the differential operators*

$$
D_{ij} := \partial_{x_i}\partial_{u_j} - \partial_{x_j}\partial_{u_i}\ ,
$$

then these operators satisfy $[D_{ij},\langle x,\partial_u\rangle] = 0 = [D_{ij},\langle u,\partial_x\rangle]$. Their duals,

$$
X_{ij} := x_iu_j - x_ju_i
$$

also commute with the skew-Euler operators.

It is clear that our inversion does not affect the transformation behaviour under $\mathfrak{so}(m-2)$, as it is defined in terms of invariants, and the same can be said about D_{12} (recall that $\mathfrak{so}(m-2)$ fixes the directions e_1 and e_2). We can then conclude that the same holds for the operator

$$
\mathrm{X} := \mathcal{J}_{x,u}\left(\partial_{x_1}\partial_{u_2} - \partial_{x_2}\partial_{u_1}\right)\mathcal{J}_{x,u}\ .
$$

From now on we will assume that we are acting on polynomials with equal degree of homogeneity in the variables (x,u). We also adopt the notation $\mathbb{E} := \mathbb{E}_x = \mathbb{E}_u$ such that our formulas reflect this symmetry. Taking into account that X preserves the space $\ker(\Delta_x,\Delta_u,\langle x,\partial_u\rangle,\langle u,\partial_x\rangle)$, we have thus found an operator

$$
\mathrm{X} : \mathcal{H}_{k,k}(\mathbb{R}^{2m},\mathbb{C}) \to \mathcal{H}_{k+1,k+1}(\mathbb{R}^{2m},\mathbb{C}).
$$

The fact that this operator indeed raises the degree in (x,u) is easily seen, that it also maps polynomials to polynomials is not a priori clear from its definition in terms of the inversion. One way to see it, is to look at its explicit form, given by the following theorem:

Theorem 4.1.7. *We have that the raising operator* X *is given by:*

$$\mathrm{X} = (x_1u_2 - x_2u_1)(3-m-2\mathbb{E})(4-m-2\mathbb{E}) + \mathrm{R}(3-m-2\mathbb{E}) + |x\wedge u|^2 D_{12}$$

where we have defined an operator

$$\begin{aligned}\mathrm{R} :=& |x|^2(u_2\partial_{x_1} - u_1\partial_{x_2}) + |u|^2(x_1\partial_{u_2} - x_2\partial_{u_1})\\ &+ \langle x,u\rangle (x_1\partial_{x_2} - x_2\partial_{x_1} + u_2\partial_{u_1} - u_1\partial_{u_2}).\end{aligned}$$

Proof. We may assume that we are acting on polynomials that are homogeneous of degree k in both variables and thus our inversion is a left multiplication with $|x\wedge u|^{3-m-2k}$. To lighten the notation we define $a := 3-m-2k$ and obtain the following operator identity:

$$\begin{aligned}&\partial_{x_1}\partial_{u_2}|x\wedge u|^a\\ =&\partial_{x_1}\left(a|x\wedge u|^{a-2}(|x|^2u_2 - \langle x,u\rangle x_2) + |x\wedge u|^a\right)\\ =&a\Bigg((a-2)|x\wedge u|^{a-4}(|u|^2x_1 - \langle x,u\rangle u_1)(|x|^2u_2 - \langle x,u\rangle x_2)\\ &+ |x\wedge u|^{a-2}(2x_1u_2 - u_1x_2) + |x\wedge u|^{a-2}(|x|^2u_2 - \langle x,u\rangle x_2)\partial_{x_1}\Bigg)\\ &+ a|x\wedge u|^{a-2}(|u|^2x_1 - \langle x,u\rangle u_1)\partial_{u_2} + |x\wedge u|^a\partial_{x_1}\partial_{u_2}.\end{aligned}$$

Using the symmetry of our expression we simply switch the roles of 1 and 2 in the previous expression to find:

$$\begin{aligned}&(\partial_{x_1}\partial_{u_2} - \partial_{x_2}\partial_{u_1})|x\wedge u|^a\\ =&a|x\wedge u|^{a-2}\Bigg((a+1)(x_1u_2 - x_2u_1) + |x|^2(u_2\partial_{x_1} - u_1\partial_{x_2})\\ &+ |u|^2(x_1\partial_{u_2} - x_2\partial_{u_1}) + \langle x,u\rangle (x_1\partial_{x_2} - x_2\partial_{x_1} + u_2\partial_{u_1} - u_1\partial_{u_2})\Bigg)\\ &+ |x\wedge u|^a(\partial_{x_1}\partial_{u_2} - \partial_{x_2}\partial_{u_1})\end{aligned}$$

where we have used the fact that

$$\begin{aligned}&(|u|^2x_1 - \langle x,u\rangle u_1)(|x|^2u_2 - \langle x,u\rangle x_2) - (|u|^2x_2 - \langle x,u\rangle u_2)(|x|^2u_1 - \langle x,u\rangle x_1)\\ &= |x\wedge u|^2(x_1u_2 - x_1u_1).\end{aligned}$$

Recall that our operator X acts on homogeneous polynomials of degree k in both variables and thus each term in the above expression is then of degree $k+a-1$ in each variable. This means that the action of $\mathcal{J}_{x,u}$ is simply a multiplication with

$$|x\wedge u|^{3-m-2(k+a-1)} = |x\wedge u|^{2-a}$$

which leads us to the desired result. ☐

Remark 4.1.8. Another way to arrive at this raising operator X is to start from the multiplication operator $(x_1u_2 - x_2u_1)$, and to apply a suitable projection which ensures that the result still belongs to the appropriate kernel space. By this we mean the following: as the multiplication with $(x_1u_2 - x_2u_1)$ does not preserve being harmonic in x and u, one has to perform a projection in order to arrive at the desired raising operator. In the classical case (one vector variable $x \in \mathbb{R}^m$), one can either use the extremal projection operator, see definition 6.1.3, for the Lie algebra $\mathfrak{sl}(2)$, or invoke the Fischer decomposition.

In order to be able to tackle the action of X on more then just the element $1 \in \mathbb{C}$ (which will turn out to be useful later), we also introduce the following 'deformed' raising operator:

Definition 4.1.9. Let $\mu \in \mathbb{C}$, we then define the following operator:

$$\mathrm{X}_\mu := -2(x_1u_2 - x_2u_1)(\mu + \mathbb{E})(1 - 2\mu - 2\mathbb{E}) - 2\,\mathrm{R}(\mu + \mathbb{E}) + |x \wedge u|^2 D_{12}$$

with the operator R defined as above. If $\mu = \frac{m-3}{2}$ then this operator reduces to the raising operator we constructed above in theorem 4.1.7.

Just like in the harmonic case, where the repeated action of the raising operator on the constant 1 led to a special function, our goal here is to characterize the repeated action of X_μ on 1 (in particular if $2\mu = m - 3$), which one should see as an element of $\mathcal{H}_{0,0}(\mathbb{R}^{2m}, \mathbb{C})$.

Example 4.1.10. To find out what we can expect $\mathrm{X}_\mu^k[1]$ to look like, we look at low values of k:

$$\begin{aligned}
\mathrm{X}_\mu[1] &= 2\mu(2\mu - 1)(x_1u_2 - x_2u_1)\\
\mathrm{X}_\mu^2[1] &= 2\mu(2\mu - 1)\,\mathrm{X}_\mu[x_1u_2 - x_2u_1]\\
&= 4\mu(\mu + 1)(2\mu - 1)(2\mu + 1)(x_1u_2 - x_2u_1)^2 + 4\mu(2\mu - 1)|x \wedge u|^2\\
&- 4\mu(\mu + 1)(2\mu - 1)\,\mathrm{R}[x_1u_2 - x_2u_1]\ ,
\end{aligned}$$

where lemma 4.1.1 can be used to rewrite

$$\mathrm{R}[x_1u_2 - x_2u_1] = |x \wedge u|^2 - |\underline{x} \wedge \underline{u}|^2 + (x_1u_2 - x_2u_1)^2.$$

All together we find that:

$$X_\mu^2[1] = 2\mu(2\mu - 1)\left(4\mu(\mu + 1)(x_1u_2 - x_2u_1)^2 - 2\mu|x \wedge u|^2 + 2(\mu + 1)|\underline{x} \wedge \underline{u}|^2\right).$$

These examples inspire us to adopt the following notation:

Definition 4.1.11. For $x = \underline{x}_2 + \underline{x} \in \mathbb{R}^m$, we define

$$\begin{aligned}
\tau &:= \frac{x_1u_2 - x_2u_1}{|x \wedge u|}\\
\sigma &:= \frac{|\underline{x} \wedge \underline{u}|}{|x \wedge u|}.
\end{aligned}$$

We can now rewrite the examples in terms of these scalar variables as

$$
\begin{aligned}
\mathrm{X}_\mu[1] &= 2\mu(2\mu-1)|x\wedge u|\tau \\
\mathrm{X}^2_\mu[1] &= 2\mu(2\mu-1)|x\wedge u|^2\bigg(4\mu(\mu+1)\tau^2+2(\mu+1)\sigma^2-2\mu\bigg).
\end{aligned}
$$

4.1.2 Explicit expressions

The main goal of this section is to arrive at an explicit expression for $\mathrm{X}^k_\mu[1]$, inspired by the examples from the previous section. To do so we will conjecture its final form, which will be a polynomial in two scalar variables (σ,τ), and determine the coefficients of this polynomial in terms of a recursive argument.

Theorem 4.1.12. *For each $k\in\mathbb{Z}^+$ we have that:*

$$
\mathrm{X}^k_\mu[1] = |x\wedge u|^k\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=0}^{i} c_{k,\mu}(i,j)\tau^{k-2i}\sigma^{2i-2j}
$$

with $c_{k,\mu}(i,j)\in\mathbb{R}$. These coefficients satisfy the following recursive relation:

$$
\begin{aligned}
c_{k+1,\mu}(i,j) &= (-2\mu-2k+2j)(-2\mu+1-k-2i+2j)c_{k,\mu}(i,j) \\
&\quad + (k-2i+2)(-2\mu+1-k-2i+2j)c_{k,\mu}(i-1,j-1) \\
&\quad - (k-2i+2)(-2\mu-2k+2j)c_{k,\mu}(i-1,j)
\end{aligned}
$$

where we define $c_{0,\mu}(0,0)=1$ and we adopt the convention that the coefficient $c_{k,\mu}(a,b)=0$ if one of the following conditions is met:

$$
a<b\ ,\ a<0\ ,\ a>\left\lfloor\frac{k}{2}\right\rfloor\ ,\ b<0\ .
$$

To keep this rather technical proof as clear as possible we have combined some results into the following lemmas, which follow from direct calculations:

Lemma 4.1.13. *For $a,b\in\mathbb{R}$ we have that:*

$$
\begin{aligned}
\mathrm{R}[(x_1u_2-x_2u_1)^a|x\wedge u|^b] &= a(x_1u_2-x_2u_1)^{a-1}\big(|x\wedge u|^{b+2}-|x\wedge u|^b|\underline{x}\wedge\underline{u}|^2\big) \\
&\quad + (a+2b)(x_1u_2-x_2u_1)^{a+1}|x\wedge u|^b.
\end{aligned}
$$

Lemma 4.1.14. *For $a,b\in\mathbb{R}$ we also have that:*

$$
\begin{aligned}
&(\partial_{x_1}\partial_{u_2}-\partial_{x_2}\partial_{u_1})[(x_1u_2-x_2u_1)^a|x\wedge u|^b] \\
&= (x_1u_2-x_2u_1)^{a-1}|x\wedge u|^{b-2} \\
&\qquad\times\bigg(a(a+b+1)|x\wedge u|^2-ab|\underline{x}\wedge\underline{u}|^2+b(a+b+1)(x_1u_2-x_2u_1)^2\bigg)
\end{aligned}
$$

Let us then return to the proof of our theorem:

Proof of theorem 4.1.12. We will proceed by induction. For $k \in \{0,1\}$, the result is trivial, so assume it holds for $k \in \mathbb{Z}^+$. Then

$$\begin{aligned} \mathrm{X}_\mu^{k+1}[1] &= \mathrm{X}_\mu[\mathrm{X}_\mu^k[1]] \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} c_{k,\mu}(i,j)\, \mathrm{X}_\mu[|x \wedge u|^k \tau^{k-2i} \sigma^{2i-2j}] \end{aligned}$$

and using the two lemmas above one can show that

$$\begin{aligned} &\mathrm{X}_\mu[(x_1u_2 - x_2u_1)^{k-2i}|x \wedge u|^{2j}|\underline{x} \wedge \underline{u}|^{2i-2j}] \\ =&|\underline{x} \wedge \underline{u}|^{2i-2j}(x_1u_2 - x_2u_1)^{k+1-2i}|x \wedge u|^{2j} \times \\ &\Big(-2(\mu+k)(1-2\mu-2k) - 2(k+\mu)(k-2i+4j) + 2j(k-2i+2j+1)\Big) \\ &+ (x_1u_2 - x_2u_1)^{k-2i-1}|x \wedge u|^{2j+2}(k+2i)(-2\mu+1-k-2i+2j) \\ &- (x_1u_2 - x_2u_1)^{k-2i-1}|x \wedge u|^{2j}|\underline{x} \wedge \underline{u}|^2(k-2i)(-2\mu-2k+2j). \end{aligned}$$

First of all, the numerical constant between brackets simplifies as:

$$\begin{aligned} &-2(\mu+k)(1-2\mu-2k) - 2(k+\mu)(k-2i+4j) + 2j(k-2i+2j+1) \\ =&(-2\mu-2k+2j)(1-2\mu-k-2i+2j). \end{aligned}$$

Next, rewriting the variables in terms of $|x \wedge u|$ and (σ, τ), we arrive at the following result for $\mathrm{X}_\mu^{k+1}[1]$:

$$\begin{aligned} &|x \wedge u|^{k+1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} c_{k,\mu}(i,j) \Big[(2\mu+2k-2j)(2\mu+k+2i-2j-1)\tau^{k+1-2i}\sigma^{2i-2j} \\ &+ (k-2i)(1-2\mu-k-2i+2j)\tau^{k+1-2(i+1)}\sigma^{2(i+1)-2(j+1)} \\ &- (k-2i)(-2\mu-2k+2j)\tau^{k+1-2(i+1)}\sigma^{2(i+1)-j} \Big] \\ =\ & |x \wedge u|^{k+1} \Bigg[\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor + 1} \sum_{j=0}^{i-1} (2\mu+2k-2j)(k-2i+2) c_{k,\mu}(i-1,j)\tau^{k+1-2i}\sigma^{2i-2j} \\ &+ \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor + 1} \sum_{j=1}^{i} (k-2i+2)(1-2\mu-k-2i+2j) c_{k,\mu}(i-1,j-1)\tau^{k+1-2i}\sigma^{2i-2j} \\ &+ \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} (2\mu+2k-2j)(2\mu+k+2i-2j-1) c_{k,\mu}(i,j)\tau^{k+1-2i}\sigma^{2i-2j} \Bigg]. \end{aligned}$$

This proves our recursive relation. All that remains now is to verify that this summation is indeed of the correct form, as one of the indices appears to be out of bounds: the index i ranges from 0 to $\left\lfloor \frac{k}{2} \right\rfloor + 1$. We want the upper bound of this index to be at most $\left\lfloor \frac{k+1}{2} \right\rfloor$ and thus we have to prove that the terms where $i = \left\lfloor \frac{k}{2} \right\rfloor + 1$ are equal to zero. We will have to make a distinction based on the parity of k:

- k odd: trivial since in this case $\left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor + 1$.
- k even: this means that $\left\lfloor \frac{k+1}{2} \right\rfloor = \frac{k}{2}$ so the term with $i = \frac{k}{2} + 1$ is out of bounds. However this is fixed because this term has a factor $k - 2i + 2$ which is of course 0 when $i = \frac{k}{2} + 1$.

This shows that our form is preserved and finishes our proof. □

An explicit form for our coefficients is given by the following theorem:

Theorem 4.1.15. *Let $\mu > 0$ and $k \in \mathbb{Z}^+$. The following formula for our coefficients solves the recursive relation found in theorem 4.1.12:*

$$c_{k,\mu}(i,j) := \frac{(-1)^j 2^{k-2i} k!(2\mu - 1)^{(k)}}{(i-j)!j!(k-2i)!} \frac{\Gamma(\mu + k - j)}{\Gamma(\mu + i - j)},$$

where $(z)^{(k)}$ denotes the rising factorial, i.e.

$$(z)^{(k)} = \begin{cases} 1 & k = 0 \\ z(z+1)\cdots(z+k-1) & \textit{otherwise.} \end{cases}$$

Proof. We will proceed by induction: if $k = 0$ then the result is trivial. Assume now that our formula is correct for $k \in \mathbb{Z}^+$. Let us start by proving the theorem for $0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$ and take an arbitrary $0 \leq j \leq i$. For these i and j the following expressions are well-defined:

$$\begin{aligned} c_{k,\mu}(i-1, j-1) &= -\frac{2j(2\mu + 2k - 2j)}{(k - 2i + 1)(k - 2i + 2)} c_{k,\mu}(i,j) \\ c_{k,\mu}(i-1, j) &= \frac{2(i-j)(2\mu - 2 + 2i - 2j)}{(k - 2i + 1)(k - 2i + 2)} c_{k,\mu}(i,j). \end{aligned}$$

This means that our recursive relation gives us the following formula:

$$\begin{aligned} c_{k+1,\mu}(i,j) = \Bigg(& (-2\mu - 2k + 2j)(1 - 2\mu - k - 2i + 2j) \\ & - \frac{4j(1 - 2\mu - k - 2i + 2j)(\mu + k - j)}{k - 2i + 1} \\ & \quad - \frac{4(i-j)(\mu + i - j - 1)(-2\mu - 2k + 2j)}{k - 2i + 1} \Bigg) c_{k,\mu}(i,j) \end{aligned}$$

and, using straightforward calculations, we can simplify this to:

$$c_{k+1,\mu}(i,j) = \frac{(k+1)(2\mu - 1 + k)(2\mu + 2k - 2j)}{k - 2i + 1} c_{k,\mu}(i,j).$$

From induction we obtain for $c_{k+1,\mu}(i,j)$ the desired expression. Now we will have to make a distinction between the parities of k. First assume that k is even then $\left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor$ and in that case we find the desired result. However, if k is odd, we have to point out that this approach no longer works due to the factor $k-2i+1$ in the denominator. So let $i = \frac{k+1}{2}$ then we first have to look at the case where $j = i$. Our recursion reduces to:

$$c_{k+1,\mu}(i,j) = (1-2\mu-k)c_{k,\mu}(i-1,j-1)$$

from which the desired result follows. If $j < i$ then our recursion tells us that:

$$c_{k+1,\mu}(i,j) = (-2\mu-2k+2j)(c_{k,\mu}(i-1,j-1) - c_{k,\mu}(i-1,j)).$$

From our induction hypothesis we can conclude that

$$\begin{aligned} &c_{k,\mu}(i-1,j-1) - c_{k,\mu}(i-1,j) \\ =& -2\frac{(-1)^j k!(2\mu-1)^{(k)}}{(i-j)!j!}\frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)}\left(j+(i-j)\frac{2\mu-2+2i-2j}{2\mu+2k-2j}\right). \end{aligned}$$

Using the fact that $2i = k+1$ we find that

$$j+(i-j)\frac{2\mu-2+2i-2j}{2\mu+2k-2j} = \frac{(k+1)(2\mu-1+k)}{2(2\mu+2k-2j)}.$$

All together we are left with:

$$c_{k+1,\mu}(i,j) = \frac{(-1)^j(k+1)!(2\mu-1)^{(k+1)}}{(i-j)!j!}\frac{\Gamma(\mu+k+1-j)}{\Gamma(\mu+i-j)},$$

which is precisely what we wanted and this finishes our proof. $\square$

4.1.3 Connection with hypergeometric functions

In this section, we introduce a special function in two scalar variables which can be seen as a generalisation of the classical Gegenbauer polynomials in one scalar variable. As the latter can be rewritten in terms of a hypergeometric function, using a suitable change of variables, one can also expect the former to satisfy a similar property. This will be shown as well.

Definition 4.1.16. For $\mu > 0$ and $k \in \mathbb{Z}^+$, we define the J-polynomial $J_k^\mu(x,y)$ as:

$$J_k^\mu(x,y) := \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\sum_{j=0}^{i} \frac{(-1)^j}{(i-j)!j!(k-2i)!}\frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)}(2x)^{k-2i}y^{i-j}.$$

First of all we point out that these polynomials can be seen as a generalisation of the standard Gegenbauer polynomials, see theorem 2.5.2, since one has that $J_k^\mu(x,0) = C_k^\mu(x)$. In order to show that these polynomials are, up to a constant and a suitable substitution, of hypergeometric type, we need the following identities:

Lemma 4.1.17. *Let $\mu > 0$. For all $j, k \in \mathbb{Z}^+$ with $j \leq k$ we have that:*

$$\sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \frac{(-1)^i}{i!} \frac{2^{k-2i}\Gamma(k+\mu-i)}{(k-2i-j)!\Gamma(\mu)} = \frac{\Gamma(k+2\mu+j)\Gamma(\mu+\frac{1}{2})}{2^j(k-j)!\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+j)}.$$

Proof. From the definition of the Gegenbauer polynomial of degree $k \in \mathbb{Z}^+$ it follows that

$$C_k^\mu(-2x+1) = \frac{\Gamma(2\mu+k)}{\Gamma(2\mu)k!} {}_2F_1(-k, k+2\mu, \mu+\frac{1}{2}; x). \tag{4.2}$$

We now have an equality between two polynomials of degree k and this means that the coefficients of both polynomials have to be equal. We start by rewriting the left-hand side of the expression:

$$\begin{aligned} C_k^\mu(-2x+1) &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i 2^{k-2i}\Gamma(k+\mu-i)}{i!(k-2i)!\Gamma(\mu)} (-2x+1)^{k-2i} \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i 2^{k-2i}\Gamma(k+\mu-i)}{i!(k-2i)!\Gamma(\mu)} \sum_{j=0}^{k-2i} \frac{(k-2i)!}{j!(k-2i-j)!}(-2x)^j \\ &= \sum_{j=0}^{k} \frac{(-1)^j 2^j}{j!} \left(\sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \frac{(-1)^i 2^{k-2i}\Gamma(k+\mu-i)}{i!(k-2i-j)!\Gamma(\mu)} \right) x^j. \end{aligned}$$

The right-hand side of expression (4.2) is given by

$$\frac{\Gamma(2\mu+k)}{\Gamma(2\mu)k!} {}_2F_1(-k, k+2\mu, \mu+\frac{1}{2}; x) = \sum_{j=0}^{k} \frac{(-1)^j \Gamma(k+2\mu+j)\Gamma(\mu+\frac{1}{2})}{j!(k-j)!\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+j)} x^j,$$

which means that for all $0 < j < k$ we have that:

$$\sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \frac{(-1)^i 2^{k-2i}\Gamma(k+\mu-i)}{i!(k-2i-j)!\Gamma(\mu)} = \frac{\Gamma(k+2\mu+j)\Gamma(\mu+\frac{1}{2})}{2^j(k-j)!\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+j)} .$$

This is exactly what we wanted to prove. □

Using this lemma we can easily prove the following theorem, which is another useful identity involving numerical constants:

Theorem 4.1.18. *Let $\mu > 0$. For all $a, k \in \mathbb{Z}^+$ with $a \leq k$, and for all $b \in \mathbb{Z}^+$ such that $b \leq \lfloor \frac{k-a}{2} \rfloor$ the following holds:*

$$\begin{aligned} &\sum_{i=b}^{\lfloor \frac{k-a}{2} \rfloor} \frac{(-1)^i}{(i-b)!} \frac{2^{k-2i}\Gamma(k+\mu+b-i)}{(k-2i-a)!\Gamma(\mu+b)} \\ &\qquad = \frac{(-1)^b \Gamma(\mu)\Gamma(k+2\mu+a+2b)\Gamma(\mu+\frac{1}{2})}{2^{a+4b}\Gamma(\mu+b)(k-a-2b)!\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+a+2b)}. \end{aligned}$$

Proof. In order to prove this, we start by putting $c = i - b$ in the expression at the left-hand side. Next, putting $j := a + 2b$, it follows from the restrictions on a and b that $0 \leq j \leq k$. This expression thus becomes

$$\frac{(-1)^b 2^{-2b}\Gamma(\mu)}{\Gamma(\mu+b)} \sum_{c=0}^{\lfloor \frac{k-j}{2} \rfloor} \frac{(-1)^c}{c!} \frac{2^{k-2c}\Gamma(k+\mu-c)}{(k-j-2c)!\Gamma(\mu)} ,$$

and using our previous lemma leads to the desired result. $\square$

We have seen that, up to a change of variable, the classical Gegenbauer polynomials can be written as a hypergeometric series with certain parameters, see 4.2. In what follows, we will obtain a similar result for the generalised Gegenbauer polynomials which were introduced in definition 4.1.16. First of all, the fact that $J_k^\mu(x,0) = C_k^\mu(x)$ strongly suggests to look at $J_k^\mu(-2x+1, cy)$, where we will determine $c \in \mathbb{R}$:

$$\begin{aligned} J_k^\mu(-2x+1,y) &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} \frac{(-1)^j 2^{k-2i}}{(i-j)!j!(k-2i)!} \frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)} (-2x+1)^{k-2i} y^{i-j} \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} \sum_{a=0}^{k-2i} \frac{(-1)^{j+a} 2^{a+k-2i}}{(i-j)!j!a!(k-2i-a)!} \frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)} x^a y^{i-j}. \end{aligned}$$

Defining $b := i - j$ turns this into:

$$\begin{aligned} &\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{b=0}^{i} \sum_{a=0}^{k-2i} \frac{(-1)^{i-b+a} 2^{a+k-2i}}{b!(i-b)!a!(k-2i-a)!} \frac{\Gamma(\mu+k+b-i)}{\Gamma(\mu+b)} x^a y^b \\ =&\sum_{a=0}^{k} \sum_{b=0}^{\lfloor \frac{k-a}{2} \rfloor} (-1)^{a-b} 2^a \left(\sum_{i=b}^{\lfloor \frac{k-a}{2} \rfloor} \frac{(-1)^i 2^{k-2i}}{(i-b)!(k-2i-a)!} \frac{\Gamma(\mu+k+b-i)}{\Gamma(\mu+b)} \right) \frac{x^a}{a!} \frac{y^b}{b!}. \end{aligned}$$

Using theorem 4.1.18, this can then be written as:

$$= \sum_{a=0}^{k} \sum_{b=0}^{\lfloor \frac{k-a}{2} \rfloor} \frac{(-1)^a \Gamma(\mu)\Gamma(k+2\mu+a+2b)\Gamma(\mu+\frac{1}{2})}{2^{4b}\Gamma(\mu+b)(k-a-2b)!\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+a+2b)} \frac{x^a}{a!} \frac{y^b}{b!}.$$

Finally, introducing the raising factorial by means of

$$(-k)^{(a+2b)} = (-1)^a \frac{k!}{(k-a-2b)!} ,$$

we are led to the expression

$$\sum_{a=0}^{k} \sum_{b=0}^{\lfloor \frac{k-a}{2} \rfloor} \frac{(-k)^{(a+2b)}\Gamma(k+2\mu+a+2b)}{(\mu)^{(b)} k! \Gamma(2\mu)(\mu+\frac{1}{2})^{(a+2b)}} \frac{x^a}{a!} \frac{y^b}{2^{4b} b!}$$

$$= \frac{\Gamma(k+2\mu)}{\Gamma(2\mu)k!}\sum_{a=0}^{k}\sum_{b=0}^{\lfloor\frac{k-a}{2}\rfloor}\frac{(-k)^{(a+2b)}(k+2\mu)^{(a+2b)}}{(\mu)^{(b)}(\mu+\frac{1}{2})^{(a+2b)}}\frac{x^a}{a!}\frac{y^b}{2^{4b}b!}.$$

From this we can thus conclude that

$$J_k^\mu(-2x+1,16y) = \frac{\Gamma(k+2\mu)}{\Gamma(2\mu)k!}\sum_{a=0}^{k}\sum_{b=0}^{\lfloor\frac{k-a}{2}\rfloor}\frac{(-k)^{(a+2b)}(k+2\mu)^{(a+2b)}}{(\mu)^{(b)}(\mu+\frac{1}{2})^{(a+2b)}}\frac{x^a}{a!}\frac{y^b}{b!}.$$

The advantage of this expression is that it lends itself well to being compared with the definition for the hypergeometric function in two variables, as it appears in the literature (e.g. [32, 49, 57]):

Definition 4.1.19. For $\beta, \gamma \notin \mathbb{Z}^-$, we define the following hypergeometric function in two variables:

$$\Phi(\alpha,\alpha',\beta,\gamma;x,y) = \sum_{a,b=0}^{\infty}\frac{(\alpha)^{(a+2b)}(\alpha')^{(a+2b)}}{(\beta)^{(b)}(\gamma)^{(a+2b)}}\frac{x^a}{a!}\frac{y^b}{b!}\ ,$$

for all x, y for which this series converges.

Remark 4.1.20. While the area of convergence could be an interesting topic, we point out that for our current purposes it is not that important. Due to the fact that our parameter $\alpha \in \mathbb{Z}^-$ we have that the series reduces to a polynomial (just as for the classical Gegenbauer polynomials).

Theorem 4.1.21. *Let $\mu > 0$ and $k \in \mathbb{Z}^+$ then*

$$J_k^\mu(x,y) = \frac{\Gamma(k+2\mu)}{\Gamma(2\mu)k!}\Phi\left(-k, k+2\mu, \mu, \mu+\frac{1}{2}; \frac{1-x}{2}, \frac{y}{16}\right).$$

Proof. Follows from all the above and the observation that for $\alpha = -k$ we can conclude that $a \leq k$ and $b \leq \lfloor\frac{k-a}{2}\rfloor$, which means that the bounds of summation are indeed correct. $\square$

In the classical case one has that the equation $\Delta_x H_k(x) = 0$, together with the requirement that $H_k(x)$ depends on the norm $|x|$ and the inner product with a fixed unit vector only, leads to the second order differential equation for the Gegenbauer polynomials, see section 2.5.2. To end this section we therefore also give the so-called Horn system of differential equations for which this hypergeometric series in two variables defines a solution. In order to define this system, one first introduces the following rational polynomials (see [49])

$$A_{a,b}(\alpha,\alpha',\beta,\gamma) := \frac{(\alpha)^{(a+2b)}(\alpha')^{(a+2b)}}{(\beta)^{(b)}(\gamma)^{(a+2b)}}$$

where we will omit the parameters α, α', β and γ at the left-hand side to lighten the notation. It is then easily seen that

$$\frac{A_{a+1,b}}{A_{a,b}} = \frac{(\alpha+a+2b)(\alpha'+a+2b)}{\gamma+a+2b}$$

$$\frac{A_{a,b+1}}{A_{a,b}} = \frac{(\alpha+a+2b+1)(\alpha+a+2b)(\alpha'+a+2b+1)(\alpha'+a+2b)}{(\beta+b)(\gamma+a+2b+1)(\gamma+a+2b)}.$$

These fractions are rational polynomials in a, b and the highest degree occurring is 4 which means that Φ is a hypergeometric series of order 4. Moreover, these fractions lead to the following definitions:

$$\begin{aligned}
P_1(a,b) &:= (\alpha+a+2b)(\alpha'+a+2b)\\
Q_1(a,b) &:= a(\gamma+a+2b-1)\\
P_2(a,b) &:= (\alpha+a+2b+1)(\alpha+a+2b)(\alpha'+a+2b+1)(\alpha'+a+2b)\\
Q_2(a,b) &:= b(\beta+b-1)(\gamma+a+2b-1)(\gamma+a+2b-2).
\end{aligned}$$

This means that if we write:

$$\Phi(\alpha,\alpha',\beta,\gamma;x,y) = \sum_{a,b=0}^{\infty} B_{a,b}x^a y^b$$

that

$$\frac{B_{a+1,b}}{B_{a,b}} = \frac{P_1(a,b)}{Q_1(a+1,b)} \quad \text{and} \quad \frac{B_{a,b+1}}{B_{a,b}} = \frac{P_2(a,b)}{Q_2(a,b+1)}.$$

Using this we can come to the following conclusion:

Proposition 4.1.22. *The hypergeometric series Φ that is defined above satisfies the following system of partial differential equations:*

$$\begin{cases} Q_1(x\partial_x, y\partial_y)\Phi - xP_1(x\partial_x, y\partial_y)\Phi = 0 \\ Q_2(x\partial_x, y\partial_y)\Phi - yP_2(x\partial_x, y\partial_y)\Phi = 0 \end{cases}$$

This means that the J-polynomials are determined by two differential equations of order two and four respectively. This might seem odd at first as they are the unique $\mathfrak{so}(m-2)$-invariant polynomials in the space of simplicial harmonics, which is defined as the kernel of three operators, with order no greater than two. We will address this in section 4.2.2 when we will relate our special functions from this chapter to functions on the Grassmannian $\mathrm{Gr}(2,m)$.

4.1.4 Associated Appell sequences

Now that we have introduced our special functions and sketched their connection with the theory of hypergeometric functions in several variables, we return to the characterisation of our Gegenbauer-type solutions for the simplicial harmonic system.

Theorem 4.1.23. *Let $\mu > 0$ and $k \in \mathbb{Z}^+$. One then has that*

$$\mathrm{X}_\mu^k[1] = k!(2\mu-1)^{(k)}|x \wedge u|^k J_k^\mu(\tau, \sigma^2).$$

This means that the right hand side becomes zero if $\mu = \frac{1}{2}$ (i.e. $m = 4$, which we will study in more detail in section 4.1.6). One can solve this is by starting from the J-polynomial of degree one, i.e.

$$\mathrm{X}_\mu^k[x_1u_2 - x_2u_1] = k!(2\mu)^{(k)}|x \wedge u|^{k+1} J_{k+1}^\mu(\tau, \sigma^2).$$

Similarly, one can also study the repeated action of our raising operator on a polynomial instead of the constant 1. This can be used when looking at the branching problem for spaces of simplicial harmonics, as it realises several of the embeddings.

Theorem 4.1.24. *Let $k, j \in \mathbb{Z}^+$ and $\mu \in \mathbb{R}$ with $\mu + j > \frac{1}{2}$. If $P_j(\underline{x}, \underline{u})$ is an arbitrary polynomial in the space $\mathcal{P}_{j,j}(\mathbb{R}^{2(m-2)}, \mathbb{C})$, then the following holds:*

$$\mathrm{X}_\mu^k[P_j(\underline{x}, \underline{u})] = P_j(\underline{x}, \underline{u}) \frac{k!\Gamma(2\mu + 2j - 1 + k)}{\Gamma(2\mu + 2j - 1)} |x \wedge u|^k J_k^{\mu+j}(\tau, \sigma^2).$$

Proof. This follows immediately from the fact that:

$$\mathrm{X}_\mu[P_j(\underline{x}, \underline{u})] = P_j(\underline{x}, \underline{u})\, \mathrm{X}_{\mu+j}[1],$$

which can be seen from the definition of these raising operators. □

In particular we can conclude that for $\mu = \frac{m-3}{2}$ (and thus $\mathrm{X}_\mu = \mathrm{X}$) there exists a mapping:

$$\mathrm{X}^{k-j} : \mathcal{H}_{j,j}(\mathbb{R}^{2(m-2)}, \mathbb{C}) \to \mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})$$

which is essentially a multiplication with a generalised Gegenbauer polynomial with a suitable parameter μ. This generalises a branching property for the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ in terms of Gegenbauer harmonics, see e.g. [64] and theorem 2.5.6.

We end this section by constructing Appell sequences using these generalised Gegenbauer polynomials. These sequences are defined as an indexed family of functions $\{\psi_k : k \in \mathbb{Z}^+\}$, together with a pair of operators (M, P), for which one has that $P\psi_k(x) = k\psi_{k-1}(x)$ and $M\psi_k(x) = \psi_{k+1}(x)$. As a result, one has that $[P, M] = 1$ which turns the Appell sequence into a representation for the Heisenberg algebra. For this purpose we need the following lemma:

Lemma 4.1.25. *Let $k \in \mathbb{Z}^+$ then*

$$D_{12}|x \wedge u|^k J_k^\mu(\tau, \sigma^2) = (k+1)(2\mu + k - 1)|x \wedge u|^{k-1} J_{k-1}^\mu(\tau, \sigma^2).$$

Proof. Let us write:

$$\begin{aligned} &D_{12}|x \wedge u|^k J_k^\mu(\tau, \sigma^2) \\ &\qquad = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{i} \alpha_k(i,j) |\underline{x} \wedge \underline{u}|^{2i-2j} D_{12}\left[|x \wedge u|^{2j}(x_1u_2 - x_2u_1)^{k-2i}\right] \end{aligned}$$

with

$$\alpha_k(i,j) := \frac{(-1)^j 2^{k-2i}}{(i-j)!j!(k-2i)!}\frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)}.$$

The lemma then follows from

$$\begin{aligned}
D_{12}&\left[|x\wedge u|^{2j}(x_1u_2-x_2u_1)^{k-2i}\right]\\
&= (k-2i)(k-2i+2j+1)|x\wedge u|^{2j}(x_1u_2-x_2u_1)^{k-2i-1}\\
&\quad + 2j(k-2i+2j+1)|x\wedge u|^{2j-2}(x_1u_2-x_2u_1)^{k-2i+1}\\
&\quad\quad - 2j(k-2i)|x\wedge u|^{2j-2}(x_1u_2-x_2u_1)^{k-2i-1}|\underline{x}\wedge\underline{u}|^2
\end{aligned}$$

and straightforward calculations. □

For all $\mu > \frac{1}{2}$ we then introduce the following family of polynomials:

$$\left\{\psi_k^\mu(x,u) := \frac{1}{(k+1)(2\mu)^{(k)}}|x\wedge u|^k J_k^\mu(\tau,\sigma^2) \mid k\in\mathbb{Z}^+\right\}.$$

From lemma 4.1.25 we can then conclude that:

$$\begin{aligned}
D_{12}\psi_k^\mu(x,u) &= \frac{1}{(2\mu)^{(k-1)}}|x\wedge u|^{k-1}J_{k-1}^\mu(\tau,\sigma^2)\\
&= k\psi_{k-1}^\mu(x,u).
\end{aligned}$$

For the construction of the raising operator, which can also be seen as 'an integration operator', we point out that

$$\mathrm{X}_\mu[|x\wedge u|^k J_k^\mu(\tau,\sigma^2)] = (k+1)(2\mu-1+k)|x\wedge u|^{k+1}J_{k+1}^\mu(\tau,\sigma^2),$$

which means that

$$\mathrm{X}_\mu[\psi_k^\mu(x,u)] = (2\mu+k)(2\mu+k-1)(k+2)\psi_{k+1}^\mu(x,u).$$

Let us now define

$$M_\mu := \frac{\mathrm{X}_\mu}{(\mathbb{E}+1)(2\mu-2+\mathbb{E})(2\mu-1+\mathbb{E})}$$

where the right hand side A/B has to be interpreted as $B^{-1}A$ (note that their relative position matters, as X_μ does not commute with the Euler operators). By construction we then have that

$$M_\mu\psi_k^\mu(x,u) = \psi_{k+1}^\mu(x,u).$$

Combining these results yields the following theorem:

Theorem 4.1.26. *For $\mu > \frac{1}{2}$, the family of polynomials*

$$\left\{\psi_k^\mu(x,u) := \frac{1}{(k+1)(2\mu)^{(k)}}|x\wedge u|^k J_k^\mu(\tau,\sigma^2) \mid k\in\mathbb{Z}^+\right\}$$

defines an Appell sequence. The associated raising and lowering operators M_μ and $P_\mu := D_{12}$ are hereby defined as above.

4.1.5 Invariants in the general case

Thus far we have focused on finding the (necessarily unique) $\mathfrak{so}(m-2)$-invariant in $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$, when the degrees in x and u coincide ($\ell = k$). In this section we will consider the more general case $\ell \geq k$ and construct the invariants in $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$. From the abstract branching rules we can conclude that there is no longer a unique scalar component but there are exactly $(\ell - k + 1)$. We are thus looking for an $(\ell - k + 1)$-dimensional subspace of $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ in which all the elements transform trivially under $\mathfrak{so}(m-2)$. In the theorem below, we show that the invariants determined in the previous sections can actually be used as a starting point, in the sense that they generate the invariants for $\ell > k$.

