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A Microstructure Sheaf and Quantum Sections over a Projective Scheme

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INTRODUCTION

Microlocalization at graded prime ideals of the associated graded ring of a Zariski filtered (noncommutative) ring appear in the analytic theory of rings of differential operators. Algebraic approaches to this theory have appeared in recent publications, e.g., [1, 6, 18, 17], and in this paper we view them as the completions of the stalks of a microstructure sheaf defined on a Zariski filtered ring. For some general results we consider an unrestricted noncommutative situation but as far as applications of the general techniques are concerned we will restrict attention to the so-called almost commutative filtered rings; that is, the class of Zariski filtered rings having a commutative associated graded ring and usually we assume the latter to be positively graded (although this is nowhere essential if one defines the projective scheme Proj suitably). Taking sections of a coherent sheaf of graded modules over Zariski open sets of $\mathrm{Proj}(G(R))$ yields graded localizations of the graded ring $G(R)$ associated to the Zariski filtered R. However, taking sections on the level of the microstructure sheaf we obtain microlocalizations at Gabriel filters not necessarily stemming from multiplicatively closed sets. Using the relations between the category of filtered left R-modules, denoted by R-filt, and the categories of graded left $G(R)$-modules, denoted by $G(R)$-gr, over the associated graded ring $G(R)$ as well

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as graded left $\tilde{R}$-modules, denoted by $\tilde{R}$-gr, over the Rees ring $\tilde{R}$ of the filtration FR on R, we can treat microlocalization at Gabriel filters rather elegantly, at least when the localizations considered are perfect. The latter restriction does not harm generality in treating the microstructure sheaf theory because the scheme $\mathrm{Proj}(G(R))$ may be covered by affine open subschemes corresponding to perfect localizations in a natural way. In Section 2 we expound the general theory of localization of filtered rings and their associated graded rings and in Section 3 we study microlocalization at perfect localizations. Both these sections have been treated in the generality of Zariski filtered rings, that is, allowing noncommutative associated graded rings. This generality is not needed in Section 4 but it is essential in the construction of so-called gauge algebras (cf. [21]), providing many interesting examples (cf. loc. cit.). The main result in Section 2 is Proposition 2.6 and in Section 3 the main conclusions are summed up in Theorem 3.6, Theorem 3.8, and Theorem 3.12 but Proposition 3.14 is also very useful in the sheaf theory.

The consideration of microlocalizations at Gabriel functors is necessary because the microstructure sheaf has to be constructed by describing the sections over Zariski open subsets of $\mathrm{Proj}(G(R))$; for the sheaf theory we may restrict to a basis of affine open subsets and correspondingly work with perfect localizations exclusively. In the almost commutative situation considered in Section 4 the microlocalization at a perfect localization may be viewed as a deformation of a graded localization of the commutative associated graded ring. In this way the sheaf defined on the microlevel corresponding to the structure-sheaf of $\mathrm{Proj}(G(R))$ may be thought of as being a deformation of the latter, so following the philosophy of [6] we call it the sheaf of quantum sections over the structure sheaf $\mathbf{O}_X$ of $X = \mathrm{Proj}(G(R))$. We provide some properties regarding the coherence (Corollary 4.5) of the sheaf of quantum sections of a good filtered R-module as well as the strictness of the sheaf map of quantum sections associated to a strict morphism in R-filt. Finaly we introduce the quantum sections over the characteristic variety and we hope this will be useful for pure (or even holonomic) modules over certain rings of differential operators.

1. Preliminaries

All rings considered are associative and have a unit element, modules are left modules but ideal means two-sided ideal. For the reader's convenience we recall some basic notions from the general theory of localization at kernel functors or torsion theories. First let us point out that we really do need only minimal knowledge about this theory because the applications

we have in mind allow us to restrict to localizations of $G(R)$ given by homogeneous multiplicative sets $\{1, f, f^2, \ldots\}$ for some homogeneous $f \in G(R)$. Nevertheless the localization of R defined by this set ($\tilde{\kappa}$ in the terminology of Lemma 2.1) is *not* of this type so the consideration of more general Gabriel filters cannot be completely avoided. A Gabriel filter (or topology) is a filter $\mathscr{L}$ of left ideals of the ring considered, say A, satisfying the following conditions:

(i) If $I \in \mathscr{L}$ an $a \in A$ then $(I : a) = \{x \in A,\ xa \in I\}$ is in $\mathscr{L}$.

(ii) If $I \in \mathscr{L}$ and $J \subset I$ is such that for every $x \in I$, $(J : x) \in \mathscr{L}$ then $J \in \mathscr{L}$ (the four conditions of [8] are equivalent to these two).

To a Gabriel filter $\mathscr{L}$ we may associate a functor κ: A-mod $\to$ A-mod, which is a subfunctor of the identity functor satisfying:

(i) κ is a left exact functor.

(ii) $\kappa(M/\kappa M) = 0$ for every $M \in A$-mod.

The functor $\kappa = \kappa(\mathscr{L})$ is called an idempotent kernel functor (cf. [8]) and it may be defined by putting $\kappa M = \{m \in M,\ Im = 0 \text{ for some } I \in \mathscr{L}\}$. Conversely to any idempotent kernel functor κ we may associate a Gabriel filter $\mathscr{L}(\kappa) = \{I \text{ left ideal of } R,\ \kappa(R/I) = R/I\}$, and κ and $\mathscr{L}(\kappa)$ determine each other uniquely. Furthermore the κ-torsion class in A-mod is $T_\kappa = \{M \in A\text{-mod},\ \kappa M = M\}$, the κ-torsionfree class in A-mod is $F_\kappa = \{M \in A\text{-mod},\ \kappa M = 0\}$, and the pair (T_κ, F_κ) determines an hereditary torsion theory in A-mod (cf. [7]). We stick to the viewpoint of [8]. To κ one associates the localization functor $Q_\kappa(-)$: A-mod $\to$ A-mod defined by $Q_\kappa(M) = \varinjlim_{I \in \mathscr{L}(\kappa)} (I, M/\kappa M)$. This localization is in general left exact but not necessarily exact.

Many conditions equivalent to the exactness of Q_κ have been given in [8, 7, ...]; just recall that κ is called perfect when Q_κ is exact and commutes with direct sums, equivalently when for every A-module $M : Q_\kappa(M) = Q_\kappa(R) \otimes_R M$, or equivalently when the canonical ring morphism j_κ: $R \to Q_\kappa(R)$ is an epimorphism in the category of rings such that $Q_\kappa(R)$ is a right flat R-module via j_κ (note that $\ker j_\kappa = \kappa(R)$), or equivalently if for every left ideal $I \in \mathscr{L}(\kappa)$ we have $Q_\kappa(R)\, I = Q_\kappa(R)$.

Note that for an Ore set S of A the filter $\mathscr{L}(\kappa_S)$ generated by the left ideals L of A such that $L \cap S \neq \varnothing$ is a Gabriel filter and the localization $Q_{\kappa_S}(-)$ is just the usual localization functor associated to S; moreover κ_S is perfect as is well known.

When A is a $\mathbb{Z}$-graded ring and $\mathscr{L}(\kappa)$ has a filterbasis $\mathscr{L}(\kappa)^g$ consisting of graded left ideals then we may define the graded localization functor $Q_\kappa^g(-)$: A-gr $\to$ A-gr, by putting $Q_\kappa^g(M) = \varinjlim_{I \in \mathscr{L}(\kappa)^g} \mathrm{HOM}(I, M/\kappa M)$, where $\mathrm{HOM}(-,-) = \bigoplus_{p \in \mathbb{Z}} \mathrm{HOM}_p(-,-)$ and $\mathrm{HOM}_p(-,-)$ denoting the

A-linear morphisms of degree p (between graded objects). Detail on graded localization may be found in [19, 13]. Let us just point out that for $\mathscr{L}(\kappa)$ having a cofinal system of graded left ideals of finite type (e.g., in case A is Noetherian) then $Q_\kappa^g(M) = Q_\kappa(M)$ for every $M \in A$-gr because $\mathrm{HOM}(I, M/\kappa M) = \mathrm{Hom}(I, M/\kappa M)$ when I is finitely generated (cf. [13]). The latter of course applies when κ corresponds to a *homogeneous* Ore set S of A; therefore in the applications we have in mind (Sect. 4) the usual localizations of graded modules coincide with the graded localizations. All kernel functors considered in this paper have *finite type* in the sense that $\mathscr{L}(\kappa)$ will always have a cofinal subset consisting of finitely generated left ideals.