Theorem 4.1.27. *Let $\ell \geq k$, $m > 4$ and put $\mu = \frac{m-3}{2}$. If $H_{\ell,k}(x,u) \in \mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ is an $\mathfrak{so}(m-2)$-invariant polynomial, then there exists a unique polynomial $P_{\ell-k}(\underline{u}_2) = P_{\ell-k}(u_1,u_2) \in \mathcal{P}_{\ell-k}(\mathbb{R}^2,\mathbb{C})$ in two variables such that:*

$$H_{\ell,k}(x,u) = P_{\ell-k}(\partial_{u_1},\partial_{u_2})|x \wedge u|^\ell J_\ell^\mu(\tau,\sigma^2).$$

Proof. From the fact that the operator $P_{\ell-k}(\partial_{u_1},\partial_{u_2})$ commutes with Δ_x, Δ_u and $\langle x, \partial_u\rangle$, we can conclude that

$$P_{\ell-k}(\partial_{u_1},\partial_{u_2})|x \wedge u|^\ell J_\ell^\mu(\tau,\sigma^2) \in \mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$$

for all $P_{\ell-k}(u_1,u_2) \in \mathcal{P}_{\ell-k}(\mathbb{R}^2,\mathbb{C})$, where moreover the $\mathfrak{so}(m-2)$-invariance is preserved. Since $\dim_{\mathbb{C}} \mathcal{P}_{\ell-k}(\mathbb{R}^2,\mathbb{C}) = (\ell-k+1)$, it now suffices to show that for each $0 \leq a \leq \ell - k$ the polynomials

$$H_{\ell,k}^{(a)}(x,u) = \partial_{u_1}^{\ell-k-a}\partial_{u_2}^a |x \wedge u|^\ell J_\ell^\mu(\tau,\sigma^2)$$

are linearly independent. Let us therefore consider a linear combination

$$\sum_{a=0}^{\ell-k} \gamma_a \partial_{u_1}^{\ell-k-a}\partial_{u_2}^a |x \wedge u|^\ell J_\ell^\mu(\tau,\sigma^2) = 0\ ,$$

from which it should follow that $\gamma_a = 0$ for all a. First of all, we note that the derivation with respect to u_1 and u_2 does not affect the variable $\underline{u} \in \mathbb{R}^{m-2}$. In order for this polynomial to be identically zero, it must disappear for all values $(x,u) \in \mathbb{R}^{2m}$. In particular, we may thus choose $\underline{x} = \underline{u} = \underline{0}$, which means that $|x \wedge u|^\ell J_\ell^\mu(\tau,\sigma^2)$ reduces to $C_\ell^\mu(1)(x_1u_2 - x_2u_1)^\ell$. We are thus left with showing that

$$(-1)^{\ell-k}\frac{\ell!}{k!}(x_1u_2 - x_2u_1)^k \sum_{a=0}^{\ell-k}(-1)^a \gamma_a x_2^{\ell-k-a}x_1^a = 0 \implies \gamma_a = 0\ ,$$

but this follows easily from the linear independence of the monomials $x_1^\alpha x_2^\beta$. $\square$

Remark 4.1.28. Another way to formulate theorem 4.1.27 is to say that:

$$\mathcal{H}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})^{\mathrm{SO}(m-2)} \cong \mathcal{P}_{\ell-k}(\mathbb{R}^2,\mathbb{C})$$

and the latter space contains a unique special element if $\ell - k \in 2\mathbb{Z}^+$. Indeed, if $\ell - k \in 2\mathbb{Z}^+$ then there exists exactly one SO(2)-invariant polynomial in $\mathcal{P}_{\ell-k}(\mathbb{R}^2, \mathbb{C})$, namely

$$\left(|x|^2\right)^{\frac{\ell-k}{2}} \in \mathcal{P}_{\ell-k}(\mathbb{R}^2, \mathbb{C}).$$

From this we can conclude that one can find a unique $\mathrm{SO}(2) \times \mathrm{SO}(m-2)$-invariant in $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})$ if and only if $\ell - k$ is even, something that will be useful in proposition 6.1.23.

4.1.6 The case $m = 4$

Throughout this chapter (and most of this thesis), we assume that $m > 4$ because otherwise, the space of simplicial harmonics is no longer irreducible. If $m = 4$ then one can show, see e.g. [15], that

$$\begin{aligned} \mathcal{H}_{\ell,k}(\mathbb{R}^8, \mathbb{C}) &\cong \mathcal{H}^+_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \oplus \mathcal{H}^-_{\ell,k}(\mathbb{R}^8, \mathbb{C}). \\ &\cong (\ell, k) \oplus (\ell, -k), \end{aligned}$$

where $\mathbb{R}^8$ should be seen as $\mathbb{R}^{4\times 2}$ (i.e. 2 variables in $\mathbb{R}^4$). If we define the orthogonal projection operators

$$P_\pm := \frac{1}{2}(1 \mp e_{1234}),$$

i.e. these operators satisfy $P_\pm^2 = P_\pm$ and $P_+P_- = 0 = P_-P_+$, then the components in this decomposition are given by solutions to $P_\mp(\partial_x \wedge \partial_u)$:

$$\mathcal{H}^\pm_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \cap \ker P_\mp(\partial_x \wedge \partial_u) \cong (\ell, \pm k).$$

If we let $P_\mp(\partial_x \wedge \partial_u)$ act on $\mathcal{H}^\pm_{\ell,k}(\mathbb{R}^8, \mathbb{C})$, then this will yield a scalar system of equations from which the following characterisation follows:

$$\mathcal{H}^\pm_{\ell,k}(\mathbb{R}^8, \mathbb{C}) = \mathcal{H}_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \cap \ker(D_{12} \pm D_{34}, D_{13} \mp D_{24}, D_{14} \pm D_{23}). \tag{4.3}$$

Using the branching rules we find that:

$$\begin{aligned} (\ell, \pm k)\Big|^{\mathrm{SO}(4)}_{\mathrm{SO}(2)} \cong \bigoplus_{i=k}^{\ell} (i)\Big|^{\mathrm{SO}(3)}_{\mathrm{SO}(2)} &\cong \bigoplus_{i=k}^{\ell} \bigoplus_{j=-i}^{i} (j)_{\mathrm{SO}(2)} \\ &= \bigoplus_{i=-\ell}^{\ell} (\ell - \max(i,k) + 1)(i)_{\mathrm{SO}(2)}. \end{aligned}$$

This means that if $\ell = k$, both components $(k, \pm k)$ contain a unique SO(2)-invariant polynomial and we know that a suitable linear combination of these two invariants must be equal to the J-polynomial of degree k, i.e.

$$|x \wedge u|^k J_k^{\frac{1}{2}}(\tau, \sigma^2) := J_k(x,u) = J_k^+(x,u) + J_k^-(x,u),$$

with $J_k^\pm(x,u) \in \mathcal{H}^\pm_{k,k}(\mathbb{R}^8,\mathbb{C})$. As these polynomials belong to the kernel of both skew-Euler operators, they have to depend on wedge-variables from which we can conclude that $J_k^\pm(x,u) = P_k^\pm(X_{12}, X_{34}, |x \wedge u|^2)$. Note that, unlike for $m > 4$, odd powers of the scalar variable σ can occur here because $|\underline{x} \wedge \underline{u}|^2 = X_{34}^2$, something that is reflected in theorem 4.1.33. Let us start with two examples to see what the polynomials $P_k^\pm$ will look like.

Example 4.1.29. Let $k = 1$, then

$$P_1^\pm(X_{12}, X_{34}, |x \wedge u|^2) = X_{12} + a_\pm X_{34} \in \mathcal{H}_{1,1}(\mathbb{R}^8,\mathbb{C}),$$

where $a_\pm \in \mathbb{C}$ has to be determined such that $P_1^\pm$ satisfies the extra equations that define the components of $\mathcal{H}^\pm_{1,1}(\mathbb{R}^8,\mathbb{C})$. Due to the fact that $P_1^\pm \in \ker(D_{13}, D_{24}, D_{14}, D_{23})$, two of the equations are trivial and thus we only have to look at the action of $\ker(D_{12} \pm D_{34})$:

$$(\ker(D_{12} \pm D_{34})P_1^\pm = 1 \pm a_\pm,$$

which means that $a_\pm = \mp 1$. It thus follows that

$$X_{12} \mp X_{34} \in \mathcal{H}^\pm_{1,1}(\mathbb{R}^8,\mathbb{C}).$$

This also follows from symmetry considerations: the choice (1,2/3,4) is arbitrary, and should thus lead to something which is 'invariant' (up to possible minus signs) under swapping these pairs of indices.

Example 4.1.30. Now, let $k = 2$ then

$$P_2^\pm(X_{12}, X_{34}, |x \wedge u|^2) = X_{12}^2 + a_\pm X_{12}X_{34} + b_\pm X_{34}^2 + c_\pm |x \wedge u|^2.$$

Letting $D_{13} \mp D_{24}$ act on this polynomial yields

$$0 = (D_{13} \mp D_{24})P_2^\pm = (a_\pm \mp 6c_\pm)X_{24} + (6c_\pm \mp a_\pm)X_{13}$$

and thus $a_\pm = \pm 6c_\pm$. The action of the operator $(D_{14} \pm D_{23})$ results in the same equation and after applying $D_{12} \pm D_{34}$ we find

$$0 = (D_{12} \pm D_{34})P_2^\pm = (6 + 6c_\pm \pm 2a_\pm)X_{12} + (2a_\pm \pm 6b_\pm \pm 6c_\pm)X_{34}.$$

From $6 + 6c_\pm \pm 2a_\pm = 0$ and $a_\pm = \pm 6c_\pm$ we find that $c_\pm = -\frac{1}{3}$, $a_\pm = \mp 2$ and from this it follows that $b_\pm = 1$. This means that:

$$P_2^\pm = X_{12}^2 \mp 2X_{12}X_{34} + X_{34}^2 - \frac{1}{3}|x \wedge u|^2 = (X_{12} \mp X_{34})^2 - \frac{1}{3}|x \wedge u|^2,$$

and it is an easy verification that $P_2^\pm \in \ker \Delta_x$.

This seems to suggest that the invariant polynomials in $\mathcal{H}^\pm_{k,k}(\mathbb{R}^8,\mathbb{C})$ only differ in argument, i.e. they are essentially given by the same polynomial albeit with a different variable, which is confirmed by the following theorem:

Theorem 4.1.31. *Let $k \in \mathbb{Z}^+$ and define*

$$\tau^{\pm} := \frac{X_{12} \mp X_{34}}{|x \wedge u|}.$$

Then we have that

$$|x \wedge u|^k C_k^{\frac{1}{2}}(\tau^{\pm}) \in \mathcal{H}^{\pm}_{k,k}(\mathbb{R}^8, \mathbb{C}).$$

Proof. Let $P_k(x) := \sum_{i=0}^{k} \alpha_i x^{k-i}$, then we look for all coefficients α_i such that

$$H_k^{\pm}(x,u) := |x \wedge u|^k P_k(\tau^{\pm}) \in \mathcal{H}^{\pm}_{k,k}(\mathbb{R}^8, \mathbb{C}).$$

First of all, because the result has to be a polynomial in x and u, and since $H_k^{\pm}(x,u)$ consists of terms $(X_{12} \mp X_{34})^{k-i}|x \wedge u|^i$, the coefficient $\alpha_i = 0$ whenever i is odd. This means that $H_k^{\pm}(x,u)$ has to be of the form

$$H_k^{\pm}(x,u) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \alpha_i (X_{12} \mp X_{34})^{k-2i} |x \wedge u|^{2i}.$$

Applying the Laplace operator Δ_x to each of these individual terms yields

$$\begin{aligned}(k-2i)(k-2i-1)|u|^2(X_{12} \mp X_{34})^{k-2i-2}|x \wedge u|^{2i} \\ + 2i(2k+1-2i)|u|^2(X_{12} \mp X_{34})^{k-2i}|x \wedge u|^{2i-2}.\end{aligned}$$

We then have that

$$\Delta_x H_k^{\pm}(x,u) = 0 \iff \alpha_i = -\frac{(k-2i+1)(k-2i+2)}{2i(2k+1-2i)}\alpha_{i-1},$$

or equivalently

$$\alpha_i = \frac{k!}{i!(k-2i)!}\frac{(-1)^i}{2^{2i}}\frac{\Gamma(k+\frac{1}{2}-i)}{\Gamma(k+\frac{1}{2})}\alpha_0,$$

where $\alpha_0 \in \mathbb{C}$ can be chosen freely. If we choose

$$\alpha_0 = \frac{2^k}{k!}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

then we find the Gegenbauer polynomials and this proves that $|x \wedge u|^k C_k^{\frac{1}{2}}(\tau^{\pm})$ belongs to $\mathcal{H}_{k,k}(\mathbb{R}^8, \mathbb{C})$. All that remains to be checked is whether or not these polynomials also satisfy the extra equations that define $\mathcal{H}^{\pm}_{k,k}(\mathbb{R}^8, \mathbb{C})$, i.e. the ones coming from $P_{\mp}(\partial_x \wedge \partial_u)$, but this follows from straightforward, but rather tedious, calculations. □

Remark 4.1.32. There is an alternative interpretation for the variables $\tau^{\pm}$ by looking at the decomposition of $\mathcal{H}_{1,1}(\mathbb{R}^8, \mathbb{C})$. First of all, note that

$$X_{ab} := \langle x \wedge u, e_{ab} \rangle = -[(x \wedge u)e_{ab}]_0$$

which means that for $s \in \mathrm{Spin}(4)$:

$$H(s)[X_{ab}] = \langle x \wedge u, \overline{s} e_{ab} s \rangle,$$

where the action on e_{ab} is simply the Ad-action of Spin(4) on the space of bivectors. When descending to the level of Lie algebras we have to take into consideration that $\mathfrak{so}(4)$ is not simple, it decomposes as

$$\begin{aligned}\mathfrak{so}(4) &\cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)\\ &\cong \mathrm{Lie}(e_{12} - e_{34}, e_{13} + e_{24}, e_{14} - e_{23}) \oplus \mathrm{Lie}(e_{12} + e_{34}, e_{13} - e_{24}, e_{14} + e_{23}).\end{aligned}$$

Because each of these Lie subalgebras is closed under the action of $\mathfrak{so}(4)$, we can conclude that both

$$\mathcal{H}^{\pm}_{1,1}(\mathbb{R}^8, \mathbb{C}) = \mathrm{Span}(X_{12} \mp X_{34}, X_{13} \pm X_{24}, X_{14} \mp X_{23})$$

are irreducible $\mathfrak{so}(4)$-submodules of $\mathcal{H}_{1,1}(\mathbb{R}^8, \mathbb{C})$. The equations from 4.3 then follow from the fact that $\mathcal{H}^{+}_{1,1}(\mathbb{R}^8, \mathbb{C})$ and $\mathcal{H}^{-}_{1,1}(\mathbb{R}^8, \mathbb{C})$ are orthogonal with respect to the Fischer innner product.

The connection between these two invariants and the J-polynomials found before can be expressed using an identity between (generalised) Gegenbauer polynomials:

Theorem 4.1.33. *The polynomials $J^{\mu}_k(x, y)$, as defined in definition 4.1.16, satisfy the following identity:*

$$J^{\frac{1}{2}}_k(x, y^2) = \frac{1}{2}\left(C^{\frac{1}{2}}_k(x - y) + C^{\frac{1}{2}}_k(x + y)\right)$$

Proof. Recall from theorem 2.5.2 that:

$$C^{\mu}_k(x) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{\Gamma(k - i + \mu)}{\Gamma(\mu) i! (k - 2i)!} (2x)^{k-2i} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \alpha^{\mu}_k(i) x^{k-2i}$$

and thus

$$\begin{aligned}C^{\mu}_k(x - y) + C^{\mu}_k(x + y) &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{k-2i} (-1)^j \binom{k-2i}{j} \alpha^{\mu}_k(i) x^{k-2i-j} y^j\\ &\quad + \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=0}^{k-2i} \binom{k-2i}{j} \alpha^{\mu}_k(i) x^{k-2i-j} y^j.\end{aligned}$$

Note that for every odd j the terms cancel each other out and thus we have

$$\begin{aligned}C_k^\mu(x-y)+C_k^\mu(x+y) &= 2\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor-i}\binom{k-2i}{2j}\alpha_k^\mu(i)x^{k-2(i+j)}y^{2j}\\ &= 2\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor-i}\binom{k-2i}{2i-2j}\alpha_k^\mu(i)x^{k-2i}y^{2i-2j}.\end{aligned}$$

If we compare this to definition 4.1.16, then we just have to show that

$$\binom{k-2i}{2i-2j}\alpha_k^\mu(i)=\frac{(-1)^j2^{k-2i}}{(i-j)!j!(k-2i)!}\frac{\Gamma(\mu+k-j)}{\Gamma(\mu+i-j)}$$

with $\mu=\frac{1}{2}$, which follows from

$$(2n)!=2^{2n}n!\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

and straightforward calculations. □

Combining theorem 4.1.31 and theorem 4.1.33 yields the following decomposition:

$$|x\wedge u|^kJ_k^{\frac{1}{2}}(\tau,\sigma^2)=\frac{1}{2}\left(|x\wedge u|^kC_k^{\frac{1}{2}}(\tau^+)+|x\wedge u|^kC_k^{\frac{1}{2}}(\tau^-)\right).$$

If $\ell\neq k$ then we can use a similar reasoning to theorem 4.1.27 to show the following:

Theorem 4.1.34. *Let $\ell\geq k$ then*

$$\mathcal{H}^{\pm}_{\ell,k}(\mathbb{R}^8,\mathbb{C})^{\mathrm{SO}(2)}\cong\mathcal{P}_{\ell-k}(\mathbb{R}^2,\mathbb{C}),$$

where the isomorphism is explicitly given by

$$P_{\ell-k}(u_1,u_2)\mapsto P_{\ell-k}(\partial_{u_1},\partial_{u_2})|x\wedge u|^\ell C_\ell^{\frac{1}{2}}(\tau^\pm).$$

4.1.7 Reproducing property

In the introduction to this chapter, we said that the construction of the J-polynomials was based on the fact that we want them to reproduce certain solution spaces and in this section we show that they do exactly that, up to a normalising constant.

First of all, let us get back to the case of the Gegenbauer harmonic polynomials, and their role as reproducing kernels for spaces of homogeneous harmonics. One way to see why these special solutions are said to be 'reproducing', is that there exist integral formulas which essentially come from a Green or Cauchy formula:

$$f(x)=\int_{\mathbb{R}^m}\mathcal{K}(x,y)f(y)dy$$

where $\mathcal{K}(x, y)$ is harmonic, but with a singularity at the origin (or, after translation, at any given point). The idea is then that decomposing the kernel leads to a series representation, such as

$$\frac{1}{|x-y|^{m-2}} = \frac{1}{|x|^2 + |y|^2 - 2\langle x, y\rangle} = \frac{1}{|y|^2} \sum_{k=0}^{\infty} C_k^{\frac{m}{2}-1} \left(\frac{\langle x, y\rangle}{|x||y|} \right) \left(\frac{|x|}{|y|} \right)^k ,$$

and plugging this into the reproducing formula for a spherical harmonic singles out the desired component in the degree of that harmonic. Another way to arrive at such a formula, and one that readily generalises to our case, is the following: one has (for instance) that

$$\langle x, \partial_y \rangle^k H_k(y) = k! H_k(x).$$

The upshot is that the formula above can be formulated as a Fischer inner product:

$$H_k(x) = \frac{1}{k!} \left[\langle x, y\rangle^k, H_k(y) \right]_F ,$$

which is not yet the formula we are after as the left argument is not harmonic (in either variable, it is symmetric anyway). What does the trick is to project the kernel above on the space of harmonics, as this then indeed gives you a reproducing kernel for the space of harmonics. In this case, this amounts to projecting the k-homogeneous polynomial $\langle x, y\rangle^k$ (considered as a polynomial in the variable $y \in \mathbb{R}^m$, with $x \in \mathbb{R}^m$ as some kind of 'parameter') onto the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$, again in y. The result will be harmonic, but — and this is the crucial observation — it will also transform trivially under the subgroup of $\mathrm{SO}(m)$ which leaves the vector $x \in \mathbb{R}^m$ (our 'parameter') invariant. This means that we easily recover the Gegenbauer harmonic as a reproducing kernel.
A similar observation can now be done for the case of 2 vector variables. One still has formulas of the following form:

$$\langle x, \partial_y \rangle^\ell \langle u, \partial_v \rangle^k H_{\ell,k}(y, v) = \ell! k! H_{\ell,k}(x, u) .$$

Note that the polynomial $H_{\ell,k}$ is allowed to be an *arbitrary* polynomial here (apart from its degree of homogeneity, as this fixes the factorials), but we do focus our attention on the case where $\ell = k$ and $H_{k,k}(y, v) \in \mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})$. Rewriting this as an inner product, one finds that

$$H_k(x, u) = \frac{1}{(k!)^2} \left[\langle x, y\rangle^k \langle u, v\rangle^k, H_k(y, v) \right]_F .$$

A full-blown reproducing formula is thus obtained if we succeed in projecting the left argument above onto the space of simplicial harmonics. This will give us an element of $H_{k,k}(\mathbb{R}^{2m}, \mathbb{C})$ with 2 additional 'parameters', and the result will be invariant with respect to the subgroup of $\mathrm{SO}(m)$ which leaves the plane spanned by these vectors invariant. If x and u are linearly independent, then the reproducing kernel will be given, up to a constant, by the J-function of degree

k as this is the only SO$(m-2)$-invariant polynomial in the space of simplicial harmonics of a given degree. In the classical case, one has the following chain of 'identifications' to make:

$$\begin{aligned} |y|^k C_k^{\frac{m}{2}-1}\left(\frac{y_1}{|y|}\right) = |y|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle e_1, y\rangle}{|y|}\right) &\xrightarrow{e_1=\omega} |y|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle \omega, y\rangle}{|y|}\right) \\ &\xrightarrow{\times |x|^k} |x|^k |y|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle x, y\rangle}{|y||x|}\right) \ . \end{aligned}$$

So in a sense, one chooses the coordinate system in such a way that e_1 becomes the unit vector along the direction of the arbitrary vector $x \in \mathbb{R}^m$. The idea is that we want to do the same thing in the case of the J-functions, where (e_1, e_2) will be considered as vectors along the additional vectors (x, u). This time, one could argue that this comes with a problem, since $e_1 \perp e_2$ and the same is not necessarily true for x and u in $\mathbb{R}^m$. This is not a problem as one could have used the a priori knowledge that

$$\langle y, \partial_v\rangle H_{k,k}(y,v) = 0 = \langle v, \partial_y\rangle H_{k,k}(y,v) \ .$$

This tells us that we could have started from a projection of the reproducing kernel $\langle x, y\rangle^k \langle u, v\rangle^k$ on the space of polynomials in the kernel of the wedge operators, which is then obviously given by

$$\langle x\wedge u, y \wedge v\rangle^k \in \mathcal{P}_{k,k}(\mathbb{R}^{2m},\mathbb{C}) \cap \ker(\langle x, \partial_u\rangle \,, \langle u, \partial_x\rangle) \ .$$

Moreover, we have that

$$\mathrm{Span}(x,u) = \mathrm{Span}\left(x, -\frac{\langle x, u\rangle}{|x|^2}x + u\right),$$

where the latter basis is orthogonal. Finally, because

$$x \wedge \left(-\frac{\langle x, u\rangle}{|x|^2}x + u\right) = x \wedge u,$$

we can assume that x and u were orthogonal to begin with. A further projection will indeed give a reproducing kernel for $\mathcal{H}_{k,k}(\mathbb{R}^{2m},\mathbb{C})$ and by making a similar chain of 'identifications' as found above, we define

$$\begin{aligned} \tau(x,u) &:= \frac{\langle y, x\rangle\langle v, u\rangle - \langle y, u\rangle\langle v, x\rangle}{|x\wedge u| \ |y \wedge v|} \\ \sigma(x,u) &:= \frac{|\,(y - \langle y, x\rangle x - \langle y, u\rangle u) \wedge (v - \langle v, x\rangle x - \langle v, u\rangle u)\,|}{|x\wedge u| \ |y \wedge v|}, \end{aligned}$$

which means that we can write our reproducing kernel as:

$$|x\wedge u|^k |y\wedge v|^k J_k^{\frac{m-3}{2}}(\tau(x,u), \sigma(x,u)^2).$$

4.2 Zonality of the J-polynomials

Now that we have obtained the generalisations of the Gegenbauer solutions for the ordinary Laplace operator, we may ask ourselves whether these also satisfy similar properties. One of these is the fact that the polynomials from proposition 2.5.5, for each $k \in \mathbb{Z}^+$, define a spherical function on the sphere $S^{m-1} \cong \mathrm{SO}(m)/\mathrm{SO}(m-1)$. We are thus interested in the following question:

Are the generalised Gegenbauer solutions spherical functions for a pair of Lie groups (G, H)?

We will begin by giving the definition of spherical functions, where we follow the approach of [48] and we use the classic harmonic Gegenbauer polynomials as an example, as these are the spherical functions belonging to the pair $(\mathrm{SO}(m), \mathrm{SO}(m-1))$. There are two possible ways to generalise this: either one looks at functions on the Stiefel manifold, or one considers functions on (oriented) Grassmannians. The latter will prove to be of greater use to us as there is a natural one to one connection between the Casimir operators for $\mathfrak{so}(m)$ and $\mathrm{SO}(m)$-invariant operators on the Grassmannian, see e.g. [43]. Moreover, when considering functions on these manifolds the so-called skew wedge-system appears (also called the Cayley-Laplace equation), which we will study in chapter 6.

4.2.1 Spherical functions: general theory

Let G be a connected Lie group and $H \subset G$ be a compact subgroup and denote by $\pi : G \to G/H$ the natural mapping of G unto G/H. Furthermore, let $D(G/H)$ denote the algebra of differential operators on G/H that are invariant under all the translations $\tau(g) : xH \mapsto gxH$ of G/H, i.e. the action of G.

Definition 4.2.1. Consider a manifold M and a diffeomorphism $\psi : M \to M$ of M. Let $\phi : M \to \mathbb{C}$, we then define

$$\phi^\psi := \phi \circ \psi^{-1},$$

i.e. ϕ^ψ is the function that makes the following scheme commute:

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & M \arrow[dl, "\phi^\psi"] \\ & \mathbb{C} & \end{tikzcd}$$

Let D be a differential operator acting on functions $\phi : M \to \mathbb{C}$ then we define:

$$D^\psi : \phi \mapsto (D\phi^{\psi^{-1}})^\psi = (D(\phi \circ \psi)) \circ \psi^{-1}.$$

An operator D is then called invariant under ψ if $D^\psi = D$.

We will use the definition of a spherical function given by Helgason in [48]:

Definition 4.2.2. Let ϕ be a complex-valued function on G/H of class C^∞ which satisfies $\phi(\pi(e)) = 1$ (i.e. ϕ is non-trivial and is normalised). Such a function ϕ is called a spherical function if

1. ϕ is invariant under the action of H (i.e. $\phi^\psi = \phi$, for $\psi = \tau(h)$ with $h \in H$ arbitrary).
2. ϕ is an eigenfunction for all the invariant differential operators on G/H, i.e. $\forall D \in D(G/H), \exists \lambda_D \in \mathbb{C} : D\phi = \lambda_D \phi$.

It is no trivial matter to determine the algebra $D(G/H)$ in full generality. However, in case G/H is reductive, there is a way to construct this algebra by looking at $\mathrm{Ad}(H)$-invariants in the symmetric algebra of a particular subspace of the Lie algebra $\mathfrak{g}$ associated to G.

Definition 4.2.3. Let V be a finite-dimensional vector space over $\mathbb{R}$. The symmetric algebra $S(V)$ over V is defined as the algebra of complex-valued polynomial functions on the dual space V^*. In particular, if $X_1, \ldots, X_n$ is a basis of V, $S(V)$ can be identified with the (commutative) algebra of polynomials

$$\sum_{(k_1,\ldots,k_n)} a_{k_1\ldots k_n} X_1^{k_1} \ldots X_n^{k_n},$$

where the index runs over all n-tuples $(k_1, \ldots, k_n)$.

Suppose G is a connected Lie group and $H \subset G$ a closed subgroup. Let $\mathfrak{g} \supset \mathfrak{h}$ be their respective Lie algebras and $\mathfrak{m}$ a complementary subspace, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Let $\pi : G \to G/H$ be the natural projection and f be a function on G/H then we define $\tilde{f} := f \circ \pi$. The coset space G/H is called reductive if the subspace $\mathfrak{m} \subset \mathfrak{g}$ can be chosen in such a way that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \quad \text{and } \mathrm{Ad}_G(h)\mathfrak{m} \subset \mathfrak{m} \quad (h \in H),$$

i.e. for all $X \in \mathfrak{m}$ and $\forall h \in H$: $hXh^{-1} \in \mathfrak{m}$. If H is compact, then G/H will be reductive.

Theorem 4.2.4. *Let G/H be a reductive homogeneous space and let $I(\mathfrak{m})$ denote the set of $\mathrm{Ad}_G(H)$-invariants in $S(\mathfrak{m})$. If $(X_1, \ldots, X_r)$ is a basis of $\mathfrak{m}$ then the following defines a linear bijection λ of $I(\mathfrak{m})$ onto $D(G/H)$:*

$$(D_{\lambda(Q)} f)(g \cdot o) := \left[Q\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right) \tilde{f}(g \exp(t_1 X_1 + \cdots + t_r X_r)) \right](0)$$

where $Q \in I(\mathfrak{m})$ and $o = \{H\}$ is the origin in G/H.

Corollary 4.2.5. *If $I(\mathfrak{m})$ has a finite system of generators $P_1, \ldots, P_l$ and we put $D_i := D_{\lambda(P_i)}$, then each $D \in D(G/H)$ can be written as*

$$D = \sum_{(n_1,\ldots,n_l)} a_{n_1,\ldots,n_l} D_1^{n_1} \cdots D_l^{n_l}.$$

Example 4.2.6. Let $G = \mathrm{SO}(m)$ and $H = \mathrm{SO}(m-1)$, then $\mathfrak{m}$ is generated by the matrices $X_i := E_{1i} - E_{i1}$, $1 < i \leq m$, where $E_{ij} = (\delta_{ik}\delta_{lj})_{kl}$ and $\mathrm{Ad}_G(H)$ acts transitively on the variables X_i, meaning that

$$S(\mathfrak{m}) \cong \mathcal{P}(\mathbb{R}^{m-1}, \mathbb{C}).$$

The only H-invariants in this space are powers of $X_2^2 + \cdots + X_m^2$, which means that $I(\mathfrak{m})$ is generated by a single polynomial of degree 2 and from this we can conclude that $D(S^{m-1})$ is generated by one differential operator of order 2. We know that the Laplace-Beltrami operator is such an invariant operator and thus, $D(S^{m-1})$ consists of polynomials in the Laplace-Beltrami operator. Eigenfunctions $\varphi : S^{m-1} \to \mathbb{C}$ of the Laplace-Beltrami are precisely the homogeneous harmonic polynomials on $\mathbb{R}^m$, i.e. the spherical functions for the pair $(\mathrm{SO}(m), \mathrm{SO}(m-1))$ are the restrictions of $\mathrm{SO}(m-1)$-invariant harmonic polynomials to S^{m-1}. As we know from section 2.5.2, the Gegenbauer polynomials are the only harmonic polynomials with the right invariance property, proving that the spherical functions for the pair $(\mathrm{SO}(m), \mathrm{SO}(m-1))$ are given by $C_k^{\frac{m}{2}-1}(\omega_1)$, for all $k \in \mathbb{Z}^+$.

4.2.2 Spherical functions on Gr(2,m)

The J-polynomials were constructed as the family of unique $\mathrm{SO}(m-2)$-invariant polynomials in the spaces $\mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})$ of simplicial harmonics with equal degrees of homogeneity and because of this invariance it would be natural to consider the pair $(\mathrm{SO}(m), \mathrm{SO}(m-2))$. In other words we would be considering polynomials on the Stiefel manifold

$$V_{m,2} = {}^{\mathrm{SO}(m)}\!/_{\mathrm{SO}(m-2)}.$$

This manifold can be identified with couples of vectors (x, u) in $\mathbb{R}^{2m}$ such that $|x|^2 = |u|^2 = 1$ and $\langle x, u\rangle = 0$. However, one would ignore part of the invariance of the J-polynomials as they are not just $\mathrm{SO}(m-2)$-invariant, but also $\mathrm{SO}(2)$-invariant. Indeed, the action of $\mathrm{SO}(2)$ does not affect $|\underline{x} \wedge \underline{u}|^2$ and $|x \wedge u|^2$ (the latter is even $\mathrm{SO}(m)$-invariant) and for $A \in \mathrm{SO}(2)$ one has that

$$A \cdot X_{12} = \det A^{-1} X_{12} = X_{12}$$

(recall that the regular action of $A \in \mathrm{SO}(m)$ on polynomials $P(x, u)$ is given by $P(A^{-1}x, A^{-1}u)$, hence the inverse). To encode this additional invariance, we introduce the following (related) manifold:

Definition 4.2.7. The (real) Grassmannian $\mathrm{Gr}(r, m)$ is defined as the set of all r-dimensional subspaces of $\mathbb{R}^m$. This can be expressed as a homogeneous space by

$$\mathrm{Gr}(r, m) \cong {}^{\mathrm{O}(m)}\!/_{\mathrm{O}(r)\times\mathrm{O}(m-r)} \cong {}^{\mathrm{SO}(m)}\!/_{\mathrm{S}(\mathrm{O}(r)\times\mathrm{O}(m-r))}$$

where $\mathrm{S}\left(\mathrm{O}(r) \times \mathrm{O}(m-r)\right)$ is the set of all matrices $H \in \mathrm{SO}(m)$ of the form

$$H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with $A \in \mathrm{O}(r), B \in O(m-r)$. The (real) oriented Grassmannian $\mathrm{Gr}_o(r,m)$ is the set of all oriented r-dimensional subspaces of $\mathbb{R}^m$. This is a double cover of $\mathrm{Gr}(r,m)$ and is realised as a homogeneous space by means of

$$\mathrm{Gr}_o(r,m) \cong {}^{\mathrm{SO}(m)}\big/_{\mathrm{SO}(r)\times\mathrm{SO}(m-r)}.$$

Finding the invariant differential operators on these manifolds is equivalent with constructing the Casimir operators of $\mathfrak{so}(m)$, see [43] for the following result:

Theorem 4.2.8. *Let $l = \min(r, m-r)$ then $D(\mathrm{Gr}(r,m))$, or $D(\mathrm{Gr}_o(r,m))$, is a commutative algebra with l algebraically independent generators. Moreover, if we denote ρ for the left regular action of $\mathrm{O}(m)$ (or $\mathrm{SO}(m)$) on $\mathcal{C}^\infty(\mathrm{Gr}(r,m),\mathbb{C})$, then these generators are precisely the images of the Casimir operators of order $2,4,\ldots,2l$ of $\mathfrak{so}(m)$ under the derived action $d\rho$.*

Example 4.2.9. Let $r = 1$, then the corresponding Grassmannian is the sphere S^{m-1} in $\mathbb{R}^m$. As we have seen in section 2.3, the only $\mathrm{SO}(m)$-invariant differential operator on $\mathbb{R}^m$ is the Laplace operator. When restricting this operator to functions on the sphere, one recovers the Laplace-Beltrami operator Δ^x_{LB}, which is precisely the Casimir operator of order 2.

Let $r = 2$, then we are looking for a realisation of $\mathcal{C}^\infty(\mathrm{Gr}_o(2,m),\mathbb{C})$ within our framework of higher spin Clifford analysis. By this we mean the following: we want to relate functions in this space to functions $f(x,u)$ on $\mathbb{R}^{2m}$ satisfying certain conditions. Let $V \in \mathrm{Gr}_o(2,m)$ and choose an orthonormal basis x, u for V, we then define a map

$$\varphi : \mathrm{Gr}_o(2,m) \to S^{m-1} \wedge S^{m-1} : V \mapsto x \wedge u$$

and the inverse $\varphi^{-1} : w \mapsto \{z \in \mathbb{R}^m : w \wedge z = 0\}$. Note that this map is well-defined because for any other orthonormal basis y, v of V with the same orientation, the base change is given by a matrix $A \in \mathrm{SO}(2)$ and then:

$$y \wedge v = \det A \; x \wedge u = x \wedge u.$$

Moreover, for $M \in \mathrm{SO}(m)$, $V \in \mathrm{Gr}_o(2,m)$ and any basis (x,u) of V:

$$M \cdot V = \mathrm{Span}(Mx, Mu) = \varphi^{-1}(Mx \wedge Mu)$$

and this means that we can see $\mathcal{C}^\infty(\mathrm{Gr}_o(2,m),\mathbb{C})$ as the space of functions that depend on wedges of orthonormal vectors, i.e. restrictions of the minors X_{ij} from lemma 4.1.6 to the Stiefel manifold $V_{m,2}$.
One can also see that functions on the Grassmannian are functions on the Stiefel manifold that satisfy an extra invariance condition. Indeed, for $f(x,u) : V_{m,2} \to \mathbb{C}$, one can see that f is only well-defined on the Grassmannian if and only if it depends on $\mathrm{Span}_o(x,u)$ (the oriented plane spanned by x and u). From this it follows that

$$\mathcal{C}^\infty(\mathrm{Gr}_o(2,m),\mathbb{C}) = \mathcal{C}^\infty(V_{m,2},\mathbb{C})^{\mathrm{SO}(2)},$$

where superscript denotes the invariance with respect to the right action of SO(2), which represents the base change in $\mathrm{Span}_o(x, u)$.
There is a slightly different, but equivalent (see remark 6.1.24), way of describing $\mathcal{C}^\infty(\mathrm{Gr}_o(2, m), \mathbb{C})$ by choosing another way to normalise the basis of an element $V \in \mathrm{Gr}_o(2, m)$ which lies closer to the wedge structure. Let us consider a basis (x, u) of V that satisfies $|x \wedge u|^2 = 1$, which is particularly true if (x, u) is orthonormal, then the map φ is still well-defined. To see this, let (y, v) be a different basis of V with the same orientation such that $|y \wedge v|^2 = 1$ and let A be the matrix that represents the base change. Then, from

$$1 = |y \wedge v|^2 = (\det A)^2 |x \wedge u|^2,$$

and the fact $\det A > 0$ (as it preserves the orientation), it follows that $y \wedge v = x \wedge u$ as desired. This means that we can also see $\mathcal{C}^\infty(\mathrm{Gr}_o(2, m), \mathbb{C})$ as the space of functions that depend on restrictions of the minors X_{ij} to $|x \wedge u|^2 = 1$, i.e. functions depending on

$$\Omega_{ij} := \frac{X_{ij}}{|x \wedge u|}.$$

This is of course similar to functions on the sphere S^{m-1}, that depend on the variables $\omega_i = \frac{x_i}{|x|}$.

Remark 4.2.10. Note that these variables Ω_{ij} also appear naturally when looking at the polar decomposition for matrices $M = (x, u)$ in $\mathbb{R}^{m \times 2}$ (it can be done in greater generality, for matrices $M \in \mathbb{R}^{m \times k}$, but we will stick to the case $k = 2$ seen our case of interest). One has that

$$\mathrm{rank}(M) = 2 \implies M = V R^{\frac{1}{2}}$$

with $M = (x, u) \in \mathbb{R}^{m \times 2}$, $V \in V_{m,2}$ and $R = M^T M \in \mathcal{P}_2$. The first space is the Stiefel manifold $V_{m,2}$, here realised as $\{S \in \mathbb{R}^{m \times 2} : M^T M = \mathrm{Id}_2\}$, and the second one is the open convex cone of positive definite real symmetric matrices in $\mathbb{R}^{2 \times 2}$. In general, one can write

$$\begin{pmatrix} x_1 & u_1 \\ \vdots & \vdots \\ x_m & u_m \end{pmatrix} = \begin{pmatrix} \phi_1 & \varphi_1 \\ \vdots & \vdots \\ \phi_m & \varphi_m \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix},$$

where $V = (\phi, \varphi)$ and $S = R^{\frac{1}{2}}$, but in our particular case of interest it is not necessary to determine these matrices explicitly. The reason for this is that we consider functions defined on the Grassmann manifold $\mathrm{Gr}_o(2, m)$, which amounts to saying that we consider functions depending on $(x, u) \in \mathbb{R}^{m \times 2}$ which can be written in terms of the minor variables $X_{pq} := x_p u_q - x_q u_p$. Using the matrix relations from above, we thus have that

$$X_{pq} = (s_{11}\phi_p + s_{12}\varphi_p)(s_{12}\phi_q + s_{22}\varphi_q) - (s_{11}\phi_q + s_{12}\varphi_q)(s_{12}\phi_p + s_{22}\varphi_p)$$

$$= (s_{11}s_{22} - s_{12}^2)(\phi_p\varphi_q - \phi_q\varphi_p)$$
$$= |x \wedge u|(\phi_p\varphi_q - \phi_q\varphi_p) \ ,$$

where we have used the fact that $\det R^{\frac{1}{2}} = \sqrt{\det R} = |x \wedge u|$. From this it readily follows that the anti-symmetrised combinations of variables $(\phi, \varphi) \in V$ can be identified with

$$\phi_p\varphi_q - \phi_q\varphi_p = \frac{X_{pq}}{|x \wedge u|} = \Omega_{pq} \ ,$$

on the analogy of the angular variables in the spherical case.