A filtered ring is a ring R with a filtration FR given as an ascending chain $\{F_n R, n \in \mathbb{Z}\}$ of additive subgroups of R satisfying $1 \in F_0 R$, $F_n R F_m R \subset F_{n+m} R$ for any $n, m \in \mathbb{Z}$. A filtered module is an R-module M with a filtration FM given as an ascending chain of additive subgroups of M, $\{F_n M, n \in \mathbb{Z}\}$ such that $F_n R F_m M \subset F_{n+m} M$ for every $n, m \in \mathbb{Z}$.

All filtrations considered will be exhaustive in the sense that $\bigcup_n F_n M = M$. We write R-filt for the category of filtered left R-modules and filtration-degree preserving R-linear maps. The filtration FR determines an associated graded ring $G(R) = \bigoplus_{i \in \mathbb{Z}} F_i R / F_{i-1} R$ and FM determines a graded left $G(R)$-module $G(M) = \bigoplus_{i \in \mathbb{Z}} F_i M / F_{i-1} M$. If $x \in F_n M - F_{n-1} M$ then we call n the filtration-degree of x and the principal symbol $\sigma(x) \in G(M)$ is defined to be the class of x in $F_n M / F_{n-1} M = G(M)_n$.

Detail on graded rings and modules may be found in [13, 14]. To a filtered ring R we may associate the Rees ring $\tilde{R} = \bigoplus_{n \in \mathbb{Z}} F_n R$ that may be viewed as the subring $\sum_n F_n R \cdot X^n$ in $R[X, X^{-1}]$ where X is a central variable, homogeneous of degree one. To $M \in R$-filt we correspond a graded $\tilde{R}$-module $\tilde{M} = \bigoplus_{n \in \mathbb{Z}} F_n M$ that may be viewed as $\sum_{n \in \mathbb{Z}} F_n M \cdot X^n$ in $M[X, X^{-1}]$. We write $\tilde{R}$-gr for the category of graded $\tilde{R}$-modules and gradation-preserving $\tilde{R}$-linear maps. A graded $\tilde{R}$-module is said to be X-torsionfree if none of its nonzero elements is annihilated by X. We let $\mathscr{F}_X$ be the full subcategory of $\tilde{R}$-gr consisting of the X-torsionfree objects. Clearly, for each $M \in R$-filt we have that $\tilde{M} \in \mathscr{F}_X$. Recall from [1] the following lemma.

1.1. LEMMA. *With conventions and notation as above*:

(a) $\tilde{R}/\tilde{R}X \cong G(R)$, $\tilde{M}/\tilde{M}X \cong G(M)$, *as graded objects.*

(b) $\tilde{R}/\tilde{R}(1-X) \cong R$, $\tilde{M}/\tilde{M}(1-X) \cong M$.

(c) $\tilde{R}_X \cong R[X, X^{-1}]$, $\tilde{M}_X \cong M[X, X^{-1}]$, *where* $(-)_X$ *denotes the localization at the multiplicative central set* $\{1, X, X^2, \ldots\}$.

(d) *The functor* $\sim : R\text{-filt} \to \tilde{R}\text{-gr}$ *defines an equivalence of categories between* R-filt *and* $\mathscr{F}_X$.

For further properties relating the Rees ring and the Rees modules to the corresponding filtered objects we refer to [1]. In the latter paper an algebraic approach to microlocalization of filtered rings is given. Starting from a multiplicatively closed set S in a filtered ring R with separated (i.e., $\bigcap_n F_n R = 0$) filtration FR, such that $\sigma(S)$ is multiplicatively closed and a left Ore set in $G(R)$, we define a multiplicatively closed $\tilde{S}$ and $\tilde{R}$ consisting of homogeneous elements sX^n with $s \in S$, $s \in F_n R - F_{n-1} R$. It is easily seen that $\tilde{S}$ maps to a homogeneous left Ore set in $\tilde{R}/\tilde{R}X^p$ for every $p \in \mathbb{N}$. Therefore we may define a graded ring $Q_{\tilde{S}}^{\mu}(\tilde{R}) = \varprojlim_n^g Q_{\tilde{S}}^{g}(\tilde{R}/X^n\tilde{R})$ where $\varprojlim^g$ denotes the inverse limit in the graded category, and we have written $\tilde{S}$ again for each image of $\tilde{S}$ in $\tilde{R}/X^n\tilde{R}$. In a similar way we may define a graded $\tilde{R}$-module (and in fact a graded $Q_{\tilde{S}}^{\mu}(\tilde{R})$-module) $Q_{\tilde{S}}^{\mu}(\tilde{M}) = \varprojlim_n^g Q_{\tilde{S}}^{g}(\tilde{M}/X^n\tilde{M})$. The microlocalization as filtered objects are then given by: $Q_S^{\mu}(R) = Q_{\tilde{S}}^{\mu}(\tilde{R})/(X-1)\,Q_{\tilde{S}}^{\mu}(\tilde{R})$; $Q_S^{\mu}(M) = Q_{\tilde{S}}^{\mu}(\tilde{M})/(X-1)\,Q_{\tilde{S}}^{\mu}(\tilde{M})$. We summarize some results of [1] in a theorem.

1.2. THEOREM. *With notation as introduced above*:

(a) $Q_{\tilde{S}}^{\mu}(\tilde{M})$ *is in* $\mathscr{F}_X$.

(b) *The filtration* $FQ_S^{\mu}(M)$ *is separated.*

(c) *For* $M \in R$-filt, $FQ_S^{\mu}(M)$ *is complete.*

(d) $G(Q_S^{\mu}(M)) = \sigma(S)^{-1}\,G(M)$.

(e) *The functor* Q_S^{μ} *is exact.*

(f) $Q_S^{\mu}(R)$ *is a flat right* R-*module.*

(g) *Define the saturation of* S, $S_{\text{sat}} = \{r \in R,\ \sigma(r) \in \sigma(S)\}$ *and let* F_{sat} *be the localized filtration defined on* $S_{\text{sat}}^{-1}M$ (*note that this makes sense because* S_{sat} *is a left Ore set of* R!). *Then* $Q_S^{\mu}(M) = Q_{S_{\text{sat}}}^{\mu}(M) = (S_{\text{sat}}^{-1}M)^{\wedge F_{\text{sat}}}$ *or in other words every microlocalization at a multiplicative set* S *as before* (*note that* S *need not be a left Ore set*) *is obtained as a completion of a localization at a left Ore set.*

In this paper, it is one of our aims to obtain similar results for microlocalization at perfect κ not necessarily associated to Ore sets because these will appear as quantum sections of the microstructure sheaf.

Finally, let us recall that FR is said to be faithful if all good filtrations FM are separated (see [9, 10] or [11] for full detail on good filtrations on finitely generated modules and the general theory of Zariskain filtrations) and this is equivalent to $F_{-1}R$ being included in the Jacobson radical of $J_0 R$. We say that FR is Zariskian whenever FR is faithful and $\tilde{R}$ is (left and right) Noetherian. When FR is Zariskian $\tilde{R}$, R, and $G(R)$ are Noetherian

rings but if FR is faithful and both R and $G(R)$ are Noetherian then FR need not be Zariskian of course; when FR is complete it will be Zariskian exactly then when $G(R)$ is Noetherian.

2. Localization of Filtered Rings and the Associated Graded Rings

Throughout FR is a faithful filtration and all kernel functors considered have finite type; these conditions hold trivially when FR is Zariskian and for the examples we have in mind we will *restrict attention to the Zariskian filtrations.*

We start from a kernel functor κ on R-mod with Gabriel filter $\mathscr{L}(\kappa)$ and we define $\mathscr{L}(\tilde{\kappa}) = \{J \text{ left ideal of } \tilde{R},\ J \supset \tilde{L} \text{ for some } L \in \mathscr{L}(\kappa)\}$.

2.1. Lemma. *$\mathscr{L}(\tilde{\kappa})$ is a Gabriel filter; hence $\tilde{\kappa}$ is an idempotent kernel functor.*

Proof. (a) If $J \in \mathscr{L}(\tilde{\kappa})$ and $H \supset J$ then $H \in \mathscr{L}(\tilde{\kappa})$ is obvious.

(b) If $I, J \in \mathscr{L}(\tilde{\kappa})$ then $I \cap J \supset \tilde{H}_1 \cap \tilde{H}_2 \supset (H_1 \cap H_2)^{\sim}$ for some $H_1, H_2 \in \mathscr{L}(\kappa)$ and then $H_1 \cap H_2 \in \mathscr{L}(\kappa)$.

(c) If $L \in \mathscr{L}(\tilde{\kappa})$ and $\tilde{y} \in \tilde{R}$ then we have to find $K \in \mathscr{L}(\tilde{\kappa})$ such that $K\tilde{y} \subset L$ and it is clear that if it suffices to do this for homogeneous $\tilde{y}$, say $\tilde{y} = yX^{m(y)}$ for some $y \in R$, $m(y) \in \mathbb{Z}$. Since $L \in \mathscr{L}(\tilde{\kappa})$, $L \supset \tilde{I}$ for some $I \in \mathscr{L}(\kappa)$ and there exists an $H \in \mathscr{L}(\kappa)$ such that $Hy \subset I$. Now, if $\tilde{h} \in \tilde{H}$ is homogeneous, say $\tilde{h} = hX^{m(h)}$ where $h \in F_{m(h)}H - F_{m(h)-1}H$, then $\tilde{h}\tilde{y} = hyX^{m(h)+m(y)}$. But $hy \in F_{m(hy)}R - F_{m(hy)-1}R$ with $m(hy) \leqslant m(h) + m(y)$ and thus $\tilde{h}\tilde{y} = \widetilde{hy}X^{m(h)+m(y)-m(hy)}$ where the exponent of X is at least zero. Therefore $\tilde{H}\tilde{y} \subset \widetilde{Hy} \subset \tilde{I} \subset L$ with $\tilde{H} \in \mathscr{L}(\tilde{\kappa})$ as desired.

(d) Let $H \subset L$ with $L \in \mathscr{L}(\tilde{\kappa})$ be such that L/H is a $\tilde{\kappa}$-torsion $\tilde{R}$-module. We have to show that $H \in \mathscr{L}(\tilde{\kappa})$; clearly we may assume that $L = \tilde{I}$, $I \in \mathscr{L}(\tilde{\kappa})$. Since H is X-torsionfree $H = \tilde{K}$ for some left ideal K of R. For each $i \in I$ there is a $J_i \in \mathscr{L}(\tilde{\kappa})$ such that $J_i\tilde{i} \subset \tilde{K}$ and it is not restrictive to assume that $J_i = \tilde{E}_i$ for some $E_i \in \mathscr{L}(\kappa)$. Since $\mathscr{L}(\kappa)$ is a Gabriel filter, $E_i i \subset K$ for every $i \in I \in \mathscr{L}(\tilde{\kappa})$ yields $K \in \mathscr{L}(\kappa)$ and so we arrive at $\tilde{K} = H \in \mathscr{L}(\tilde{\kappa})$. ∎

The definition of $\mathscr{L}(\tilde{\kappa})$ entails that $\mathscr{L}(\tilde{\kappa})$ has a cofinal system of graded left ideals, so we may consider the graded filter $\mathscr{L}^g(\tilde{\kappa}) = \{\tilde{L},\ L \in \mathscr{L}(\tilde{\kappa})\}$ and we write $\tilde{\kappa}$ again for the graded kernel functor on $\tilde{R}$-gr associated to $\mathscr{L}^g(\tilde{\kappa})$. Full detail on graded localizations (rigid torsion theories) may be found in [14, 19]. If $\pi\colon \tilde{R} \to G(R)$ is the canonical ring epimorphism then

we define $G\kappa$ on $G(R)$-mod by $\mathscr{L}(G\kappa) = \{L, L$ a left ideal of $G(R)$ such that $L \supset \pi(\tilde{H})$ for some $\tilde{H} \in \mathscr{L}(\tilde{\kappa})\}$.

2.2. LEMMA. *$\mathscr{L}(G\kappa)$ is a Gabriel filter.*

Proof. Easy. ∎

The definition of $\mathscr{L}(G\kappa)$ entails that it has a cofinal system of graded left ideals of $G(R)$; so we consider the graded filter $\mathscr{L}^g(G\kappa)$ and write $G\kappa$ again for the graded kernel functor on $G(R)$-gr associated to $\mathscr{L}^g(G\kappa)$. We use $Q^g_{\tilde{\kappa}}$, resp. $Q^g_{G\kappa}$, to denote the graded localization functor at $\tilde{\kappa}$, resp. $G\kappa$, in $\tilde{R}$-gr, resp. $G(R)$-gr.

At first sight it may not seem to be the natural thing to do to start from a κ in R-mod and construct a $\tilde{\kappa}$ on $\tilde{R}$-gr and a $G\kappa$ on $G(R)$-gr. Indeed, when looking at microlocalization it would be most plausible to start from some $\bar{\kappa}$ on $G(R)$-gr and then lift it to a $\tilde{\bar{\kappa}}$ on $\tilde{R}$-gr. However, if we start from an Ore set $\sigma(S)$ in $G(R)$ then S need not be an Ore set but S_{sat} is. On the level of more abstract localization we may define $\tilde{\bar{\kappa}}$ by taking $\beta^g = \{\tilde{L} \subset \tilde{R},\ \pi(\tilde{L}) \in \mathscr{L}(\bar{\kappa})\}$ as a filter basis. If this set is indeed a filter basis then we may define a κ on R-mod by letting $\mathscr{L}(\tilde{\kappa})$ be generated by those left ideals L of R such that $L[X, X^{-1}] \cong \tilde{L}_X$ for some $\tilde{L} \in \beta^g$. In general it seems that β^g need not be a filter basis for an idempotent filter in $\tilde{R}$-mod. We say that $\bar{\kappa}$ *is saturated* if β^g is a filter basis for an idempotent (graded) filter, $\mathscr{L}(\tilde{\bar{\kappa}})$ say. In the sequel we shall only consider saturated $\bar{\kappa}$ and so we will write $\tilde{\kappa}$ for $\tilde{\bar{\kappa}}$ and we have $\bar{\kappa} = G\kappa$ in this case. Following notation of [1] we write $\tilde{G}$ for the functor $\tilde{R}\text{-gr} \to G(R)\text{-gr}$ given by $\tilde{M} \mapsto \tilde{M}/X\tilde{M}$.

2.3. LEMMA 1. *For $\tilde{M} \in \tilde{R}$, $\tilde{\kappa}(\tilde{M}) = \widetilde{\kappa(M)}$, $\tilde{M}/\tilde{\kappa}(\tilde{M}) = (M/\kappa(M))^{\sim}$.*

2. *For $M \in R$-filt, $G(\kappa(M)) = \bar{\kappa}(G(M))$, $G(M/\kappa M) = G(M)/\bar{\kappa}G(M)$.*

3. *For $\tilde{M} \in \tilde{R}$-gr, $\tilde{G}(\tilde{\kappa}(\tilde{M})) = \tilde{\kappa}(\tilde{G}(\tilde{M}))$, $\tilde{G}(\tilde{M}/\tilde{\kappa}\tilde{M}) = \tilde{G}(\tilde{M})/\tilde{\kappa}\tilde{G}(\tilde{M})$.*

Proof. 1. Since $\tilde{\kappa}(\tilde{M}) \subset \tilde{M}$ it is X-torsionfree and thus $\tilde{\kappa}(\tilde{M}) = \tilde{N}$. Clearly N is κ-torsion and thus $\tilde{N} \subset \widetilde{\kappa(M)}$. On the other hand, if $\tilde{m}$ is homogeneous in $\widetilde{\kappa(M)}$ then $\tilde{m} = mX^d$ for some $d \in \mathbb{N}$, $m \in F_dM - F_{d-1}M$ where $m \in \kappa(M)$. Thus $Lm = 0$ for some $L \in \mathscr{L}(\kappa)$ and then it follows from $\widetilde{LM} \supset \tilde{L}\tilde{m}$ (see proof of Lemma 2.1) that $\tilde{m} \in \tilde{\kappa}(\tilde{M})$; hence $\tilde{\kappa}(\tilde{M}) = \widetilde{\kappa(M)}$. Next we check that $\tilde{M}/\tilde{\kappa}(\tilde{M})$ is X-torsionfree; if $\tilde{y}$ is homogeneous in $\tilde{M}$ such that $X\tilde{y} \subset \tilde{\kappa}(\tilde{M})$ then $\tilde{I}X\tilde{y} = 0$ for some $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$ and therefore $\tilde{I}\tilde{y} = 0$ or $\tilde{y} \in \tilde{\kappa}(\tilde{M})$. So we may write $\tilde{M}/\tilde{\kappa}(\tilde{M}) = \tilde{M}/\widetilde{\kappa(M)} = \tilde{N}$ for some $N \in R$-mod and one easily checks that $N = M/\kappa(M)$ (e.g., localize at X: $N[X, X^{-1}] = \tilde{M}_X/\tilde{\kappa}(\tilde{M})_X = M[X, X^{-1}]/\kappa(M)[X, X^{-1}] = (M/\kappa M)[X, X^{-1}]$).