We are thus interested in functions that depend on the minors X_{ij} and there exists a suitable characterisation to express that functions on $\mathbb{R}^{2m}$ satisfy this, as is given by e.g. [15, 42]:

Theorem 4.2.11. *Let $P_k(x,u) \in \mathcal{P}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})$ then the following are equivalent:*

1. *The polynomial $P_k(x,u)$ only depends on the wedge-variables X_{ij}.*
2. *The polynomial $P_k(x,u)$ belongs to $\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$.*

This means that every polynomial

$$P_k(x,u) \in \mathcal{P}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$$

leads to a well-defined function on the Grassmannian by either restricting it to the Stiefel manifold, or by looking at

$$f(x,u) = \frac{P_k(x,u)}{|x \wedge u|^k},$$

where the latter is then a function depending on Ω_{ij}. In chapter 6, more precisely in theorem 6.1.9, we will obtain a full decomposition for the space

$$\mathcal{P}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$$

into irreducible $\mathrm{SO}(m)-$ components. In particular we have that the space of simplicial harmonics is a special subspace of $\mathcal{P}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$. This means that the Casimir operators of $\mathfrak{so}(m)$ will act as a constant multiple of the identity on this space, or in other words, using theorem 4.2.8, we have that:

$$\forall D \in D(\mathrm{Gr}_o(2,m)) : D\bigg|_{\mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})} = \lambda_D(k) 1_{\mathcal{H}_{k,k}(\mathbb{R}^{2m}, \mathbb{C})}$$

Combining these results allows us to conclude that the restriction of the generalised Gegenbauer solutions to the Grassmann manifold are indeed spherical functions:

Theorem 4.2.12. *Let $k \in \mathbb{Z}^+$ and $m > 4$. The function*

$$\left(|x \wedge u|^k J_k^{\frac{m-3}{2}}(\tau^2, \sigma^2)\right)\Big|_{\mathrm{Gr}_o(2,m)}$$

is a spherical function for the pair $\big(\mathrm{SO}(m), \mathrm{SO}(2) \times \mathrm{SO}(m-2)\big)$. If $m = 4$ then we can conclude the same for the polynomials

$$\left(|x \wedge u|^k C_k^{\frac{1}{2}}(\tau^{\pm})\right)\Big|_{\mathrm{Gr}_o(2,m)}.$$

The algebra $D(\mathrm{Gr}_o(2,m))$ is generated by two operators of order two and four, and spherical functions on $\mathrm{Gr}_o(2,m)$ are uniquely determined by their eigenvalues under the action of these operators. This means that the J-polynomials, as a family of special functions, are the unique (invariant) solution to a system of two differential equations, as was found in proposition 4.1.22.

Remark 4.2.13. While there exists a general theory of hypergeometric functions on Grassmann manifolds, see e.g. [3, 40, 41], the connection between our spherical polynomials and said hypergeometric functions is still under investigation.

CHAPTER 5

Cauchy-Kovalevskaya extension for simplicial monogenic spaces

If it's good enough for Kirk, Crunch and Kangaroo it's good enough for me.
Dr. Sheldon Cooper

The Cauchy-Kovalevskaya extension for polynomials is a crucial result in Clifford analysis which describes theoretically how to construct monogenics in m dimensions starting from certain polynomial spaces in $m-1$ dimensions, hereby using the branching rules, see theorem 1.4.1. In particular its connection with the construction of orthonormal bases for spaces of polynomial solutions has been exploited.
The chapter is structured as follows: we start with the classical case before we generalise the result to polynomials of two variables. The latter case was treated in [78] where it was proven that not every polynomial will admit a CK extension. Only solutions to the so called skew-wedge system have to be considered and in chapter 6 we will take a closer look at this system. While the CK map we obtain is the same as the one from [78], we focus on simplicial polynomials, i.e. polynomials depending on x and $x \wedge u$. We will prove that when acting on such polynomials, the CK extension map is onto on the spaces of simplicial monogenics. We end the chapter with an explicit example related to our generalised Gegenbauer polynomials.

5.1 Classical CK extension

In the classical case, one has that the space $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}^{\pm})$ is isomorphic with the space of $\mathbb{S}^{\pm}$-valued polynomials of degree k on $\mathbb{R}^{m-1}$. Note that the latter space has one dimension less here. In order to make a clear distinction between vectors in $\mathbb{R}^m$ and $\mathbb{R}^{m-1}$, we use the notation $x = x_1 e_1 + \tilde{x}$ with $\tilde{x} \in \mathbb{R}^{m-1}$. One way to see why these spaces are isomorphic, is to use abstract branching rules for $\mathfrak{so}(m)$-modules (see theorem 1.4.1), as these tell you which summands to expect when restricting the action to the subalgebra $\mathfrak{so}(m-1)$. For instance, starting from the space $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ in case $m = 2n+1$, this theorem tells us that

$$\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})\Bigg|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong (k)'\Bigg|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong \bigoplus_{j=0}^{k} (j)'_{\pm}$$

The sum at the right-hand side can be seen as the *full* space of polynomials of a degree k, in view of theorem 2.3.20. We are stressing the fact that there are no conditions to be imposed here, as this will turn out to be different for the higher spin case considered in this chapter. The Fischer decomposition indeed shows that $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S}) \cong \mathcal{P}_k(\mathbb{R}^{m-1}, \mathbb{S})$ and the CK-extension can then be seen as the *explicit* ismorphism, which thus turns an arbitrary polynomial on $\mathbb{R}^{m-1}$ into a monogenic polynomial on $\mathbb{R}^m$. It can for instance be obtained as follows: consider $M_k(x) \in \mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$ and write it as

$$M_k(x) = \sum_{i=0}^{k} x_1^i P_{k-i}(\tilde{x}),$$

where $P_{k-i}(\tilde{x}) \in \mathcal{P}_{k-i}(\mathbb{R}^{m-1}, \mathbb{S})$. Because $M_k(x)$ is a monogenic polynomial we find the following system of equations:

$$\begin{aligned} \partial_x M_k(x) = 0 &\iff \partial_{x_1} M_k(x) = e_1 \partial_{\tilde{x}} M_k(x) \\ &\iff \sum_{i=0}^{k-1} (i+1) x_1^i P_{k-i-1}(\tilde{x}) = \sum_{i=0}^{k-1} x_1^i e_1 \partial_{\tilde{x}} P_{k-i}(\tilde{x}) \end{aligned}$$

with $\partial_{\tilde{x}}$ the Dirac operator on $\mathbb{R}^{m-1}$. This yields the following system,

$$\begin{cases} P_{k-1}(\tilde{x}) & = e_1 \partial_{\tilde{x}} P_k(\tilde{x}) \\ 2P_{k-2}(\tilde{x}) & = e_1 \partial_{\tilde{x}} P_{k-1}(\tilde{x}) \\ \quad\cdots & \\ (k-1)P_1(\tilde{x}) & = e_1 \partial_{\tilde{x}} P_2(\tilde{x}) \\ kP_0(\tilde{x}) & = e_1 \partial_{\tilde{x}} P_1(\tilde{x}) \end{cases}$$

from which we can conclude that $M_k(x)$ is completely determined by $P_k(\tilde{x})$. Moreover, given $P(\tilde{x}) \in \mathcal{P}(\mathbb{R}^{m-1}, \mathbb{S})$ one can find a formal solution by means of:

$$M(x) = M(x_1, \tilde{x}) := \exp(e_1 x_1 \partial_{\tilde{x}}) P(\tilde{x})$$

which allows us to define an expression which is independent of the degree $k \in \mathbb{Z}^+$, given by

$$\begin{aligned}\mathrm{CK}: \mathcal{P}(\mathbb{R}^{m-1},\mathbb{S}) &\to \mathcal{P}(\mathbb{R}^m,\mathbb{S}) \cap \ker \partial_x : \\ P(\tilde{x}) &\mapsto \mathrm{CK}[P(\tilde{x})] := \exp\big(x_1 e_1 \partial_{\tilde{x}}\big) P(\tilde{x}).\end{aligned} \tag{5.1}$$

Note that the exponential operator in the formula has to be understood through its power series, which always reduces to a finite sum on polynomials (so there are no convergence issues) but for infinite sums there exists a certain neighbourhood of $\Omega \subset \mathbb{R}^{m-1}$ such that this converges, see e.g. [18] for more details.

5.2 Simplicial monogenic CK extension

In this section we will do a similar analysis for the spaces $\mathcal{S}_{\ell,k}(\mathbb{R}^{2m},\mathbb{S})$ with $k \neq 0$. In sharp contrast to the previous case, we will have to impose extra conditions on the space $\mathcal{P}_{\ell,k}(\mathbb{R}^{2m},\mathbb{S})$ in order to arrive at an isomorphism. First of all, simply applying the abstract branching rules for the orthogonal Lie algebra $\mathfrak{so}(m)$ to the highest weight $(\ell,k)'$ tells us that

$$\mathcal{S}_{\ell,k}\left(\mathbb{R}^{2m},\mathbb{S}\right)\Big|^{\mathfrak{so}(m)}_{\mathfrak{so}(m-1)} \cong \bigoplus_{a=0}^{\ell-k}\bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}\left(\mathbb{R}^{2(m-1)},\mathbb{S}\right).$$

Remark 5.2.1. As is common in the context of spinors, the parity of the dimension m plays a crucial role here. If $m = 2n+1$, the spinor space at the left-hand side is the irreducible space $\mathbb{S}$ of Dirac spinors and the right-hand side then contains both $\mathbb{S}^+$ and $\mathbb{S}^-$. If $m = 2n$, the spinor space at the left-hand side should come with a sign (one needs Weyl spinors to have an irreducible module) but then the spaces at the right-hand side bear no extra sign. This technicality can be ignored in a sense, if one works with Dirac spinors $\mathbb{S}$ only and takes into account that these may decompose either at the left-hand side or the right-hand side of the isomorphism.

This time, it is not immediately clear how to relate the summands appearing at the right-hand side to the *full* space of polynomials in $(\tilde{x},\tilde{u}) \in \mathbb{R}^{2(m-1)}$. One explanation for this is simply because the equivalent of the Fischer decomposition (see theorem 2.3.8 and theorem 2.3.20) in the setting of 2 vector variables is much more complicated. We refer to e.g. [16], where this was done using the notion of transvector algebras. As a matter of fact, the method used below will indeed prove that not all polynomials in $\mathcal{P}_{\ell,k}(\mathbb{R}^{2(m-1)},\mathbb{S})$ can be CK-extended to a simplicial monogenic polynomial on $\mathbb{R}^{2m}$, see also [78].

Again choosing the Lie algebra $\mathfrak{so}(m-1)$ as the subalgebra fixing the direction $e_1 \in \mathbb{R}^m$, we can use the fact that the coordinates x_1 and u_1 are constant to write an arbitrary element of the space at the left-hand side as

$$S_{\ell,k}(x,u) = \sum_{a=0}^{\ell}\sum_{b=0}^{k} x_1^a u_1^b P_{\ell-a,k-b}(\tilde{x},\tilde{u}) \in \mathcal{S}_{\ell,k}\left(\mathbb{R}^{2m},\mathbb{S}\right).$$

Again the notation $x = x_1 e_1 + \tilde{x} \in \mathbb{R}^m$, with $\tilde{x} \in \mathbb{R}^{m-1}$ was used (similarly for the vector $u \in \mathbb{R}^m$), and $P_{\ell-a,k-b}(\tilde{x},\tilde{u}) \in \mathcal{P}_{\ell-a,k-b}\left(\mathbb{R}^{2m-2},\mathbb{S}\right)$. If we now express the fact that $S_{\ell,k}(x,u)$ is simplicial monogenic in the variables x and u, we are led to the following systems of equations:

$$\begin{aligned}
\partial_x S_{\ell,k} &= 0 \\
&\Longleftrightarrow \sum_{a=1}^{\ell}\sum_{b=0}^{k} a x_1^{a-1} u_1^b P_{\ell-a,k-b} = \sum_{a=0}^{\ell}\sum_{b=0}^{k} x_1^a u_1^b e_1 \partial_{\tilde{x}} P_{\ell-a,k-b} \qquad (5.2)\\
\partial_u S_{\ell,k} &= 0 \\
&\Longleftrightarrow \sum_{a=0}^{\ell}\sum_{b=1}^{k} b x_1^{a} u_1^{b-1} P_{\ell-a,k-b} = \sum_{a=0}^{\ell}\sum_{b=0}^{k} x_1^a u_1^b e_1 \partial_{\tilde{u}} P_{\ell-a,k-b} \qquad (5.3)\\
\langle x, \partial_u \rangle S_{\ell,k} &= 0 \\
&\Longleftrightarrow \sum_{a=0}^{\ell}\sum_{b=1}^{k} b x_1^{a+1} u_1^{b-1} P_{\ell-a,k-b} = -\sum_{a=0}^{\ell}\sum_{b=0}^{k} x_1^a u_1^b \langle \tilde{x}, \partial_{\tilde{u}} \rangle P_{\ell-a,k-b}\,. \\
&\qquad (5.4)
\end{aligned}$$

These systems of equations can be represented diagrammatically (see figure 5.1), hereby using the notation $\{i,j\}$ for $P_{i,j}(\tilde{x},\tilde{u})$ and drawing an arrow between nodes $\{a,b\} \to \{c,d\}$ whenever there is an equation in the system above which says that $P_{c,d}(\tilde{x},\tilde{u})$ can be obtained from $P_{a,b}(\tilde{x},\tilde{u})$ using any of the operators $\partial_{\tilde{x}}$, $\partial_{\tilde{u}}$ or $\langle \tilde{x}, \partial_{\tilde{u}} \rangle$. System (5.4) contains equations of the form $\langle \tilde{x}, \partial_{\tilde{u}} \rangle P_{\ell,k-j} = 0$ for $0 \leq j \leq k$ and $P_{0,k-j} = 0$ for $1 \leq j \leq k$. This is then represented by the right and lower boundary of the diagram. One can move horizontally in the diagram by acting with $\langle \tilde{x}, \partial_{\tilde{u}} \rangle$ and down to the left (resp. right) by acting with $e_1 \partial_{\tilde{x}}$ (resp. $e_1 \partial_{\tilde{u}}$) on any given node.

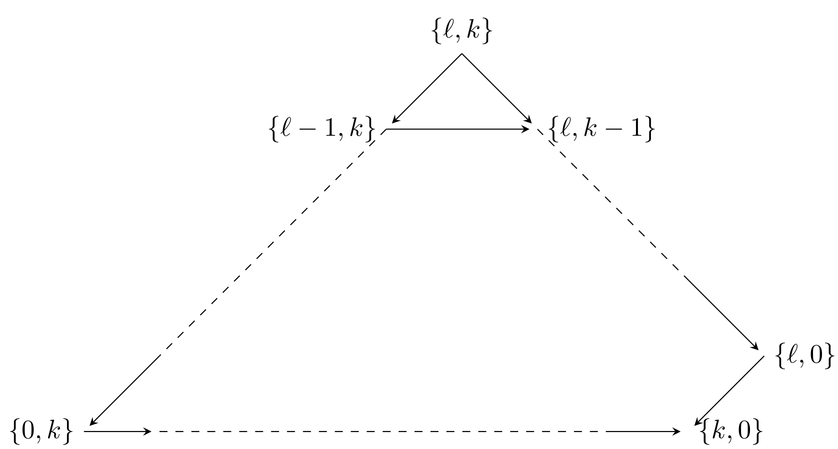


Figure 5.1: *The CK system for* $\mathcal{S}_{\ell,k}(\mathbb{R}^{2m},\mathbb{S})$. *Here* $\{i,j\}$ *refers to* $P_{i,j}(\tilde{x},\tilde{u})$.

Example 5.2.2. Let us for example consider the case $(\ell, k)' = (3, 2)'$ in odd dimensions $m = 2n+1$, for which $\mathcal{S}_{3,2}(\mathbb{R}^{2m}, \mathbb{S})$ provides a model. Restricting the action to the subalgebra $\mathfrak{so}(m-1)$ fixing the direction e_1, the classical branching rules give the following:

$$(3,2)'\Big|_{\mathfrak{so}(m-1)}^{\mathfrak{so}(m)} \cong (3,2)'_{\pm} \oplus (2,2)'_{\pm} \oplus (3,1)'_{\pm} \oplus (2,1)'_{\pm} \oplus (3,0)'_{\pm} \oplus (2,0)'_{\pm}.$$

Writing

$$S_{3,2}(x,u) = \sum_{a=0}^{3}\sum_{b=0}^{2} x_1^a u_1^b P_{3-a,2-b}(\tilde{x},\tilde{u}) \in \mathcal{S}_{3,2}\left(\mathbb{R}^{2m}, \mathbb{S}\right)$$

and expressing that $S_{3,2}(x,u)$ belongs to the kernel of ∂_x, ∂_u and $\langle x, \partial_u \rangle$ leads to a system of equations which can be visualised in the diagram below:

$$\begin{tikzcd}
 & & & \{3,2\} \arrow[dl] \arrow[dr] & & & \\
 & & \{2,2\} \arrow[rr] \arrow[dl] \arrow[dr] & & \{3,1\} \arrow[dl] \arrow[dr] & & \\
 & \{1,2\} \arrow[rr] \arrow[dl] \arrow[dr] & & \{2,1\} \arrow[rr] \arrow[dl] \arrow[dr] & & \{3,0\} \arrow[dl] & \\
\{0,2\} \arrow[rr] & & \{1,1\} \arrow[rr] & & \{2,0\} & &
\end{tikzcd}$$

In order to reduce the diagram to its essence, we will first prove a few lemmas. The first one states that every subdiagram in figure 5.1 is a commutative one. By this we mean that if one has two different paths in a subdiagram (a single arrow versus a composition of two arrows), the corresponding equations are compatible (equivalent). Put differently, the chain of equations described by (5.2 - 5.4) contains no inconsistencies.

Lemma 5.2.3.

(i) For all $0 \leq j \leq \ell$ and $0 \leq i \leq k-1$, the following diagram commutes:

$$\begin{tikzcd}
\{\ell - j, k - i\} \arrow[rr] \arrow[dr] & & \{\ell - j + 1, k - i - 1\} \arrow[dl] \\
 & \{\ell - j, k - i - 1\} &
\end{tikzcd}$$

(ii) For all $0 \leq j \leq \ell - 1$ and $0 \leq i \leq k$, the following diagram commutes:

$$\begin{tikzcd}
& \{\ell - j, k - i\} \arrow[dl] \arrow[dr] & \\
\{\ell - j - 1, k - i\} \arrow[rr] & & \{\ell - j, k - i - 1\}
\end{tikzcd}$$

Proof. From systems (5.2-5.4) above we have the following set of equations, corresponding to the first triangle above (hereby using our notation $\{a, b\}$ for the polynomials $P_{a,b}(\tilde{x}, \tilde{u})$ appearing in the system):

$$\begin{aligned}
e_1 \partial_{\tilde{x}} \{\ell - j + 1, k - i - 1\} &= j \{\ell - j, k - i - 1\}, \\
e_1 \partial_{\tilde{u}} \{\ell - j, k - i\} &= (i + 1) \{\ell - j, k - i - 1\}, \\
\langle \tilde{x}, \partial_{\tilde{u}} \rangle \{\ell - j, k - i\} &= -(i + 1) \{\ell - j + 1, k - i - 1\},
\end{aligned}$$

which represent the given subdiagram. On the one hand, applying the last and the first equation (which amounts to the composition of two arrows), one arrives at

$$e_1 \partial_{\tilde{x}} \left(\langle \tilde{x}, \partial_{\tilde{u}} \rangle \{\ell - j, k - i\} \right) = -(i + 1) j \{\ell - j, k - i - 1\} \ .$$

On the other hand, one can also invoke the Leibniz rule:

$$\begin{aligned}
& e_1 \partial_{\tilde{x}} \left(\langle \tilde{x}, \partial_{\tilde{u}} \rangle \{\ell - j, k - i\} \right) \\
&= e_1 \partial_{\tilde{u}} \{\ell - j, k - i\} + \langle \tilde{x}, \partial_{\tilde{u}} \rangle \left(e_1 \partial_{\tilde{x}} \{\ell - j, k - i\} \right) \\
&= (i + 1) \{\ell - j, k - i - 1\} + \langle \tilde{x}, \partial_{\tilde{u}} \rangle \left(e_1 \partial_{\tilde{x}} \{\ell - j, k - i\} \right) \ ,
\end{aligned}$$

where we have used the second equation (the arrow pointing down to the right). In order to calculate the remaining term, we switch our attention to the lower triangle (where $\{\ell - j, k - i\}$ appears at the top). The equations expressing these arrows then give:

$$\begin{aligned}
\langle \tilde{x}, \partial_{\tilde{u}} \rangle \left(e_1 \partial_{\tilde{x}} \{\ell - j, k - i\} \right) &= (j + 1) \langle \tilde{x}, \partial_{\tilde{u}} \rangle \{\ell - j - 1, k - 1\} \\
&= -(i + 1)(j + 1) \{\ell - j, k - i - 1\} \ .
\end{aligned}$$

Adding this to what was already found after the previous paragraph, this indeed proves that the equations are compatible. $\square$

Next, we will look for the necessary conditions one has to impose on $P_{\ell,k}(\tilde{x}, \tilde{u})$, the polynomial with maximal degree in $(\tilde{x}, \tilde{u})$, such that it can provide a solution to our system of equations. This makes sense, as it immediately follows from the diagram that all components $P_{\ell-a,k-b}(\tilde{x}, \tilde{u})$ with $(a, b) \neq (0, 0)$ are determined by $P_{\ell,k}(\tilde{x}, \tilde{u})$, which has to satisfy the equation $\langle \tilde{x}, \partial_{\tilde{u}} \rangle P_{\ell,k}(\tilde{x}, \tilde{u}) = 0$, as is clear from system (5.4). A second condition follows from the following lemma:

Lemma 5.2.4. *The polynomials $P_{\ell,k}(\tilde{x}, \tilde{u}) \in \mathcal{P}_{\ell,k}\left(\mathbb{R}^{2(m-1)}, \mathbb{S}\right)$ which can lead to a simplicial monogenic in m dimensions through a CK-extension must satisfy*

$$\left(\partial_{\tilde{x}} \wedge \partial_{\tilde{u}} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) = 0 = \langle \tilde{x}, \partial_{\tilde{u}} \rangle P_{\ell,k}(\tilde{x}, \tilde{u}),$$

where $\partial_{\tilde{x}} \wedge \partial_{\tilde{u}} = \partial_{\tilde{x}} \partial_{\tilde{u}} + \langle \partial_{\tilde{x}}, \partial_{\tilde{u}} \rangle$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \{\ell, k\} \arrow[dl] \arrow[dr] & \\
\{\ell-1, k\} \arrow[rr] \arrow[dr] & & \{\ell, k-1\} \arrow[dl] \\
 & \{\ell-1, k-1\} &
\end{tikzcd}$$

From this diagram we obtain two equations:

$$\begin{aligned}
(e_1\partial_{\tilde{x}})(e_1\partial_{\tilde{u}})\,\{\ell, k\} &= \{\ell-1, k-1\} \\
(e_1\partial_{\tilde{u}})(e_1\partial_{\tilde{x}})\,\{\ell, k\} &= \{\ell-1, k-1\}\,.
\end{aligned}$$

Subtracting them leads to the desired result. □

The analysis of the CK-problem can thus be seen as a motivation to study the following system of rotationally invariant equations, whereby $P_{\ell,k}(\tilde{x}, \tilde{u}) \in \mathcal{P}_{\ell,k}(\mathbb{R}^{2(m-1)}, \mathbb{S})$:

$$\begin{cases}
\langle \tilde{x}, \partial_{\tilde{u}}\rangle P_{\ell,k}(\tilde{x}, \tilde{u}) & = \quad 0 \\[2ex]
(\partial_{\tilde{x}} \wedge \partial_{\tilde{u}})\, P_{\ell,k}(\tilde{x}, \tilde{u}) & = \quad 0
\end{cases}$$

We will from now on refer to this system as *the skew wedge system* and this will be discussed further in the next chapter. What is still missing here, is the explicit isomorphism, known as the CK-extension map. This map thus generalises the role of the map CK[$\bullet$] defined in (5.1). We claim the following:

Theorem 5.2.5. *The CK-extension map*

$$\mathrm{CK}[\bullet] : \mathcal{P}(\mathbb{R}^{2(m-1)}, \mathbb{S}) \cap \ker(\langle \tilde{x}, \partial_{\tilde{u}}\rangle, \partial_{\tilde{x}} \wedge \partial_{\tilde{u}}) \to \bigoplus_{\ell \geq k \geq 0} \mathcal{S}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S})$$

is explicitly given by

$$\mathrm{CK}[\bullet] := \exp\big(e_1 x_1 \partial_{\tilde{x}}\big) \exp\big(e_1 u_1 \partial_{\tilde{u}}\big) = \exp\big(e_1 u_1 \partial_{\tilde{u}}\big) \exp\big(e_1 x_1 \partial_{\tilde{x}}\big)\ .$$

In particular, the order in which the exponential maps are written is irrelevant. Moreover, the CK-extension map is an isomorphism.

Proof. First of all, in view of the fact that the map CK[$\bullet$] does not affect the total degree of homogeneity, one can focus on fixed degrees ℓ and k of homogeneity (for the variables x and u). So, let us consider an arbitrary element

$$P_{\ell,k}(\tilde{x}, \tilde{u}) \in \mathcal{P}(\mathbb{R}^{2(m-1)}, \mathbb{S}) \cap \ker(\langle \tilde{x}, \partial_{\tilde{u}}\rangle, \partial_{\tilde{x}} \wedge \partial_{\tilde{u}})$$

and prove that

$$\partial_x\Big(\mathrm{CK}[P_{\ell,k}(\tilde{x},\tilde{u})]\Big) = 0 = \langle x,\partial_u\rangle\Big(\mathrm{CK}[P_{\ell,k}(\tilde{x},\tilde{u})]\Big)\ .$$

Note that in each operator identity we will assume that we are acting on $\ker(\langle\tilde{x},\partial_{\tilde{u}}\rangle,\partial_{\tilde{x}}\wedge\partial_{\tilde{u}})$, which means that in general they will not necessarily hold. We start by showing that the order in which the exponential maps are written has no effect on the final result. An arbitrary term in the Taylor expansion of $\mathrm{CK}[\bullet]$ is (up to a constant) given by $(e_1x_1\partial_{\tilde{x}})^i(e_1u_1\partial_{\tilde{u}})^j$ with $i,j\in\mathbb{Z}^+$. For even powers, the term $(e_1x_1\partial_{\tilde{x}})^i$ is scalar and will thus commute with $(e_1u_1\partial_{\tilde{u}})^j$. We therefore only have to consider the case where both i and j are odd. As one can always peel away the even powers, one merely has to consider the combination

$$e_1x_1\partial_{\tilde{x}}e_1u_1\partial_{\tilde{u}} = -e_1^2x_1u_1\partial_{\tilde{x}}\partial_{\tilde{u}} = x_1u_1\big(-\langle\partial_{\tilde{x}},\partial_{\tilde{u}}\rangle + \partial_{\tilde{x}}\wedge\partial_{\tilde{u}}\big)\ .$$

Invoking the fact that we are acting on $\ker(\partial_{\tilde{x}}\wedge\partial_{\tilde{u}})$, we are left with an expression which is clearly symmetric in both variables. Using the classical result, we can thus immediately conclude that

$$\partial_x\Big(\mathrm{CK}[P_{\ell,k}(\tilde{x},\tilde{u})]\Big) = 0 = \partial_u\Big(\mathrm{CK}[P_{\ell,k}(\tilde{x},\tilde{u})]\Big).$$

Indeed, it suffices to move the desired exponential map to the left. To prove that the CK-extension image of the polynomial $P_{\ell,k}(\tilde{x},\tilde{u})$ belongs to $\ker\langle x,\partial_u\rangle$ we introduce the following notation:

$$(e_1x_1\partial_{\tilde{x}})^i = \epsilon_ie_1^ix_1^i\partial_{\tilde{x}}^i$$

with ϵ_i a sign factor which is given by:

$$\epsilon_i = \begin{cases} 1 & \text{for } i\equiv 0,1 \bmod 4\\ -1 & \text{for } i\equiv 2,3 \bmod 4\end{cases}$$

Using this notation we have that:

$$\begin{aligned}
x_1\partial_{u_1}(e_1u_1\partial_{\tilde{u}})^j &= x_1\partial_{u_1}\epsilon_je_1^ju_1^j\partial_{\tilde{u}}^j\\
&= je_1x_1\epsilon_je_1^{j-1}u_1^{j-1}\partial_{\tilde{u}}^{j-1}\partial_{\tilde{u}}\\
&= \begin{cases} je_1x_1\partial_{\tilde{u}}(-\epsilon_j)e_1^{j-1}u_1^{j-1}\partial_{\tilde{u}}^{j-1} & \text{for } j\in 2\mathbb{Z}^+\\ je_1x_1\partial_{\tilde{u}}\epsilon_je_1^{j-1}u_1^{j-1}\partial_{\tilde{u}}^{j-1} & \text{for } j\in 2\mathbb{Z}^++1\end{cases}\\
&= je_1x_1\partial_{\tilde{u}}(e_1u_1\partial_{\tilde{u}})^{j-1}.
\end{aligned}$$

On the other hand one has that:

$$\begin{aligned}
\langle\tilde{x},\partial_{\tilde{u}}\rangle(e_1x_1\partial_{\tilde{x}})^i &= \epsilon_ie_1^ix_1^i\langle\tilde{x},\partial_{\tilde{u}}\rangle\partial_{\tilde{x}}^i\\
&= \epsilon_ie_1^ix_1^i\left[\langle\tilde{x},\partial_{\tilde{u}}\rangle,\partial_{\tilde{x}}^i\right]
\end{aligned}$$

and, via induction, one can show that:

$$\left[\langle \tilde{x}, \partial_{\tilde{u}} \rangle, \partial_{\tilde{x}}^i\right] = -i \partial_{\tilde{u}} \partial_{\tilde{x}}^{i-1} + (-1)^{i-1} \left\lfloor \frac{i}{2} \right\rfloor \partial_{\tilde{x}}^{i-2} \partial_{\tilde{x}} \wedge \partial_{\tilde{u}},$$

where $\lfloor \cdot \rfloor$ represents the floor function. Therefore, when acting on the space $\ker(\partial_{\tilde{x}} \wedge \partial_{\tilde{u}}, \langle \tilde{x}, \partial_{\tilde{u}} \rangle)$, we have the following operator identity:

$$\begin{aligned}
\langle \tilde{x}, \partial_{\tilde{u}} \rangle (e_1 x_1 \partial_{\tilde{x}})^i &= \epsilon_i e_1^i x_1^i \left(-i \partial_{\tilde{u}} \partial_{\tilde{x}}^{i-1} + (-1)^{i-1} \left\lfloor \frac{i}{2} \right\rfloor \partial_{\tilde{x}}^{i-2} \partial_{\tilde{x}} \wedge \partial_{\tilde{u}} \right) \\
&= \begin{cases} -i e_1 x_1 \partial_{\tilde{u}} (-\epsilon_i) e_1^{i-1} x_1^{i-1} \partial_{\tilde{x}}^{i-1} & \text{for } i \in 2\mathbb{Z}^+ \\ -i e_1 x_1 \partial_{\tilde{u}} \epsilon_i e_1^{i-1} x_1^{i-1} \partial_{\tilde{x}}^{i-1} & \text{for } i \in 2\mathbb{Z}^+ + 1 \end{cases} \\
&= -i e_1 x_1 \partial_{\tilde{u}} (e_1 x_1 \partial_{\tilde{x}})^{i-1}.
\end{aligned}$$

Since $\left[\langle \tilde{x}, \partial_{\tilde{u}} \rangle, (e_1 u_1 \partial_{\tilde{u}})^j\right] = 0$ and $P_{\ell,k}(\tilde{x}, \tilde{u}) \in \ker(\partial_{u_1}, \langle \tilde{x}, \partial_{\tilde{u}} \rangle, \partial_{\tilde{x}} \wedge \partial_{\tilde{u}})$ we have that

$$\begin{aligned}
&\langle x, \partial_u \rangle \, \mathrm{CK}[P_{\ell,k}(\tilde{x}, \tilde{u})] \\
&= (x_1 \partial_{u_1} + \langle \tilde{x}, \partial_{\tilde{u}} \rangle) \left(\sum_{j=0}^{\infty} \frac{(e_1 u_1 m \partial_{\tilde{u}})^j}{j!} \right) \left(\sum_{i=0}^{\infty} \frac{(e_1 x_1 \partial_{\tilde{x}})^i}{i!} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) \\
&= \left(\sum_{j=1}^{\infty} \frac{j e_1 x_1 \partial_{\tilde{u}} (e_1 u_1 \partial_{\tilde{u}})^{j-1}}{j!} \right) \left(\sum_{i=0}^{\infty} \frac{(e_1 x_1 \partial_{\tilde{x}})^i}{i!} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) \\
&\qquad + \left(\sum_{j=0}^{\infty} \frac{(e_1 u_1 \partial_{\tilde{u}})^j}{j!} \right) \left(\sum_{i=1}^{\infty} \frac{-i e_1 x_1 \partial_{\tilde{u}} (e_1 x_1 \partial_{\tilde{x}})^{i-1}}{i!} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) \\
&= e_1 x_1 \partial_{\tilde{u}} \left(\sum_{j=1}^{\infty} \frac{(e_1 u_1 \partial_{\tilde{u}})^{j-1}}{(j-1)!} \right) \left(\sum_{i=0}^{\infty} \frac{(e_1 x_1 \partial_{\tilde{x}})^i}{i!} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) \\
&\qquad - e_1 x_1 \partial_{\tilde{u}} \left(\sum_{j=0}^{\infty} \frac{(e_1 u_1 \partial_{\tilde{u}})^j}{j!} \right) \left(\sum_{i=1}^{\infty} \frac{(e_1 x_1 \partial_{\tilde{x}})^{i-1}}{(i-1)!} \right) P_{\ell,k}(\tilde{x}, \tilde{u}) \\
&= 0,
\end{aligned}$$

which shows that $\mathrm{CK}[P_{\ell,k}(\tilde{x}, \tilde{u})]$ is indeed simplicial monogenic. The injectivity of the CK-extension map follows from

$$\mathrm{CK}[P_{\ell,k}(\tilde{x}, \tilde{u})] \Big|_{x_1 = u_1 = 0} = P_{\ell,k}(\tilde{x}, \tilde{u}), \tag{5.5}$$

whereas the surjectivity is a direct consequence of the fact that a simplicial monogenic polynomial $S_{\ell,k}(x, u)$ is completely determined by its restriction to $\mathbb{R}^{2(m-1)}$ and lemma 5.2.4. $\square$

An immediate consequence of this theorem is the following:

Corollary 5.2.6. *Let $\ell, k \in \mathbb{Z}^+$ with $\ell \geq k$ and $m \geq 4$. If m is odd then*

$$\mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}) \cap \ker(\partial_x \wedge \partial_u, \langle x, \partial_u \rangle) \cong \bigoplus_{a=0}^{\ell-k} \bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}).$$

If m is even then

$$\begin{aligned}&\mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^+ \oplus \mathbb{S}^-) \cap \ker(\partial_x \wedge \partial_u, \langle x, \partial_u \rangle)\\ &\qquad \cong \bigoplus_{a=0}^{\ell-k} \bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^+) \oplus \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^-).\end{aligned}$$

Proof. Using the abstract branching rules we find that, for $m = 2n$ even:

$$\mathcal{S}_{\ell,k}(\mathbb{R}^{2m+2}, \mathbb{S}) \cong \bigoplus_{a=0}^{\ell-k} \bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^+) \oplus \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^-)$$

and for m odd:

$$\mathcal{S}_{\ell,k}(\mathbb{R}^{2m+2}, \mathbb{S}^\pm) \cong \bigoplus_{a=0}^{\ell-k} \bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}).$$

The CK-extension map provides the explicit isomorphism between the simplicial monogenics $\mathcal{S}_{\ell,k}(\mathbb{R}^{2m+2}, \mathbb{S}^\pm)$ and $\ker(\partial_w, \langle x, \partial_u \rangle)$ on $\mathbb{R}^{2m}$, from which the desired result follows. ∎

In the next chapter we will provide embeddings that turn these isomorphisms into an equality if $\ell = k$. Finally, we turn our attention to the following example, which generalises the following situation from the classical case: the action of the CK-map on $\tilde{x}^k$ leads to the Gegenbauer solutions, which follows from the fact that both $\mathrm{CK}[\tilde{x}^k]$ and the Gegenbauer solutions are invariant w.r.t. the action of $\mathfrak{so}(m-1)$. If we use the notation of lemma 2.5.7:

$$\mathcal{G}_k^m(x) := |x|^k \left(C_k^{\frac{m}{2}} \left(\frac{x_1}{|x|} \right) + \frac{x e_1}{|x|} C_{k-1}^{\frac{m}{2}} \left(\frac{x_1}{|x|} \right) \right),$$

where the $C_k^\alpha(t)$ are the classic Gegenbauer polynomials, we have (for a constant $c_k \in \mathbb{C}$) that:

$$\exp(e_1 x_1 \partial_{\tilde{x}})[\tilde{x}^k] = c_k \mathcal{G}_k^m(x).$$

Something similar can be found here but for this we need the following result, which will be proven in the next chapter:

$$\pi u^k : \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}) \hookrightarrow \ker_{k,k}(\langle x, \partial_u \rangle, \partial_x \wedge \partial_u) : M_k(x) \to \pi u^k M_k(x).$$

Here π represents a projection onto $\ker \langle x, \partial_u \rangle$ that is $\mathfrak{so}(m)$-invariant.

Example 5.2.7. Let $m = 2n$ be even. Consider the generalised Gegenbauer solutions from theorem 4.1.23 then, we will show that, for each $k \in \mathbb{Z}^+$, there exists a constant $\alpha_k \in \mathbb{C}$ such that:

$$\alpha_k \partial_w |x \wedge u|^{k+1} J_{k+1}^{\frac{m-3}{2}}(\tau, \sigma^2) I = \begin{cases} \mathrm{CK}[\pi \tilde{u}^k \mathcal{G}_k^{m-1}(\tilde{x}) I] & k \text{ even} \\ \mathrm{CK}[\pi \tilde{u}^k \mathcal{G}_k^{m-1}(\tilde{x}) \mathfrak{f}_1^\dagger I] & k \text{ odd} \end{cases}$$

where $\mathfrak{f}_1^\dagger = -\frac{e_1 + ie_2}{2}$. We have to make the distinction between the parities of k because of the multiplication with $\tilde{u}^k$, which changes the parity of the spinors if k is odd and thus one has to start from an element of $\mathbb{S}^-$. The other factors in the formula above can be written as a combination of scalars and bivectors and thus it is clear that all of those polynomials are indeed elements of $\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)$. From the branching rules it follows that:

$$\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+_{2n})\Big|_{\mathfrak{so}(m-2)}^{\mathfrak{so}(m)} \cong \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-i} (j+1) \Big(\mathcal{S}_{k-i,k-i-j}(\mathbb{R}^{2(m-2)}, \mathbb{S}^+_{2n-2}) \oplus \mathcal{S}_{k-i,k-i-j}(\mathbb{R}^{2(m-2)}, \mathbb{S}^-_{2n-2}) \Big)$$

and in particular one can conclude that there exists a unique polynomial in $\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)$ that, under the action of $\mathfrak{so}(m-2)$, acts as the highest weight vector for the copy of $\mathbb{S}^+_{2n-2}$. As we have mentioned before, the action of $\mathfrak{so}(m)$ can be expressed in terms of the operators

$$M_{ij} := L_{ij}^x + L_{ij}^u - \frac{1}{2} e_{ij}$$

and it follows from straightforward computations that, for $2 < i < j \leq m$, the operator M_{ij} acts solely on the idempotent I in our polynomials, i.e.:

$$M_{ij} \mathrm{CK}[\pi \tilde{u}^k \mathcal{G}_k^{m-1}(\tilde{x}) \mathfrak{f}_1^\dagger I] = \mathrm{CK}[\pi \tilde{u}^k \mathcal{G}_k^{m-1}(\tilde{x}) \mathfrak{f}_1^\dagger M_{ij} I].$$

And therefore, under the action of $\mathfrak{so}(m-2)$, they act as the highest weight vector for $\mathbb{S}^+$ and since only one such copy can exist in $\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)$, the polynomials have to be equal (up to a constant α_k).

CHAPTER 6

Wedge-Fischer theorem

Wait, I'm having one of those things...
Like a headache, with pictures.
Philip J. Fry

In chapter 4, more specifically in section 4.2, we have related the special polynomials we constructed there to functions on the Grassmannian $\mathrm{Gr}_o(2, m)$. This meant looking at restrictions of functions on $\mathbb{R}^{2m}$ that belong to the kernel space

$$\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle).$$

A homogeneous polynomial $P(x, u)$ can only belong to this kernel space if the degree of homogeneity in both variables is equal, and thus we assume this to be the case at the start of this chapter. We will then study this system in more detail, prompting the appearance of the scalar skew-wedge system, i.e. solutions to $\langle x, \partial_u\rangle, \langle u, \partial_x\rangle$ and the Cayley-Laplace operator Δ_w. Moreover, this system also occurs naturally when considering polynomials that are invariant w.r.t. a right SL(2)-action, see e.g. [42], and then decomposing these polynomial spaces into irreducible components for the action of $\mathrm{SO}(m)$. Because, for a fixed degree of homogeneity k, the space $\mathcal{P}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$ is an irreducible $\mathrm{GL}(m)$-module, we will do so by looking at a branching problem from $\mathrm{GL}(m)$ to $\mathrm{SO}(m)$, the rules for which are found in section 1.4.2. Hereby we will generalise theorem 2.3.8, where the operator Δ_w will play a similar role to the Laplace operator Δ_x in the classic theorem. While the abstract decomposition in terms of weights was already studied in [52], we build on their result by introducing suitable embeddings for the irreducible $\mathrm{SO}(m)$-components and show that in our setting, the Cayley-Laplace operator is surjective.
Next we will consider a spinor refinement of this, which will lead to the skew-wedge system that was also encountered in chapter 5. Finally, we see what

happens if $\ell \neq k$, where things get rather complicated due to the fact that the branching problem we consider will no longer be multiplicity free. Nevertheless we show that the Cayley-Laplace operator is still of great importance by starting from its spinor version and then working our way back from there.

6.1 Scalar skew-wedge system

Again, we let $\mathcal{P}_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})$ denote the space of scalar-valued polynomials on $\mathbb{R}^{2m}$ which are homogeneous of degree ℓ (resp. k) in the variable x (resp. u) and for now we assume that $\ell = k$. Despite being a special case, it is interesting on its own because of its connection with $\mathbb{C}^\infty(\mathrm{Gr}_o(2,m))$ and already contains the main ideas of what to expect in the general case (which will be treated in section 6.3). On the vector space $\mathcal{P}_{k,k}(\mathbb{R}^{2m},\mathbb{C})$, there exists a canonical action of the Lie algebra $\mathfrak{gl}(m)$, which can be realised inside the full Weyl algebra on $\mathbb{R}^{2m}$ by means of

$$\mathfrak{gl}(m) = \mathrm{Span}\big(x_a\partial_{x_b} + u_a\partial_{u_b} : 1 \leq a,b \leq m\big)\ .$$

In view of the commutation relation

$$[\langle u,\partial_x\rangle\,, x_a\partial_{x_b} + u_a\partial_{u_b}] = 0 = [\langle x,\partial_u\rangle\,, x_a\partial_{x_b} + u_a\partial_{u_b}]\ ,$$

which says that the operators $\langle u,\partial_x\rangle$ and $\langle x,\partial_u\rangle$ are invariant under $\mathfrak{gl}(m)$, it makes sense to study the kernel of both operators in order to obtain $\mathfrak{gl}(m)$-modules:

Definition 6.1.1. For all $k \in \mathbb{Z}^+$, we define the space

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m},\mathbb{C}) := \mathcal{P}_{k,k}(\mathbb{R}^{2m},\mathbb{C}) \cap \ker(\langle x,\partial_u\rangle\,, \langle u,\partial_x\rangle).$$

This space can be identified with the homogeneous coordinate ring of $\mathrm{Gr}(2,m)$, see e.g. [37, 52], which illustrates the deep connection to polynomials on Grassmannians.

The notation SL(2) hereby refers to the fact that the operators $\langle u,\partial_x\rangle$ and $\langle x,\partial_u\rangle$ are actually coming from the derived action of $\mathfrak{sl}(2)$. Indeed, for all matrices $M \in \mathrm{SL}(2)$ the regular action is given by

$$P_{k,k}(x,u) \mapsto M[P_{k,k}](x,u) := P_{k,k}\big((x\ u)M\big),$$

whereby $(x\ u)M$ is an ordinary matrix multiplication. If we then let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2),$$

then the derived action of X on a function $f(x,u)$ is given by:

$$\frac{d}{dt} f\big((x\ u)\exp(tX)\big)\bigg|_{t=0} = \frac{d}{dt} f(x, tx+u)\bigg|_{t=0}$$

$$= \sum_{i=1}^{m} \left(\partial_t (tx_i + u_i) \partial_{tx_i + u_i} f(x, tx + u) \right) \Big|_{t=0}$$
$$= \langle x, \partial_u \rangle f(x, u).$$

The derived action of Y and H in $\mathfrak{sl}(2)$ is calculated similarly.
For instance in [42] it was then shown that the space $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$ with $k \in \mathbb{Z}^+$ defines a model for the irreducible module for the Lie algebra $\mathfrak{gl}(m)$ (or the Lie group $\mathrm{GL}(m)$) with highest weight (k, k) and highest weight vector X_{12}^k, where one can consider the Cartan algebra $\mathfrak{h} = \mathrm{Span}(x_a \partial_{x_a} + u_a \partial_{u_a} : 1 \leq a \leq m)$ for our realisation from above. It was also shown that polynomials in $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$ essentially depend on the minors $x_i u_j - x_j u_i$ $(1 \leq i < j \leq m)$, see theorem 4.2.11. Just like in the case of one vector variable $x \in \mathbb{R}^m$, we will now look at what happens with these spaces when the action of $\mathfrak{gl}(m)$ is restricted to $\mathfrak{so}(m)$. In the classical case (one vector variable), the answer is essentially provided by the Fischer decomposition: the space $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$, an irreducible representation for $\mathfrak{gl}(m)$ with highest weight (k) (and highest weight vector x_1^k), decomposes into harmonic polynomials, see theorem 2.3.8. In the present situation these will be replaced with spaces of simplicial harmonics and in order to know which of these spaces will occur as a summand inside the orthogonal decomposition for $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$, one can look at the branching rules for $\mathrm{SO}(m) \subset \mathrm{GL}(m)$, see section 1.4.2:

Theorem 6.1.2. *For arbitrary $k \in \mathbb{Z}^+$ and for $m > 4$ we have that:*

$$(k, k) \Big|_{\mathrm{SO}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} (k - 2i, k - 2i - 2j)_{\mathrm{SO}(m)}.$$

If $m = 4$ then

$$(k, k) \Big|_{\mathrm{SO}(m)}^{\mathrm{GL}(m)} \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} (k - 2i, k - 2i - 2j)_{\mathrm{SO}(m)} \oplus (k - 2i, 2i + 2j - k)_{\mathrm{SO}(m)}.$$

Proof. Recall the result from [53], see section 1.4.2, which says that:

$$F_\lambda \Big|_{\mathrm{O}(m)}^{\mathrm{GL}(m)} = \bigoplus_{\mu} m_\mu E_\mu$$

where F_λ is an irreducible $\mathrm{GL}(m)$-module and the multiplicity of the irreducible $\mathrm{O}(m)$-module E_μ is given by

$$m_\mu = \sum_{2\delta} c_{\mu, 2\delta}^{\lambda},$$

which is a sum of Littlewood-Richardson coefficients over all Young tableaux with even parts (hence the 2δ in the summation). In our case $\lambda = (k, k)$ and we have to consider modules E_μ such that there exists a 2δ where λ occurs in the Young product of μ and 2δ (how to construct this product was explained in definition 1.4.6). We can immediately draw the following conclusions:

- It is clear that $\mu = (\mu_1, \mu_2)$ because if the third entry is different from zero we can never obtain λ since here the third row has to be empty (and we can only add boxes, not remove them).

- One can also conclude that $2\delta = (2\delta_1, 2\delta_2)$ because of the third rule: if we assume the third entry is different from zero, then after adding the a's and b's we need to add c's to the diagram μ. We cannot add c to the first row: in that case the sequence of added letters would begin with $ca\cdots$ which violates the third rule. For a similar reason one cannot add c to the second row, in this case the sequence would read $a\cdots acb\cdots$ which contradicts the third rule as well. Therefore, the first row we can add c to is the third one, and unless the third entry in 2δ is zero this cannot result in λ.

As a result we need to consider $(\mu_1, \mu_2)\cdot(2\delta_1, 2\delta_2)$ and see what the multiplicity of λ in this product is. To begin one has to add $(k - \mu_1)$ a's to the first row because there have to be k boxes and one cannot use b's for that (3th rule). We then need to add $(\mu_1 - \mu_2)$ a's to the second row: we cannot add more because of the first rule and if we add less then filling the row up to k boxes with b's will break the third rule. If there are any a left after this then we cannot reach λ and thus we find a restriction on μ and 2δ, given by $2\delta_1 = k - \mu_2$. Adding b's is an even simpler task: we have to add $k - \mu_2$ b's to the second row (there are $k - \mu_1$ entries left to reach k boxes) and if there are any b's left after that then λ will not be a part of the product $(\mu \cdot 2\delta)$. This leads us to the restriction that $2\delta_2 = k - \mu_1$, implying that $\mu_1 = k - 2\delta_2$ and $\mu_2 = k - 2\delta_1$. Under these conditions we have that λ occurs exactly once in $(\mu \cdot 2\delta)$, as there is only one way to stack the boxes. Using the dominant weight condition we can write

$$2\delta = (2i + 2j, 2i)$$

and thus the only μ that occur in our decomposition are

$$\mu = (k - 2i, k - 2i - 2j).$$

If $m > 4$ then we can, from section 1.4.1, conclude that the restriction of E_μ to $\mathrm{SO}(m)$ is still irreducible. If $m = 4$, then E_μ decomposes into two $\mathrm{SO}(m)$-irreducible components with highest weight (μ_1, μ_2) and $(\mu_1, -\mu_2)$ resp. which finishes the proof. $\square$

In other words, if $m > 4$, we have that:

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

Note that this isomorphism, and the results that follow, are still valid if $m = 4$ but then one has to take into account that the spaces of simplicial harmonics on the right hand side further decompose into two irreducible components.
We would like to turn this isomorphism into an equality (the polynomials at the right-hand side are not necessarily homogeneous of degree (k, k) in x and u) and for this one needs to use $\mathfrak{so}(m)$ invariants which preserve $\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$. This will happen in two steps, reflecting the occurrence of indices i and j in the decomposition above. First of all, the invariant $|x \wedge u|^2 = |x|^2|u|^2 - \langle x, u\rangle^2$ raises the degree in (x, u) by $(2, 2)$ and clearly preserves $\ker(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle)$. This leads to the refined isomorphism

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

To also raise the degree in u by $2j$ one could multiply with $|u|^{2j}$, which is clearly $\mathfrak{so}(m)$-invariant, but it does not preserve the kernel of the generators of $\mathfrak{sl}(2)$ as $[\langle x, \partial_u\rangle, |u|^2] \neq 0$. To remedy this we introduce the following operator, for which we refer to the work of Zhelobenko and Tolstoy [80]:

Definition 6.1.3. The extremal projection operator for the Lie algebra $\mathfrak{sl}(2)$, written in terms of $\mathrm{Alg}(\langle x, \partial_u\rangle, \langle u, \partial_x\rangle, \mathbb{E}_x - \mathbb{E}_u)$, is given by

$$\pi := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\mathbb{E}_x - \mathbb{E}_u + 2)}{\Gamma(\mathbb{E}_x - \mathbb{E}_u + n + 2)} \langle u, \partial_x\rangle^n \langle x, \partial_u\rangle^n \quad . \tag{6.1}$$

This operator, a formal power series in a localisation of $\mathcal{U}\big(\mathfrak{sl}(2)\big)$ with respect to the algebra $\mathcal{U}\big(\mathfrak{h}\big)$ associated to the Cartan algebra $\mathfrak{h} \subset \mathfrak{sl}(2)$, satisfies $\pi^2 = \pi$ and $\pi \langle u, \partial_x\rangle = 0 = \langle x, \partial_u\rangle \pi$. This operator thus projects on extremal vectors for any representation of $\mathfrak{sl}(2)$.

Combining the multiplication operator $|u|^{2j}$ with this projection operator, we arrive at the following:

Definition 6.1.4. For all $j \in \mathbb{Z}^+$, we define the operator $R_j := \pi|u|^{2j}$, where one first has to multiply with $|u|^{2j}$ and then project using the operator π.

In our setting these operators act on $\ker(\langle x, \partial_u\rangle, \Delta_x, \Delta_u)$, so we can also find an alternative (more explicit) expression for these operators R_j. Let us begin by rewriting $R_1 = \pi|u|^2$ as an example. First of all, since $\langle x, \partial_u\rangle^n |u|^2 = 0$ on $\ker(\langle x, \partial_u\rangle)$ for all $n \geq 3$, easy manipulations show that R_1 can be written as

$$\begin{aligned} R_1 = &\frac{\mathbb{E}_x - \mathbb{E}_u + 1}{\mathbb{E}_x - \mathbb{E}_u + 3}|u|^2 - \frac{2(\mathbb{E}_x - \mathbb{E}_u + 1)}{(\mathbb{E}_x - \mathbb{E}_u + 3)(\mathbb{E}_x - \mathbb{E}_u + 2)} \langle x, u\rangle \langle u, \partial_x\rangle \\ &+ \frac{1}{(\mathbb{E}_x - \mathbb{E}_u + 3)(\mathbb{E}_x - \mathbb{E}_u + 2)}|x|^2 \langle u, \partial_x\rangle^2 . \end{aligned}$$

If we take a closer look at this expression we can see that:

$$R_1 = \sum_{i=0}^{2} P_i(\mathbb{E}_x - \mathbb{E}_u)\big(\langle x, \partial_u\rangle^i |u|^2\big) \langle u, \partial_x\rangle^i$$

where $P_i(\mathbb{E}_x - \mathbb{E}_u)$ is a rational expression in $(\mathbb{E}_x - \mathbb{E}_u)$, but because R_1 acts on simplicial harmonics $H_{k,k-2}$, we can replace these with $P_i(0)$. Note that the factor $\langle x, \partial_u\rangle^i |u|^2$ gives a polynomial $Q_{i,2-i}(x,u)$ and is not meant to act on other variables u, hence the extra brackets. So R_1 consists of the action of the operator $\langle u, \partial_x\rangle^i$, followed by the multiplication with a polynomial in (x,u) and an overall constant coming from the action of $P_i(\mathbb{E}_x - \mathbb{E}_u)$. One can also do this in general, i.e.

Lemma 6.1.5. *Let $j \in \mathbb{Z}^+$, then on $\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ we have:*

$$R_j = \sum_{i=0}^{2j} \frac{(-1)^i (2j-i)!}{(2j+1)!i!}\big(\langle x, \partial_u\rangle^i |u|^{2j}\big) \langle u, \partial_x\rangle^i .$$

Proof. The operator $R_j = \pi |u|^{2j}$ acts on $P(x,u) \in \ker(\Delta_x, \Delta_u, \langle x, \partial_u\rangle)$ and thus $|u|^{2j} P(x,u) \in \ker \langle x, \partial_u\rangle^{2j+1}$, which means that the index in definition 6.1.3 runs from 0 to $2j$. If we write

$$F_n(\mathbb{E}_x - \mathbb{E}_u) = \frac{\Gamma(\mathbb{E}_x - \mathbb{E}_u + 2)}{\Gamma(\mathbb{E}_x - \mathbb{E}_u + n + 2)}$$

then

$$\begin{aligned} R_j &= \sum_{n=0}^{2j} \frac{(-1)^n}{n!} F_n(\mathbb{E}_x - \mathbb{E}_u) \langle u, \partial_x\rangle^n \langle x, \partial_u\rangle^n |u|^{2j} \\ &= \sum_{n=0}^{2j} \frac{(-1)^n}{n!} F_n(\mathbb{E}_x - \mathbb{E}_u) \sum_{i=0}^{n} \binom{n}{i} \Big(\langle u, \partial_x\rangle^{n-i} \langle x, \partial_u\rangle^n |u|^{2j}\Big) \langle u, \partial_x\rangle^i . \end{aligned}$$

Using the fact that, for $i \in \mathbb{Z}^+$:

$$\Big[\langle u, \partial_x\rangle, \langle x, \partial_u\rangle^i\Big] = i(\mathbb{E}_u - \mathbb{E}_x + i - 1) \langle x, \partial_u\rangle^{i-1} ,$$

we can write this as

$$\begin{aligned} R_j &= \sum_{n=0}^{2j} \frac{(-1)^n}{n!} F_n(\mathbb{E}_x - \mathbb{E}_u) \sum_{i=0}^{n} \binom{n}{i} \left(\frac{n!}{i!} \frac{\Gamma(2j-i+1)}{\Gamma(2j-n+1)} \langle x, \partial_u\rangle^i |u|^{2j}\right) \langle u, \partial_x\rangle^i \\ &= \sum_{i=0}^{2j} \left(\sum_{n=i}^{2j} \frac{(-1)^n}{i!} \binom{n}{i} \frac{\Gamma(2j-i+1)}{\Gamma(2j-n+1)} F_n(\mathbb{E}_x - \mathbb{E}_u)\right) \Big(\langle x, \partial_u\rangle^i |u|^{2j}\Big) \langle u, \partial_x\rangle^i \\ &= \sum_{i=0}^{2j} c_j(i) \Big(\langle x, \partial_u\rangle^i |u|^{2j}\Big) \langle u, \partial_x\rangle^i , \end{aligned}$$

where we have use the fact that R_j acts on polynomials $H_{k,k-2j}(x,u)$, and this proves that R_j is of the correct form. To find a more explicit expression for $c_j(i)$ we will obtain a recurrence relation for the coefficients by looking at the identity $\langle x, \partial_u \rangle R_j = 0$:

$$\langle x, \partial_u \rangle R_j = \sum_{i=0}^{2j-1} c_j(i) \left(\langle x, \partial_u \rangle^{i+1} |u|^{2j} \right) \langle u, \partial_x \rangle^i$$
$$+ \sum_{i=1}^{2j} c_j(i) \left(\langle x, \partial_u \rangle^{i} |u|^{2j} \right) i(\mathbb{E}_x - \mathbb{E}_u + i - 1) \langle u, \partial_x \rangle^{i-1}$$

from which it follows that

$$c_j(i) = -(i+1)(2j-i)c_j(i+1),$$

for $0 \leq i \leq 2j-1$. This means that all the coefficients are determined once $c_j(2j)$ is known and from our previous calculations we know that

$$c_j(2j) = \frac{1}{(2j+1)!(2j)!},$$

which leads to

$$c_j(i) = \frac{(-1)^i(2j-i)!}{(2j+1)!i!}.$$

□

From the definition of R_j it is clear that

$$R_j : \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}) \hookrightarrow \mathcal{P}^{SL(2)}_{k-2i}(\mathbb{R}^{2m}, \mathbb{C}).$$

While we prefer to work with the definition in terms of the extremal projection operator, the expression from lemma 6.1.5 is particularly useful to prove that these embeddings are non-trivial. To do so, we need an explicit expression for the polynomials $\langle x, \partial_u \rangle^i |u|^{2j}$:

Lemma 6.1.6. *Let $i, j \in \mathbb{Z}^+$ such that $i \leq 2j$, then*

$$\langle x, \partial_u \rangle^i |u|^{2j} = \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} c_n(i,j)|u|^{2j-2i+2n}|x|^{2n} \langle x, u \rangle^{i-2n},$$

where

$$c_n(i,j) = \frac{(-1)^{i-n}2^{i-2n}(-i)^{(2n)}(-j)^{(i-n)}}{n!}$$

Note that if $i > j$, then $c_n(i,j) = 0$ for $0 \leq n < i-j$, meaning that not all terms will occur.

Proposition 6.1.7. *Let $j, k \in \mathbb{Z}^+$ such that $k \geq 2j$, then the operator R_j is non-trivial on $\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$.*

Proof. Assume that $m > 4$, the case $m = 4$ is completely similar but there one has to work with the individual irreducible components, i.e. one can show that R_j is non-trivial on both $\mathcal{H}^{\pm}_{k,k-2j}(\mathbb{R}^8, \mathbb{C})$. Let

$$\ker R_j = \{H_{k,k-2j} \in \mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C}) : R_j H_{k,k-2j} = 0\}$$

then, because R_j is an $\mathfrak{so}(m)$-invariant operator, we know that $\ker R_j$ is an $\mathfrak{so}(m)$-invariant subspace of $\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ and thus, since $\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ is irreducible, it follows that either $\ker R_j = \mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ or $\ker R_j = 0$. It thus suffices to prove that R_j is non-zero on the highest weight vector, see proposition 2.4.2:

$$\begin{aligned} w_{k,k-2j} &= (x_1 - ix_2)^{2j} \left((x_1 - ix_2)(u_3 - iu_4) - (x_3 - ix_4)(u_1 - iu_2)\right)^{k-2j} \\ &= (x_1 - ix_2)^{2j} w_{k-2j,k-2j}. \end{aligned}$$

To do so we will fill in a specific value for (x, u) and use the expression for R_j found in lemma 6.1.5. Moreover, we will write $Q_{n,j}(x, u) := \langle x, \partial_u \rangle^n |u|^{2j}$, then

$$\begin{aligned} &R_j w_{k,k-2j} \\ &= \sum_{n=0}^{2j} \frac{(-1)^n (2j-n)!}{(2j+1)!n!} Q_{n,j}(x,u)(2j)_n (u_1 - u_2 i)^n (x_1 - x_2 i)^{2j-n} w_{k-2j,k-2j}, \end{aligned}$$

where we have used the fact that $w_{k-2j,k-2j} \in \ker \langle u, \partial_x \rangle$. It then follows that

$$(R_j w_{k,k-2j})\Big|_{x=e_3, u=e_1} = \frac{(-1)^{k-2j} Q_{2j,j}(e_3, e_1)}{(2j+1)!}$$

and from lemma 6.1.6 we can conclude that $Q_{2j,j}(e_3, e_1) \neq 0$, which finishes the proof. $\square$

Remark 6.1.8. There exists an alternative interpretation for the operator R_j. First of all, note that the polynomial $Q_{n,j}(x, u) := \langle x, \partial_u \rangle^n |u|^{2j}$ can be written differently:

$$\begin{aligned} Q_{n,j}(x,u) &= \frac{1}{(2j)!} \langle x, \partial_u \rangle^n \langle u, \partial_x \rangle^{2j} |x|^{2j} \\ &= \frac{(-1)^n n! (-2j)^{(n)}}{(2j)!} \langle u, \partial_x \rangle^{2j-n} |x|^{2j}, \end{aligned}$$

meaning that

$$R_j = \sum_{i=0}^{2j} \frac{(-1)^i}{(2j+1)!} \left(\langle u, \partial_x \rangle^{2j-i} |x|^{2j} \right) \langle u, \partial_x \rangle^i .$$

Consider an arbitrary $H_{k,k-2j} \in \mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$, then we can construct the irreducible representation $\mathbb{V}_{2j}$ of

$$\mathfrak{sl}(2) \cong \operatorname{Span}(\langle x, \partial_u \rangle, \langle u, \partial_x \rangle, \mathbb{E}_x - \mathbb{E}_u) = \operatorname{Span}(X, Y, H)$$

with highest weight vector $H_{k,k-2j}$. In theorem 1.3.2 we have seen that such representations are of the form

$$\mathbb{V}_{2j} = \bigoplus_{i=0}^{2j} \mathbb{C} \langle u, \partial_x \rangle^i H_{k,k-2j}.$$

On the other hand, we can also consider

$$\mathbb{V}_{2j} = \bigoplus_{i=0}^{2j} \mathbb{C} \langle u, \partial_x \rangle^i |x|^{2j},$$

and by the Clebsch-Gordan rule, theorem 1.3.3, we can conclude that there exists a unique component $\mathbb{V}_0$ in the tensorproduct $\mathbb{V}_{2j} \otimes \mathbb{V}_{2j}$. Most importantly, such a component $\mathbb{V}_0 \subset \ker(X, Y, H)$, meaning that it belongs to $\ker(\langle x, \partial_u \rangle, \langle u, \partial_x \rangle)$. One can easily see that a suitable linear combination

$$\sum_{i=0}^{2j} c_{i,j} \left(\langle u, \partial_x \rangle^{2j-i} |x|^{2j} \right) \langle u, \partial_x \rangle^i H_{k,k-2j},$$

indeed belongs to $\mathbb{V}_0$, i.e. a combination of products $v_\alpha \otimes v_{-\alpha}$ with $\alpha \in R$, which is precisely our operator R_j.

Combining everything we have found thus far, leaves us with the following theorem:

Theorem 6.1.9. *For $k \in \mathbb{Z}^+$ and $m \geq 4$, the space $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$ admits the following decomposition:*

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} R_j \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

Note that this is indeed an equality now, and no longer a simple isomorphism.

One can now group certain spaces together, by means of

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} \mathcal{A}_{k-2i}(\mathbb{R}^{2m}, \mathbb{C}) = \mathcal{A}_k \oplus |x \wedge u|^2 \mathcal{P}_{k-2}^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$$

where we have introduced the spaces (for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$)

$$\mathcal{A}_{k-2i}(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} R_j \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

In the classical case, for one vector variable $x \in \mathbb{R}^m$, one has that

$$\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})/\mathcal{P}_{k-2}(\mathbb{R}^m, \mathbb{C}) \cong \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \ .$$

We now do something similar in the present setting, replacing the role of the Laplace operator by the following operator:

Definition 6.1.10. We define the wedge-Laplace operator Δ_w as

$$\Delta_w := \Delta_x \Delta_u - \langle \partial_x, \partial_u \rangle^2 = |\partial_x \wedge \partial_u|^2.$$

This operator is also known as the Cayley-Laplace operator and can also be written as:

$$\Delta_w = \sum_{i<j} (\partial_{x_i} \partial_{u_j} - \partial_{x_j} \partial_{u_i})^2.$$

Moreover, this operator also has a representation theoretical interpretation. If we let $\mathbb{V}_2$ be the irreducible 3 dimensional $\mathfrak{sl}(2)$-representation realised as

$$\mathbb{V}_2 \cong \mathrm{Span}(\Delta_x, \langle \partial_x, \partial_u \rangle, \Delta_u),$$

then, by the Clebsch-Gordan formula, their exists a unique trivial component $\mathbb{V}_0 \subset \mathbb{V}_2 \otimes \mathbb{V}_2$ which is spanned by Δ_w.

In view of the fact that $[\Delta_w, \langle x, \partial_u \rangle] = 0 = [\Delta_w, \langle u, \partial_x \rangle]$, one has that

$$\Delta_w : \mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) \to \mathcal{P}_{k-2}^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$$

and we will show that this map is surjective. On the analogy of the classical spaces of harmonics, we introduce the following subspace of $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$:

Definition 6.1.11. For $k \in \mathbb{Z}^+$, we define:

$$\mathcal{H}_k^w(\mathbb{R}^{2m}, \mathbb{C}) := \left\{ P_k(x,u) \in \mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) : \Delta_w P_k(x,u) = 0 \right\}.$$

Since Δ_w commutes with the skew-Euler operators $\langle x, \partial_u \rangle$ and $\langle u, \partial_x \rangle$, it will also commute with $\mathbb{E}_x - \mathbb{E}_u$. Therefore it will also commute with π which means that, when acting on a space of simplicial harmonics, we have the following operator identity:

$$\begin{aligned}
\Delta_w R_j = \pi \Delta_w |u|^{2j} &= \pi \Delta_u |u|^{2j} \Delta_x - \pi \langle \partial_x, \partial_u \rangle [\langle \partial_x, \partial_u \rangle, |u|^{2j}] \\
&= -2j\pi \langle \partial_x, \partial_u \rangle |u|^{2j-2} \langle u, \partial_x \rangle \\
&= -2j\pi \Delta_x |u|^{2j-2} = 0.
\end{aligned}$$

We hereby made us of the fact that $\pi \langle u, \partial_x \rangle = 0$, see definition 6.1.3. From this we can conclude that

$$\mathcal{A}_{k-2i} \subset \mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C}).$$

Remark 6.1.12. In what follows we will often need the fact that, when a polynomial belongs to the kernel of both skew-Euler operators, the degrees of homogeneity in both variables are necessarily equal (it suffices to calculate their commutator). Therefore we will introduce the notation $\mathbb{E} := \mathbb{E}_x = \mathbb{E}_u$ in order for our formulas to reflect the symmetry in x and u if the need arises.

We will now derive an analogue for the classical relation $[\Delta_x, |x|^2] = 2m + 4\mathbb{E}_x$:

Lemma 6.1.13. *When acting on the kernel of the skew-Euler operators, one has:*

$$[\Delta_w, |x \wedge u|^2] = 2(m + 2\mathbb{E})(3(2\mathbb{E} + m - 1) + T_{x,u})$$

where $T_{x,u} = |x|^2\Delta_x + |u|^2\Delta_u + 2\langle x, u\rangle \langle \partial_x, \partial_u\rangle$.

Proof. Using the fact that:

$$[\Delta_x, |x \wedge u|^2] = 2|u|^2(2\mathbb{E}_x + m - 1) - 4\langle x, u\rangle \langle u, \partial_x\rangle$$
$$[\Delta_u, |x \wedge u|^2] = 2|x|^2(2\mathbb{E}_u + m - 1) - 4\langle x, u\rangle \langle x, \partial_u\rangle$$

we find the following identity:

$$\begin{aligned}
[\Delta_x\Delta_u, |x \wedge u|^2] &= [\Delta_x, |x \wedge u|^2]\Delta_u + \Delta_x[\Delta_u, |x \wedge u|^2] \\
&= \big(2|u|^2(2\mathbb{E}_x + m - 1) - 4\langle x, u\rangle \langle u, \partial_x\rangle\big)\Delta_u \\
&\quad + \Delta_x\big(2|x|^2(2\mathbb{E}_u + m - 1) - 4\langle x, u\rangle \langle x, \partial_u\rangle\big) \\
&= 2(2\mathbb{E}_x + m - 1)|u|^2\Delta_u + 2(2\mathbb{E}_u + m - 1)|x|^2\Delta_x \\
&\quad + 4(2\mathbb{E}_u + m - 1)(m + 2\mathbb{E}_x) + 8\langle x, u\rangle \langle \partial_x, \partial_u\rangle .
\end{aligned}$$

Moreover

$$\begin{aligned}
\langle \partial_x, \partial_u\rangle |x \wedge u|^2 &= \sum_{i=1}^{m} \partial_{x_i}\big(2u_i|x|^2 - 2\langle x, u\rangle x_i + |x \wedge u|^2\partial_{u_i}\big) \\
&= \langle x, u\rangle (2 - 2m - 2\mathbb{E}_x - 2\mathbb{E}_u) + 2|x|^2\langle u, \partial_x\rangle \\
&\quad + 2|u|^2\langle x, \partial_u\rangle + |x \wedge u|^2\langle \partial_x, \partial_u\rangle
\end{aligned}$$

which leads to

$$[\langle \partial_x, \partial_u\rangle, |x \wedge u|^2] = \langle x, u\rangle (2 - 2m - 2\mathbb{E}_x - 2\mathbb{E}_u) + 2|x|^2\langle u, \partial_x\rangle + 2|u|^2\langle x, \partial_u\rangle .$$

Finally this gives us that

$$\begin{aligned}
&[\langle \partial_x, \partial_u\rangle^2, |x \wedge u|^2] \\
&= \langle \partial_x, \partial_u\rangle [\langle \partial_x, \partial_u\rangle, |x \wedge u|^2] + [\langle \partial_x, \partial_u\rangle, |x \wedge u|^2]\langle \partial_x, \partial_u\rangle \\
&= \langle \partial_x, \partial_u\rangle \langle x, u\rangle (2 - 2m - 2\mathbb{E}_x - 2\mathbb{E}_u) \\
&\quad + \big(\langle x, u\rangle (2 - 2m - 2\mathbb{E}_x - 2\mathbb{E}_u) + 2|x|^2\langle u, \partial_x\rangle + 2|u|^2\langle x, \partial_u\rangle\big)\langle \partial_x, \partial_u\rangle
\end{aligned}$$

$$
\begin{aligned}
&= 2\left(m+\mathbb{E}_x+\mathbb{E}_u\right)\left(1-m-\mathbb{E}_x-\mathbb{E}_u\right)+4\left\langle x,u\right\rangle\left\langle \partial_x,\partial_u\right\rangle\left(2-m-\mathbb{E}_x-\mathbb{E}_u\right)\\
&\qquad -2|x|^2\Delta_x-2|u|^2\Delta_u.
\end{aligned}
$$

Combining everything leaves us with

$$
\begin{aligned}
&[\Delta_w,|x\wedge u|^2]\\
=&2(2\mathbb{E}_x+m)|u|^2\Delta_u+2(2\mathbb{E}_u+m)|x|^2\Delta_x\\
&\qquad +4(2\mathbb{E}_u+m-1)(m+2\mathbb{E}_x)-2\left(m+\mathbb{E}_x+\mathbb{E}_u\right)\left(1-m-\mathbb{E}_x-\mathbb{E}_u\right)\\
&\qquad +8\left\langle x,u\right\rangle\left\langle \partial_x,\partial_u\right\rangle-4\left\langle x,u\right\rangle\left\langle \partial_x,\partial_u\right\rangle\left(2-m-\mathbb{E}_x-\mathbb{E}_u\right)\\
=&2(2\mathbb{E}_x+m)|u|^2\Delta_u+2(2\mathbb{E}_u+m)|x|^2\Delta_x+4(m+\mathbb{E}_x+\mathbb{E}_u)\left\langle x,u\right\rangle\left\langle \partial_x,\partial_u\right\rangle\\
&\qquad +4(2\mathbb{E}_u+m-1)(m+2\mathbb{E}_x)-2\left(m+\mathbb{E}_x+\mathbb{E}_u\right)\left(1-m-\mathbb{E}_x-\mathbb{E}_u\right).
\end{aligned}
$$

To finish the proof we use the fact that $\mathbb{E}=\mathbb{E}_x=\mathbb{E}_u$ to obtain the desired result. □

Remark 6.1.14. Note that there is an alternative interpretation for the operator $T_{x,u}$. Two particular models for the 3-dimensional module $\mathbb{V}_2$ for $\mathfrak{sl}(2)$, as realised in terms of the skew-Euler operators, can be given by

$$
\operatorname{span}_{\mathbb{C}}\big(|x|^2,\langle x,u\rangle,|u|^2\big)\cong\mathbb{V}_2\cong\operatorname{span}_{\mathbb{C}}\big(\Delta_x,\langle\partial_x,\partial_u\rangle,\Delta_u\big)\ .
$$

The unique trivial component $\mathbb{V}_0\subset\mathbb{V}_2\otimes\mathbb{V}_2$, as predicted by the Clebsch-Gordan theorem, is then precisely equal to $\mathbb{C}T_{x,u}$.

Remark 6.1.15. Earlier we mentioned that the Casimir operator of order 2, in the case of two vector variables, was given by:

$$
\frac{1}{4}\mathcal{C}_2=\Delta_{LB}^x+\Delta_{LB}^u+2\left(\left\langle x,u\right\rangle\left\langle \partial_x,\partial_u\right\rangle-\left\langle u,\partial_x\right\rangle\left\langle x,\partial_u\right\rangle+\mathbb{E}_u\right).
$$

Using the connection between Δ_{LB}^x and Δ_x, we can also write this as:

$$
\frac{1}{4}\mathcal{C}_2=T_{x,u}-\mathbb{E}_x(\mathbb{E}_x+m-2)-\mathbb{E}_u(\mathbb{E}_u+m-4)-2\left\langle u,\partial_x\right\rangle\left\langle x,\partial_u\right\rangle.
$$

This means that on $\ker(\left\langle x,\partial_u\right\rangle,\left\langle u,\partial_x\right\rangle)$, the operator $T_{x,u}$ is given by a shifted Casimir operator $\mathcal{C}_2$.

This operator $T_{x,u}$ satisfies a few remarkable properties. First of all, it commutes with the skew-Euler operators and it satisfies a Leibniz-type rule:

Lemma 6.1.16. *For $f,g\in\mathcal{C}^2(\mathbb{R}^{2m},\mathbb{C})$, we have the following identity:*

$$
\begin{aligned}
T_{x,u}\big(fg\big)=(T_{x,u}f)g+f(T_{x,u}g)+2\sum_{j=1}^m\big(|x|^2f_{x_j}g_{x_j}+|u|^2f_{u_j}g_{u_j}\big)\\
+2\left\langle x,u\right\rangle\sum_{j=1}^m(f_{x_j}g_{u_j}+f_{u_j}g_{x_j})
\end{aligned}
$$

where $f_{u_j}:=\partial_{u_j}f$ and $f_{x_j}:=\partial_{x_j}f$.