2. Note that $G(\kappa(M)) = \pi(\widetilde{\kappa(M)}) = \pi(\tilde{\kappa}(\tilde{M}))$. Hence $G(\kappa(M))$ is $\bar{\kappa}(M))$ is $\bar{\kappa}$-torsion or $G(\kappa(M)) \subset \bar{\kappa}(G(M))$. Conversely if $\bar{z} \in \bar{\kappa}(G(M))$ is homogeneous then there is an $\tilde{H} \in \mathscr{L}(\tilde{\kappa})$ such that $\pi(\tilde{H})\bar{z} = 0$. Take a homogeneous $z \in \tilde{M}$ representing $\bar{z}$. Then $\tilde{H}z$ is in $\operatorname{Ker} \pi$ and since $\tilde{H}z$ is X-torsionfree we have that $\tilde{H}z = \tilde{N}$ for some filtered submodule N of M (not necessarily having the induced filtration though). Since $\tilde{H}$ is finitely generated by the Zariski hypothesis, it follows that $\tilde{N}$ has finite type and hence FN is a good filtration. Again by the Zariski hypothesis FN is then separated and $\pi(\tilde{N}) = G(N) = 0$ yields $N = 0$ or $z \in \tilde{\kappa}(\tilde{H})$ and $z \in (\tilde{\kappa}(\tilde{M})) = \pi(\kappa(M)^{\sim}) = G(\kappa(M))$. The second statement follows from the *strict* exactness of the sequence in R-filt: $0 \to \kappa(M) \to M \to M(\kappa(M) \to 0$, and the fact that G is exact on strict sequences.

3. Easy, using the saturatedness assumption on $\bar{\kappa}$ in the description of $\mathscr{L}(\bar{\kappa})$ and $\mathscr{L}(\tilde{\kappa})$. ∎

2.4. LEMMA. *With assumptions as before, $Q_{\tilde{\kappa}}^{g}(\tilde{M})$ is X-torsionfree.*

Proof. As observed before $\tilde{M}/\tilde{\kappa}(\tilde{M})$ is X-torsionfree. Now take a homogeneous $z \in Q_{\tilde{\kappa}}^{g}(\tilde{M})$ such that $X \cdot z = 0$. There is a $\tilde{J} \in \mathscr{L}(\tilde{\kappa})$ such that $\tilde{J}z \subset \tilde{M}/\tilde{\kappa}(\tilde{M})$ and $XJz = \tilde{J}Xz = 0$. Hence $\tilde{J}z = 0$ but that contradicts the fact that $Q_{\tilde{\kappa}}^{g}(\tilde{M})$ is $\tilde{\kappa}$-torsionfree. ∎

2.5. PROPOSITION. *With assumptions as before, consider $M \in R$-filt with good filtration FM. Then $Q_{\kappa}(M)$ has a natural filtration $FQ_{\kappa}(M)$ making the localization morphism $j_{\kappa}: M \to Q_{\kappa}(M)$ into a strict filtered morphism.*

1. *If $Q_{\kappa}(M)^{\sim}$ is constructed with respect to the filtration $FQ_{\kappa}(M)$ then we have $Q_{\tilde{\kappa}}^{g}(\tilde{M}) = Q_{\kappa}(M)^{\sim}$.*

2. *Letting κ, $\tilde{\kappa}$, and $\bar{\kappa}$ be as before (in particular $\bar{\kappa}$ is assumed to be saturated) then we have*

$$G(Q_{\kappa}(M)) = \tilde{G}(Q_{\tilde{\kappa}}^{g}(\tilde{M})) \subset Q_{\bar{\kappa}}^{g}(G(M))$$

$$Q_{\kappa}(M) = Q_{\tilde{\kappa}}^{g}(\tilde{M})/(1 - X)\, Q_{\tilde{\kappa}}^{g}(\tilde{M}).$$

Proof. If FM is good then the quotient filtration on $M/\kappa(M)$ is good and in view of Lemma 2.3(1) we may reduce the problem to the case where $\kappa(M) = \tilde{\kappa}(\tilde{M}) = \bar{\kappa}(G(M)) = 0$ and $j_{\kappa}: M \to Q_{\kappa}(M)$ is injective. For a non-zero $x \in Q_{\kappa}(M)$ there is an $I \in \mathscr{L}(\kappa)$ such that $Ix \subset M$. Since the filtration induced by FR on I, FI say, is good we have, for all $n \in \mathbb{Z}$, $F_n I = \sum_i F_{n-d_i} R \cdot \xi_i$ for certain $\xi_i \in I$ of degree $d_i \in \mathbb{Z}$. For some $\gamma_i \in \mathbb{Z}$ we have $\xi_i x \in F_{d_i + \gamma_i} M$ for each i and we may take $\gamma = \max \gamma_i$. Hence there exists a $\gamma \in \mathbb{Z}$ such that $Ix \subset M$ and for all $n \in \mathbb{Z}$, $F_n Ix \subset F_{n+\gamma} M$ $(*)$.

Let us first check that this γ depends on x but not on I provided we assume γ is taken to be minimal such that $(*)$ holds (note that a γ exists because otherwise $Ix \subset \bigcap_n F_n M = 0$ contradicts $\kappa(M) = 0$ unless $x = 0$ but the latter is excluded by the choice of x). This assumption on γ entails that there is an $n \in \mathbb{Z}$ such that $F_n I_x \subset F_{n+\gamma} M - F_{n+\gamma-1} M$. Now assume that for some $J \in \mathscr{L}(\kappa)$ we have $Jx \subset M$ and for all $n \in \mathbb{Z}$, $F_n Jx \subset F_{n+\gamma-1} M$. Pick $\xi \in F_n I$ such that $\xi x \in F_{n+\gamma} M - F_{n+\gamma-1} M$ and look at $(J : \xi) \in \mathscr{L}(\kappa)$. We now have for all $m \in \mathbb{Z}$:

$$F_m(J : \xi)\, \xi x \subset (F_{m+n} J)\, x \subset F_{m+n+\gamma-1} M.$$

For $\sigma(\xi x) \in G(M)_{n+\gamma}$ this means that $G(J : \xi)\, \sigma(\xi x) = 0$ but since $\sigma(M)$ is $\bar{\kappa}$-torsionfree it then follows that $\sigma(\xi x) = 0$; hence $\xi x = 0$, a contradiction because $\xi x \notin F_{n+\gamma-1} M$. So we may define a filtration on $Q_\kappa(M)$ by the filtration degree function v given by $v(x) = \gamma$, where γ is as above, i.e., minimal with respect to the property $(*)$, putting $F_\gamma Q_\kappa(M) = \{x \in Q_\kappa(M),\ v(x) \leqslant \gamma\}$. It is clear that $F_\gamma Q_\kappa(M)$, $\gamma \in \mathbb{Z}$, define an ascending chain of abelian subgroups of $Q_\kappa(M)$. Now look at $x, y \in Q_\kappa(R)$ such that $I_1 x \subset R$, $I_2 y \subset R$ satisfying property $(*)$ and put $J = (I_2 : x)_k$. Then, for all $n \in \mathbb{Z}$ we have for all $n \in \mathbb{Z} : F_n(J \cap I_1) x \subset F_{n+v(x)} R \cap I_2 = F_{n+v(x)} I_2$, $F_n(J \cap I_1)\, xy \subset (F_{n+v(x)} I_2)\, y \subset F_{n+v(x)+v(y)} R$. Hence it follows that $FQ_\kappa(R)$ makes $Q_\kappa(R)$ into a filtered ring. That $F_n R \subset F_n Q_\kappa(R)$ is obvious. On the other hand $F_n Q_\kappa(R) \cap R$ is the κ-closure of $F_n R$ in R but since $G(R)$ is $\bar{\kappa}$-torsionfree it follows directly that $F_n Q_\kappa(R) \cap R = F_n R$. Similarly for $x \in Q_\kappa(R)$, $y \in Q_\kappa(M)$ and $I_1 x \subset R$, $I_2 y \subset M$ satisfying $(*)$ we may put $J = (I_2 : x)_R$ and argue as before in the ring case. This proves the first statement.