Moreover, the following lemma shows that the wedge norm is an eigenfunction:

Lemma 6.1.17. *For $n \in \mathbb{Z}$ we have that:*

$$T_{x,u}|x \wedge u|^n = 2n(m+n-3)|x \wedge u|^n.$$

Proof. We know that

$$\begin{aligned}
\Delta_x|x \wedge u|^n &= n(m+n-3)|x \wedge u|^{n-2}|u|^2 \\
\Delta_u|x \wedge u|^n &= n(m+n-3)|x \wedge u|^{n-2}|x|^2
\end{aligned}$$

and we also have that

$$\langle \partial_x, \partial_u \rangle |x \wedge u|^n = n(3-m-n)|x \wedge u|^{n-2} \langle x, u \rangle .$$

Using this we can conclude that:

$$\begin{aligned}
&T_{x,u}|x \wedge u|^n \\
&= |x|^2 n(m+n-3)|x \wedge u|^{n-2}|u|^2 + |u|^2 n(m+n-3)|x \wedge u|^{n-2}|x|^2 \\
&\qquad + 2 \langle x, u \rangle n(3-m-n)|x \wedge u|^{n-2} \langle x, u \rangle \\
&= 2n(m+n-3)|x \wedge u|^{n-2}|x|^2|u|^2 - 2(m+n-3) \langle x, u \rangle^2 |x \wedge u|^{n-2} \\
&= 2n(m+n-3)|x \wedge u|^n.
\end{aligned}$$

□

Note that these are not the only eigenfunctions. We will show that each of the irreducible components of $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$ is an eigenspace for the operator $T_{x,u}$. Combining the previous lemma and the Leibniz-type rule, we obtain the following:

Lemma 6.1.18. *For $n \in \mathbb{Z}$, we have the following operator identity when acting on the kernel of the skew-Euler operators:*

$$[T_{x,u}, |x \wedge u|^n] = 2n|x \wedge u|^n(m+n-3+\mathbb{E}_x+\mathbb{E}_u).$$

Proof. This follows from the Leibniz-type rule, lemma 6.1.17 and straightforward calculations. □

Theorem 6.1.19. *For $i \in \mathbb{Z}^+$, we have the following operator identity when acting on the kernel of the skew-Euler operators:*

$$\begin{aligned}
[\Delta_w, |x \wedge u|^{2i}] = |x \wedge u|^{2i-2} 2i(m+2\mathbb{E}+2i-2) \times& \\
\Big((2i+1)(2\mathbb{E}+m-3+2i) + T_{x,u}\Big)&.
\end{aligned}$$

Proof. Repeatedly using lemma 6.1.13 we can conclude that:

$$[\Delta_w, |x \wedge u|^{2i}] = \sum_{n=0}^{i-1} |x \wedge u|^{2n} 2(m + 2\mathbb{E})(3(2\mathbb{E} + m - 1) + T_{x,u})|x \wedge u|^{2i-2n-2}.$$

Using lemma 6.1.18 we can rewrite this as:

$$\begin{aligned}
&\sum_{n=0}^{i-1} |x \wedge u|^{2n} 2(m + 2\mathbb{E})|x \wedge u|^{2i-2n-2} \\
&\qquad \times \Big(3(2(\mathbb{E} + 2i - 2n - 2) + m - 1) \\
&\qquad\qquad + 2(2i - 2n - 2)(m + 2i - 2n - 5 + 2\mathbb{E}) + T_{x,u}\Big) \\
=&|x \wedge u|^{2i-2} \sum_{n=0}^{i-1} 2(m + 2\mathbb{E} + 4i - 4n - 4) \\
&\qquad \times \Big(3(2\mathbb{E} + 4i - 4n - 4 + m - 1) \\
&\qquad\qquad + 2(2i - 2n - 2)(m + 2i - 2n - 5 + 2\mathbb{E}) + T_{x,u}\Big).
\end{aligned}$$

Furthermore we need two identities. The first one is:

$$\begin{aligned}
\sum_{n=0}^{i-1} 2(m + 2\mathbb{E} + 4i - 4n - 4) &= 2i(m + 2\mathbb{E} + 4i - 4) - 4i(i - 1) \\
&= 2i(m + 2\mathbb{E} + 2i - 2)
\end{aligned}$$

but the second one requires more work. Using tedious, but straightforward calculations one can show that:

$$\begin{aligned}
&\sum_{n=0}^{i-1} 2(m + 2\mathbb{E} + 4i - 4n - 4)\Big(3(2\mathbb{E} + 4i - 4n - 4 + m - 1) \\
&\qquad\qquad + 2(2i - 2n - 2)(m + 2i - 2n - 5 + 2\mathbb{E})\Big) \\
=&2i(m + 2\mathbb{E} + 2i - 2)(2i + 1)(2\mathbb{E} + m - 3 + 2i)
\end{aligned}$$

Finally, using these two identities yields us the desired result. □

The next lemma shows that the components of $\mathcal{H}_k^w(\mathbb{R}^{2m}, \mathbb{C})$ are eigenspaces for the operator $T_{x,u}$, albeit with different eigenvalues (the fact that they are different allows us to decompose $\mathcal{H}_k^w$ into $\mathfrak{so}(m)$-irreducible components):

Lemma 6.1.20. *Let* $H(x, u) \in \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C})$*, then*

$$T_{x,u} R_j H(x, u) = 2j(m - 4 - 2j + 2\mathbb{E}_u) R_j H(x, u)\ .$$

Proof. To prove this lemma we look at the commutator between $T_{x,u}$ and $|u|^{2j}$ when it acts on $\ker(\Delta_x, \Delta_u, \langle x, \partial_u\rangle)$:

$$\begin{aligned}
[T_{x,u}, |u|^{2j}] &= |u|^2[\Delta_u, |u|^{2j}] + 2\langle x, u\rangle [\langle \partial_x, \partial_u\rangle, |u|^{2j}] \\
&= 2j|u|^{2j}(m + 2j - 2 + 2\mathbb{E}_u) + 4j\langle x, u\rangle \langle u, \partial_x\rangle |u|^{2j-2} \\
&= 2j|u|^{2j}(m + 2j - 2 + 2\mathbb{E}_u) + 4j\left(\langle u, \partial_x\rangle \langle x, u\rangle - |u|^2\right)|u|^{2j-2} \\
&= 2j|u|^{2j}(m + 2j - 4 + 2\mathbb{E}_u) + 4j\langle u, \partial_x\rangle \langle x, u\rangle .
\end{aligned}$$

Using the fact that $T_{x,u}$ commutes with π we can conclude that:

$$\begin{aligned}
T_{x,u}R_jH &= T_{x,u}\pi|u|^{2j}H \\
&= \pi T_{x,u}|u|^{2j}H \\
&= \pi[T_{x,u}, |u|^{2j}]H \\
&= \pi\left(2j|u|^{2j}(m + 2j - 4 + 2\mathbb{E}_u) + 4j\langle u, \partial_x\rangle \langle x, u\rangle\right)H \\
&= \pi 2j|u|^{2j}(m + 2j - 4 + 2\mathbb{E}_u)H \\
&= 2j(m + 2j - 4 + 2(\mathbb{E}_u - 2j))\pi|u|^{2j}H \\
&= 2j(m - 2j - 4 + 2\mathbb{E}_u)R_jH.
\end{aligned}$$

This proves the assertion. $\square$

We are finally ready to prove the surjectivity of the operator Δ_w on the space $\mathcal{P}^{\mathrm{SL}(2)}_{k-2}(\mathbb{R}^{2m}, \mathbb{C})$. Let us consider an arbitrary polynomial

$$P_{k-2}(x, u) \in \mathcal{P}^{\mathrm{SL}(2)}_{k-2}(\mathbb{R}^{2m}, \mathbb{C})$$

which can be written as a combination of polynomials of the form $|x \wedge u|^{2i}R_jH$ (with $H \in \mathcal{H}_{k-2-2i,k-2-2i-2j}(\mathbb{R}^{2m}, \mathbb{C})$). Looking at each of these generators individually, using theorem 6.1.19 and lemma 6.1.20, lets us conclude that

$$\begin{aligned}
&\Delta_w|x \wedge u|^{2i+2}R_jH \\
=&|x \wedge u|^{2i}(2i + 2)(m + 2\mathbb{E}_x + 2i)\Big((2i + 3)(2\mathbb{E}_x + m - 1 + 2i) + T_{x,u}\Big)R_jH \\
=&|x \wedge u|^{2i}(2i + 2)(m + 2\mathbb{E} + 2i)(2i + 2j + 3)(2\mathbb{E} + m - 1 + 2i - 2j)R_jH
\end{aligned}$$

where we have written $\mathbb{E}_u$ and $\mathbb{E}_x$ as $\mathbb{E}$ because it acts on a polynomial with equal degrees of homogeneity. From the restrictions on i and j we can conclude that:

$$\Delta_w|x \wedge u|^2P_{k-2}(x, u) = c_{i,j}(\mathbb{E})|x \wedge u|^{2i}R_jH$$

where $c_{i,j}(\mathbb{E}) \neq 0$ and thus we have that Δ_w is surjective on $\mathcal{P}^{\mathrm{SL}(2)}_{k-2}(\mathbb{R}^{2m}, \mathbb{C})$. This proves the following theorem:

Theorem 6.1.21. *Let $k \in \mathbb{Z}^+$ and $m \geq 4$. We have the following decomposition:*

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} \mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C})$$

where $\mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C})$ decomposes as:

$$\mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} R_j \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

Remark 6.1.22. Note that the decomposition in theorem 6.1.21 is orthogonal with respect to the Fischer inner product

$$[P(x,u), Q(x,u)]_F = \overline{P(\partial_x, \partial_u)} Q(x,u) \Big|_{x=u=0}.$$

Indeed, from theorem 6.1.19 we can conclude that for $i_1 \neq i_2$

$$|x \wedge u|^{2i_1} \mathcal{H}_{k-2i_1}^w(\mathbb{R}^{2m}, \mathbb{C}) \perp_F |x \wedge u|^{2i_2} \mathcal{H}_{k-2i_2}^w(\mathbb{R}^{2m}, \mathbb{C})$$

because

$$[|x \wedge u|^2 P(x,u), Q(x,u)]_F = [P(x,u), \Delta_w Q(x,u)]_F.$$

All that remains to be shown is that all components in $\mathcal{H}_k^w(\mathbb{R}^{2m}, \mathbb{C})$ are also orthogonal w.r.t. the Fischer inner product. Let therefore $a, b \in \mathbb{Z}^+$, with $a \neq b$, such that $2a \leq k, 2b \leq k$, then we want to show that

$$[R_a H_{k,k-2a}, R_b H_{k,k-2b}]_F = 0.$$

One can show this by looking at the operator $T_{x,u}$ and lemma 6.1.20 which states that

$$T_{x,u} R_a H_{k,k-2a} = 2a(m - 4 - 2a + 2k) R_a H_{k,k-2a} := \lambda_a R_a H_{k,k-2a}.$$

One of the properties of $T_{x,u}$ is the fact that it is a self-dual operator w.r.t. the Fischer inner product, which leads to:

$$\begin{aligned} \lambda_a [R_a H_{k,k-2a}, R_b H_{k,k-2b}]_F &= [T_{x,u} R_a H_{k,k-2a}, R_b H_{k,k-2b}]_F \\ &= [R_a H_{k,k-2a}, T_{x,u} R_b H_{k,k-2b}]_F \\ &= \lambda_b [R_a H_{k,k-2a}, R_b H_{k,k-2b}]_F. \end{aligned}$$

Note that $\lambda_a \neq \lambda_b$ if $a \neq b$ and from this we can conclude that

$$(\lambda_a - \lambda_b)[R_a H_{k,k-2a}, R_b H_{k,k-2b}]_F = 0$$

which proves the desired result.

Similar to the classical Fischer decomposition one can also look at a graphical representation, by decomposing the space of polynomials

$$\mathcal{P}^{\mathrm{SL}(2)}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^m, \mathbb{C})$$

into irreducible modules under the action of the Spin group. This yields:

$$\begin{array}{ccccccc}
\mathcal{P}_0^{\mathrm{SL}(2)} & \mathcal{P}_1^{\mathrm{SL}(2)} & \mathcal{P}_2^{\mathrm{SL}(2)} & \mathcal{P}_3^{\mathrm{SL}(2)} & \mathcal{P}_4^{\mathrm{SL}(2)} & \mathcal{P}_5^{\mathrm{SL}(2)} & \cdots \\
|| & || & || & || & || & || & \\
 & & & & & & \\
\mathcal{H}_0^w & & |x \wedge u|^2 \mathcal{H}_0^w & & |x \wedge u|^4 \mathcal{H}_0^w & & \cdots \\
 & \mathcal{H}_1^w & \oplus & |x \wedge u|^2 \mathcal{H}_1^w & \oplus & |x \wedge u|^4 \mathcal{H}_1^w & \cdots \\
 & & \mathcal{H}_2^w & \oplus & |x \wedge u|^2 \mathcal{H}_2^w & \oplus & \cdots \\
 & & & \mathcal{H}_3^w & \oplus & |x \wedge u|^2 \mathcal{H}_3^w & \cdots \\
 & & & & \mathcal{H}_4^w & \oplus & \cdots \\
 & & & & & \mathcal{H}_5^w & \cdots \\
 & & & & & & \ddots
\end{array}$$

When one restricts to $|x \wedge u|^2 = 1$ one finds functions on the Grassmannian $\mathrm{Gr}(2, m)$, see section 4.2.2. However, this restriction is not an isomorphism, each row in the diagram is mapped to the same space, unless one works with a subspace of $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^m, \mathbb{C})$. Similar to proposition 2.3.11, the previous diagram reduces to the diagonal when restricting to $|x \wedge u|^2 = 1$, allowing for a characterisation of L^2-functions on the Grassmann manifold $\mathrm{Gr}(2, m)$:

Proposition 6.1.23. *Let $m \in \mathbb{Z}^+$ with $m \geq 4$ then*

$$L^2(\mathrm{Gr}_o(2, m)) \cong \bigoplus_{k \in \mathbb{Z}^+} \mathcal{H}_k^w(\mathbb{R}^{2m}, \mathbb{C}).$$

Proof. Assume $m > 4$, the case $m = 4$ is treated similarly. We will use the Peter-Weyl theorem which states that:

$$L^2(G/H) \cong \bigoplus_{\lambda} \mathbb{V}_\lambda^{\otimes m(\lambda)}$$

where we sum over all irreducible finite-dimensional G-representations $\mathbb{V}_\lambda$ and $m(\lambda)$ is equal to the multiplicity of the trivial representation in the decomposition $\lambda\big|_H$. This means that we are interested in the following branching problem

$$\lambda\Big|_{\mathrm{SO}(2)\times\mathrm{SO}(m-2)}^{\mathrm{SO}(m)} = \bigoplus_{\rho} \mathbb{V}_\rho^{\eta_\lambda(\rho)}.$$

There are two possible approaches: one can either use the branching rules found in [77] (because $m(\lambda) = \eta_\lambda((0))$) or one can use the invariant polynomials that where constructed in theorem 4.1.27, we will pursue the latter approach. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ then one can immediately conclude that $\lambda_i = 0$ if $i \geq 3$ because

$$\lambda\Big|_{\mathrm{SO}(m-2)}^{\mathrm{SO}(m)} = \left(\lambda\Big|_{\mathrm{SO}(2)\times\mathrm{SO}(m-2)}^{\mathrm{SO}(m)}\right)\Big|_{\mathrm{SO}(m-2)}^{\mathrm{SO}(2)\times\mathrm{SO}(m-2)}$$

and the decomposition on the left hand side cannot contain the trivial representation unless the third entry of λ is equal to zero, see theorem 1.4.1. We thus only have to consider the weights $\lambda = (\ell, k)$ and for these we have a polynomial model $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})$. This means that the multiplicity of the trivial representation in $\lambda\big|_{\mathrm{SO}(2)\times\mathrm{SO}(m-2)}$ is equal to the dimension of the subspace $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})^{\mathrm{SO}(2)\times\mathrm{SO}(m-2)}$ of simplicial harmonics that are $\mathrm{SO}(2)\times\mathrm{SO}(m-2)$-invariant. In particular, these polynomials are also $\mathrm{SO}(m-2)$-invariant and from theorem 4.1.27 we know that

$$\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})^{\mathrm{SO}(m-2)} \cong \mathcal{P}_{l-k}(\mathbb{R}^2, \mathbb{C}),$$

where the isomorphism is explicitly given by:

$$P_{\ell-k}(u_1, u_2) \mapsto P_{\ell-k}(\partial_{u_1}, \partial_{u_2})|x \wedge u|^{\ell} J_{\ell}^{\frac{m-3}{2}}(\tau, \sigma^2).$$

The $\mathrm{SO}(2)$-transformational behaviour is solely determined by $P_{\ell-k}(u_1, u_2)$ and such invariants can only be powers of the Laplacian $\Delta_{\underline{u}_2}$. This is only possible if $\ell - k$ is even, in which case there exists a unique one, which leads to

$$m\big((\ell, k)\big) = \begin{cases} 1 & \ell - k \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that

$$L^2(\mathrm{Gr}_o(2, m)) \cong \bigoplus_{k\in\mathbb{Z}^+} \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} (k, k-2i)$$

which, along with theorem 6.1.21, finishes the proof. □

Remark 6.1.24. In section 4.2.2 we introduced two ways of looking at functions on the Grassmannian $\mathrm{Gr}_o(2, m)$ and claimed that they are equivalent. While restricting simplicial harmonic polynomials to the Stiefel manifold is a bijection, i.e.

$$\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C}) \cong \mathcal{H}_{k,k-2j}(V_{m,2}, \mathbb{C}),$$

see e.g. [66], it is a priori not clear that it maps $\mathrm{SL}(2)$-invariant polynomials to $\ker(\langle x, \partial_u\rangle - \langle u, \partial_x\rangle)$, which is equivalent to $\mathrm{SO}(2)$-invariance. We will show that, on the Stiefel manifold, the embedding of the space of simplicial harmonics $\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ is also written in terms of the operator R_j, i.e.

$$\mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C}) \xrightarrow{R_j} \mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) \xrightarrow{\text{Res.}} \mathcal{C}^{\infty}(V_{m,2}, \mathbb{C})^{\mathrm{SO}(2)}.$$

First of all, we take a closer look at the SO(2)-invariance and for this we introduce another realisation of $\mathfrak{sl}(2) \cong \mathrm{Span}(X, Y, H)$ by defining:

$$\begin{aligned} 2X_N &:= X + Y + iH \\ 2Y_N &:= X + Y - iH \\ H_N &:= [X_N, Y_N] = i(X - Y). \end{aligned}$$

We then have that

$$\mathfrak{sl}(2) \cong \mathrm{Span}(X_N, Y_N, H_N) = \mathfrak{sl}_N(2).$$

Moreover, given an arbitrary $\mathfrak{sl}(2)$-irreducible representation

$$\mathbb{V}_n = \mathrm{Span}(v_0, \dots, v_n)$$

where

$$Xv_0 = 0 \qquad Yv_i = (1 - \delta_{in})v_{i+1} \qquad Hv_i = (n - 2i)v_i,$$

it is clear that $\mathbb{V}_n$ is also a $\mathfrak{sl}_N(2)$-representation (in fact, it will still be irreducible). In particular, we are interested in the zero weight space of H_N, as elements of this weight space are SO(2)-invariant. The latter space is one-dimensional with basis

$$w_0 = \sum_{j=0}^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2} - j\right)}{2^{2j} j! \Gamma\left(\frac{n-1}{2}\right)} Y^{2j} v_0$$

if n is even, and zero-dimensional otherwise. Consider a simplicial harmonic polynomial $H_{k,k-2j}(x, u) \in \mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C})$ then we can consider the following representation $\mathbb{V}_{2j}$ of the $\mathfrak{sl}(2)$ realisation in terms of the skew-Euler operators from remark 6.1.8:

$$\mathbb{V}_{2j} = \bigoplus_{i=0}^{2j} \mathbb{C} \left\langle u, \partial_x \right\rangle^i H_{k,k-2j}.$$

We have thus found a way to embed the space $\mathcal{H}_{k,k-2i}(\mathbb{R}^{2m}, \mathbb{C})$ into

$$\ker(\langle x, \partial_u \rangle - \langle u, \partial_x \rangle)$$

by means of:

$$H_{k,k-2j}(x, u) \mapsto \sum_{i=0}^{j} \frac{\Gamma\left(j - i + \frac{1}{2}\right)}{2^{2i} i! \Gamma\left(j - \frac{1}{2}\right)} \left\langle u, \partial_x \right\rangle^{2i} H_{k,k-2j}(x, u).$$

Restricting this polynomial to the Stiefel manifold $V_{m,2}$ yields the desired map

$$\varphi_j : \mathcal{H}_{k,k-2j}(\mathbb{R}^{2m}, \mathbb{C}) \to \mathcal{C}^\infty(V_{m,2}, \mathbb{C})^{\mathrm{SO}(2)}.$$

On the other hand, one could also have started by applying the operator R_j to $H_{k,k-2j}(x,u)$ and restricting afterwards. From lemma 6.1.5 we know that this operator consists of the action of powers of $\langle u, \partial_x \rangle$ and a multiplication with polynomials of the form $\langle x, \partial_u \rangle^i |u|^{2j}$. From lemma 6.1.6, we can see that the latter polynomial will be zero if i is odd because $\langle x, u \rangle = 0$ on $V_{m,2}$, for even i we have find the following expression:

$$\left(\langle x, \partial_u \rangle^{2i} |u|^{2j} \right)\Big|_{V_{m,2}} = \frac{(-1)^i (-2i)^{(2i)} (-j)^{(i)}}{i!} = \frac{(2i)! j!}{(j-i)! i!}.$$

Combining this with lemma 6.1.5 yields

$$\begin{aligned} (R_j H_{k,k-2j})\Big|_{V_{m,2}} &= \left(\sum_{i=0}^{j} \frac{2^{2j-2i} \Gamma\left(j - i + \frac{1}{2}\right)) j!}{(2j+1)! i! \Gamma\left(\frac{1}{2}\right)} \langle u, \partial_x \rangle^{2i} H_{k,k-2j}(x,u) \right)\Bigg|_{V_{m,2}} \\ &= \frac{j!}{(2j+1)!} \frac{\Gamma\left(j - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \varphi_j(H_{k,k-2j}), \end{aligned}$$

which proves the assertion.

6.1.1 The Cayley-Laplace operator

The classical Laplace operator Δ_x could be written as

$$\Delta_x = \sum_{j=1}^{m} \partial_{x_j}^2 = \partial_r^2 + (m-1)\partial_r + \frac{1}{r^2} \Delta_{LB}^x,$$

whereby ∂_r denotes derivation with respect to the radius $r = |x|$ (for $x \in \mathbb{R}^m$), and with Δ_{LB} the Laplace-Beltrami operator. The latter is a purely angular operator, in that it acts trivially on radial functions $f(r)$. Put differently, this operator acts solely on the angular coordinates on the sphere S^{m-1}. Our goal is to extend these decompositions and its related properties to the Cayley-Laplace operator, where part of the inspiration for our calculations comes from a paper by B. Rubin [71]. A function $f(x,u)$ is called 'radial' if it depends on $(x,u) \in \mathbb{R}^{m \times 2}$ through the combination $(x,u)^T(x,u)$ only, and we introduce the following coordinates t, b, a:

$$\begin{pmatrix} t & a \\ a & b \end{pmatrix} := (x,u)^T(x,u) = \begin{pmatrix} |x|^2 & \langle x, u \rangle \\ \langle x, u \rangle & |u|^2 \end{pmatrix}.$$

Recall that matrices $M = (x,u) \in \mathbb{R}^{m \times 2}$ of rank 2 had a polar decomposition $M = VR^{\frac{1}{2}}$, where $R = M^T M \in \mathcal{P}_2$, with $\mathcal{P}_2$ the open convex cone of positive definite real symmetric matrices. In other words, the coordinates (t, b, a), referring to 'top, bottom and anti-diagonal' respectively, are the 3 (independent) coordinates used to characterise elements in $\mathcal{P}_2$.

In [71] it was already found that for *radial* functions $f(x,u) = f_r(t,b,a)$ on $\mathbb{R}^{m \times 2}$, the action of the operator Δ_w reduces to

$$\Delta_w f(x,u) = L f_r(t,b,a) = 16(tb - a^2)^{\frac{3-m}{2}} D (tb - a^2)^{\frac{m-1}{2}} D f_r(t,b,a) ,$$

with $D = \partial_t \partial_b - \frac{1}{4}\partial_a^2$. In remark 4.2.10, we saw that the variables

$$\frac{X_{pq}}{|x \wedge u|} := \Omega_{pq} \ ,$$

play the same role as the angular variables in the spherical case. The specific result that we are after here is the following: we would like to show that $D\Omega_{pq} = 0$, as this implies that the action of Δ_w on functions $f(\Omega)$ reduces to the non-radial part (just like the action of the Laplace operator on functions $f(\omega)$ reduces to the action of Laplace-Beltrami). In the case of one vector variable $x \in \mathbb{R}^m$, this amounts to showing that

$$\partial_r \omega_j = \partial_r \frac{x_j}{r} = 0 \ .$$

To do so, we will rely on the fact that for functions depending on $\rho = r^2 = |x|^2$ (this variable ρ is the one that generalises to $|x \wedge u|^2 = tb - a^2$), one has that

$$\partial_{x_j} = \frac{\partial \rho}{\partial x_j}\partial_\rho = 2x_j \partial_\rho \ \Rightarrow \ \mathbb{E}_x = \sum_j x_j \partial_{x_j} = 2\rho\partial_\rho \ .$$

Using this operator identity, it is then easily verified that $\partial_\rho \omega_j = 0$ (for all j). In order to mimic this approach in the case of 2 vector variables, we first note that on radial functions $f_r(t, b, a)$:

$$\begin{aligned}
\partial_{x_i} &= \frac{\partial a}{\partial x_i}\partial_a + \frac{\partial t}{\partial x_i}\partial_t = u_i \partial_a + 2x_i \partial_t \\
\partial_{u_j} &= \frac{\partial a}{\partial u_j}\partial_a + \frac{\partial b}{\partial u_j}\partial_b = x_j \partial_a + 2u_j \partial_b \ ,
\end{aligned}$$

from which we get the following identities:

$$\begin{aligned}
\langle u, \partial_x \rangle &= 2a\partial_t + b\partial_a \\
\langle x, \partial_u \rangle &= 2a\partial_b + t\partial_a \\
\mathbb{E}_x &= 2t\partial_t + a\partial_a \\
\mathbb{E}_u &= 2b\partial_b + a\partial_a
\end{aligned}$$

By suitable linear combinations of these operator identities, one easily finds that

$$\begin{aligned}
2|x \wedge u|^2 \partial_t &= |u|^2 \mathbb{E}_x - \langle x, u \rangle \langle u, \partial_x \rangle \\
2|x \wedge u|^2 \partial_b &= |x|^2 \mathbb{E}_u - \langle x, u \rangle \langle x, \partial_u \rangle \\
|x \wedge u|^2 \partial_a &= |u|^2 \langle x, \partial_u \rangle - \langle x, u \rangle \mathbb{E}_u = |x|^2 \langle u, \partial_x \rangle - \langle x, u \rangle \mathbb{E}_x \ .
\end{aligned}$$

In this form, it is absolutely not trivial to see that the variables $(|x|^2, |u|^2, \langle x, u \rangle)$ are independent. However, using the formulas above, one can indeed verify that for instance $\partial_a b = 0 = \partial_a t$, and also

$$\partial_a a = \frac{1}{|x \wedge u|^2}\big(|u|^2 \langle x, \partial_u \rangle - \langle x, u \rangle \mathbb{E}_u\big)\langle x, u \rangle = \frac{|x \wedge u|^2}{|x \wedge u|^2} = 1 \ .$$

Similar calculations confirm that $\partial_t b = \partial_b t = 0 = \partial_t a = \partial_b a$ and $\partial_t t = 1 = \partial_b b$. Also, although this requires a lengthier calculation, one can show that (for instance) $\partial_t \partial_b = \partial_b \partial_t$, *when acting on an appropriate class of functions*. One for instance has that

$$
\begin{aligned}
4\partial_t\partial_b &= \frac{1}{|x\wedge u|^2}\Big(|u|^2\mathbb{E}_x - \langle x,u\rangle\langle u,\partial_x\rangle\Big)\frac{1}{|x\wedge u|^2}\Big(|x|^2\mathbb{E}_u - \langle x,u\rangle\langle x,\partial_u\rangle\Big)\\
&= \frac{1}{|x\wedge u|^4}\Big(|u|^2(\mathbb{E}_x-2) - \langle x,u\rangle\langle u,\partial_x\rangle\Big)\Big(|x|^2\mathbb{E}_u - \langle x,u\rangle\langle x,\partial_u\rangle\Big)\\
&= \frac{1}{|x\wedge u|^4}\Big(|x|^2|u|^2\mathbb{E}_u\mathbb{E}_x - |u|^2\langle x,u\rangle\langle x,\partial_u\rangle\mathbb{E}_x\\
&\qquad - \langle x,u\rangle\langle u,\partial_x\rangle|x|^2\mathbb{E}_u + \langle x,u\rangle\langle u,\partial_x\rangle\langle x,u\rangle\langle x,\partial_u\rangle\Big)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
4\partial_b\partial_t &= \frac{1}{|x\wedge u|^2}\Big(|x|^2\mathbb{E}_u - \langle x,u\rangle\langle x,\partial_u\rangle\Big)\frac{1}{|x\wedge u|^2}\Big(|u|^2\mathbb{E}_x - \langle x,u\rangle\langle u,\partial_x\rangle\Big)\\
&= \frac{1}{|x\wedge u|^4}\Big(|x|^2(\mathbb{E}_u-2) - \langle x,u\rangle\langle x,\partial_u\rangle\Big)\Big(|u|^2\mathbb{E}_x - \langle x,u\rangle\langle u,\partial_x\rangle\Big)\\
&= \frac{1}{|x\wedge u|^4}\Big(|x|^2|u|^2\mathbb{E}_u\mathbb{E}_x - |x|^2\langle x,u\rangle\langle u,\partial_x\rangle\mathbb{E}_u\\
&\qquad - \langle x,u\rangle\langle x,\partial_u\rangle|u|^2\mathbb{E}_x + \langle x,u\rangle\langle x,\partial_u\rangle\langle x,u\rangle\langle u,\partial_x\rangle\Big)\ .
\end{aligned}
$$

Subtracting both expressions, one arrives at

$$
4[\partial_t,\partial_b] = \frac{\langle x,u\rangle}{|x\wedge u|^4}\Big((\mathbb{E}_x-\mathbb{E}_u) + |u|^2\langle x,\partial_u\rangle - |x|^2\langle u,\partial_x\rangle\Big)\ ,
$$

which means that these operators commute when acting on

$$
f(x,u) \in \ker\big(\langle u,\partial_x\rangle, \langle x,\partial_u\rangle\big)\ .
$$

We hereby used the fact that the commutator of these operators is equal to $(\mathbb{E}_x - \mathbb{E}_u)$, which also appears in the expression for $[\partial_t,\partial_b]$. Similarly, one for instance finds that

$$
2[\partial_t,\partial_a] = 0 = 2[\partial_b,\partial_a].
$$

In the end, this allow us to conclude that

$$
D\Omega_{pq} = \left(\partial_t\partial_b - \frac{1}{4}\partial_a^2\right)\frac{X_{pq}}{|x\wedge u|} = 0\ ,
$$

for the simple reason that both operators $\langle u,\partial_x\rangle$ and $\langle x,\partial_u\rangle$ act trivially on X_{pq} for all $p \neq q$, and the variable Ω_{pq} is homogeneous of degree $(0,0)$ in (x,u)

which means that also both Euler operators act trivially. This even allows us to conclude that

$$L\Big(\varphi(|x \wedge u|)\psi(\Omega)\Big) = \big(L\varphi(|x \wedge u|)\big)\psi(\Omega)\ ,$$

whereby Ω stands for $(\Omega_{pq})_{p<q}$. Moreover, from

$$D|x \wedge u|^k = \frac{k}{k+1}4|x \wedge u|^{k-2}$$

we find that:

$$L|x \wedge u|^k = k(k+1)(m+k-3)(m+k-2)|x \wedge u|^{k-2},$$

which coincides with what we found in theorem 6.1.19. If we now write

$$\frac{\Delta_{LB}^w}{|x \wedge u|^2} = \Delta_w - L$$

then this allows us to define an analogue of the classical spherical harmonics on the Grassmannian $\mathrm{Gr}_o(2,m)$:

Theorem 6.1.25. *Let $k \in \mathbb{Z}^+$, $m \geq 4$ and define*

$$\lambda_{k,m} := -k(k+1)(m+k-3)(m+k-2),$$

then

$$\mathcal{H}_k^w(\mathrm{Gr}_o(2,m),\mathbb{C}) = \{f : \mathrm{Gr}_o(2,m) \to \mathbb{C} \mid \Delta_w^{LB}|x \wedge u|^k f = \lambda_{k,m}|x \wedge u|^k f\},$$

where the space on the left hand side is defined as the restriction of polynomials in $\mathcal{H}_k^w(\mathbb{R}^{2m},\mathbb{C})$ to the Grassmann manifold.

Proof. By definition,

$$f \in \mathcal{H}_k^w(\mathrm{Gr}_o(2,m),\mathbb{C}) \iff |x \wedge u|^k f \in \ker \Delta_w$$

and

$$\begin{aligned} 0 = |x \wedge u|^2\Delta_w|x \wedge u|^k f &= \left(\Delta_{LB}^w + |x \wedge u|^2 L\right)|x \wedge u|^k f \\ &= \left(\Delta_{LB}^w + k(k+1)(m+k-3)(m+k-2)\right)|x \wedge u|^k f, \end{aligned}$$

from which the desired result follows. ☐

Remark 6.1.26. Note that at this point it is not clear whether or not Δ_{LB}^w does indeed commute with $|x \wedge u|$ as it is not a priori clear that it only depends on angular operators. For this a full decomposition of the Cayley-Laplace operator in polar coordinates is needed, which would then also be connected to Casimir operators on the Grassmann manifold $\mathrm{Gr}(2,m)$.

6.2 Spinor refinement

Let $m > 4$, similar to the classical case, the decomposition from theorem 6.1.21 can be refined by considering functions which take values in the aforementioned (Weyl) spinor spaces. In order to investigate the decomposition of the space $\mathcal{P}_k(\mathbb{R}^{2m}, \mathbb{S}^{\pm})^{\mathrm{SL}(2)}$ (note that there is no action on the right for $\mathbb{S}^{\pm}$) into irreducible components for the action of the orthogonal Lie algebra $\mathfrak{so}(m)$, we can start from the scalar Fischer decomposition and use properties of the tensor product:

$$\begin{aligned}\mathcal{P}_k(\mathbb{R}^{2m}, \mathbb{S}^{\pm})^{\mathrm{SL}(2)} &:= \mathcal{P}_k(\mathbb{R}^{2m}, \mathbb{C})^{\mathrm{SL}(2)} \otimes \mathbb{S}^{\pm} \\ &\cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^{\pm}.\end{aligned}$$

The tensor products $\mathcal{H}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^{\pm})$ can for all $\ell \geq k \in \mathbb{Z}^+$ be expressed in terms of simplicial monogenics, see proposition 2.4.5. This means that the spaces occurring on the right-hand side of the isomorphism are of the following type:

$$\begin{array}{rl} (k-2i, k-2i-2j)'_{\pm} & \quad (k-2i-1, k-2i-2j)'_{\mp} \\ (k-2i, k-2i-2j-1)'_{\mp} & \quad (k-2i-1, k-2i-2j-1)'_{\pm} \end{array}$$

Note that the parity changes occur whenever the entries inside the brackets have *different* parities (as positive integers), or whenever the difference $\lambda_1 - \lambda_2 = 1$ in $\mathbb{Z}_2$ (λ_1 and λ_2 being the weight entries). Moreover, it is also clear from the abstract tensor product above and theorem 6.1.9 that the decomposition in the spinor-valued case is also multiplicity-free. This leads us to the following theorem (again merely an isomorphism at this point):

Theorem 6.2.1. *For $k \in \mathbb{Z}^+$ and $m > 4$ one has that*

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}) \cong \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-i} \mathcal{S}_{k-i,k-i-j}(\mathbb{R}^{2m}, \mathbb{S}^{\sigma}).$$

The sign $\sigma \in \mathbb{Z}_2$ is hereby determined according to the following rule: for a space $\mathcal{S}_{a,b}(\mathbb{R}^{2m}, \mathbb{S}^{\sigma})$ one has that $\sigma = \pm 1$ if $(a-b) = 0 \in \mathbb{Z}_2$. The parity change (i.e. $\sigma = \mp 1$) occurs when $(a-b) = 1 \in \mathbb{Z}_2$.

Using the extremal projection operator π from (6.1), one can turn this into an equality by defining $R'_j := \pi u^j$. Note that this operator is vector-valued for j odd and $R'_{2j} = (-1)^j R_j$. Completely similar to R_j one can find a more explicit expression for R'_j and show that it is non-trivial:

Proposition 6.2.2. *Let $k, j \in \mathbb{Z}^+$ with $j \leq k$. The operator R'_j, when acting on $\mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^{\sigma})$, can be written as*

$$R'_j = \sum_{i=0}^{j} \frac{(-1)^i (j-i)!}{(j+1)! i!} \left(\langle x, \partial_u \rangle^i u^j \right) \langle u, \partial_x \rangle^i$$

and is non-trivial on $\mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^{\sigma})$.

Proof. Analogue to the proof of lemma 6.1.5 and proposition 6.1.7. $\square$

We then get that

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m},\mathbb{S}^{\pm}) = \bigoplus_{i=0}^{k}(x\wedge u)^i \bigoplus_{j=0}^{k-i} R'_j \mathcal{S}_{k-i,k-i-j}(\mathbb{R}^{2m},\mathbb{S}^{\sigma}).$$

The following operator, which factorizes the operator Δ_w (and thus plays the same role as the Dirac operator in the classic case), will be crucial:

Definition 6.2.3. We define the operator ∂_w as:

$$\partial_w := \partial_x \wedge \partial_u = \partial_x\partial_u + \langle \partial_x, \partial_u\rangle .$$

Classically, this operator is also defined as

$$\partial_w = \frac{1}{2}(\partial_x\partial_u - \partial_u\partial_x),$$

representing the anti-symmetric part of tensors. Moreover, we define the following subspace of $\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m},\mathbb{S}^{\pm})$:

$$\mathcal{M}_k^w(\mathbb{R}^{2m},\mathbb{S}^{\pm}) := \left\{P_k(x,u)\in \mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m},\mathbb{S}^{\pm}) : \partial_w P_k(x,u) = 0\right\}.$$

This space obviously generalises the role of the spaces $\mathcal{M}_k(\mathbb{R}^m,\mathbb{S}^{\pm})$.

This operator factorises the Cayley-Laplace operator $(\partial_w)^2 = -\Delta_w$ and the operator ∂_w commutes with the skew-Euler operators. Indeed, one has that

$$\begin{aligned}
\partial_w^2 &= \partial_x\partial_u\partial_x\partial_u + 2\langle\partial_x,\partial_u\rangle\partial_x\partial_u + \langle\partial_x,\partial_u\rangle^2\\
&= \partial_x(-\partial_x\partial_u - 2\langle\partial_x,\partial_u\rangle)\partial_u + 2\langle\partial_x,\partial_u\rangle\partial_x\partial_u + \langle\partial_x,\partial_u\rangle^2\\
&= -\Delta_x\Delta_u + \langle\partial_x,\partial_u\rangle^2 .
\end{aligned}$$

While it is clear that

$$\mathcal{M}_k^w(\mathbb{R}^{2m},\mathbb{S}^{\pm}) \subset \mathcal{H}_k^w(\mathbb{R}^{2m},\mathbb{C})\otimes\mathbb{S}^{\pm},$$

we will prove that

$$\mathcal{H}_k^w(\mathbb{R}^{2m},\mathbb{C})\otimes\mathbb{S}^{\pm} = \mathcal{M}_k^w(\mathbb{R}^{2m},\mathbb{S}^{\pm}) \oplus (x\wedge u)\mathcal{M}_{k-1}^w(\mathbb{R}^{2m},\mathbb{S}^{\pm}),$$

which generalises theorem 2.3.19.

Lemma 6.2.4. *Let $k\in\mathbb{Z}^+$ and $j\le k$. Then:*

$$R'_j\mathcal{S}_{k,k-j}(\mathbb{R}^{2m},\mathbb{S}^{\sigma}) \subset \mathcal{M}_k^w(\mathbb{R}^{2m},\mathbb{S}^{\pm}).$$

where $\sigma = \pm 1$ is the sign that makes the parities of the spinors on both side match (seeing as for j odd the parity on the left hand side switches).

Proof. We will have to make a distinction based on the parity of j. We will prove the statement for j odd, the case where j is even is completely similar. Let j be odd then:

$$\begin{aligned}
\partial_w R'_j &= (-1)^{\frac{j-1}{2}} \pi \partial_w |u|^{j-1} u \\
&= (-1)^{\frac{j-1}{2}} \pi (-\partial_u \partial_x - \langle \partial_x, \partial_u \rangle) |u|^{j-1} u \\
&= (-1)^{\frac{j-1}{2}+1} \pi \big(\partial_u |u|^{j-1} (-u\partial_x - 2 \langle u, \partial_x \rangle) \\
&\quad + (j-1) \langle u, \partial_x \rangle |u|^{j-3} u + |u|^{j-1} \partial_x + |u|^{j-1} u \langle \partial_x, \partial_u \rangle \big) \\
&= (-1)^{\frac{j-1}{2}} 2\pi \partial_u |u|^{j-1} \langle u, \partial_x \rangle \\
&= (-1)^{\frac{j-1}{2}} 2\pi \left([\partial_u, \langle u, \partial_x \rangle] + \langle u, \partial_x \rangle \partial_u \right) |u|^{j-1} \\
&= (-1)^{\frac{j-1}{2}} 2\pi \partial_x |u|^{j-1} \\
&= 0.
\end{aligned}$$

□

Lemma 6.2.5. *When acting on the kernel of the skew-Euler operators one has the following operator identity:*

$$[\partial_w, x \wedge u] = -(m + 2\mathbb{E}) \left(m + 2\mathbb{E} - 1 + T'_{x,u} \right)$$

where $T'_{x,u} := x\partial_x + u\partial_u$.

Proof. One has that

$$\begin{aligned}
[\partial_w, x \wedge u] &= [\partial_x \partial_u + \langle \partial_x, \partial_u \rangle, xu + \langle x, u \rangle] \\
&= [\partial_x \partial_u, xu] + [\partial_x \partial_u, \langle x, u \rangle] + [\langle \partial_x, \partial_u \rangle, xu] + [\langle \partial_x, \partial_u \rangle, \langle x, u \rangle]
\end{aligned}$$

and using straightforward calculations, hereby invoking the relation

$$[A, BC] = B[A, C] + [A, B]C,$$

one can see that

$$\begin{aligned}
[\partial_x \partial_u, \langle x, u \rangle] &= \partial_x [\partial_u, \langle x, u \rangle] + [\partial_x, \langle x, u \rangle] \partial_u \\
&= \partial_x x + u \partial_u \\
[\langle \partial_x, \partial_u \rangle, xu] &= x[\langle \partial_x, \partial_u \rangle, u] + [\langle \partial_x, \partial_u \rangle, x] u \\
&= x\partial_x + \partial_u u \\
[\langle \partial_x, \partial_u \rangle, \langle x, u \rangle] &= m + \mathbb{E}_x + \mathbb{E}_u.
\end{aligned}$$

The remaining commutator requires slightly more work and for this we will rewrite $\partial_x \partial_u x u$, as an operator acting on the kernel of the skew-Eulers, using the relations:

$$\partial_x x = -x\partial_x - m - 2\mathbb{E}_x$$

$$\partial_u x = -x\partial_u - 2\langle x, \partial_u\rangle$$

which yields:

$$\begin{aligned}\partial_x\partial_u xu = &-(m+2\mathbb{E}_u)x\partial_x - (m+2\mathbb{E}_x)u\partial_u\\ &+ (m+2\mathbb{E}_x)(2-m-2\mathbb{E}_x) + xu\partial_x\partial_u.\end{aligned}$$

Combining these identities give us that:

$$\begin{aligned}[\partial_w, x\wedge u] = &(m+2\mathbb{E}_x)(2-m-2\mathbb{E}_u) - (m+2\mathbb{E}_u)x\partial_x - (m+2\mathbb{E}_x)u\partial_u\\ &- (m+\mathbb{E}_x+\mathbb{E}_u).\end{aligned}$$

Finally, using that $\mathbb{E} = \mathbb{E}_x = \mathbb{E}_u$ brings us to the desired result. $\square$

Remark 6.2.6. Also the operator $T'_{x,u}$ can be seen from a different perspective. This time we start from the 2-dimensional module $\mathbb{V}_1$ for $\mathfrak{sl}(2)$, which can be realised as

$$\text{span}_{\mathbb{C}}\big(x, u\big) \cong \mathbb{V}_1 \cong \text{span}_{\mathbb{C}}\big(\partial_x, \partial_u\big)\ .$$

The unique trivial component $\mathbb{V}_0 \subset \mathbb{V}_1 \otimes \mathbb{V}_1$, as predicted by the Clebsch-Gordan theorem, is then precisely equal to $\mathbb{C}T'_{x,u}$.

One can easily see that $T'_{x,u}$ preserves the kernel of the skew-Euler operators and the following lemma gives us a nice operator identity. Note that the identity depends on the parity of $j \in \mathbb{Z}^+$, but this often happens in the framework of Dirac operators.

Lemma 6.2.7. *Let $k, j \in \mathbb{Z}^+$ such that $k \geq j$. We have the following operator identities when acting on $\mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^{\pm})$:*

$$\begin{aligned} j \in 2\mathbb{Z}^+ : &\qquad T'_{x,u}R'_j = -jR'_j\\ j \notin 2\mathbb{Z}^+ : &\qquad T'_{x,u}R'_j = (3+j-m-2\mathbb{E}_u)R'_j.\end{aligned}$$

Proof. We will prove the lemma for j odd (the case j even is completely similar):

$$\begin{aligned} T'_{x,u}R'_j &= (-1)^{\frac{j-1}{2}}\pi T'_{x,u}u|u|^{j-1}\\ &= (-1)^{\frac{j-1}{2}}\pi x[\partial_x, u]|u|^{j-1} + (-1)^{\frac{j-1}{2}}\pi u[\partial_u, u|u|^{j-1}]\\ &= (-1)^{\frac{j-1}{2}}\pi x(-u\partial_x - 2\langle u, \partial_x\rangle)|u|^{j-1}\\ &\qquad + (-1)^{\frac{j-1}{2}}\pi u\left(u[\partial_u, |u|^{j-1}] + [\partial_u, u]|u|^{j-1}\right)\\ &= (-1)^{\frac{j-1}{2}}2\pi[\langle u, \partial_x\rangle, x]|u|^{j-1}\\ &\qquad + (-1)^{\frac{j-1}{2}}\pi u\left(u(j-1)u|u|^{j-3} - (m+2u\partial_u+2\mathbb{E}_u)|u|^{j-1}\right)\\ &= (-1)^{\frac{j-1}{2}}\pi u\left(-(j-1)|u|^{j-1} - (m+2\mathbb{E}_u)|u|^{j-1} - 2u[\partial_u, |u|^{j-1}]\right)\\ &\qquad + (-1)^{\frac{j-1}{2}}2\pi u|u|^{j-1}\\ &= (2-j+1+2-m-2\mathbb{E}_u)R'_j + 2(-1)^{\frac{j-1}{2}}\pi|u|^2(j-1)u|u|^{j-3}\\ &= (3+j-m-2\mathbb{E}_u)R'_j.\end{aligned}$$

$\square$

We can now prove the following theorem:

Theorem 6.2.8. *Let $k \in \mathbb{Z}^+$, then*

$$\mathcal{M}_k^w(\mathbb{R}^{2m}, \mathbb{S}^\pm) = \bigoplus_{j=0}^{k} R'_j \mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^\sigma).$$

where $\sigma = \pm 1$ such that the parity of the spinors on the right matches the one on the left.

Proof. We start by using the fact that $\mathcal{M}_k^w(\mathbb{R}^{2m}, \mathbb{S}^\pm) \subset \mathcal{H}_k^w(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm$ and the latter space can be decomposed as:

$$\bigoplus_{j=0}^{k} R'_j \mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^\sigma) \oplus (x \wedge u) \bigoplus_{i=0}^{k-1} R'_j \mathcal{S}_{k-1,k-1-j}(\mathbb{R}^{2m}, \mathbb{S}^\sigma).$$

We have already proven that

$$\bigoplus_{j=0}^{k} R'_j \mathcal{S}_{k,k-j}(\mathbb{R}^{2m}, \mathbb{S}^\sigma) \subset \ker_k \mathcal{M}_k^w(\mathbb{R}^{2m}, \mathbb{S}^\pm)$$

and from lemma 6.2.5 and lemma 6.2.7 we can conclude that, for all $j \in \mathbb{Z}^+$ with $j \leq k-1$:

$$\Big((x \wedge u) R'_j \mathcal{S}_{k-1,k-1-j}(\mathbb{R}^{2m}, \mathbb{S}^\sigma)\Big) \cap \mathcal{M}_k^w(\mathbb{R}^{2m}, \mathbb{S}^\pm) = 0.$$

This finishes the proof. $\square$

All of the above leads to our main theorem, which is the analogue of the classical Fischer decomposition theorem 2.3.8 and theorem 2.3.20 for polynomials depending on a matrix variable, which belong to the kernel of the skew-Euler operators:

Theorem 6.2.9 (The Wedge Fischer decomposition)**.** *Let $k \in \mathbb{Z}^+$ and $m > 4$. We then have the following decompositions:*

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} \mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C})$$

with

$$\mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C}) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} R_j \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C}).$$

In case of spinor-values, this becomes

$$\mathcal{P}_k^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}) = \bigoplus_{i=0}^{k} (x \wedge u)^i \mathcal{M}_{k-i}^{w}(\mathbb{R}^{2m}, \mathbb{S}^{\pm})$$

with

$$\mathcal{M}_{k-i}^{w}(\mathbb{R}^{2m}, \mathbb{S}^{\pm}) = \bigoplus_{j=0}^{k-i} R_j' \mathcal{S}_{k-i,k-i-j}(\mathbb{R}^{2m}, \mathbb{S}^{\sigma})$$

where the sign σ is determined by the parity of j (the parity of the spinors on both sides has to be equal).

6.2.1 $m = 4$

Note that similarly to the scalar case, all these results also hold for $m = 4$, except for the decomposition of $\mathcal{H}^{\pm}_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \otimes \mathbb{S}_4^{\pm}$, for which we can no longer use proposition 2.4.5. Moreover, the extra equations that define the polynomial spaces $\mathcal{H}^{\pm}_{\ell,k}(\mathbb{R}^8, \mathbb{C})$ no longer appear when considering simplicial monogenics. Indeed, the difference there lies in the parity of the spinors that we take value in, i.e.

$$\mathcal{H}^{\pm}_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \cong (\ell, \pm k) \text{ and } \mathcal{S}_{\ell,k}(\mathbb{R}^8, \mathbb{S}_4^{\pm}) \cong \left(\ell + \frac{1}{2}, \pm(k + \frac{1}{2})\right),$$

while the restrictions on the polynomial part vanish. Nevertheless we will show that something similar to proposition 2.4.5 still holds in this case, from which it follows that the wedge-Fischer theorem also holds when $m = 4$. For this we take a closer look at the representation theory of $\mathfrak{so}(4)$, the proofs can be found in [2, 15, 56]. In e.g. example 2.1.14 we provided a model for $\mathfrak{so}(4)$ using Witt-vectors and found the following:

$$\begin{aligned} \mathfrak{so}(4) &\cong \mathfrak{sl}(2)_1 \oplus \mathfrak{sl}(2)_2 \\ &\cong \mathrm{Span}\{\mathfrak{f}_1\mathfrak{f}_1^{\dagger} + \mathfrak{f}_2\mathfrak{f}_2^{\dagger} - 1, \mathfrak{f}_1\mathfrak{f}_2, \mathfrak{f}_2^{\dagger}\mathfrak{f}_1^{\dagger}\} \oplus \mathrm{Span}\{\mathfrak{f}_1\mathfrak{f}_1^{\dagger} - \mathfrak{f}_2\mathfrak{f}_2^{\dagger}, \mathfrak{f}_1\mathfrak{f}_2^{\dagger}, \mathfrak{f}_2\mathfrak{f}_1^{\dagger}\}. \end{aligned}$$

The isomorphism is given by

$$\Phi : \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \to \mathfrak{so}(4) : (A, B) \mapsto A + B,$$

and there exists a canonical action of $\mathfrak{so}(4)$ on the tensor product of two representation $\mathbb{V}$ and $\mathbb{W}$ of $\mathfrak{sl}(2)$:

$$(A, B).(v \otimes w) = (A \otimes 1 + 1 \otimes B)(v \otimes w). \tag{6.2}$$

Note that all irreducible representations of $\mathfrak{so}(4)$ are of this form:

Theorem 6.2.10. *The irreducible representations $\mathbb{V}$ of $\mathfrak{so}(4)$ are given by*

$$\mathbb{V} = \mathbb{V}_r \otimes \mathbb{V}_s,$$

where $\mathbb{V}_r$ and $\mathbb{V}_s$ are irreducible representations for the two commuting copies $\mathfrak{sl}(2)_1$ and $\mathfrak{sl}(2)_2$ defined above and the action is given by 6.2.

All these representations can be modelled using polynomial spaces:

Proposition 6.2.11. *Let $r, s \in \mathbb{Z}^+$ and let $\mathbb{V} = \mathbb{V}_r \otimes \mathbb{V}_s$ be a finite-dimensional irreducible $\mathfrak{so}(4)$-representation. Then $\mathbb{V}$ has highest weight*

$$\lambda = \frac{1}{2}(r+s, r-s)$$

and is isomorphic to

$$\begin{array}{llll} r = \ell + k & s = \ell - k & \rightarrow & \mathcal{H}^+_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \\ r = \ell - k & s = \ell + k & \rightarrow & \mathcal{H}^-_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \\ r = \ell + k + 1 & s = \ell - k & \rightarrow & \mathcal{S}_{\ell,k}(\mathbb{R}^8, \mathbb{S}^+_4) \\ r = \ell - k & s = \ell + k + 1 & \rightarrow & \mathcal{S}_{\ell,k}(\mathbb{R}^8, \mathbb{S}^-_4). \end{array}$$

We can now prove that proposition 2.4.5 still holds if $m = 4$:

Theorem 6.2.12. *Let $\ell \geq k$, then*

$$\begin{aligned} \mathcal{H}_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \otimes \mathbb{S}^\pm_4 \cong \mathcal{S}_{\ell,k}(\mathbb{R}^8, \mathbb{S}^\pm_4) &\oplus \mathcal{S}_{\ell,k-1}(\mathbb{R}^8, \mathbb{S}^\mp_4) \\ &\oplus \mathcal{S}_{\ell-1,k}(\mathbb{R}^8, \mathbb{S}^\mp_4) \oplus \mathcal{S}_{\ell-1,k-1}(\mathbb{R}^8, \mathbb{S}^\pm_4) \end{aligned}$$

Proof. We will prove the result for the tensor product with $\mathbb{S}^+_4$, the product with the negative Weyl spinors is completely similar. Let us first look at $\mathcal{H}^+_{\ell,k}(\mathbb{R}^8, \mathbb{C})$, where the superscript 1 or 2 denotes the copy of $\mathfrak{sl}(2)$ for which it is a representation:

$$\begin{aligned} \mathcal{H}^+_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \otimes \mathbb{S}^+_4 &\cong \left(\mathbb{V}^1_{\ell+k} \otimes \mathbb{V}^2_{\ell-k}\right) \otimes \left(\mathbb{V}^1_1 \otimes \mathbb{V}^2_0\right) \\ &\cong \left(\mathbb{V}^1_{\ell+k} \otimes \mathbb{V}^1_1\right) \otimes \mathbb{V}^2_{\ell-k} \\ &\cong \left(\mathbb{V}^1_{\ell+k+1} \otimes \mathbb{V}^2_{\ell-k}\right) \oplus \left(\mathbb{V}^1_{\ell+k-1} \otimes \mathbb{V}^2_{\ell-k}\right) \\ &\cong \mathcal{S}_{\ell,k}(\mathbb{R}^8, \mathbb{S}^+_4) \oplus \mathcal{S}_{\ell-1,k-1}(\mathbb{R}^8, \mathbb{S}^+_4), \end{aligned}$$

where we have used theorem 1.3.3 to decompose the tensor products of the $\mathfrak{sl}(2)$-representations. On the other hand, we have that:

$$\begin{aligned} \mathcal{H}^-_{\ell,k}(\mathbb{R}^8, \mathbb{C}) \otimes \mathbb{S}^+_4 &\cong \left(\mathbb{V}^1_{\ell-k} \otimes \mathbb{V}^2_{\ell+k}\right) \otimes \left(\mathbb{V}^1_1 \otimes \mathbb{V}^2_0\right) \\ &\cong \left(\mathbb{V}^1_{\ell-k} \otimes \mathbb{V}^1_1\right) \otimes \mathbb{V}^2_{\ell+k} \\ &\cong \left(\mathbb{V}^1_{\ell-k+1} \otimes \mathbb{V}^2_{\ell+k}\right) \oplus \left(\mathbb{V}^1_{\ell-k-1} \otimes \mathbb{V}^2_{\ell+k}\right) \\ &\cong \mathcal{S}_{\ell,k-1}(\mathbb{R}^8, \mathbb{S}^-_4) \oplus \mathcal{S}_{\ell-1,k}(\mathbb{R}^8, \mathbb{S}^-_4), \end{aligned}$$

where the last component only appears when $\ell \neq k$. Combining both findings yields the desired result. ☐

6.2.2 Symmetry of the wedge operator ∂_w

The story behind the symmetries of the wedge systems considered in this thesis (both the scalar case $\Delta_w f(x,u) = 0$ as well as the spinor-valued case $\partial_w f(x,u) =$

0) could be interesting and, as we do not yet fully understand it, warrants future research. We start from the observation that the wedge-Dirac operator actually appears in the resolution for the 2-Dirac operator. This is a system of equations defined on the analogy of holomorphic functions in several vector variables, hereby seeing the Dirac operator as a generalised Cauchy-Riemann operator:

$$\left\{ \begin{array}{rcl} \partial_x f(x,u) & = & h_1 \\ \partial_u f(x,u) & = & h_2 \end{array} \right.$$

As this is an over-determined system, one is typically faced with the question of compatibility conditions for the associated minimal resolution. This is a deep question, for which 2 possible approaches were developed in the literature. On the one hand there is the work done by e.g. Colombo et al, see e.g. [14, 72], in which techniques coming from algebraic geometry and Clifford analysis were used to describe the resolutions in matrix language. On the other hand, there is the work done by Souček et al, see e.g. [62, 75], whereby Penrose transforms are used to tackle this problem using representation theory (this then involves the machinery of BGG-resolutions). The general problem, involving k Dirac operators ∂_{x_j} with $1 \leq j \leq k$ turns out to be a highly complicated problem, but luckily enough the case $k = 2$ is relatively well understood. In particular, there exists a minimal resolution of the form

$$\mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) \xrightarrow{A_1} \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) \xrightarrow{B_2} \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) \xrightarrow{C_1} \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) \ ,$$

whereby the index stands for the order of the operator. In terms of Dirac operators ∂_x and ∂_u, these operators are given by

$$\begin{array}{rcl} A_1 : \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) & \to & \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) \\ f(x,u) & \mapsto & \left(\begin{array}{c} \partial_x f \\ \partial_u f \end{array} \right) \\ B_2 : \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) & \to & \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) \\ \left(\begin{array}{c} g_1 \\ g_2 \end{array} \right) & \mapsto & \left(\begin{array}{c} \partial_u \partial_x g_1 - \partial_x^2 g_2 \\ \partial_u^2 g_1 - \partial_x \partial_u g_2 \end{array} \right) \\ C_1 : \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S} \otimes \mathbb{C}^2) & \to & \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) \\ \left(\begin{array}{c} h_1 \\ h_2 \end{array} \right) & \mapsto & \partial_x h_2 - \partial_u h_1 \ . \end{array}$$

As we are dealing with symmetries here, we will explicitly verify that each of the operators above is invariant with respect to the action of $\mathrm{Spin}(m) \otimes \mathrm{SL}(2)$. First of all, since each of the operators above is expressed in terms of Dirac operators, we immediately have that the operators commute with the action of $\mathrm{Spin}(m)$, which was given by

$$f(x,u) \mapsto L(s)[f(x,u)] = sf(\bar{s}xs, \bar{s}us) \ .$$

As for the action of $g \in \mathrm{SL}(2)$, we note that the action on $\mathbb{V}_\lambda$-valued functions $f(x,u)$, with $\mathbb{V}_\lambda$ an irreducible representation for $\mathfrak{sl}(2)$, will be given by

$$g \bullet \big(f(x,u) \otimes v_\lambda \big) := f((x,u)g) \otimes \big(g \bullet v_\lambda \big) \ .$$

Note that the modules $\mathbb{V}_\lambda$ appearing above are either the trivial representation $\mathbb{V}_0 \cong \mathbb{C}$ or the defining representation $\mathbb{V}_1 \cong \mathbb{C}^2$, for which the action reduces to left matrix multiplication. From now on, we will choose

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$$

as a generic element, which means that $ad - bc = 1$. We then also introduce the following notation:

$$f((x,u)g) = f(ax + cu, bx + du) := f(x,u)' \ ,$$

whereby we sometimes simply write f' and omit the variables (x,u). In other words: adding a prime to a function on $\mathbb{R}^{2m}$ means that this function is evaluated in $(x,u)g$. A simple application of the chain rule then gives that

$$\begin{aligned} \partial_x f' &= a(\partial_x f)' + b(\partial_u f)' \\ \partial_u f' &= c(\partial_x f)' + d(\partial_u f)'. \end{aligned}$$

In order to verify the SL(2)-invariance of the first operator A_1, we need to show that

$$A_1(g \bullet f) = g \bullet (A_1 f)$$

The left-hand side is given by

$$A_1 f((x,u)g) = \begin{pmatrix} \partial_x f' \\ \partial_u f' \end{pmatrix} = \begin{pmatrix} a(\partial_x f)' + b(\partial_u f)' \\ c(\partial_x f)' + d(\partial_u f)' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (\partial_x f)' \\ (\partial_u f)' \end{pmatrix} \ ,$$

which is indeed equal to the right-hand side. A similar consideration holds for the other first-order operator: we need to show that

$$C_1\left(g \bullet \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) = g \bullet \left(C_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right)$$

The right-hand side is given by

$$g \bullet \left(C_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) = g \bullet (\partial_x f_2 - \partial_u f_1) = (\partial_x f_2)' - (\partial_u f_1)'$$

The left-hand side is equal to

$$C_1 \begin{pmatrix} af_1' + bf_2' \\ cf_1' + df_2' \end{pmatrix} = \partial_x\big(cf_1' + df_2'\big) - \partial_u\big(af_1' + bf_2'\big) \ .$$

Making use of the derivation rules from above, we find that

$$\begin{aligned} \partial_x\big(cf_1' + df_2'\big) &= c\big(a(\partial_x f_1)' + b(\partial_u f_1)'\big) + d\big(a(\partial_x f_2)' + b(\partial_u f_2)'\big) \\ \partial_u\big(af_1' + bf_2'\big) &= a\big(c(\partial_x f_1)' + d(\partial_u f_1)'\big) + b\big(c(\partial_x f_2)' + d(\partial_u f_2)'\big) \end{aligned}$$

which means that the left-hand side reduces to

$$(ad - bc)\big((\partial_x f_2)' - (\partial_u f_1)'\big) \ ,$$

as was to be shown. Finally, for the middle operator B_2 one needs to show that

$$B_2\left(g \bullet \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) = g \bullet \left(B_2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right)$$

whereby the action of $g \in \mathrm{SL}(2)$ will this time involve left matrix multiplication on $\mathbb{C}^2$ on both sides of the equation. The right-hand side immediately gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (\partial_u\partial_x g_1)' - (\partial_x^2 g_2)' \\ (\partial_u^2 g_1)' - (\partial_x\partial_u g_2)' \end{pmatrix}$$

but the left-hand side involves more computations as it is given by

$$B_2 \begin{pmatrix} ag_1' + bg_2' \\ cg_1' + dg_2' \end{pmatrix} = \begin{pmatrix} \partial_u\partial_x(ag_1' + bg_2') - \partial_x^2(cg_1' + dg_2') \\ \partial_u^2(ag_1' + bg_2') - \partial_x\partial_u(cg_1' + dg_2') \end{pmatrix}$$

In order to calculate and simplify these terms, we invoke the chain rule formula from above, which leads to

$$\begin{aligned} \partial_u\partial_x(ag_1' + bg_2') &= a\partial_u\big(a(\partial_x g_1)' + b(\partial_u g_1)'\big) + b\partial_u\big(a(\partial_x g_2)' + b(\partial_u g_2)'\big) \\ &= a^2\big(c(\partial_x^2 g_1)' + d(\partial_u\partial_x g_1)'\big) + ab\big(c(\partial_x\partial_u g_1)' + d(\partial_u^2 g_1)'\big) \\ &\quad + ab\big(c(\partial_x^2 g_2)' + d(\partial_u\partial_x g_2)'\big) + b^2\big(c(\partial_x\partial_u g_2)' + d(\partial_u^2 g_2)'\big) \end{aligned}$$

and similarly one finds that

$$\begin{aligned} \partial_x^2(cg_1' + dg_2') &= ac\big(a(\partial_x^2 g_1)' + b(\partial_u\partial_x g_1)'\big) + bc\big(a(\partial_x\partial_u g_1)' + b(\partial_u^2 g_1)'\big) \\ &\quad + ad\big(a(\partial_x^2 g_2)' + b(\partial_u\partial_x g_2)'\big) + bd\big(a(\partial_x\partial_u g_2)' + b(\partial_u^2 g_2)'\big) \ . \end{aligned}$$

Subtracting the latter from the former then gives

$$a\big((\partial_u\partial_x g_1)' - (\partial_x^2 g_2)'\big) + b\big((\partial_u^2 g_1)' - (\partial_x\partial_u g_2)'\big) \ ,$$

hereby using the fact that $ad - bc = 1$. The remaining term can be calculated in a completely similar way and reduces to

$$c\big((\partial_u\partial_x g_1)' - (\partial_x^2 g_2)'\big) + d\big((\partial_u^2 g_1)' - (\partial_x\partial_u g_2)'\big) \ ,$$

as was to be shown. This means that each of the operators A_1, B_2 and C_1 is indeed invariant with respect to the indicated action of $\mathrm{Spin}(m) \times \mathrm{SL}(2)$. What is crucial for our thesis is the following observation:

$$C_1 A_1 : \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{S}) : f \mapsto (\partial_x\partial_u - \partial_u\partial_x)f$$

is, up to a factor 2, equal to the wedge Dirac operator studied earlier. This seems to suggest 2 things:

(i) In order to determine the symmetries of the wedge system (be it scalar or spinor, for Laplace or Dirac), one has to consider a generalisation of the conformal Lie algebra $\mathfrak{so}(1, m+1)$. This new Lie algebra turns out to be $\mathfrak{so}(2, m+2)$, as this is the one containing $\mathfrak{so}(m) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}$ as a Levi subalgebra. Note that the summand $\mathbb{C}$, which comes from $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{C}$ plays the role of a conformal weight (just like in the classical case for 1 vector variable).

(ii) There is a hope to rediscover the wedge operators using a geometrical framework, hereby starting from a parabolic geometry G/P for suitable Lie groups G and P. As a matter of fact, we would like to investigate the following claim: we expect the (integer) powers of $\partial_x \wedge \partial_u$ to correspond to so-called 'long operators in a singular character'.

Both open up possibilities for future research.

6.3 General ℓ, k: polynomials depending on flag variables

In theorem 6.2.9 we have assumed that the degrees of homogeneity are equal, we will try to find a similar result for $\ell \neq k$. In [15] it was shown that

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) := \mathcal{P}_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker \langle x, \partial_u \rangle \cong (\ell, k)$$

as a $\mathrm{GL}(m)$-representation. Polynomials $P_{\ell,k}$ in this space depend on flag variables, i.e. their only dependence on the variable u is through the wedge variables encountered earlier. When looking at the decomposition into irreducible $\mathrm{SO}(m)$-components it turns out that the decomposition will no longer be multiplicity-free. Let us thus start by finding a formula that gives us the multiplicity of $\mathcal{H}_{a,b}(\mathbb{R}^{2m}, \mathbb{C})$ in $\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C})$ in terms of a, b, ℓ and k.

Definition 6.3.1. Define the following map:

$$A_E : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ : (a, b) \mapsto |\{n : a \leq 2n \leq b\}|,$$

which counts the amount of even numbers between a and b.

Using this map we can find:

Lemma 6.3.2. *Let $\ell, k \in \mathbb{Z}^+$ such that $\ell \geq k$ and $m \geq 4$ then:*

$$(\ell, k)\Big|^{\mathrm{GL}(m)}_{\mathrm{O}(m)} \cong \bigoplus_{\mu \in W_{\ell,k}} c_{\ell,k}(\mu) E_\mu$$

with E_μ an irreducible $\mathrm{O}(m)$-module,

$$W_{\ell,k} := \{(\mu_1, \mu_2) : \ell + k \equiv \mu_1 + \mu_2 \ \mathit{mod}\ 2 \ \mathit{and}\ \mu_1 \leq \ell, \mu_2 \leq k \ \mathit{and}\ \mu_1 \geq \mu_2\}$$

and

$$c_{\ell,k}(\mu) = A_E(\max(k - \mu_2, \ell - \mu_1), \min(\ell - \mu_2, \ell + k - \mu_1 - \mu_2)).$$

Proof. Similar to the proof of theorem 6.1.2, we use section 1.4.2 which states that:

$$F_\lambda\Big|_{\mathrm{O}(m)}^{\mathrm{GL}(m)} = \bigoplus_\mu m_\mu E_\mu$$

where the multiplicity of the irreducible $\mathrm{O}(m)$-module E_μ is given by

$$m_\mu = \sum_{2\delta} c^\lambda_{\mu,2\delta},$$

the sum of Littlewood-Richardson coeffcients over all Young tableaux with even parts. One can immediately conclude that neither μ nor 2δ can have more then two entries different from zero. Let us write

$$\begin{aligned} \mu &= (\mu_1, \mu_2) \\ 2\delta &= (2\delta_1, 2\delta_2) \end{aligned}$$

then one cannot add any of the $2\delta_2$ b-labelled boxes to the first line of μ after adding the a-labelled boxes since this would break rule 3. Therefore, one always has to add $\ell - \mu_1$ a-labelled boxes on the first line and thus $2\delta_1 \geq \ell - \mu_1$. Now there can only be at most $\mu_1 - \mu_2$ a-labelled boxes left because one can add at most $\mu_1 - \mu_2$ a-labelled boxes to the second line before one has to put them on the third line (which needs to remain zero to obtain (ℓ, k)). This means that, after adding the a-labelled boxes we are left with ℓ boxes on the first line, and $\mu_2 + 2\delta_1 - \ell + \mu_1$ boxes on the second line. This means that there have to be precisely $k - (\mu_2 + 2\delta_1 - \ell + \mu_1)$ b-labelled boxes to find the desired Young diagram. Combined we have the following:

$$\ell + k - \mu_1 - \mu_2 - 2\delta_1 - 2\delta_2 = 0 \tag{6.3}$$

$$k - \mu_2 \leq 2\delta_1 \leq \ell - \mu_2 \tag{6.4}$$

$$0 \leq 2\delta_2 \leq \ell - \mu_1. \tag{6.5}$$

From (6.3) we can conclude that the only possible μ that can occur are those for which $\ell + k \equiv \mu_1 + \mu_2 \mod 2$. Moreover, given μ and $2\delta_1$ we can determine $2\delta_2$ from (6.3). The question that remains is, for how many values of $2\delta_1$ that satisfy (6.4), does $2\delta_2$ satisfy (6.5). In other words:

$$\begin{aligned} 0 \leq \ell + k - \mu_1 - \mu_2 - 2\delta_1 &\leq \ell - \mu_1 \\ k - \mu_2 \leq 2\delta_1 &\leq \ell - \mu_2 \end{aligned}$$

and thus for all $2\delta_1$ that satisfy

$$\max(k - \mu_2, \ell - \mu_1) \leq 2\delta_1 \leq \min(\ell - \mu_2, \ell + k - \mu_1 - \mu_2)$$

we can find a $2\delta_2$ that satisfies equation (6.5), which finishes the proof. □

Before we continue, let us verify that for $\ell = k$ we indeed recover our earlier results. Here, from the definition of $W_{k,k}$, we know that we have to consider highest weights of the form

$$\mu = (k - 2a, k - 2a - 2b) \qquad \text{or} \qquad \mu = (k - 2a - 1, k - 2a - 2b - 1)$$

Let us look at the second option first: for these μ we have that:

$$\begin{aligned} c_{k,k}(\mu) &= A_E(\max(2a + 2b + 1, 2a + 1), \min(2a + 2b + 1, 4a + 2 + 2b)) \\ &= A_E(2a + 2b + 1, 2a + 2b + 1) \\ &= 0. \end{aligned}$$

On the other hand, if $\mu = (k - 2a, k - 2a - 2b)$ then:

$$\begin{aligned} c_{k,k}(\mu) &= A_E(\max(2a + 2b, 2a), \min(2a + 2b, 4a + 2b)) \\ &= A_E(2a + 2b, 2a + 2b) \\ &= 1. \end{aligned}$$

Before we take a look at another example, we introduce the following spaces:

Definition 6.3.3. Let $\ell, k \in \mathbb{Z}^+$ such that $\ell \geq k$. We then define

$$\mathcal{H}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) := \mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) \cap \ker \Delta_w.$$

Similarly to the definitions in section 6.2 we introduce

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) := \mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm$$

and

$$\mathcal{M}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) := \mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cap \ker \partial_w.$$

6.3.1 Example: $k + 1, k$

Let us take a closer look at $\ell = k + 1$, as this shows how we will tackle the general case. We need to look at all μ such that:

$$\mu = (k + 1 - 2a, k - 2a - 2b) \qquad \text{or} \qquad \mu = (k - 2a, k - 2a - 2b - 1).$$

This means that:

$$\begin{aligned} &c_{k+1,k}((k + 1 - 2a, k - 2a - 2b)) \\ &\qquad = A_E(\max(2a + 2b, 2a), \min(2a + 2b + 1, 4a + 2b)) \\ &\qquad = A_E(2a + 2b, \min(2a + 2b + 1, 4a + 2b)) \end{aligned}$$

and the latter is, for $a \neq 0$, equal to $A_E(2a + 2b, 2a + 2b + 1) = 1$. If $a = 0$ then this is equal to $A_E(2b, 2b) = 1$. The other case is treated similarly:

$$c_{k+1,k}((k - 2a, k - 2a - 2b - 1))$$

$$
\begin{aligned}
&= A_E(\max(2a+2b+1, 2a+1), \min(2a+2b+2, 4a+2b+2)) \\
&= A_E(2a+2b+1, 2a+2b+2) \\
&= 1.
\end{aligned}
$$

Everything combined we have found the following:

Theorem 6.3.4. *Let $k \in \mathbb{Z}^+$ and $m \geq 4$ then:*

$$
\mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m}, \mathbb{C}) \cong \left(\bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{b=0}^{\lfloor \frac{k}{2} \rfloor - a} \mathcal{H}_{k+1-2a,k-2a-2b}(\mathbb{R}^{2m}, \mathbb{C}) \right) \oplus \left(\bigoplus_{a=0}^{\lfloor \frac{k-1}{2} \rfloor} \bigoplus_{b=0}^{\lfloor \frac{k-1}{2} \rfloor - a} \mathcal{H}_{k-2a,k-2a-2b-1}(\mathbb{R}^{2m}, \mathbb{C}) \right).
$$

In contrast to the case $\ell = k$, we first take a look at the spinor refinement:

Proposition 6.3.5. *Let $k \in \mathbb{Z}^+$ and $m \geq 4$, the map*

$$
\partial_w : \mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \to \mathcal{P}^w_{k,k-1}(\mathbb{R}^{2m}, \mathbb{S}^\pm)
$$

is surjective and

$$
\mathcal{M}^w_{k+1,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cong \bigoplus_{a=0}^{1} \bigoplus_{b=0}^{k} \mathcal{S}_{k+1-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^\sigma),
$$

with $\sigma = \pm$ if $a+b$ is even, and $\sigma = \mp$ otherwise.

Proof. From theorem 6.3.4 we can conclude that

$$
\mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \big/ \mathcal{P}^w_{k,k-1}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cong \bigoplus_{b=0}^{\lfloor \frac{k}{2} \rfloor} \mathcal{H}_{k+1,k-2b}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm.
$$

The latter sum can be decomposed using proposition 2.4.5 (or theorem 6.2.12 if $m = 4$) into

$$
\bigoplus_{b=0}^{\lfloor \frac{k}{2} \rfloor} \mathcal{H}_{k+1,k-2b}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm \cong \bigoplus_{a=0}^{1} \bigoplus_{b=0}^{k} \mathcal{S}_{k+1-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^\sigma)
$$

where $\sigma = \pm$ if $a+b$ is even, and $\sigma = \mp$ otherwise. Finally, applying corollary 5.2.6 finishes the proof. □

From this proposition it immediately follows that

$$
\Delta_w = -\partial_w^2 : \mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm \to \mathcal{P}^w_{k-1,k-2}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^\pm
$$

is surjective and

$$\mathcal{H}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C})\otimes\mathbb{S}^\pm \cong \mathcal{M}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{S}^\pm)\oplus\mathcal{M}^w_{k,k-1}(\mathbb{R}^{2m},\mathbb{S}^\pm).$$

One can say even more, because Δ_w is scalar and therefore does not affect $\mathbb{S}^\pm$, we can conclude that

$$\Delta_w : \mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C}) \to \mathcal{P}^w_{k-1,k-2}(\mathbb{R}^{2m},\mathbb{C})$$

is surjective as well.

Proposition 6.3.6. *Let $k\in\mathbb{Z}^+$ and $m\geq 4$ then*

$$\mathcal{H}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C})\cong\bigoplus_{i=0}^{\lfloor\frac{k}{2}\rfloor}\mathcal{H}_{k+1,k-2i}(\mathbb{R}^{2m},\mathbb{C})\oplus\bigoplus_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\mathcal{H}_{k,k-2i-1}(\mathbb{R}^{2m},\mathbb{C}).$$

Proof. Because Δ_w is surjective, this immediately follows from

$$\mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C})\big/\mathcal{P}^w_{k-1,k-2}(\mathbb{R}^{2m},\mathbb{C})\cong\mathcal{H}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C}).$$

□

Combining everything yields:

Theorem 6.3.7. *Let $k\in\mathbb{Z}^+$ and $m\geq 4$, then*

$$\mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{C})\cong\bigoplus_{i=0}^{\lfloor\frac{k}{2}\rfloor}\mathcal{H}^w_{k+1-2i,k-2i}(\mathbb{R}^{2m},\mathbb{C}).$$

When considering spinor-valued polynomials this becomes

$$\mathcal{P}^w_{k+1,k}(\mathbb{R}^{2m},\mathbb{S}^\pm)\cong\bigoplus_{i=0}^{k}\mathcal{M}^w_{k+1-i,k-i}(\mathbb{R}^{2m},\mathbb{S}^\pm).$$

6.3.2 General case

Our goal is to prove a generalisation of theorem 6.2.9 for the space of polynomials depending on flag variables. These spaces are irreducible $\mathrm{GL}(m)$-modules and we start by obtaining a decomposition into irreducible $\mathrm{O}(m)$-components:

Theorem 6.3.8. *Let $\ell\geq k$, then $(\ell,k)_{\mathrm{GL}(m)}$ can be decomposed into irreducible $\mathrm{O}(m)$-modules as follows:*

$$(\ell,k)_{\mathrm{GL}(m)}\cong\bigoplus_{a=0}^{\lfloor\frac{k}{2}\rfloor}\left(\bigoplus_{b=0}^{\lfloor\frac{\ell-k}{2}\rfloor}\bigoplus_{c=0}^{\lfloor\frac{k}{2}\rfloor-a}(\ell-2a-2b,k-2a-2c)_{\mathrm{O}(m)}\oplus\right.$$
$$\left.\bigoplus_{b=0}^{\lfloor\frac{\ell-k-1}{2}\rfloor}\bigoplus_{c=0}^{\lfloor\frac{k-1}{2}\rfloor-a}(\ell-2a-1-2b,k-2a-1-2c)_{\mathrm{O}(m)}\right).$$

Proof. We know from our previous theorem that we only need to consider μ such that $\ell + k + \mu_1 + \mu_2$ is even. Therefore

$$\mu = (\ell - 2a, k - 2b) \qquad \text{or} \qquad \mu = (\ell - 2a - 1, k - 2b - 1).$$

However, one can rewrite these μ by defining $i := \min(a, b)$ and $j := \max(a, b) - \min(a, b)$ to arrive at the following four possibilities:

$$\begin{aligned} \mu_1 &= (\ell - 2i, k - 2i - 2j) & \mu_2 &= (\ell - 2i - 2j, k - 2i) \\ \mu_3 &= (\ell - 2i - 1, k - 2i - 2j - 1) & \mu_4 &= (\ell - 2i - 2j - 1, k - 2i - 1). \end{aligned}$$

where i, j are chosen such that the dominant weight condition is satisfied. We will calculate the multiplicity of each if these weights μ and compare it to the multiplicity of the same module in the decomposition above, hereby finding the amount of (a, b, c) within bounds that can yield a specific weight. Moreover, all the weights on the right hand side are clearly all of the form μ_p for some $p \in \{1, 2, 3, 4\}$.
Consider $\ell - k$ to be even, as the other case is treated similarly, then

$$\begin{aligned} c_{\ell,k}(\mu_1) &= A_E(\max(2i, 2i + 2j), \min(\ell - k + 2i + 2j, 4i + 2j)) \\ &= A_E(2i + 2j, \min(\ell - k + 2i + 2j, 4i + 2j)) \\ &= \begin{cases} i + 1 & \text{for } 2i \leq \ell - k \\ \dfrac{\ell - k}{2} + 1 & \text{for } \ell - k \leq 2i. \end{cases} \end{aligned}$$

These are the weights $(\ell - 2a - 2b, k - 2a - 2c)$ where $a + b = i$ and $a + c = i + j$. Let $2i \leq \ell - k$, this means that $b \leq i \leq \frac{\ell - k}{2}$ meaning that for each $0 \leq b \leq i$ we can find an a and c that satisfy the equations and it is easy to see that these a, c will be within the right bounds, i.e. there are $i + 1$ different possibilities for (a, b, c) that yield the same module. Assume now that $\ell - k \leq 2i$, then for each value $0 \leq b \leq \frac{\ell - k}{2}$ we find an a and c such that $a + b = i$ and $a + c = i + j$, meaning that $\frac{\ell - k}{2} + 1$ different solutions (a, b, c) exist. From this we can conclude that the multiplicity of μ_1 is equal on both sides of the suggested isomorphism. The next weight is slightly more complicated:

$$\begin{aligned} c_{\ell,k}(\mu_2) &= A_E(\max(2i + 2j, 2i), \min(4i + 2j, \ell - k + 2i)) \\ &= A_E(2i + 2j, \min(4i + 2j, \ell - k + 2i)) \\ &= \begin{cases} i + 1 & \text{for } 2j \leq \ell - k - 2i \\ \dfrac{\ell - k}{2} - j + 1 & \text{for } \ell - k - 2i \leq 2j. \end{cases} \end{aligned}$$

These are precisely the weights $(\ell - 2a - 2b, k - 2a - 2c)$ where

$$\begin{cases} a + b = i + j \\ a + c = i \end{cases}$$

meaning $a \leq i$. Assume that $2j \leq \ell - k - 2i$, then all such a will yield a solution. Indeed $c = i - a$ will always be within bounds, as $i \leq \lfloor \frac{k}{2} \rfloor$ and for $b = j + c$:

$$2(j+c) \leq 2(j+i) \leq \ell - k$$

as desired. This means that there are exactly $i+1$ different (a,b,c) that satisfy our equations if $2j \leq \ell - k - 2i$. If $2j \geq \ell - k - 2i$, then we focus on the possible values for c first. We claim that we can find a solution (a,b,c) if and only if $0 \leq c \leq \frac{\ell-k}{2} - j$. Assume that c does not satisfy this then:

$$2c > \ell - k - 2j \iff 2(j+c) > \ell - k \iff 2b > \ell - k,$$

which means that b is out of bounds. Therefore if $0 \leq c \leq \frac{\ell-k}{2} - j$, then b will be within bounds and because

$$2c \leq \ell - k - 2j \leq 2i,$$

we can find an $a \geq 0$ to find a solution (a,b,c), meaning that there exist exactly $\frac{\ell-k}{2} - j + 1$ different ones. In other words: the multiplicity of μ_2 is the same on both sides of the suggested isomorphism.
Next, we consider

$$\begin{aligned}
c_{\ell,k}(\mu_3) &= A_E(\max(2i+1, 2i+2j+1), \min(\ell - k + 2i + 2j + 1, 4i + 2j + 2)) \\
&= A_E(2i+2j+1, \min(\ell - k + 2i + 2j + 1, 4i + 2j + 2)) \\
&= \begin{cases} \dfrac{\ell - k}{2} & \text{for } \ell - k - 1 \leq 2i \\ i + 1 & \text{for } 2i \leq \ell - k - 1 \end{cases}
\end{aligned}$$

meaning that we are again looking for (a,b,c) such that $(\ell - 2a - 1 - 2b, k - 2a - 1 - 2c)$ yields μ_3. Similarly to μ_1, using $a + b = i$ and $2i \leq \ell - k - 1$, we can, for each $0 \leq b \leq i$, find a, c such that (a,b,c) is a solution. If $2i \geq \ell - k - 1$, then for each $0 \leq b \leq \frac{\ell-k}{2} - 1$ we let $a = i - b$ and $c = b + j$. It is clear that $2a \leq 2i \leq k - 1$ and because all weight entries of μ_3 have to be positive, we know that $2i + 2j \leq k - 1$, leading to

$$2c \leq 2i + 2j - 2a \leq k - 1 - 2a$$

as desired.
Finally, we consider the weight μ_4 :

$$\begin{aligned}
c_{\ell,k}(\mu_4) &= A_E(\max(2i+2j+1, 2i+1), \min(\ell - k + 2i + 1, 4i + 2j + 2)) \\
&= A_E(2i+2j+1, \min(\ell - k + 2i + 1, 4i + 2j + 2)) \\
&= \begin{cases} \dfrac{\ell - k}{2} - j & \text{for } \ell - k - 2i - 1 \leq 2j \\ i + 1 & \text{for } 2j \leq \ell - k - 2i - 1 \end{cases}
\end{aligned}$$

Similarly to our approach for μ_2, if $2j \leq \ell - k - 2i - 1$ then all $0 \leq a \leq i$ determine a triple (a,b,c) such that $(\ell - 2a - 1 - 2b, k - 2a - 1 - 2c)$ is equal to μ_4. On the other hand, if $2j \geq \ell - k - 2i - 1$, then $0 \leq c \leq \frac{\ell-k}{2} - 1 - j$, as otherwise $b = j + c$ will be out of bounds. This leads to the desired result. $\square$

Proposition 6.3.9. *Let $\ell, k \in \mathbb{Z}^+$ with $\ell \geq k$ and $m \geq 4$. The operator*

$$\partial_w : \mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \to \mathcal{P}^w_{\ell-1,k-1}(\mathbb{R}^{2m}, \mathbb{S}^\pm)$$

is surjective and

$$\mathcal{M}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cong \bigoplus_{a=0}^{\ell-k} \bigoplus_{b=0}^{k} \mathcal{S}_{\ell-a,k-b}(\mathbb{R}^{2m}, \mathbb{S}^\sigma),$$

where $\sigma = \pm$ if $a + b$ is even, and $\sigma = \mp$ otherwise.