1. Since $Q^g_{\tilde{\kappa}}(\tilde{M})/\tilde{M}$ is $\tilde{\kappa}$-torsion it follows that $Q_\kappa(M)/M$ is κ-torsion and $Q_\kappa(M)^\sim/\tilde{M}$ is $\tilde{\kappa}$-torsion; hence $Q_\kappa(M)^\sim \subset Q^g_{\tilde{\kappa}}(\tilde{M})$. For the converse note that $Q^g_{\tilde{\kappa}}(\tilde{M})$ is X-torsionfree (Lemma 2.4); hence $Q^g_{\tilde{\kappa}} = \tilde{N}$ for some $N \supset M$ in R-filt. Since $\tilde{N}/\tilde{N}$ is $\tilde{\kappa}$-torsion it follows that N/M is κ-torsion and so $N \subset Q_\kappa(M)$, but then $\tilde{N} \subset Q_\kappa(M)^\sim$ and this leads to $Q^g_{\tilde{\kappa}}(\tilde{M}) \subset Q_\kappa(M)^\sim$ and the equality $Q^g_{\tilde{\kappa}}(\tilde{M}) = Q_\kappa(M)^\sim$.

2. That $G(Q_\kappa(M)) = \tilde{G}(Q^g_{\tilde{\kappa}}(\tilde{M}))$ follows from 1. In view of Lemma 2.3(2) and (3) it follows that we may assume that $\kappa(M) = \tilde{\kappa}(\tilde{M}) = \bar{\kappa}(G(M)) = 0$. By the first part $j_\kappa : M \hookrightarrow Q_\kappa(M)$ is a strict filtered morphism. Exactness of G on strict exact sequences yields that $G(Q_\kappa(M))/G(M)$ is $\bar{\kappa}$-torsion (again using Lemma 2.3(2)) and thus $G(Q_\kappa(M)) \subset Q^g_{\tilde{\kappa}}(G(M))$. The second statement in 2 follows from $Q_\kappa(M)^\sim = Q^g_{\tilde{\kappa}}(\tilde{M})$ and $Q_\kappa(M) = Q_\kappa(M)^\sim/(1 - X)\, Q_\kappa(M)^\sim$. ∎

2.6. Proposition. *With notations as before, if $\tilde{\kappa}$ is perfect then we have:* $\tilde{G}(Q^g_{\tilde{\kappa}}(\tilde{R})) = Q^g_\kappa(G(R))$, $\tilde{G}(Q^g_{\tilde{\kappa}}(\tilde{M})) = Q^g_\kappa(G(M))$ *for* $M \in R$-filt.

Proof. We have: $Q^g_{\tilde\kappa}(\tilde M)/XQ^g_{\tilde\kappa}(\tilde M) = Q^g_{\tilde\kappa}(\tilde M/X\tilde M) = Q^g_{\tilde\kappa}(G(M))$ by the exactness of $Q^g_{\tilde\kappa}(-)$. It is clear that $Q^g_{\tilde\kappa}(G(M)) = Q^g_{\bar\kappa}(G(M))$. ∎

2.7. PROPOSITION. *If $\bar\kappa$ is saturated and $\tilde\kappa$ is perfect then $\bar\kappa$ is perfect.*

Proof. Follows from $\tilde G(Q^g_{\tilde\kappa}(\tilde M)) = Q^g_{\bar\kappa}(G(M))$ and the exactness properties of $\tilde G$ and $Q^g_{\tilde\kappa}$ or else by applying this formula to $\tilde M = \tilde I \in \mathscr{L}(\tilde\kappa)$ to obtain $\tilde G(Q^g_{\tilde\kappa}(\tilde R)) = Q^g_{\bar\kappa}(\pi(\tilde I))$ for all $\pi(\tilde I) \in \mathscr{L}^g(\bar\kappa)$ (note that perfectness of a graded kernel functor may be checked on graded modules). ∎

2.8. COROLLARY. *Let us write $\tilde\kappa(n)$ for the kernel functor induced by $\tilde\kappa$ on $\tilde R/X^n\tilde R$-mod. Then $Q^g_{\tilde\kappa(n)}(\tilde M/X^n\tilde M) = Q^g_{\tilde\kappa}(\tilde M/X^n\tilde M)$ for all $n \in \mathbb{N}$, and if $\tilde\kappa$ is perfect then all the $\tilde\kappa(n)$ are perfect too.*

Proof. Similar to the one above but replacing $\tilde G$ by the functor $(-)/X^n(-)$ that enjoys similar exactness properties. ∎

It would be nice to obtain the perfectness of $\tilde\kappa$ from the perfectness of $\bar\kappa$ but as far as we could see the extra assumption $\tilde G Q^g_{\tilde\kappa}(\tilde R) = Q^g_{\bar\kappa}(G(R))$ has to be added in order to obtain results. This property does hold in situations where for $\pi(\tilde I)$ in a filter basis of $\mathscr{L}(\bar\kappa)$ any $G(R)$-linear map $\pi(\tilde I) \to G(R)$ may be lifted to an $\tilde R$-linear $\tilde I \to \tilde R$; this happens in case $\bar\kappa$ is a localization at a (left) Ore set $\sigma(S)$ and $\tilde\kappa$ is then the localization at $\tilde S_{\text{sat}}$ as described earlier.

2.9. PROPOSITION. *Let $\bar\kappa$ be saturated and perfect and assume that $\tilde G(Q^g_{\tilde\kappa}(\tilde R)) = Q^g_{\bar\kappa}(G(R))$. If either $X \in J^g(Q^g_{\tilde\kappa}(\tilde R))$ or $Q^g_{\tilde\kappa}(\tilde R)$ is (left) Noetherian then $\tilde\kappa$ is perfect.*

Proof. In case $X \in J^g(Q^g_{\tilde\kappa}(\tilde R))$ then for $\tilde I \in \mathscr{L}(\tilde\kappa)$ we have that $1 \in Q^g_{\bar\kappa}(G(R)) \cdot \pi(\tilde I)$ and this leads to the existence of a homogeneous unit in $Q^g_{\tilde\kappa}(\tilde R)\tilde I$ or $Q^g_{\tilde\kappa}(\tilde R) \cdot \tilde I = Q^g_{\tilde\kappa}(\tilde R)$ for every $\tilde I \in \mathscr{L}(\tilde\kappa)$. So let us consider the case where $Q^g_{\tilde\kappa}(\tilde R)$ is Noetherian. Recall that for a normalizing element x of a left Noetherian ring A and a finitely generated A-module M we have that either $\bigcap_n x^n M = 0$ or $\bigcap_n x^n M$ is an x-torsionfree A-submodule of M. Now first consider a finitely generated graded $Q^g_{\tilde\kappa}(\tilde R)$-module M. Then M/XM is a $\tilde G(Q^g_{\tilde\kappa}(\tilde R)) = Q^g_{\bar\kappa}(G(R))$-module and by the perfectness of $\bar\kappa$ we know that $\bar\kappa(M/XM) = 0$ or $\tilde\kappa(M) = \tilde\kappa(XM)$ is contained in XM. Replacing M by XM we obtain $\tilde\kappa(M) \subset X^2M$ and so on; hence $\tilde\kappa(M) \subset \bigcap_n X^n M$. When the latter intersection is zero then $\tilde\kappa(M) = 0$. So let us look at the case where $\bigcap_n X^n M$ is an X-torsionfree $Q^g_{\tilde\kappa}(\tilde R)$-module and note that it is again a finitely generated and graded $Q^g_{\tilde\kappa}(\tilde R)$-module, so we write M again for $\bigcap_n X^n M$. Since now M is X-torsionfree we may write $M = \tilde N$ for some $N \in R$-filt with good filtration FN. By the standing assumption (FR is faithful) we know that the filtration FN is separated and therefore $\tilde N$ is

separated in the X-adic topology, i.e., $\bigcap_n X^n\tilde{N}=0$. Since $\tilde{\kappa}(M)\subset\bigcap_n X^n\tilde{N}$ follows again from the fact that each $X^n\tilde{N}/X^{n+1}\tilde{N}$ is $\bar{\kappa}$-torsionfree because they are $Q^g_{\bar{\kappa}}(G(R))$-modules, we arrive at $\tilde{\kappa}(M)=0$. Now consider a graded $Q^g_{\bar{\kappa}}(R)$-module not necessarily of finite type, M again say. If $z\in h(\tilde{\kappa}(M))$ then $Q^g_{\bar{\kappa}}(\tilde{R})z=L$ is a finitely generated graded $Q^g_{\bar{\kappa}}(\tilde{R})$-module and so $\tilde{\kappa}(L)=0$ by the first part of the proof. However, $z\in\tilde{\kappa}(L)$ then yields $z=0$ and since $\tilde{\kappa}$ is a graded kernel functor and M being graded we may conclude from $h(\tilde{\kappa}(M))=0$ that $\tilde{\kappa}(M)=0$. ∎

2.10. *Question.* 1. If FR is Zariskian and $\bar{\kappa}$ is saturated and perfect does it follow that $Q^g_{\bar{\kappa}}(\tilde{R})$ is Noetherian?