Proof. Assume $\ell - k$ to be odd, then

$$\left\lfloor \frac{\ell - k}{2} \right\rfloor = \frac{\ell - k - 1}{2},$$

meaning that both of the summations over b in theorem 6.3.8 have the same upper bound. In this case it immediately follows that

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \big/ \mathcal{P}^w_{\ell-1,k-1}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cong \bigoplus_{b=0}^{\lfloor \frac{\ell-k}{2} \rfloor} \bigoplus_{c=0}^{\lfloor \frac{k}{2} \rfloor} (\ell - 2b, k - 2c) \otimes \mathbb{S}^\pm,$$

the desired result then follows from proposition 2.4.5 (or from theorem 6.2.12 if $m = 4$) and corollary 5.2.6.

If $\ell - k$ is even then it is slightly more complicated, and to clarify matters we have included an explicit example below the proof (example 6.3.10). Nevertheless one can easily find that

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \big/ \mathcal{P}^w_{\ell-1,k-1}(\mathbb{R}^{2m}, \mathbb{S}^\pm) \cong \mathcal{A}_{\ell,k} \big/ \mathcal{B}_{\ell,k},$$

where

$$\mathcal{A}_{\ell,k} \cong \bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{c=0}^{\lfloor \frac{k}{2} \rfloor - a} (k - 2a, k - 2a - 2c) \otimes \mathbb{S}^\pm \oplus \bigoplus_{b=0}^{\lfloor \frac{\ell-k}{2} \rfloor - 1} \bigoplus_{c=0}^{\lfloor \frac{k}{2} \rfloor} (\ell - 2b, k - 2c) \otimes \mathbb{S}^\pm$$

$$\mathcal{B}_{\ell,k} \cong \bigoplus_{a=0}^{\lfloor \frac{k-1}{2} \rfloor} \bigoplus_{c=0}^{\lfloor \frac{k-1}{2} \rfloor - a} (k - 1 - 2a, k - 1 - 2a - 2c) \otimes \mathbb{S}^\pm.$$

Note that the second part of $\mathcal{A}_{\ell,k}$ is only partially what we are after as the components $(k, k-2c)'_\pm$ and $(k, k-2c-1)'_\mp$ appear to be missing. Again from proposition 2.4.5 (or theorem 6.2.12), one can conclude that

$$\begin{aligned} \mathcal{A}_{\ell,k} \big/ \mathcal{B}_{\ell,k} \cong & \bigoplus_{c=0}^{\lfloor \frac{k}{2} \rfloor} (k, k - 2c)'_\pm \oplus \bigoplus_{c=0}^{\lfloor \frac{k-1}{2} \rfloor} (k, k - 2c - 1)'_\mp \\ & \oplus \bigoplus_{b=0}^{\lfloor \frac{\ell-k}{2} \rfloor - 1} \bigoplus_{c=0}^{\lfloor \frac{k}{2} \rfloor} (\ell - 2b, k - 2c) \otimes \mathbb{S}^\pm, \end{aligned}$$

which, along with corollary 5.2.6, finishes the proof. $\square$

Example 6.3.10. Consider the representations $(5,3)_{\mathrm{GL}(m)}$ and $(4,2)_{\mathrm{GL}(m)}$, then

$$\begin{aligned}(5,3)_{\mathrm{GL}(m)} &\cong (5,3)\oplus(5,1)\oplus(4,2)\oplus(4,0)\oplus(3,3)\oplus 2(3,1)\oplus(2,0)\oplus(1,1)\\ (4,2)_{\mathrm{GL}(m)} &\cong (4,2)\oplus(4,0)\oplus(3,1)\oplus(2,2)\oplus 2(2,0)\oplus(0,0).\end{aligned}$$

Assume, for the sake of clarity, that m is odd so we do not have to keep track of the signs of the spinors, then

$$\begin{aligned}&(5,3)_{\mathrm{GL}(m)}\otimes\mathbb{S}\big/(4,2)_{\mathrm{GL}(m)}\otimes\mathbb{S}\\ &\cong \big((5,3)\oplus(5,1)\oplus(3,3)\oplus(3,1)\oplus(1,1)\big)\otimes\mathbb{S}\big/\big((2,2)\oplus(2,0)\oplus(0,0)\big)\otimes\mathbb{S}\end{aligned}$$

i.e. we started by removing all weights that appeared in both spaces. The first space is then given by:

$$\begin{aligned}\Big(\big((5,3)\oplus(5,1)\big)\otimes\mathbb{S}\Big)&\oplus(3,3)'\oplus(3,2)'\oplus(2,2)'\oplus(3,1)'\oplus(3,0)'\\ &\oplus(2,1)'\oplus(2,0)'\oplus(1,1)'\oplus(1,0)'\oplus(0,0)',\end{aligned}$$

whereas the second space can be written as

$$(2,2)'\oplus(2,1)'\oplus(1,1)'\oplus(2,0)'\oplus(1,0)'\oplus(0,0)'.$$

Eliminating these components from the first space finally yields:

$$\begin{aligned}&\cong\Big(\big((5,3)\oplus(5,1)\big)\otimes\mathbb{S}\Big)\oplus(3,3)'\oplus(3,2)'\oplus(3,1)'\oplus(3,0)'\\ &\cong\bigoplus_{i=0}^{2}\bigoplus_{j=0}^{3}(5-i,3-j)'\end{aligned}$$

as is desired.

Similarly to the case $(k+1,k)$, we can conclude that

$$\Delta_w:\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})\to\mathcal{P}^w_{\ell-2,k-2}(\mathbb{R}^{2m},\mathbb{C})$$

is surjective and from $-\partial_w^2=\Delta_w$ we immediately obtain:

Corollary 6.3.11. *Let $\ell\geq k$ and $m\geq 4$ then:*

$$\mathcal{H}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})\otimes\mathbb{S}^{\pm}\cong\mathcal{M}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{S}^{\pm})\oplus\mathcal{M}^w_{\ell-1,k-1}(\mathbb{R}^{2m},\mathbb{S}^{\pm})$$

Moreover, we also find the following decomposition:

Proposition 6.3.12.

$$\begin{aligned}\mathcal{H}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})\cong&\bigoplus_{i=0}^{\lfloor\frac{\ell-k}{2}\rfloor}\bigoplus_{j=0}^{\lfloor\frac{k}{2}\rfloor}\mathcal{H}_{\ell-2i,k-2j}(\mathbb{R}^{2m},\mathbb{C})\\ &\oplus\bigoplus_{i=0}^{\lfloor\frac{\ell-k-1}{2}\rfloor}\bigoplus_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\mathcal{H}_{\ell-1-2i,k-1-2j}(\mathbb{R}^{2m},\mathbb{C}).\end{aligned}$$

Proof. Because Δ_w is surjective, we have that

$$\mathcal{H}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C}) \cong {}^{\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C})}\big/_{\mathcal{P}^w_{\ell-2,k-2}(\mathbb{R}^{2m},\mathbb{C})}$$

from which the desired result immediately follows. $\square$

Combining all these results proves the following theorem, which is the generalisation of theorem 6.2.9:

Theorem 6.3.13 (General Wedge-Fischer theorem)**.** *Let $\ell \geq k$, $m \geq 4$, then*

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{C}) \cong \bigoplus_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \mathcal{H}^w_{\ell-2i,k-2i}(\mathbb{R}^{2m},\mathbb{C}).$$

Moreover, we have that

$$\mathcal{P}^w_{\ell,k}(\mathbb{R}^{2m},\mathbb{S}^{\pm}) \cong \bigoplus_{i=0}^{k} \mathcal{M}^w_{\ell-i,k-i}(\mathbb{R}^{2m},\mathbb{S}^{\pm}).$$

CHAPTER 7

A higher spin Fueter theorem

I may not have gone where I intended to go,
but I think I've ended up where I needed to be.
Dirk Gently

The aim of this chapter is to introduce a higher spin version of Fueter's theorem, an elegant result in classical Clifford analysis, which says that given a holomorphic function $f(z)$ on $\Omega \subset \mathbb{C}$, the function

$$F(x) := \Delta_x^{\mu}\big[f(\bar{e}_1 x)\big] \qquad \Big(\mu = \frac{m}{2} - 1\Big)$$

is monogenic, in case $m \in 2\mathbb{Z}^+$ is even. For odd m there exists a similar result, but this requires making use of Fourier multipliers to give meaning to fractional powers of the operator Δ_x (see for instance [69]). Note that it makes sense to write $f(\bar{e}_1 x)$ as one has that

$$z \in \mathbb{C} \mapsto \bar{e}_1 x = \bar{e}_1(\tilde{x} + x_1 e_1) = x_1 + \tilde{x} e_1 \in \mathbb{R}_m^{(0)} \oplus \mathbb{R}_m^{(2)} \ ,$$

where $x \in \mathbb{R}^m$ and $\tilde{x} \in \mathbb{R}^{m-1}$. One can then see that $\bar{e_1} x = x_1 + \omega e_1 |\tilde{x}|$, which behaves like a complex variable z.
The classical Fueter theorem can be seen as a connection between solutions for one fixed operator (albeit in different dimensions: a 'minimal' dimension $m = 2$ and an 'arbitrary' dimension $m \in 2\mathbb{N}$), which can both be expressed in terms of the same family of orthogonal polynomials. Indeed, there is a natural connection between powers z^k and the Gegenbauer polynomials, i.e. if $z = a + ib$ then

$$z^k = |z|^k \left(\frac{k}{2} C_k^0 \left(\frac{a}{|z|} \right) + i \frac{b}{|z|} C_{k-1}^1 \left(\frac{a}{|z|} \right) \right).$$

This leads to:

Theorem 7.0.1. *Let $k \in \mathbb{Z}^+$ then:*

$$z^{k+m-2} \mapsto \Delta_x^\mu \big[(\bar{e}_1 x)^{k+m-2}\big] \in \ker \partial_x.$$

Moreover, this monogenic polynomial can be written as:

$$\Delta_x^\mu \big[(\bar{e}_1 x)^{k+m-2}\big] = c_{k,m}|x|^k \left(C_k^{\frac{m}{2}} \left(\frac{x_1}{|x|}\right) + \frac{xe_1}{|x|} C_{k-1}^{\frac{m}{2}} \left(\frac{x_1}{|x|}\right)\right),$$

with $c_{k,m}$ a non-trivial real constant.

One way to see that the Gegenbauer polynomials should indeed appear is by noting that the substitution $z \mapsto \bar{e}_1 x$ has the property that one direction plays a preferential role (it maps a complex number to a Clifford number of a rather special form, being the sum of a scalar and a bivector). This means that the Fueter image of the holomorphic power z^{k+m-2}, i.e. the polynomial $\Delta_x^\mu\big[(\bar{e}_1 x)^{k+m-2}\big]$, gives a monogenic polynomial which remains invariant under the action of the subgroup of the Spin group which leaves this direction e_1 invariant, this then leads to the Gegenbauer solutions found in section 2.5.3.

7.1 The classical theorem revisited

We will give another proof for the Fueter theorem in the classical sense. Although there exist many proofs for this well-known fact already, we have not found any reference to our new approach in the current literature. Moreover, it is a method which will allow a generalisation to the case we are interested in (and, by extension, to any function theory in which one has an inversion and a Fischer decomposition at its disposal). In order to reobtain the classical version of the Fueter theorem, we will heavily rely on the Kelvin inversion and its role as a conformal symmetry for the Dirac operator, see theorem 2.3.25, which was defined in terms of a parameter $\alpha \in \mathbb{R}$ by means of

$$\mathcal{I}_m^{(\alpha)}\big[f(x)\big] := \frac{x}{|x|^{m-\alpha}} f\left(\frac{x}{|x|^2}\right).$$

One can rewrite the monogenic inversion for $\alpha = 0$ as the composition of a multiplication operator involving the norm $|x|$ and an inversion which generates powers of $\bar{e}_m x$ under a suitable conjugation. Indeed, first of all note that $\mathcal{I}_m^{(0)}$ maps $\ker \partial_x$ to itself (barring a possible singularity at the origin), whereas $\mathcal{I}_m^{(m-2)}$ clearly amounts to multiplying $\ker \partial_x$ with a polynomial power of $|x|^2$ (for m even):

$$\mathcal{I}_m^{(m-2)} = |x|^{m-2}\mathcal{I}_m^{(0)}\ . \tag{7.1}$$

However, the case $\alpha = m-2$ does has it advantages, as it was shown in e.g. [28], this operator satisfies the operator identity

$$\mathcal{I}_m^{(m-2)}\partial_{x_1}\mathcal{I}_m^{(m-2)} = -|x|^2\partial_{x_1} + x_1(2\mathbb{E}_x + 1) + \tilde{x}e_1.$$

The value $\alpha = m-2$ is indeed special here, as it reduces the inversion in $\mathbb{R}^m$ to the classical inversion with respect to a circle in the complex plane. The operator above can now be seen as a raising operator (it will raise the degree of homogeneity when acting on a polynomial of degree k), whose repeated action on $1 \in \mathbb{C}$ is given by

$$\big(\mathcal{I}_m^{(m-2)} \partial_{x_1}^k \mathcal{I}_m^{(m-2)}\big)[1] = k!(\bar{e}_1 x)^k \ .$$

This thus means that this raising operator generates holomorphic powers for $m = 2$, because in that case $(\bar{e}_2 x)$ behaves like a complex variable z. Recalling equation (7.1), one finds that

$$k!(\bar{e}_1 x)^k I = \big(|x|^{m-2}\mathcal{I}_m^{(0)} \partial_{x_1}^k |x|^{m-2}\mathcal{I}_m^{(0)}\big)[I], \tag{7.2}$$

where I is the idempotent used to realise the spinor spaces from section 2.2. This idempotent is not crucial for calculations, it basically reduces to a right multiplication, but one needs it in order to obtain spinor-valued polynomials (our argument involves irreducibility). The left hand side of equation 7.2 is a spinor-valued polynomial and can thus be decomposed into monogenic polynomials by means of the Fischer decomposition, see theorem 2.3.20:

$$(\bar{e}_1 x)^k I = \sum_{i=0}^{k} x^i M_{k-i}(x),$$

with $M_{k-i}(x) \in \mathcal{M}_{k-i}(\mathbb{R}^m, \mathbb{S}^{\sigma_i})$, where $\sigma_i = (-1)^i$. We will now show that $(\bar{e}_m x)^k I$ does not have all the Fischer components, as a matter of fact, there will be a *fixed* maximal number of components, and this allows us to generate monogenic polynomials using an operator independent of the value for k.

Remark 7.1.1. Each polynomial $(\bar{e}_1 x)^k I$ also contains a monogenic component (for $i = 0$ in the Fischer decomposition, but the projection on this summand heavily depends on the value for k. The component we are after here, is the 'last term' which will always have the same index (for $k \geq m-2$).

To see why the number of summands is fixed (i.e. why this last term always has the same index i), we will use the fact that the operator $\partial_{x_1}^k$ can itself be decomposed into a sum of differential operators, using a dual version of the (scalar) Fischer decomposition in terms of harmonics, see theorem 2.3.8. This is inspired by the fact that $\mathrm{span}_{\mathbb{C}}\{\partial_{x_j} : 1 \leq j \leq m\} \cong \mathbb{C}^m$ provides another model for the defining representation (just like the coordinates x_j themselves), which means that polynomial expressions can be written in terms of 'harmonics'. In particular, this means that

$$x_1^k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} |x|^{2j} H_{k-2j}(x) \implies \partial_{x_1}^k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \Delta_x^j H_{k-2j}(\partial_x) \ ,$$

where we have thus replaced each variable x_j by its associated partial derivative. Plugging this into (7.2), we find that

$$(\bar{e}_1 x)^k I = \left(|x|^{m-2} \mathcal{I}_m^{(0)} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \Delta_x^j H_{k-2j}(\partial_x) |x|^{m-2} \mathcal{I}_m^{(0)} \right) [I] \ ,$$

but we can restrict this summation to a *fixed* number of terms, regardless the value for k (at least when $k \geq m-2$). This is due to the fact that

$$j \geq \frac{m}{2} \ \Rightarrow \ \Delta_x^j |x|^{m-2} \mathcal{I}_m^{(0)} [I] = 0 \ .$$

In other words, for $k \geq m-2$ one has that

$$(\bar{e}_1 x)^k I = \left(|x|^{m-2} \mathcal{I}_m^{(0)} \sum_{j=0}^{\frac{m}{2}-1} \Delta_x^j H_{k-2j}(\partial_x) |x|^{m-2} \mathcal{I}_m^{(0)} \right) [I] \ , \tag{7.3}$$

where the upper index of the summation between brackets has now become a fixed constant depending on the value for m only (which, we recall, is supposed to be an even number here). We thus have found a polynomial of degree k for which the transformation behaviour under the Spin group can easily be determined. Indeed, it is built in terms of invariants (such as Δ_x and $|x|^2$) and pieces transforming as an element of $\mathcal{H}_a \otimes \mathbb{S}^+$, with $a \in \{k, k-2, \ldots, k-(m-2)\}$. This is due to the fact that on the level of representations, the information is encoded in the behaviour of $H_{k-2j}(\partial_x)$ under the action of the Spin group (and this operator behaves as an element of a space of harmonic polynomials). Put differently, we have that

$$(\bar{e}_1 x)^k I \in \left(\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \oplus \ldots \oplus |x|^{m-2} \mathcal{H}_{k-(m-2)}(\mathbb{R}^m, \mathbb{C}) \right) \otimes \mathbb{S}^+ \ ,$$

where the spinor space at the right appears because of the idempotent $I \in \mathbb{S}^+$. This thus means that regardless the value for $k \in \mathbb{Z}^+$ (for $k < m-2$ we adopt the convention that the space of harmonics is trivial, which makes sense as the action of the Laplace power will give zero anyway), one always has that

$$\Delta_x^{\frac{m}{2}-1} \big((\bar{e}_1 x)^k I \big) \in \mathcal{H}_{k-(m-2)}(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S}^+ \ .$$

Note that this space of $\mathbb{S}^+$-valued harmonics still decomposes as

$$\mathcal{H}_{k-(m-2)}(\mathbb{R}^m, \mathbb{S}^+) = \mathcal{M}_{k-(m-2)}(\mathbb{R}^m, \mathbb{S}^+) \oplus x \mathcal{M}_{k-(m-2)-1}(\mathbb{R}^m, \mathbb{S}^-) \ ,$$

see theorem 2.3.19, but the projection of $(\bar{e}_1 x)^k$ on the last summand is trivial. Indeed, for $j = \frac{m}{2} - 1$ (the maximal index in the summation (7.3) above), we get that

$$(\bar{e}_1 x)^k I = \text{L.O.T.} + \frac{c_m}{k!} |x|^{m-2} \mathcal{I}_m^{(0)} H_{k-(m-2)}(\partial_x) \mathcal{I}_m^{(0)} [I] \ ,$$

with $c_m \in \mathbb{R}_0$ a constant coming from the action of $\Delta_m^{\frac{m}{2}-1}$ on $|x|^{m-2}$ and with 'L.O.T.' the lower order terms (referring to the exponent of $|x|^2$). From this it is clear that the action of the power of Laplace gives a monogenic, since both the inversion and the differential operator $H_{k-(m-2)}(\partial_x)$ act as endomorphisms on the kernel of the Dirac operator. We have thus obtained the following conclusion:

$$f(z) \in \ker \bar{\partial}_z \;\mapsto\; \Delta_x^{\mu}\big(f(\bar{e}_1 x)\big) \in \ker \partial_x \qquad \Big(\mu = \frac{m}{2} - 1\Big) \ .$$

To recapitulate, we needed 3 basic ingredients:

(i) A suitable substitution, mapping $z \in \mathbb{C}$ to Clifford numbers $(\bar{e}_1 x)$ of a certain special form (which already reveals the invariance properties).

(ii) An inversion operator (indexed by a parameter), which can be related to both a 'raising operator' (in a special dimension, $m = 2$ in the classical Fueter theorem, this describes the starting point for the substitution) and an endomorphism on the kernel of an operator on $\mathbb{R}^m$ (the operator ∂_x).

(iii) A Fischer decomposition theorem, which then essentially tells us how to decompose polynomials in terms of solutions for this operator on $\mathbb{R}^m$, i.e. into irreducible representation spaces for the Spin group.

We will repeat this story for the operator $\partial_w = \partial_x \wedge \partial_u$, whereby solutions for ∂_w can be obtained from generalisations of Gegenbauer solutions for the Dirac operator (i.e. the Fueter images of the holomorphic powers) in dimension $m = 4$. terms of an inversion.

7.2 Fueter theorem for ∂_w

Seen the importance of the inversion operator in the proof for Fueter's theorem, we start from the following definition:

Definition 7.2.1. For arbitrary $\alpha \in \mathbb{R}$, we define the following inversion on $\mathbb{C}_m$-valued functions $f(x, u)$ depending on two vector variables in $\mathbb{R}^m$:

$$\mathcal{I}_{x,u}^{(m,\alpha)} f(x, u) := (x \wedge u)|x \wedge u|^{1-m+\alpha} f\left(\frac{x}{|x \wedge u|}, \frac{u}{|x \wedge u|}\right) .$$

Just like in the classical case, this inversion preserves an important subspace of $\ker \partial_w$ (for a suitable choice of the parameter α):

Lemma 7.2.2. *The inversion $\mathcal{I}_{x,u}^{(m,\alpha)}$ preserves $\ker(\partial_x, \partial_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle)$ if and only if $\alpha = 0$.*

Proof. It is sufficient to prove this statement for polynomials and since they belong to the kernel of $\langle x, \partial_u \rangle$ and $\langle u, \partial_x \rangle$ one can conclude that the degrees of homogeneity in (x, u) are equal. We can therefore use a single index $k \in \mathbb{Z}^+$ to denote the degree of these polynomials. Let $P_k(x,u) \in \mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{C}_m)$ then we look for a value a, depending on k, such that

$$(x \wedge u)|x \wedge u|^a P_k(x,u) \in \ker(\partial_x, \partial_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle).$$

It is trivial that the kernel of the skew-Euler operators is preserved and by symmetry it is sufficient to verify that it also holds for ∂_x. One can easily find the following commutators by straightforward calculations:

$$\begin{aligned}
[\partial_x, |x \wedge u|^a] &= -a|x \wedge u|^{a-2}(x \wedge u)u \\
[\partial_x, x \wedge u] &= (-m - 2\mathbb{E}_x + 1)u + 2x \langle u, \partial_x \rangle .
\end{aligned}$$

This implies that

$$\begin{aligned}
[\partial_x, (x \wedge u)|x \wedge u|^a] =& \big((-m - 2\mathbb{E}_x + 1)u + 2x \langle u, \partial_x \rangle \big)|x \wedge u|^a \\
& - a|x \wedge u|^{a-2}(x \wedge u)^2 u.
\end{aligned}$$

Acting on our polynomial $P_k(x,u)$ yields:

$$\partial_x (x \wedge u)|x \wedge u|^a P_k(x,u) = (-m - 2k + 1 - a)|x \wedge u|^a u P_k(x,u)$$

which is zero if and only if $a = 1 - m - 2k$ which corresponds to $\alpha = 0$. □

Remark 7.2.3. Note that, in contrast to the one-variable case, the inversion does not act as an endomorphism on the full kernel of the operator $\partial_x \wedge \partial_u$. However, we still have that the resulting images are rational functions.

In analogy to the classic case, we will start from special solutions on a space of 'minimal dimension' and then use a substitution to end up with functions defined on spaces of higher dimension. The operator

$$D_{12} = \partial_{x_1}\partial_{u_2} - \partial_{x_2}\partial_{u_1} \ ,$$

will hereby generalise the role played by ∂_{x_1} in the classical case. Note also that we have singled out 2 preferred directions here, which has its repercussions on the invariance properties under the action of the Spin group. The analogue of the complex holomorphic powers z^k on $\mathbb{R}^2 \cong \mathbb{C}$ is now given by the special functions

$$\mathcal{I}_{x,u}^{(4,0)} D_{12}^k \mathcal{I}_{x,u}^{(4,0)}[I] \in \mathcal{S}_{k,k}(\mathbb{R}^8, \mathbb{S}^+) \ .$$

Note that these are indeed polynomials, which follows from the explicit form of the raising operator:

Theorem 7.2.4. *Let $\alpha \in \mathbb{R}$ and $\mathrm{X}_\alpha := \mathcal{I}_{x,u}^{(m,\alpha)} D_{12} \mathcal{I}_{x,u}^{(m,\alpha)}$, then we have the following operator identity:*

$$\begin{aligned}
\mathrm{X}_\alpha = 2x \wedge u e_{12} &- (\underline{x} \wedge \underline{u} - x \wedge u - \underline{x}_2 \wedge \underline{u}_2)(1 - m + \alpha - \mathbb{E}_x - \mathbb{E}_u)e_{12} - \mathrm{X}_\alpha^H \\
&+ (x \wedge u)\left(-\underline{x}\partial_{\underline{x}_2} - \underline{u}\partial_{\underline{u}_2} + \mathbb{E}_{\underline{x}_2} + \mathbb{E}_{\underline{u}_2}\right) e_{12},
\end{aligned}$$

where X_α^H denotes the harmonic raising operator found in chapter 4:

$$\begin{aligned}\mathrm{X}_\alpha^H &:= |x\wedge u|^{1+m-\alpha+2k} D_{12} |x\wedge u|^{1-m+\alpha-2k}\\ &= (x_1u_2 - x_2u_1)(1-m+\alpha-\mathbb{E}_x-\mathbb{E}_u)(2-m+\alpha-\mathbb{E}_x-\mathbb{E}_u)\\ &\quad + R(1-m+\alpha-\mathbb{E}_x-\mathbb{E}_u) + |x\wedge u|^2 D_{12}\end{aligned}$$

with

$$\begin{aligned}R :=\ & |x|^2(u_2\partial_{x_1} - u_1\partial_{x_2}) + |u|^2(x_1\partial_{u_2} - x_2\partial_{u_1})\\ &+ \langle x,u\rangle\,(x_1\partial_{x_2} - x_2\partial_{x_1} + u_2\partial_{u_1} - u_1\partial_{u_2}).\end{aligned}$$

Proof. Because we are interested in the repeated action on I (or any element of $\mathbb{S}^\pm$) and that X_α raises both the degree in x and u by 1 it is sufficient to prove this result on $\mathcal{P}_{k,k}(\mathbb{R}^{2m},\mathbb{C}_m)$. Let $a = 1-m+\alpha-2k$ then, on $\mathcal{P}_{k,k}(\mathbb{R}^{2m},\mathbb{C}_m)$, we have that

$$\mathrm{X}_\alpha = (x\wedge u)|x\wedge u|^{-a} D_{12}(x\wedge u)|x\wedge u|^a.$$

Consider first $D_{12}(x\wedge u)|x\wedge u|^a$, then we can write this as:

$$\begin{aligned}D_{12}(x\wedge u)|x\wedge u|^a &= 2e_1e_2|x\wedge u|^a + (x\wedge u)D_{12}|x\wedge u|^a\\ &\quad + (xe_2+x_2)\left(a|x\wedge u|^{a-2}(x_1|u|^2 - \langle x,u\rangle\, u_1) + |x\wedge u|^a\partial_{x_1}\right)\\ &\quad - (xe_1+x_1)\left(a|x\wedge u|^{a-2}(x_2|u|^2 - \langle x,u\rangle\, u_2) + |x\wedge u|^a\partial_{x_2}\right)\\ &\quad + (e_1u+u_1)\left(a|x\wedge u|^{a-2}(u_2|x|^2 - \langle x,u\rangle\, x_2) + |x\wedge u|^a\partial_{u_2}\right)\\ &\quad - (e_2u+u_2)\left(a|x\wedge u|^{a-2}(u_1|x|^2 - \langle x,u\rangle\, x_1) + |x\wedge u|^a\partial_{u_1}\right)\end{aligned}$$

which is of the following form:

$$2e_{12}|x\wedge u|^a + (x\wedge u)D_{12}|x\wedge u|^a + a|x\wedge u|^{a-2}A + |x\wedge u|^a B$$

and we will find alternative expressions for A and B. After some straightforward calculations, and using the fact that $x_1e_2 - x_2e_1 = e_{12}\underline{x}_2 = -\underline{x}_2e_{12}$, we find

$$A = (x\wedge u)(\underline{x}\wedge\underline{u} - x\wedge u - \underline{x}_2\wedge\underline{u}_2)e_{12}.$$

On the other hand we have:

$$\begin{aligned}B &= (xe_2+x_2)\partial_{x_1} - (xe_1+x_1)\partial_{x_2} + (e_1u+u_1)\partial_{u_2} - (e_2u+u_2)\partial_{u_1}\\ &= (xe_2+x_2)\partial_{x_1} - (xe_1+x_1)\partial_{x_2} + (-ue_1-u_1)\partial_{u_2} - (-ue_2-u_2)\partial_{u_1}\\ &= x(e_2\partial_{x_1} - e_1\partial_{x_2}) + u(e_2\partial_{u_1} - e_1\partial_{u_2}) + x_2\partial_{x_1} - x_1\partial_{x_2} + u_2\partial_{u_1} - u_1\partial_{u_2}\\ &= -x\partial_{\underline{x}_2}e_{12} - u\partial_{\underline{u}_2}e_{12} + (\underline{x}_2\partial_{\underline{x}_2} + \mathbb{E}_{\underline{x}_2})e_{12} + (\underline{u}_2\partial_{\underline{u}_2} + \mathbb{E}_{\underline{u}_2})e_{12}\\ &= \left(-\underline{x}\partial_{\underline{x}_2} - \underline{u}\partial_{\underline{u}_2} + \mathbb{E}_{\underline{x}_2} + \mathbb{E}_{\underline{u}_2}\right)e_{12}.\end{aligned}$$

Combining everything leaves us with:

$$\mathrm{X}_\alpha = 2(x\wedge u)e_{12} - |x\wedge u|^{2-a}D_{12}|x\wedge u|^a + a(x\wedge u)|x\wedge u|^{-2}A + (x\wedge u)B$$

$$
\begin{aligned}
&= (2x \wedge u - a\underline{x} \wedge \underline{u} + ax \wedge u + a\underline{x}_2 \wedge \underline{u}_2)e_{12} - |x \wedge u|^{2-a} D_{12} |x \wedge u|^a \\
&+ (x \wedge u) \left(-\underline{x}\partial_{\underline{x}_2} - \underline{u}\partial_{\underline{u}_2} + \mathbb{E}_{\underline{x}_2} + \mathbb{E}_{\underline{u}_2} \right) e_{12}.
\end{aligned}
$$

The desired outcome now follows from theorem 4.1.7. $\square$

We now need a kind of substitution which mimics the mapping $z \mapsto \bar{e}_1 x$ from the classical case. For that purpose, we consider the following decomposition on $\mathbb{R}^4$:

$$
\mathbb{R}^4 \ni x^{(4)} = \underline{x}_2^{(4)} + \underline{x}^{(4)} \in \mathbb{R}^2 \oplus \mathbb{R}^2 \ ,
$$

where $\underline{x}_2^{(4)} = x_1 e_1 + x_2 e_2$ and $\underline{x}^{(4)} = x_3 e_3 + x_4 e_4$. The idea is that the first part, involving (e_1, e_2) will be kept, whereas the second part will be 'expanded' (notationally, this amounts to replacing 4 by an arbitrary dimension m). This mimics the classical case, where we essentially kept e_1 as a preferred direction and then 'expanded' $x_2 \mapsto \tilde{x} \in \mathbb{R}^{m-1}$. One can show that the polynomials $\mathcal{I}_{x,u}^{(m,\alpha)} D_{12}^k \mathcal{I}_{x,u}^{(m,\alpha)}[I]$ are of the form

$$
\sum_{i=0}^{k} \sum_{j=0}^{i} c_{k,i,j}(m-\alpha)(\underline{x}_2 \wedge \underline{u}_2)^{k-i} (\underline{x} \wedge \underline{u})^{i-j} (x \wedge u)^j I
$$

where the $c_{k,i,j}(m-\alpha) \in \mathbb{C}$ are polynomial expressions in $m - \alpha$. One way to see this is via induction where we will use the fact that

$$
\mathcal{I}_{x,u}^{(m,\alpha)} D_{12}^{k+1} \mathcal{I}_{x,u}^{(m,\alpha)}[I] = -\mathcal{I}_{x,u}^{(m,\alpha)} D_{12} \mathcal{I}_{x,u}^{(m,\alpha)} \left(\mathcal{I}_{x,u}^{(m,\alpha)} D_{12}^k \mathcal{I}_{x,u}^{(m,\alpha)}[I] \right).
$$

First of all, let us look at the action of our inversion $\mathcal{I}_{x,u}^{(m,\alpha)}$, which is a left multiplication with $(x \wedge u)|x \wedge u|^{1+\alpha-m-2k}$. While the variables $\underline{x}_2 \wedge \underline{u}_2$ and $\underline{x} \wedge \underline{u}$ commute, this no longer holds when looking at $x \wedge u$. Fortunately, the commutation relations can be expressed in terms of our variables:

$$
\begin{aligned}
[x \wedge u \ , \ \underline{x} \wedge \underline{u}] &= -2\underline{x} \wedge \underline{u}(x \wedge u - \underline{x}_2 \wedge \underline{u}_2 - \underline{x} \wedge \underline{u}) \\
[x \wedge u \ , \ \underline{x}_2 \wedge \underline{u}_2] &= -2\underline{x}_2 \wedge \underline{u}_2 \left(x \wedge u - \underline{x} \wedge \underline{u} - \underline{x}_2 \wedge \underline{u}_2 \right).
\end{aligned}
$$

Using these relations one can show that:

$$
\begin{aligned}
&\mathcal{I}_{x,u}^{(m,\alpha)} \sum_{i=0}^{k} \sum_{j=0}^{i} c_{k,i,j}(m-\alpha)(\underline{x}_2 \wedge \underline{u}_2)^{k-i} (\underline{x} \wedge \underline{u})^{i-j} (x \wedge u)^j \\
=&|x \wedge u|^{1+\alpha-m-2k} \sum_{i=0}^{k+1} \sum_{j=0}^{i} \tilde{c}_{k,i,j}(m-\alpha)(\underline{x}_2 \wedge \underline{u}_2)^{k+1-i} (\underline{x} \wedge \underline{u})^{i-j} (x \wedge u)^j.
\end{aligned}
$$

where not all terms occur, depending on the parity of k. To calculate the action of D_{12} one has to make a distinction based on the parities of the powers of the wedge-variables and after some tedious, but straight-forward, calculations one finds that D_{12} preserves this form (e.g. the case where all the powers are even can be found in lemma 4.1.14). Moreover, the only dependency on α and m

comes from deriving the factor $|x \wedge u|^{1+\alpha-m-2k}$ which means that our coefficients depend on k, i, j and $m-\alpha$. Finally, applying the inversion one last time, this time this simplifies to a left multiplication with $(x \wedge u)|x \wedge u|^{m+2k-1-\alpha}$, yields the desired result.
In particular we have that:

$$\begin{aligned}
&\mathcal{I}^{(4,0)}_{x,u} D^k_{12} \mathcal{I}^{(4,0)}_{x,u}[I] \\
&\qquad = \sum_{i=0}^{k} \sum_{j=0}^{i} c_{k,i,j}(4)(\underline{x}^{(4)}_2 \wedge \underline{u}^{(4)}_2)^{k-i} (\underline{x}^{(4)} \wedge \underline{u}^{(4)})^{i-j} (x^{(4)} \wedge u^{(4)})^j I.
\end{aligned}$$

If we look at the general splitting $\mathbb{R}^m = \mathbb{R}^2 \oplus \mathbb{R}^{m-2}$, then we can also write each vector $x = \underline{x}_2 + \underline{x}$ and thus, replacing the variables by $\underline{x}_2 \wedge \underline{u}_2$ (resp. $\underline{x} \wedge \underline{u}$ and $x \wedge u$), gives us a polynomial on $\mathbb{R}^{2m}$ (in other words: the substitution formally amounts to dropping the upper index 4). This means that we end up with the polynomial:

$$\mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I]$$

and thus our substitution map is given by:

$$\mathcal{I}^{(4,0)}_{x,u} D^k_{12} \mathcal{I}^{(4,0)}_{x,u}[I] \longmapsto \mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I].$$

Indeed, as

$$\mathcal{I}^{(m,\alpha)}_{x,u} D^k_{12} \mathcal{I}^{(m,\alpha)}_{x,u}[I] = \sum_{i=0}^{k} \sum_{j=0}^{i} c_{k,i,j}(m-\alpha)(\underline{x}_2 \wedge \underline{u}_2)^{k-i} (\underline{x} \wedge \underline{u})^{i-j} (x \wedge u)^j I,$$

letting $\alpha = m-4$ then yields the desired expression. Just like in the one variable case, we will now consider a dual version for the Fischer decomposition theorem 6.2.9, applied to powers of the operator D_{12}, which leads to

Proposition 7.2.5. *Let $k \in \mathbb{Z}^+$ and m be even. We then have that*

$$\mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I] \in \left(\bigoplus_{i=0}^{\frac{m}{2}-2} |x \wedge u|^{2i} \mathcal{H}^w_{k-2i}(\mathbb{R}^{2m}, \mathbb{C}) \right) \otimes \mathbb{S}^+,$$

where we adopt the convention that for $k-2i<0$ the spaces on the right hand side do not appear. Note that the range for i in this summation is again restricted by a fixed number depending on m only (not k).

Proof. Let us apply theorem 6.2.9 to the polynomial $(x_1u_2 - x_2u_1)^k$ which gives:

$$(x_1u_2 - x_2u_1)^k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} |x \wedge u|^{2i} Q_{k-2i}(x,u)$$

where $Q_{k-2i}(x,u) \in \mathcal{H}^w_{k-2i}(\mathbb{R}^{2m},\mathbb{C})$. This means that:

$$\mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I] = \left(\mathcal{I}^{(m,m-4)}_{x,u} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \Delta^i_w Q_{k-2i}(\partial_x,\partial_u) \mathcal{I}^{(m,m-4)}_{x,u} \right) [I].$$

Considering the fact that $\mathcal{I}^{(m,m-4)}_{x,u} = |x \wedge u|^{m-4} \mathcal{I}^{(m,0)}_{x,u}$ we can conclude that

$$i > \frac{m}{2} - 2 \ \Rightarrow \ \Delta^i_w \mathcal{I}^{(m,m-4)}_{x,u}[I] = 0 \ ,$$

which means that there is a fixed amount of terms in the summation. For $k \geq m-4$, we thus have that:

$$\mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I] = \left(\mathcal{I}^{(m,m-4)}_{x,u} \sum_{i=0}^{\frac{m}{2}-2} \Delta^i_w Q_{k-2i}(\partial_x,\partial_u) \mathcal{I}^{(m,m-4)}_{x,u} \right) [I] \ ,$$

where the summation is now restricted independently of k. Each of these terms consists of invariants (e.g. powers of Δ_w and $|x \wedge u|^2$) and operators $Q_{k-2i}(\partial_x,\partial_u)$ which thus encode the behaviour of the polynomial at the left hand side under transformations. This implies that each of these terms transforms as a polynomial in $\mathcal{H}^w_{k-2i}(\mathbb{R}^{2m},\mathbb{C}) \otimes \mathbb{S}^+$. In other words we have found that:

$$\mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I] \in \left(\bigoplus_{i=0}^{\frac{m}{2}-2} |x \wedge u|^{2i} \mathcal{H}^w_{k-2i}(\mathbb{R}^{2m},\mathbb{C}) \right) \otimes \mathbb{S}^+,$$

which finishes the proof. $\square$

Therefore, when applying $\Delta_w^{\frac{m}{2}-2}$, only the last component in proposition 7.2.5 will survive, i.e. $\mathcal{H}^w_{k-m+4} \otimes \mathbb{S}^+$. This thus means that

$$\Delta_w^{\frac{m}{2}-2} \mathcal{I}^{(m,m-4)}_{x,u} D^k_{12} \mathcal{I}^{(m,m-4)}_{x,u}[I] \in \mathcal{H}^w_{k-m+4}(\mathbb{R}^{2m},\mathbb{C}) \otimes \mathbb{S}^+.$$

Let us take a closer look at the only remaining term (which is the term corresponding to the index $i = \frac{m}{2} - 2$) that survives the action of $\Delta_w^{\frac{m}{2}-2}$, namely:

$$\mathcal{I}^{(m,m-4)}_{x,u} \Delta_w^{\frac{m}{2}-2} Q_{k-m+4}(\partial_x,\partial_u) \mathcal{I}^{(m,m-4)}_{x,u}[I].$$

In view of theorem 6.1.19, there exists a constant $c_m \in \mathbb{C}$, such that:

$$\Delta_w^{\frac{m}{2}-2} \mathcal{I}^{(m,m-4)}_{x,u}[I] = \Delta_w^{\frac{m}{2}-2} |x \wedge u|^{m-4} \mathcal{I}^{(m,0)}_{x,u}[I] = c_m \mathcal{I}^{(m,0)}_{x,u}[I].$$

Explicitly, we have that:

$$c_m = (m-3)!^2,$$

in particular $c_m \neq 0$. This means that :

$$\begin{aligned}\mathcal{I}_{x,u}^{(m,m-4)} \Delta_w^{\frac{m}{2}-2} Q_{k-m+4}(\partial_x, \partial_u) \mathcal{I}_{x,u}^{(m,m-4)}[I] \\ = c_m |x \wedge u|^{m-4} \mathcal{I}_{x,u}^{(m,0)} Q_{k-m+4}(\partial_x, \partial_u) \mathcal{I}_{x,u}^{(m,0)}[I].\end{aligned}$$

Recall that $\mathcal{I}_{x,u}^{(m,0)}$ preserves $\ker(\partial_x, \partial_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle)$ and the same holds for $Q_{k-m+4}(\partial_x, \partial_u)$. The latter follows from the following observation: it comes from a scalar-valued polynomial depending on the minors $x_i u_j - x_j u_i$ $(1 \leq i < j \leq m)$, which means that $Q_{k-m+4}(\partial_x, \partial_u)$ is a scalar operator that can be written in terms of $\partial_{x_i} \partial_{u_j} - \partial_{x_j} \partial_{u_i}$. As these commute with the skew-Euler operators, the conclusion follows. We can therefore conclude that

$$\mathcal{I}_{x,u}^{(m,0)} Q_{k-m+4}(\partial_x, \partial_u) \mathcal{I}_{x,u}^{(m,0)}[I] \in \mathcal{S}_{k-m+4,k-m+4}(\mathbb{R}^{2m}, \mathbb{S}^+).$$

From theorem 6.1.19 we know that the operator

$$\Delta_w^{\frac{m}{2}-2} |x \wedge u|^{m-4} \in \mathrm{End}(\mathcal{S}_{k-m+4,k-m+4}(\mathbb{R}^{2m}, \mathbb{S}^+))$$

reduces to a non-trivial multiple of the identity operator, which means that:

$$\Delta_w^{\frac{m}{2}-2} \mathcal{I}_{x,u}^{(m,m-4)} D_{12}^k \mathcal{I}_{x,u}^{(m,m-4)}[I] \in \mathcal{S}_{k-m+4,k-m+4}(\mathbb{R}^{2m}, \mathbb{S}^+).$$

Finally, we define the following function space which will serve as our analogue for the space of analytic functions:

$$\mathcal{S}_{\mathcal{F}}(\mathbb{R}^8, \mathbb{S}^+) := \mathrm{Span}\left\{ \mathcal{I}_{x,u}^{(4,0)} D_{12}^k \mathcal{I}_{x,u}^{(4,0)} \mid k \in \mathbb{Z}^+ \right\},$$

which is a subspace of

$$\left\{ f(\underline{x}_2^{(4)} \wedge \underline{u}_2^{(4)}, \underline{x}^{(4)} \wedge \underline{u}^{(4)}, x^{(4)} \wedge u^{(4)}) : \mathbb{R}^{4\times 2} \to \mathbb{S}^+ \mid \partial_x f = \partial_u f = 0 \right\}.$$

Theorem 7.2.6 (A higher spin Fueter theorem)**.** *Let $m \geq 4$ be even then, for all $f(\underline{x}_2^{(4)} \wedge \underline{u}_2^{(4)}, \underline{x}^{(4)} \wedge \underline{u}^{(4)}, x^{(4)} \wedge u^{(4)}) \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^8, \mathbb{S}^+)$, we have that:*

$$\Delta_w^{\frac{m}{2}-2} f(\underline{x}_2 \wedge \underline{u}_2, \underline{x} \wedge \underline{u}, x \wedge u) \in \ker(\partial_x, \partial_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle) \subset \ker \partial_w.$$

Classically, the image of the Fueter map is described in terms of the monogenic Gegenbauer polynomials, which can be expressed as the image of the harmonic Gegenbauer polynomials under the action of the Dirac operator, i.e.

$$\partial_x |x|^{k+1} C_{k+1}^{\frac{m}{2}-1}\left(\frac{x_1}{|x|}\right) = \mathcal{G}_k^m(x).$$

First of all, it is clear from the proof of the Fueter theorem that our image is generated by the polynomials $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[I]$ and the following theorem gives us a connection with the simplicial generalisation of the Gegenbauer polynomials as found in chapter 4 (more specifically in theorem 4.1.23):

Theorem 7.2.7. *Let $k \in \mathbb{Z}^+$ and $m > 4$, then there exists a constant $\gamma_k \in \mathbb{C}$ such that*

$$\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[I] = \gamma_k \partial_w \mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I].$$

where the inversion

$$\mathcal{J}_{x,u}[f](x,u) := |x \wedge u|^{3-m} f\left(\frac{x}{|x \wedge u|}, \frac{u}{|x \wedge u|}\right)$$

is the one from definition 4.1.4. If $m = 4$ then there exists a constant $\gamma_k \in \mathbb{C}$ such that:

$$\mathcal{I}_{x,u}^{(4,0)} D_{12}^k \mathcal{I}_{x,u}^{(4,0)}[I] = \gamma_k \partial_w \mathcal{J}_{x,u} D_{12}^k \mathcal{J}_{x,u}[(x_1 u_2 - x_2 u_1) I].$$

Proof. Assume $m > 4$ as the case $m = 4$ is similar. First of all, both polynomials belong to the same irreducible function space. The polynomial on the left hand side belongs to $\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)$ by construction, while for the right hand side we have that

$$\mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I] \in \mathcal{H}_{k+1,k+1}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^+$$

and the latter space decomposes, up to an isomorphism, as

$$\begin{aligned} &\mathcal{H}_{k+1,k+1}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^{\pm} \\ &\qquad \cong \mathcal{S}_{k+1,k+1}(\mathbb{R}^{2m}, \mathbb{S}^+) \oplus \mathcal{S}_{k+1,k}(\mathbb{R}^{2m}, \mathbb{S}^-) \oplus \mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+). \end{aligned}$$

On the other hand, we know from theorem 4.1.5 and lemma 4.1.6 that

$$\mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I] \in \mathcal{P}_{k+1}^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}) \otimes \mathbb{S}^+$$

which means that

$$\mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I] = M_{k+1}^w(x,u) + (x \wedge u) S_{k,k}(x,u)$$

where $M_{k+1}^w(x,u) \in \mathcal{M}_{k+1}^w(\mathbb{R}^{2m}, \mathbb{S}^+)$ and $S_{k,k}(x,u) \in \mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)$. From this it immediately follows that

$$\partial_w \mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I] \in \mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+).$$

In order that they are indeed equal up to a (scalar) constant, we still need to prove that they transform the same under the action of $\mathfrak{so}(m)$, which is given in terms of the operators

$$M_{ab} := x_a \partial_{x_b} - x_b \partial_{x_a} + u_a \partial_{u_b} - u_b \partial_{u_a} - \frac{1}{2} e_{ab}$$

with $a < b$. It is easy to see that:

$$[M_{ab}, \mathcal{I}_{x,u}] = [M_{ab}, \mathcal{J}_{x,u}] = [M_{ab}, \partial_w] = 0$$

for all a, b and if $2 < a, b \leq m$ then it is also true that $[M_{ab}, D_{12}] = 0$. Moreover, $[M_{12}, D_{12}] = 0$ which implies that the action of the Cartan algebra commutes with both raising operators and only acts on the spinor on which we apply these operators. This means that the polynomials on both sides transform, under $\mathfrak{so}(m-2)$, the same way as the spinor I.
We need to make a distinction based on the parity of m. Let m be even, then the abstract branching rules tell us that:

$$\begin{aligned}
\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S}^+)\Big|_{\mathfrak{so}(m-2)}^{\mathfrak{so}(m)} &\cong (k,k)'_+\Big|_{\mathfrak{so}(m-2)}^{\mathfrak{so}(m)} \\
&\cong \bigoplus_{i=0}^{k}\bigoplus_{j=0}^{k-i}\Big((j+1)(k-i, k-i-j)'_+ \\
&\qquad \oplus (j+1)(k-i, k-i-j)'_-\Big).
\end{aligned}$$

This means that under $\mathfrak{so}(m-2)$ there is exactly one subspace that transforms as $\mathbb{S}^+$. Since both polynomials behave like the highest weight vector for this representation under $\mathfrak{so}(m-2)$ they have to be equal upto an element of $\alpha \in \mathbb{C}_m^+$ that is $\mathfrak{so}(m-2)$-invariant. The only possibility is the Clifford number

$$a + be_{12} + ce_{12\cdots m} + de_{34\cdots m} \in \mathbb{C}_m^+,$$

where $a, b, c, d \in \mathbb{C}$, and thus we have:

$$\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[I] = \partial_w \mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[\alpha I].$$

However, since each of these Clifford elements reduces to a complex number when acting on I, e.g. $e_{12}I = iI$, we find the desired result. The case where m is odd is slightly more complicated. In this case the branching rules tell us that:

$$\begin{aligned}
\mathcal{S}_{k,k}(\mathbb{R}^{2m}, \mathbb{S})\Big|_{\mathfrak{so}(m-2)}^{\mathfrak{so}(m)} &\cong (k,k)'\Big|_{\mathfrak{so}(m-2)}^{\mathfrak{so}(m)} \\
&\cong \bigoplus_{i=0}^{k}\bigoplus_{j=0}^{k-i} 2(j+1)(k-i, k-i-j)'
\end{aligned}$$

and thus, under $\mathfrak{so}(m-2)$, there are two distinct copies of the spinor space $\mathbb{S}$ to be found. We claim that these two copies are spanned by $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[I]$ and $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[\mathfrak{f}_1^\dagger I]$ respectively. It is clear that $\mathfrak{f}_1^\dagger$ commutes with the action of $\mathfrak{so}(m-2)$ and thus both of these polynomials will act as the highest weight vector for the copies of $\mathbb{S}$. However both belong to different weight spaces: the polynomial $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[I]$ has weight $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ whereas $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[\mathfrak{f}_1^\dagger I]$ has weight $(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. The action of $\mathfrak{so}(m-2)$ does not change the first weight entry and thus we have shown that both of them span the distinct copies of the spinor space. Because $\partial_w \mathcal{J}_{x,u} D_{12}^{k+1} \mathcal{J}_{x,u}[I]$ acts as the highest weight vector for the first copy one can once again conclude that it has to be equal to $\mathcal{I}_{x,u}^{(m,0)} D_{12}^k \mathcal{I}_{x,u}^{(m,0)}[\alpha I]$, where $\alpha \in \mathbb{C}_m^+$ commutes with the action of $\mathfrak{so}(m-2)$. This implies that $\alpha = a + be_{12}$ (note that $e_{12\cdots m}$ and $e_{34\cdots m}$ do not appear here as these are not elements of $\mathbb{C}_m^+$ if m is odd) and we have thus finished the proof. $\square$

7.3 Fueter theorem for Δ_w

One can also make similar considerations when looking at scalar valued polynomials, however things are slightly more complicated. Nevertheless, for $m \geq 4$ even with $m \neq 6$, we will construct a Fueter map:

$$\mathcal{H}_{k+m-4,k+m-4}(\mathbb{R}^8, \mathbb{C}) \to \mathcal{H}_k^w(\mathbb{R}^m, \mathbb{C}).$$

We consider the following inversion, which is similar to the one defined in definition 4.1.4, but with a parameter α:

Definition 7.3.1. Let $f \in \mathcal{C}^\infty(\mathbb{R}^{2m}, \mathbb{C})$, then we define the following inversion:

$$\mathcal{J}_{x,u}^{(m,\alpha)} f(x,u) := |x \wedge u|^{3-m+\alpha} f\left(\frac{x}{|x \wedge u|}, \frac{u}{|x \wedge u|}\right).$$

In particular, when acting on polynomials $H_{\ell,k}(x,u)$ this inversion can be written as:

$$\mathcal{J}_{x,u}^{(m,\alpha)} H_{\ell,k}(x,u) = |x \wedge u|^{3-m+\alpha-\ell-k} H_{\ell,k}(x,u).$$

This inversion satisfies $(\mathcal{J}_{x,u}^{(m,\alpha)})^2 = \mathrm{Id}$ and it preserves

$$\ker(\Delta_x, \Delta_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle)$$

if and only if $\alpha = 0$.

If we define $X_\alpha := \mathcal{J}_{x,u}^{(m,\alpha)} D_{12} \mathcal{J}_{x,u}^{(m,\alpha)}$, then we have the following result:

Lemma 7.3.2. *Let $m - 3 - \alpha > 0$ and $k \in \mathbb{Z}^+$. One then has that*

$$X_\alpha^k[1] = k!(m-4-\alpha)^{(k)} |x \wedge u|^k J_k^{\frac{m-3-\alpha}{2}}(\tau, \sigma^2).$$

Proof. Immediately follows from theorem 4.1.23. $\square$

If $m = 4$ and $\alpha = 0$ then this raising operator should generate the special Gegenbauer solutions that will act as the starting point for the Fueter map. However, if $\alpha = 0$ and $m = 4$ then this operator acts trivially on 1. The solution is to start from the first degree solutions, which means looking at

$$X_\alpha^k[(x_1u_2 - x_2u_1)] = \frac{(k+1)!(m-3-\alpha)^{(k)}}{m-3-\alpha} |x \wedge u|^{k+1} J_{k+1}^{\frac{m-3-\alpha}{2}}(\tau, \sigma^2).$$

As a starting point for the Fueter map we take, in analogue with the classical case, the special Gegenbauer solutions on $\mathbb{R}^8$:

$$|x \wedge u|^k J_k^{\frac{1}{2}}(\tau_{(4)}, \sigma_{(4)}^2).$$

Our substitution will map $\tau_{(4)} \mapsto \tau$ and $\sigma_{(4)} \mapsto \sigma$ to find the following polynomial on $\mathbb{R}^{2m}$: $|x \wedge u|^k J_k^{\frac{1}{2}}(\tau, \sigma^2)$. Therefore, we have to take a closer look at:

$$F_{k+1}(x,u) := \left(\mathcal{J}_{x,u}^{m,m-4} D_{12}^k \mathcal{J}_{x,u}^{m,m-4}\right)[x_1u_2 - x_2u_1] \in \mathcal{P}_{k+1}^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C}).$$

Lemma 7.3.3. *Let $k \in \mathbb{Z}^+$ and m be even. We then have that:*

$$F_{k+1}(x,u) \in \left(\bigoplus_{i=0}^{\frac{m}{2}-2} |x \wedge u|^{2i} \mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C})\right) \otimes \mathcal{H}_{1,1}(\mathbb{R}^{2m}, \mathbb{C}).$$

Proof. Using the same approach as proposition 7.2.5, we can write

$$D_{12}^k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \Delta_w^i Q_{k-2i}(\partial_x, \partial_u)$$

which then leads to

$$\begin{aligned}
&\left(\mathcal{J}_{x,u}^{m,m-4} D_{12}^k \mathcal{J}_{x,u}^{m,m-4}\right)[x_1u_2 - x_2u_1] \\
&\qquad = \left(\mathcal{J}_{x,u}^{m,m-4} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \Delta_w^i Q_{k-2i}(\partial_x, \partial_u) \mathcal{J}_{x,u}^{m,m-4}\right)[x_1u_2 - x_2u_1].
\end{aligned}$$

Because $\mathcal{J}_{x,u}^{m,m-4} = |x \wedge u|^{m-4} \mathcal{J}_{x,u}^{m,0}$ and $x_1u_2 - x_2u_1 \in \mathcal{H}_{1,1}(\mathbb{R}^{2m}, \mathbb{C})$ we can see that for $i \geq \frac{m}{2} - 1$:

$$\Delta_w^i |x \wedge u|^{m-4} J_{x,u}^{m,0}[x_1u_2 - x_2u_1] = 0.$$

This means that there are a fixed amount of terms, independent of the value for k. Looking at each of the individual terms that remain, shows that they are built in terms of invariants and pieces transforming as $\mathcal{H}_{k-2i}^w \otimes \mathcal{H}_{1,1}$. This finishes the proof. $\square$

In order to proceed, we need to know how the tensor products $\mathcal{H}_a^w \otimes \mathcal{H}_{1,1}$, with $a \in \{k, k-2, \ldots, k-m+4\}$, decomposes. The following lemma shows which (simplicial) components we can expect in these tensor products:

Lemma 7.3.4. *Let $\ell, k \in \mathbb{Z}^+$, with $\ell \geq k$ and $m \geq 8$. We then have the following decomposition of the tensor product of $\mathrm{O}(m)$-representations:*

$$(\ell, k) \otimes (1,1) \cong 2(\ell, k) \oplus (\ell+1, k+1) \oplus (\ell+1, k, 1) \oplus (\ell, k, 1, 1)$$

$$\oplus(\ell-1,k-1)\oplus(\ell-1,k+1)\oplus(\ell-1,k,1)\oplus(\ell+1,k-1)$$
$$\oplus(\ell,k-1,1)\oplus(\ell,k+1,1)$$

where the spaces on the right only occur if they satisfy the dominant weight condition and have non negative integer entries.

Proof. Since $m\geq 8$ we can use the result in [53] which states that:

$$[\mathbb{V}^{(\ell,k)}\otimes\mathbb{V}^{(1,1)},\mathbb{V}^{\lambda}]=\sum_{\alpha,\beta,\gamma}c^{\lambda}_{\alpha\beta}c^{(\ell,k)}_{\alpha\gamma}c^{(1,1)}_{\beta\gamma}$$

where the sum runs over all non-negative integer partitions and the constants $c^{\lambda}_{\mu\nu}:=[F^{\mu}\otimes F^{\nu},F^{\lambda}]$ are the Littlewood-Richardson coefficients. We will now give the only possibilities for α,β,γ and λ for which the product of the three coefficients is not zero. In other words:

1. $[F^{\alpha}\otimes F^{\beta},F^{\lambda}]\neq 0$
2. $[F^{\alpha}\otimes F^{\gamma},F^{(\ell,k)}]\neq 0$
3. $[F^{\beta}\otimes F^{\gamma},F^{(1,1)}]\neq 0$

First of all, one can conclude from 2. and 3. that α,β and γ have at most two entries different from zero. Moreover, from 3., we can see that the only possibilities for β are $(0,0),(1,0)$ and $(1,1)$. Using 3. we can then find the values for γ and from 2. we can deduce what α has to be. It then suffices to calculate the product of the Young tableaux given by α and β to find the possible values for λ. The calculations are summarised in the following table:

$$\begin{array}{rrrr}
\beta=(0,0)\implies & \gamma=(1,1)\implies & \alpha=(\ell-1,k-1)\implies & \lambda=(\ell-1,k-1)\\
\beta=(1,0)\implies & \gamma=(1,0)\implies & \alpha=(\ell-1,k)\implies & \lambda=(\ell,k)\\
 & & & \lambda=(\ell-1,k+1)\\
 & & & \lambda=(\ell-1,k,1)\\
 & & \alpha=(\ell,k-1)\implies & \lambda=(\ell+1,k-1)\\
 & & & \lambda=(\ell,k)\\
 & & & \lambda=(\ell,k-1,1)\\
\beta=(1,1)\implies & \gamma=(0,0)\implies & \alpha=(\ell,k)\implies & \lambda=(\ell+1,k+1)\\
 & & & \lambda=(\ell+1,k,1)\\
 & & & \lambda=(\ell,k+1,1)\\
 & & & \lambda=(\ell,k,1,1)
\end{array}$$

This finishes the proof. □

Let us denote

$$\mathcal{H}_{\mathcal{F}}(\mathbb{R}^8,\mathbb{C}):=\operatorname{Span}\left\{|x\wedge u|^k J_k^{\frac{1}{2}}\left(\tau_{(4)},\sigma^2_{(4)}\right)\mid k\in\mathbb{Z}^+\right\},$$

then we are now ready for the actual Fueter theorem, which gives us a way to construct solutions for Δ_w starting from generalized Gegenbauer polynomials in $m = 4$.

Theorem 7.3.5. *Let $m \geq 4$ be even with $m \neq 6$ then, for all functions* $f\big(x_1u_2 - x_2u_1, |\underline{x}^{(4)} \wedge \underline{u}^{(4)}|^2, |x^{(4)} \wedge u^{(4)}|^2\big) \in \mathcal{H}_{\mathcal{F}}(\mathbb{R}^8, \mathbb{C})$:

$$\Delta_w^{\frac{m}{2}-2} f\big(x_1u_2 - x_2u_1, |\underline{x} \wedge \underline{u}|^2, |x \wedge u|^2\big) \in \ker \Delta_w.$$

Proof. Let $k \in \mathbb{Z}^+$ then we have already shown that

$$F_k(x, u) = \Big(\mathcal{J}_{x,u}^{(m,m-4)} D_{12}^k \mathcal{J}_{x,u}^{(m,m-4)}\Big)\,[x_1u_2 - x_2u_1]$$

can be written as:

$$\sum_{i=0}^{\frac{m}{2}-2} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} \Big(\mathcal{J}_{x,u}^{(m,m-4)} \Delta_w^i (R_j H_{k-2i,k-2i-2j})(\partial_x, \partial_u) \mathcal{J}_{x,u}^{(m,m-4)}\Big)\,[x_1u_2 - x_2u_1]$$

where $H_{k-2i,k-2i-2j} \in \mathcal{H}_{k-2i,k-2i-2j}(\mathbb{R}^{2m}, \mathbb{C})$ and we have used the decomposition of $\mathcal{H}_{k-2i}^w(\mathbb{R}^{2m}, \mathbb{C})$ into spaces of simplicial harmonics. The embedding operator R_j consists of a multiplication with $|u|^{2j}$ followed by a projection operator. Since both are $\mathfrak{so}(m)$-invariant, each of the terms in the summand above transforms as an element of $(k - 2i, k - 2i - 2j) \otimes (1, 1)$. Using the previous lemma we know how this tensorproduct decomposes, however not all the components will appear. We know that each term also belongs to the space $\mathcal{P}_{k+1}^{\mathrm{SL}(2)}(\mathbb{R}^{2m}, \mathbb{C})$ and in its decomposition only spaces $(k+1-2i, k+1-2i-2j)$, with $0 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{k+1}{2} \rfloor - i$, can appear. Therefore we can determine the transformational behaviour of each of the terms in the summand above, namely:

$$\begin{aligned}&(k+1-2i, k+1-2i-2j) \oplus (k+1-2(i+1), k+1-2(i+1)-2(j-1))\\ \oplus&(k+1-2(i+1), k+1-2(i+1)-2j) \oplus (k+1-2i, k+1-2i-2(j+1)).\end{aligned}$$

Note that for $m = 8$, the space $(k-2i, k-2i-2j, 1, 1)$ on the right hand side in lemma 7.3.4 decomposes when restricted to the action of $\mathrm{SO}(m)$ but since this space does not occur, this does not pose a problem. Using this result we arrive at the following expression for $F_k(x, u)$:

$$\begin{aligned}&F_k(x, u)\\ &= \sum_{i=0}^{\frac{m}{2}-2} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor - i} \Big[|x \wedge u|^{2i} \big(R_j H_{k+1-2i,k+1-2i-2j} + R_{j+1} H_{k+1-2i,k-2i-2j-1}\big)\\ &\qquad + |x \wedge u|^{2(i+1)} \big(R_{j-1} H_{k-2i-1,k-2i-2j+1} + R_j H_{k-2i-1,k-2i-2j-1}\big)\Big]\end{aligned}$$

and after grouping the summation in j we find that:

$$F_k(x,u) = \sum_{i=0}^{\frac{m}{2}-2} |x \wedge u|^{2i} \left(Q^{(1)}_{k-2i+1}(x,u) + |x \wedge u|^2 Q^{(2)}_{k-2i-1}(x,u) \right)$$

with $Q^{(a)}_b \in \mathcal{H}^w_b(\mathbb{R}^{2m}, \mathbb{C})$. From this and theorem 6.1.19 we can conclude that

$$\Delta_w^{\frac{m}{2}-2} F_k(x,u) = c_k Q^{(1)}_{k-m+5}(x,u) + d_k Q^{(2)}_{k-m+5}(x,u) + e_k |x \wedge u|^2 Q^{(2)}_{k-m+3}(x,u).$$

However, the second term does not appear as this polynomial comes from the last tensor product in lemma 7.3.3, which is given by:

$$\begin{aligned} &\left(\mathcal{J}^{m,m-4}_{x,u} \Delta_w^{\frac{m}{2}-2} Q_{k-m+4}(\partial_x, \partial_u) \mathcal{J}^{m,m-4}_{x,u} \right) [x_1 u_2 - x_2 u_1] \\ &\qquad = \alpha_m |x \wedge u|^{m-4} \left(\mathcal{J}^{m,0}_{x,u} Q_{k-m+4}(\partial_x, \partial_u) \mathcal{J}^{m,0}_{x,u} \right) [x_1 u_2 - x_2 u_1] \\ &\qquad = |x \wedge u|^{m-4} H_{k-m+5,k-m+5}(x,u), \end{aligned}$$

with $\alpha_m \in \mathbb{C}_0$ and $H_{k-m+5,k-m+5}(x,u) \in \mathcal{H}_{k-m+5,k-m+5}(\mathbb{R}^{2m}, \mathbb{C})$. From this we can conclude that

$$\Delta_w^{\frac{m}{2}-2} F_k(x,u) = c_k H_{k-m+5,k-m+5}(x,u) + d_k Q^{(2)}_{k-m+5}(x,u)$$

which finishes the proof. $\square$

Conclusion

That's what I do. I drink and I know things.
Tyrion Lannister

To obtain a generalisation for the classical Gegenbauer polynomials we pursued two approaches and while both have merit, the second approach proved to serve us best. We thus found Gegenbauer-type solutions in the spaces of simplicial harmonics by focussing on their reproducing property. This lead to simplicial harmonic polynomials on $\mathbb{R}^{2m}$ that are $\mathrm{SO}(m-2)$-invariant and we constructed a raising operator that generates these solutions by repeated action on 1. Moreover, this allowed us to generalise Appell sequences to the framework of higher spin Clifford analysis.
While the J-polynomials were constructed as the unique $\mathrm{SO}(m-2)$-invariant simplicial harmonics with equal degrees of homogeneity, they turned out to satisfy additional symmetry properties. This connected them to spherical functions on oriented Grassmann manifolds, i.e. functions depending on normalised wedge-variables, which naturally lead to polynomials belonging to the kernel of both skew-Euler operators.

Afterwards, we looked at the Cauchy-Kovalevskaya extension in two vector variables and showed that it is surjective onto the space of simplicial monogenics on $\mathbb{R}^{2m}$ if one starts from solutions to the wedge-system, i.e. solutions to both ∂_w and $\langle x, \partial_u \rangle$, on the lower dimensional space $\mathbb{R}^{2(m-1)}$. Because the CK-extension is an isomorphism, we found a decomposition of the kernel of the wedge-system into irreducible $\mathrm{Spin}(m)$-components.

The wedge-system (both the scalar and spinor-valued version) popped up multiple times in this thesis and thus warranted an in depth look. First we assumed the degrees of homogeneity in both variables to be equal, which meant that the polynomials we considered depended on wedge-variables and lead to well-defined functions on the Grassmannian by restricting to $|x \wedge u|^2 = 1$. We then obtained a wedge-Fischer decomposition for both scalar and spinor-valued polynomials where the role of the Laplace and Dirac operator is fulfilled by the Cayley-Laplace operator Δ_w and its refinement ∂_w. The case of polynomials with different degrees of homogeneity was more complicated because the decomposition was no longer multiplicity-free. Nevertheless we were able to show that both Δ_w and ∂_w are still surjective operators and that their kernel spaces can be used to efficiently describe the decomposition.

Finally, we introduced a higher spin version of Fueter's theorem, allowing us to create solutions on $\mathbb{R}^{2m}$ for both the Cayley-Laplace operator Δ_w and the wedge operator ∂_w, starting from a space of polynomials on $\mathbb{R}^{2\times 4}$. The proof relied on the transformational behaviour of polynomials that depend on wedge-variables, i.e. the wedge-Fischer decomposition, and the properties of certain inversions on spaces of simplicial polynomials.

Nederlandse samenvatting

Aan allen die dit lezen, proficiat!
Aan allen die dit niet lezen, ook proficiat!
Meneer de burgemeester

In fysica worden vele fenomenen beschreven d.m.v. differentiaalvergelijkingen en één van de meest voorkomende is de zogenaamde Laplace-vergelijking. Deze wordt beschreven door de Laplace-operator, een differentiaaloperator van tweede orde die gebruikt wordt in bijvoorbeeld potentiaaltheorie, warmteverspreiding, kwantumfysica, enz. Ook in de wiskunde is het belang van deze operator niet te onderschatten aangezien deze centraal staat in de harmonische analyse.
Een ander voorbeeld van een differentiaaloperator is de Dirac-operator die een grote rol speelt in de deeltjesfysica, waar deze operator het gedrag van het elektron beschrijft, zie [19]. De Dirac-operator is van eerste orde, factoriseert de Laplace-operator en werkt in op functies die hun waarden aannemen in welbepaalde deelruimten van Clifford-algebra's. In de wiskunde wordt deze operator voornamelijk bestudeerd binnen de Clifford-analyse, een domein dat klassieke analyse combineert met representatietheorie.

Wanneer men differentiaalvergelijkingen wil oplossen is er een belangrijk concept, namelijk symmetrie, wat beschreven wordt door groepen (en bijhorende algebra's). Als een vergelijking invariant is onder de actie van een Lie-algebra $\mathfrak{g}$, dan zal de oplossingenruimte van deze vergelijking een moduul vormen voor $\mathfrak{g}$, wat ons dan in staat stelt om nieuwe oplossingen te genereren. Een andere manier om oplossingen te vinden is door de vergelijking te beschouwen in een ruimte met lagere dimensie, en deze oplossingen nadien te transformeren. Voorbeelden hiervan zijn de Cauchy-Kovalevskaya-extensie en de stelling van Fueter.
Een voorbeeld van een invariante operator is opnieuw de Laplace-operator, die rotatie-invariant is. Meer zelfs, als men homogene polynomiale oplossingen voor deze operator beschouwd, dan zullen deze een irreducibele representatie vormen voor de rotatiegroep. Tussen al deze polynomiale oplossingen zijn er enkele die een speciale rol vervullen omdat zij extra symmetrie vertonen door bijvoorbeeld invariant te zijn onder de actie van een deelgroep van de rotatiegroep. Het volgende cruciale voorbeeld uit de klassieke harmonische analyse ligt aan de basis

van deze thesis:

Op een vermenigvuldiging met een constante na, bestaat er een unieke oplossing voor de Laplace-operator

$$\Delta_x = \sum_{i=1}^{m} \partial_{x_i}^2$$

op $\mathbb{R}^m$ *die homogeen is van graad* k *en die enkel afhangt van het inproduct van de vector* $x \in \mathbb{R}^m$ *met een vaste eenheidsvector. Als we voor de eenheidsvector* e_1 *kiezen, dan wordt deze oplossing gegeven door*

$$H_k(x) := |x|^k C_k^{\frac{m}{2}-1}\left(\frac{\langle x, e_1\rangle}{|x|}\right) \in \ker \Delta_x \ ,$$

waar $C_k^\alpha(t)$ *een klassieke Gegenbauer veelterm is van orde* α *en graad* k*.*

Het bestaan (en de uniciteit) van deze oplossing volgt uit de representatietheorie van de orthogonale Lie-algebra $\mathfrak{so}(m)$. De k-homogene polynomiale oplossingen van Δ_x, die genoteerd worden door $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$, vormen een irreducibele representatie voor deze algebra met *highest weight* $(k, 0, \ldots, 0)$ en door gebruik te maken van de *branching rules* kan men concluderen dat er onder de actie van de deelalgebra $\mathfrak{so}(m-1)$ een triviale representatie te vinden is in $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$. Deze wordt dan beschreven door de speciale oplossing van hierboven als men $\mathfrak{so}(m-1)$ bekijkt als de deelalgebra van $\mathfrak{so}(m)$ die de e_1-richting invariant laat.

Het doel van deze thesis is om de speciale Gegenbauer-oplossingen van de Laplace-operator te veralgemenen naar hogere spin Clifford-analyse, dwz. functies die afhangen van meerdere vectorveranderlijken. Hoewel de meeste (en mogelijk alle) resultaten in deze thesis veralgemenen naar een willekeurig aantal veranderlijken, stuit men al snel op zo goed als onmogelijke berekeningen. Daarom hebben we ervoor gekozen om ons te focussen op functies die afhangen van twee veranderlijken, om het onderliggende idee zo duidelijk mogelijk te maken.

De eerste twee hoofdstukken van deze thesis geven de lezer de nodige informatie om de rest te kunnen begrijpen. Het eerste hoofdstuk legt zich toe op de representatietheorie van Lie-groepen en -algebra's, waar we beginnen met nodige definities alvorens de *branching rules* te beschrijven die cruciaal zijn in deze thesis. Het tweede hoofdstuk geeft een korte inleiding in het domein Clifford-analyse, waar we eerst een dubbele overdekking van $\mathrm{SO}(m)$ construeren, namelijk de Spingroep $\mathrm{Spin}(m)$, om nadien enkele representaties van deze laatste groep nader te bekijken. Men kan zelfs voor elke eindig dimensionale irreducibele representatie van $\mathrm{Spin}(m)$ een polynomiaal model vinden binnen de hogere spin Clifford-analyse, wat nogmaals duidelijk maakt dat de twee domeinen elkaar versterken.

Als men de Gegenbauer-oplossingen wil veralgemenen dan zijn er twee verschillende aanpakken mogelijk. Men kan enerzijds focussen op het feit dat de

Laplace-operator niet alleen rotatie-invariant is, maar ook conform-invariant en dat men dus op zoek moet gaan naar invariante oplossingen van hogere spin versies van Δ_x, de operatoren $\mathcal{D}_k$. Deze aanpak volgen we in het derde hoofdstuk, maar een exacte uitdrukking vinden voor deze oplossing, in functie van speciale veeltermen, bleek een moeilijk probleem doordat de oplossingenruimten van $\mathcal{D}_k$ niet irreducibel zijn. Dit maakt onmiddelijk het verschil duidelijk met de tweede aanpak, die vertrekt van de irreducibele Spin(m)-modulen en eist dat de speciale oplossingen die we zoeken deze ruimten moeten reproduceren. In het vierde hoofdstuk vinden we een expliciete uitdrukking voor deze oplossingen, in functie van hypergeometrische polynomen die afhangen van twee veranderlijken, en geven we een manier om deze oplossingen te genereren d.m.v. een *raising operator*. Dit stelt ons nadien in staat om een *Appell sequence* te construeren, die een representatie vormt voor de Heisenberg-algebra.

Hoofdstuk 5 gaat dieper in op één van de manieren om oplossingen te genereren, namelijk de Cauchy-Kovalevskaya-extensie en we gaan deze gebruiken om simpliciaal monogenen te construeren op $\mathbb{R}^{2m}$ vertrekkende van oplossingen van het zogenaamde wedge-systeem op $\mathbb{R}^{2(m-1)}$. In het klassieke geval werd deze extensie onder meer gebruikt om orthonormale basissen voor ruimten van polynomiale oplossingen op te stellen, wat dan verband houdt met de theorie van Gelfand-Tsetlin-basissen.

In hoofdstuk 6 bekijken we het wedge-systeem uit het vorige hoofdstuk nader en gaan we op zoek naar een analogie van de klassieke Fischer-ontbinding op $\mathbb{R}^m$ die zegt:

Gegeven een homogene veelterm $q(x)$ op $\mathbb{R}^m$ en een willekeurige k-homogene veelterm $P_k(x)$, dan kunnen we deze laatste ontbinden als

$$P_k(x) = Q_k(x) + q(x)R(x),$$

waar $R(x)$ een veelterm is van de gepaste graad en $Q_k(x)$ een oplossing is voor de differentiaal-operator $q(\partial_x)$.

Als de veelterm $q(x)$ bijvoorbeeld SO(m)-invariant is, dan zegt de Fischer-ontbinding ons hoe polynomen gaan ontbinden in irreducibele componenten onder de actie van SO(m). Met dit in het achterhoofd, zijn we vertrokken van de ruimte van scalaire veeltermen die afhangen van wedge-variabelen, zijnde $X_{ij} := x_i u_j - x_j u_i$, die voor een gegeven graad van homogeniteit een irreducibel GL(m)-moduul opspannen. Door gebruik te maken van *branching rules* kunnen we de SO(m)-componenten vinden en een operator introduceren die dezelfde rol vervult als de klassieke Laplace-operator, namelijk de Cayley-Laplace-operator, en bewijzen we een wedge-Fischer-ontbinding. Nadien verfijnen we onze resultaten door te kijken naar veeltermen die spinor-waardig zijn, analoog naar de verfijning van harmonische analyse tot Clifford-analyse.

In het laatste hoofdstuk gaan we onze resultaten uit hoofdstuk 6 toepassen om een hogere spin versie van de stelling van Fueter te bewijzen. Deze stelling komt origineel uit de quaternionische analyse en zegt ons hoe we oplossingen kunnen construeren voor een veralgemeende Cauchy-Riemann-operator, door te vertrekken van holomorfe functies in het complexe vlak. Sindsdien is deze stelling uitgebreid naar onder andere Clifford-analyse, waar het ons in staat stelt om oplossingen voor de Dirac-operator te vinden, door te vertrekken van de (eenvoudigere) holomorfe functies. We beginnen met een alternatief bewijs voor de stelling van Fueter te geven die te veralgemenen is naar de hogere spin Clifford-analyse. Dit bewijs steunt op eigenschappen van inversies en het bestaan van Fischer-decomposities, wat ons in staat stelt om de wedge-Fischer-ontbinding toe te passen om zo simpliciaal monogenen te genereren.

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