2. Can one characterize a nice class of $\bar{\kappa}$ not necessarily associated to multiplicatively closed sets such that $\tilde{G}(Q^g_{\bar{\kappa}}(\tilde{R}))=Q^g_{\bar{\kappa}}(G(R))$ holds without further restrictions? In Section 3 we can approach this problem from a different angle by using microlocalizations. For our use the condition "$\bar{\kappa}$ is perfect" seems to be the most practical one, but the philosophy of the use of the Rees ring would be better served if we could give nice sufficient conditions on $\bar{\kappa}$ only.

3. In the situation of Question 1, even allowing $Q^g_{\bar{\kappa}}(\tilde{R})$ to be Noetherian, is the filtration $FQ_\kappa(R)$ then Zariskian; in other words, does it follow from these assumptions that $X\in J^g(Q^g_{\bar{\kappa}}(\tilde{R}))$. The problem may be extended to: for which $\bar{\kappa}$ is $FQ_\kappa(R)$ again a Zariskian filtration if we assume that FR is Zariskian?

3. Microlocalization at Perfect $\bar{\kappa}$

Throughout this section we assume that FR is a Zariskian filtration and $\bar{\kappa}$ is saturated such that $\bar{\kappa}$ is perfect. As in the foregoing section we write $\bar{\kappa}(n)$ for the kernel functor induced by $\bar{\kappa}$ on $\tilde{R}/X^n\tilde{R}$ and by the saturatedness condition $\mathscr{L}(\bar{\kappa}(n))$ has a filter basis consisting of $\tilde{I}/X^n\tilde{I}$ with $\tilde{I}\in\mathscr{L}(\bar{\kappa})$; let us write Q^g_n for the graded localization functor associated to $\bar{\kappa}(n)$ and recall that all $\bar{\kappa}(n)$ are perfect (see Corollary 2.8). We now define the microlocalization on the Rees ring level as follows:

$$Q^\mu_{\bar{\kappa}}(\tilde{M})=\varprojlim_n{}^g\, Q^g_n(\tilde{M}/X^n\tilde{M}).$$

Up to checking the following lemma we may then define the microlocalization of $M\in R$-filt at κ by the expected formula: $Q^\mu_\kappa(M)=Q_{\bar{\kappa}}(\tilde{M})/(1-X)Q^\mu_{\bar{\kappa}}(\tilde{M})$, meaning in particular that $FQ^\mu_\kappa(M)$ is determined by the gradation of $Q^\mu_{\bar{\kappa}}(\tilde{M})$ in the usual way (cf. [1] before Lemma 2.1.).

3.1. LEMMA. *$Q^{\mu}_{\tilde{\kappa}}(\tilde{M})$ is X-torsionfree.*

Proof. Suppose that $Xa=0$ for some $a \in Q^{\mu}_{\tilde{\kappa}}(\tilde{M})$ and let us write $a_{(n)} \in Q^{g}_{n}(\tilde{M}/X^n\tilde{M})$ representing a at a level n in the inverse limit. For some $n \in \mathbb{N}$ we have $Xa_{(n)}=0$ with $a_{(n)} \neq 0$. For some $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$ we have then $\tilde{I}a_{(n)} \neq 0$ in $j_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M})$ and $X\tilde{I}a_{(n)}=0$. For $i \in \tilde{I}$ let $b_{(n)}(i) \in \tilde{M}/X^n\tilde{M}$ represent $ia_{(n)}$. Since $Xb_{(n)}(i)$ maps to zero in $j_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M})$ (where $j_{\tilde{\kappa}(n)}$ is the canonical morphism $\tilde{M}/X^n\tilde{M} \to Q^{g}_{n}(\tilde{M}/X\tilde{M})$), then $Xb_n(i) \in \tilde{\kappa}(n)(\tilde{M}/X^n\tilde{M})$ and thus there is a $\tilde{J} \in \mathscr{L}(\tilde{\kappa})$ such that $\tilde{J}Xb_{(n)}(i)=0$ in $\tilde{M}/X^n\tilde{M}$. Consequently $\tilde{J}b_{(n)}(i) \in X^{n-1}\tilde{M}/X\tilde{M}$ or $ia_{(n)} \in Q^{g}_{n} \in Q^{g}_{n}(X^{n-1}\tilde{M}/X^n\tilde{M})$ for all $i \in \tilde{I}$. Hence $\tilde{I}a_{(n)} \in Q^{g}_{n}(X^{n-1}\tilde{M}/X^n\tilde{M})$ and thus also $a_{(n)} \in Q^{g}_{n}(X^{n-1}\tilde{M}/X^n\tilde{M})$. This means that $a_{(n-1)}=0$. Since we may start the argument at any m larger than n it follows that $a_{(n)}=0$ for all n and thus $a=0$ as desired. ∎

3.2. LEMMA. *With notation as before: $FQ^{\mu}_{\kappa}(R)$ and $FQ^{\mu}_{\kappa}(M)$ are separated (FM is assumed to be separated).*

Proof. An obvious modification of Lemma 3.7 of [1]. ∎

3.3. LEMMA. *$FQ^{\mu}_{\kappa}(M)$ is complete for $M \in R$-filt.*

Proof. One checks that $Q^{\mu}_{\kappa}(\tilde{M})$ is X-adically complete just as in the proof of Proposition 3.9 of [1] using the exactness of the functor Q^{g}_{n}, where necessary, instead of the exactness deriving from the Ore conditions used in the proposition, loc. cit. ∎

For microlocalizations $Q^{\mu}_{\kappa}(R)$ we may derive a universal property generalizing the situation of localization at Ore sets.

3.4. PROPOSITION. *Let us write $j^{\mu}_{\kappa}: R \to Q^{\mu}_{\kappa}(R)$ for the canonical microlocalization morphism. Given a complete filtered ring B and a filtered ring homomorphism $h: R \to B$ such that $Bh(I)=B$ and $G(B)\,G(I)=G(B)$ for every $I \in \mathscr{L}(\kappa)$ then there exists a unique filtered ring homomorphism $g: Q^{\mu}_{\kappa}(R) \to B$ such that $h=g \circ j^{\mu}_{\kappa}$.*

Proof. Since h is a filtered homomorphism it yields a graded morphism (of degree zero) $\tilde{h}: \tilde{R} \to \tilde{B}$. From $h(1_R)=1_B$ it follows that $\tilde{h}(X)=X_B$ and $\tilde{h}$ is a graded ring morphism. From $\tilde{h}(X^n\tilde{R})=X^n_B\tilde{h}(\tilde{R})$ it follows that $\tilde{h}$ gives rise to the composition of graded morphisms, $\tilde{h}_n: \tilde{R}/X^n\tilde{R} \to \tilde{h}(\tilde{R})/X^n_B h \neq (\tilde{R}) \to \tilde{B}/X^n_B\tilde{B}$. Since $Bh(I)=B$ for $I \in \mathscr{L}(\kappa)$ it is clear that $B=Q_{\kappa}(B)$. Localizing $\tilde{h}_n$ at $\tilde{\kappa}(n)$ we obtain a graded ring homomorphism

$$Q^{g}_{\kappa(n)}(\tilde{h}_n): Q^{g}_{\tilde{\kappa}(n)}(\tilde{R}/X^n)\tilde{R}) \to Q^{g}_{\tilde{\kappa}(n)}(\tilde{h}(\tilde{R})/X^n_B\tilde{h}(\tilde{R})) \to \tilde{B}/X^n_B\tilde{B},$$

where $\tilde{B}/X^n_B\tilde{B} = Q^g_{\tilde{\kappa}(n)}(\tilde{B}/X^n_B\tilde{B})$ by the exactness of $Q^g_{\tilde{\kappa}(n)}$ and the assumptions on B. Note that the condition $G(B)\,\pi(\tilde{I}) = G(B)$ is necessary here (even in case one inverts an Ore set S it is necessary to specify in the universal property that $h(s)^{-1} \in B$ has the correct order in the sense that if $\deg \sigma(s) = n$ then $\deg \sigma(h(s)^{-1} = -n!$). It is in fact this condition that allows one to conclude that $\tilde{B}\tilde{I} = \tilde{B}$ for $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$. Now taking inverse limits in the graded sense yields a graded morphism $\tilde{g}\colon Q^\mu_{\tilde{\kappa}}(\tilde{R}) \to \tilde{B} = \varprojlim^g_n \tilde{B}/X^n_B\tilde{B}$ (the latter equality holds because B is complete and thus $\tilde{B}$ is X_B-adically complete). From $\tilde{R} \to_{f^\mu_\kappa} Q^\mu_{\tilde{\kappa}}(\tilde{R}) \to \tilde{B}$ we derive the desired morphism $g\colon Q^\mu_\kappa(R) \to B$ factorizing h as claimed. ∎

3.5. *Note.* From $Bh(I) = B$ it follows that $N = \tilde{B}/\widetilde{Bh(I)}$ is X_B-torsion while $G(B)\,\pi(\widetilde{h(I)}) = G(B)$ yields that $N/X_BN = 0$; hence $N = X_BN = \cdots = X^t_BN$ for any t but that contradicts the fact that N is X_B-torsion unless $N = 0$.

All properties of microlocalization at Ore sets only depending on the exactness of the functor $Q^g_{\tilde{\kappa}}$ (hence also Q^g_κ is exact) carry over to the case we are considering without real changes.

3.6. THEOREM. *With assumptions and notation as before*:

1. *$Q^\mu_{\tilde{\kappa}}(\tilde{R})$ is a flat right $\tilde{R}$-module.*

2. *If FM is a good filtration on $M \in R$-*filt *then we have $Q^\mu_{\tilde{\kappa}}(\tilde{M}) = Q^\mu_{\tilde{\kappa}}(\tilde{R}) \otimes_{\tilde{R}} \tilde{M}$.*

Proof. Along the lines of Theorem 3.19 in [1]. ∎

3.7. COROLLARY. 1. *The functor $Q^\mu_\kappa(R) \otimes_R$ preserves strict filtered maps and it is exact on R-modules.*

2. *If $\tilde{M}$ is finitely generated (i.e., FM is good) then $Q^\mu_\kappa(R) \otimes_R M \cong Q^\mu_\kappa(M)$ as filtered R-modules.*

Proof. Modify the proof of Corollary 3.20 in [1]. ∎

3.8. THEOREM. *With assumptions and notation as before*: $G(Q^\mu_\kappa(M)) = Q^g_\kappa(G(M))$ *for* $M \in R$-filt.

Proof. We have:

$$Q^\mu_{\tilde{\kappa}}(\tilde{M})/XQ^\mu_{\tilde{\kappa}}(\tilde{M}) = \varprojlim_n{}^g\, Q^g_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M})/\varprojlim_n Q^g_{\tilde{\kappa}(n)}(X\tilde{M}/X^n\tilde{M}).$$

The maps in the inverse system $\{Q^g_{\tilde{\kappa}(n)}(X\tilde{M}/X^n\tilde{M}), n\}$ are all surjective (exactness of $Q^g_{\tilde{\kappa}(n)}$) and so we may calculate the latter quotient term by term and then take $\varprojlim^g$; hence

$$\begin{aligned} Q^\mu_{\tilde{\kappa}}(\tilde{M}/XQ^\mu_{\tilde{\kappa}}(\tilde{M})) &= \varprojlim_n{}^g\, [Q^g_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M})/Q^g_{\tilde{\kappa}(n)}(X\tilde{M}/X^n\tilde{M})] \\ &= \varprojlim_n{}^g\, Q^g_{\tilde{\kappa}(n)}(\tilde{M}/X\tilde{M}) \\ &= Q^g_{\tilde{\kappa}(n)}(G(M)) \\ &= Q^g_{\tilde{\kappa}}(G(M)), \end{aligned}$$

where the latter equality follows because $\tilde{\kappa}(n)$ induces $\tilde{\kappa}$ on $G(R)$-gr for every $n \in \mathbb{N}$.

3.9. *Remark.* Since $Q^g_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M})$ is a graded $Q^g_{\tilde{\kappa}(n)}(\tilde{R}/X^n\tilde{R})$-module for every n, it follows that $Q^\mu_{\tilde{\kappa}}(M)$ is a $Q^\mu_{\tilde{\kappa}}(R)$-module and so $Q^\mu_{\tilde{\kappa}}(-)$ is a functor R-filt $\to Q^\mu_{\tilde{\kappa}}(R)$-filt. Along the lines of Corollary 3.16(2) of [1] it also follows that equivalent filtrations FM and $F'M$ yield equivalent filtrations $FQ^\mu_{\tilde{\kappa}}(M)$ and $F'Q^\mu_{\tilde{\kappa}}(M)$.

3.10. LEMMA. *With notation and assumptions as before we have:* $X \in J^g((Q_{\tilde{\kappa}}(R)^\wedge)^\sim) = J^g((Q^g_{\tilde{\kappa}}(\tilde{R}))^{\wedge_X})$, *where* $\wedge_X$ *stands for the completion in the* X*-adic topology.*

Proof. From Proposition 2.5(1) we know that $Q^g_{\tilde{\kappa}}(\tilde{R}) = (Q_{\tilde{\kappa}}(R))^\sim$ and by the translation from FR to the X-adic topology on $\tilde{R}$ we know that $Q^g_{\tilde{\kappa}}(\tilde{R})^{\wedge_X} = (Q_{\tilde{\kappa}}(R)^\wedge)^\sim$. So we arrive at the equality $(Q_{\tilde{\kappa}}(R)^\wedge)^\sim = Q^g_{\tilde{\kappa}}(\tilde{R})^{\wedge_X}$ and therefore the respective graded Jacobson radicals coincide. Put $B = Q^g_{\tilde{\kappa}}(\tilde{R})^{\wedge_X}$ and take $b \in B_{-1}$. Since B is complete in the X-topology there exists an element $1 + bX + b^2X^2 + \cdots$ in B and therefore $1 - bX$ has an inverse $1 + bX + b^2X^2 + \cdots$ for every $b \in B_{-1}$ or $X \in J^g(B)$. ∎

3.11. PROPOSITION. *Let* $\tilde{\kappa}$ *be saturated such that* $\tilde{\kappa}$ *is perfect as before. Then for every* $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$ *we have* $(Q^g_{\tilde{\kappa}}(\tilde{R}))^{\wedge_X} \cdot \tilde{I} = Q^g_{\tilde{\kappa}}(\tilde{R})^{\wedge_X} \cdot \tilde{I} = B$.

Proof. Since $\tilde{\kappa}$ has finite type we may assume that $\tilde{I}$ is finitely generated. We have $B/XB = G(Q^g_{\tilde{\kappa}}(\tilde{R})^{\sim_X}) = G(Q^g_{\tilde{\kappa}}(\tilde{R})) = Q^g_{\tilde{\kappa}}(G(R))$ and since $\tilde{\kappa}$ is perfect too we have $Q^g_{\tilde{\kappa}}(G(R))\,\pi(\tilde{I}) = Q^g_{\tilde{\kappa}}(G(R))$. Consequently $B \cdot \tilde{I} + BX = B$ and the foregoing lemma allows one to use the graded Nakayama lemma in order to derive $B\tilde{I} = B$. ∎

3.12. THEOREM. *With assumptions as before:* $Q^\mu_{\tilde{\kappa}}(M) \cong (Q_{\tilde{\kappa}}(M))^\wedge$, *for* $M \in R$-filt.

Proof. It will be sufficient to establish that $(Q^{\mu}_{\kappa}(M))^{\sim} \cong (Q_{\kappa}(M)^{\wedge})^{\sim}$, or $Q^{\mu}_{\kappa}(\tilde{M}) \cong \varprojlim_n Q^{g}_{\kappa}(\tilde{M})/X^n Q^{g}_{\kappa}(\tilde{M}) = \varprojlim_n Q^{g}_{\kappa}(\tilde{M}/X^n\tilde{M})$, where the latter equality follows from the exactness of Q^{g}_{κ}. By the saturatedness condition the $\tilde{I}/X^n\tilde{I}$ with $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$ form a filter basis for $\mathscr{L}(\tilde{\kappa}(n))$ (note that $\tilde{I}/X^n\tilde{I}$ maps to $\pi(\tilde{I}) \in \mathscr{L}(\bar{\kappa})$ under the canonical $\tilde{R}/X^n\tilde{R} \to \tilde{R}/X\tilde{R} = G(R)$); hence Q^{g}_{κ} and $Q^{g}_{\tilde{\kappa}(n)}$ coincide on $\tilde{R}/X^n\tilde{R}$-modules and we arrive at:

$$\begin{aligned} Q^{\mu}_{\kappa}(\tilde{M}) &= \varprojlim_n Q^{g}_{\tilde{\kappa}(n)}(\tilde{M}/X^n\tilde{M}) \\ &\cong \varprojlim_n Q^{g}_{\kappa}(\tilde{M}/X^n\tilde{M}) \\ &\cong \varprojlim_n Q^{g}_{\kappa}(\tilde{M})/(X^n Q^{g}_{\kappa}(\tilde{M})) \\ &= (Q^{g}_{\kappa}(\tilde{M}))^{\wedge X}. \end{aligned}$$

3.13. *Remark.* 1. Theorem 3.12 also follows from Proposition 3.11 because $B \cdot \tilde{I} = B$ and clearly $G(B)\,\pi(\tilde{I}) = G(B)$ allows one to apply the universal property and derive the isomorphism from the triangle:

$$\begin{tikzcd} & Q^{g}_{\tilde{\kappa}}(\tilde{R})^{\wedge X} \arrow[d, shift left] \\ Q^{g}_{\kappa}(\tilde{R}) \arrow[ur] \arrow[dr] & \\ & Q^{\mu}_{\kappa}(\tilde{R}) \arrow[uu, shift left] \end{tikzcd}$$

2. Theorem 3.12 extends the result that $Q^{\mu}_{S}(R) = (S^{-1}_{\mathrm{sat}} R)^{\wedge}$ where S is a multiplicative set of R such that $\sigma(S)$ is an Ore set in $G(R)$.

3. From Lemma 3.10 it is clear that $FQ^{\mu}_{\kappa}(R)$ is Zariskian when FR is; indeed, the facts that $Q^{\mu}_{\kappa}(R)$ is complete and $G(Q^{\mu}_{\kappa}(R)) = Q^{g}_{\kappa}(G(R))$ is Noetherian yield that $FQ^{\mu}_{\kappa}(R)$ is Zariskian.

For use in the next section we include some results concerning the comparison of microlocalizations at different Gabriel filters. Recall that we write $\tau > \sigma$ if $\mathscr{L}(\tau) \supset \mathscr{L}(\sigma)$ and let us suppose that $\bar{\tau} > \bar{\sigma}$ are saturated kernel functors on $G(R)$-gr; then (by definition) one sees that $\tilde{\tau} > \tilde{\sigma}$ and $\tau > \sigma$. Again we only consider saturated kernel functions $\tau, \sigma, \ldots$, such that $\tilde{\tau}, \tilde{\sigma}, \ldots$ are perfect. The exact sequence of strict filtered morphisms, for a given $M \in R$-filt

$$0 \to \tau(M)/\sigma(M) \to M/\sigma(M) \to M/\tau(M) \to 0 \qquad (*)$$

yields an exact sequence:

$$0 \to Q^{\mu}_{\sigma}(\tau(M)) \to Q^{\mu}_{\sigma}(M) \to Q^{\mu}_{\sigma}(M/\tau M) \to 0.$$

Since for every $\tilde{I} \in \mathscr{L}(\tilde{\sigma})$ we have that $\tilde{I} \in \mathscr{L}(\tilde{\tau})$ and thus

$$Q^{\mu}_{\tilde{\tau}}(\tilde{R})\,\tilde{I} = Q^{\mu}_{\tilde{\tau}}(\tilde{R}), \qquad Q^{g}_{\tilde{\tau}}(G(R))\,\pi(\tilde{I}) = Q^{g}_{\tau}(G(R)),$$

we have a unique filtered ring morphism $Q^{\mu}_{\sigma}(R) \to Q^{\mu}_{\tau}(R)$ and we obtain

$$\begin{array}{ccccccccc} 0 & \to & Q^{\mu}_{\sigma}(\tau(M)) & \to & Q^{\mu}_{\sigma}(M) & \to & Q^{\mu}_{\sigma}(M/\tau(M)) & \to & 0 \\ & & & & \downarrow{\scriptstyle \rho^{\sigma}_{\tau}} & & \downarrow{\scriptstyle \gamma^{\sigma}_{\tau}} & & \\ & & & & Q^{\mu}_{\tau}(M) & = & Q^{\mu}_{\tau}(M/\tau(M)). & & \end{array} \qquad (**)^{\mu}$$

From $\tilde{\tau} > \tilde{\sigma}$ we obtain the graded exact sequence in $\mathscr{F}_X$:

$$0 \to \tilde{\tau}(\tilde{M})/\tilde{\sigma}(\tilde{M}) \to \tilde{M}/\tilde{\sigma}(\tilde{M}) \to \tilde{M}/\tilde{\tau}(\tilde{M}) \to 0. \qquad \widetilde{(*)}$$

For $\tilde{M}/X^n\tilde{M}$ we will write $\bar{\tilde{M}}$. Since $\tilde{M}/\tilde{\tau}(\tilde{M})$ is X-torsionfree we may derive from $\widetilde{(*)}$ a graded exact sequence of $\bar{\tilde{R}}$-modules:

$$0 \to \tilde{\tau}(n)(\bar{\tilde{M}})/\tilde{\sigma}(n)(\bar{\tilde{M}}) \to \bar{\tilde{M}}/\tilde{\sigma}(n)(\bar{\tilde{M}}) \to \bar{\tilde{M}}/\tilde{\tau}(n)(\bar{\tilde{M}}) \to 0. \qquad (\tilde{*})_n$$

The exactness of $Q^{g}_{\tilde{\sigma}(n)}$ then yields a graded exact:

$$\begin{array}{ccccccccc} 0 & \to & Q^{g}_{\tilde{\sigma}(n)}(\tilde{\tau}(n))(\bar{\tilde{M}})) & \to & Q^{g}_{\tilde{\sigma}(n)}(\bar{\tilde{M}}) & \to & Q^{g}_{\tilde{\sigma}(n)}(\bar{\tilde{M}}/\tilde{\tau}(n)(\bar{\tilde{M}})) & \to & 0 \\ & & \downarrow{\scriptstyle \tilde{\rho}^{\sigma}_{\tau}(n)} & & & & \downarrow{\scriptstyle \tilde{\gamma}^{\sigma}_{\tau}(n)} & & \\ & & Q^{g}_{\tilde{\tau}(n)}(\bar{\tilde{M}}) & & \xrightarrow[=]{} & & Q^{g}_{\tilde{\tau}(n)}(\bar{\tilde{M}}). & & \end{array} \qquad \widetilde{(**)}_n$$

Now we may calculate $\varprojlim_n \widetilde{(**)}_n$ and do this term-wise in the exact sequence because all morphisms in the inverse systems are surjective (again by exactness of $Q^{g}_{\tilde{\sigma}(n)}$) so the Mittag-Lefler conditions hold; we obtain

$$\begin{array}{ccccccccc} 0 & \to & Q^{\mu}_{\tilde{\sigma}}(\tilde{\tau}(\tilde{M})) & \to & Q^{\mu}_{\tilde{\sigma}}(\bar{\tilde{M}}) & \to & Q^{\mu}_{\tilde{\sigma}}(\tilde{M}/\tilde{\tau}(\tilde{M})) & \to & 0 \\ & & \downarrow{\scriptstyle (\rho^{\sigma}_{\tau})^{\mu}} & & & & \downarrow{\scriptstyle (\gamma^{\sigma}_{\tau})^{\mu}} & & \\ & & Q^{\mu}_{\tilde{\tau}}(\tilde{M}) & & \xrightarrow[=]{} & & Q^{\mu}_{\tilde{\tau}}(\tilde{M}/\tilde{\tau}(\tilde{M})). & & \end{array} \qquad \widetilde{(**)}^{\mu}$$

Finally, by the exactness of $Q^g_{\tilde\sigma}$ acting on $\widetilde{(*)}$ we also have:

$$\begin{tikzcd} 0 \arrow[r] & Q^g_{\tilde\sigma}(\tilde\tau(\tilde M)) \arrow[r] & Q^g_{\tilde\sigma}(\tilde M) \arrow[r] \arrow[d, "\tilde\rho^\sigma_\tau"] & Q^g_{\tilde\sigma}(\tilde M)/\tilde\tau(\tilde M) \arrow[r] \arrow[d, "\tilde\gamma^\sigma_\tau"] & 0 \\ & & Q^g_{\tilde\tau}(\tilde M) \arrow[r, "="'] & Q^g_{\tilde\tau}(\tilde M/\tilde\tau(\tilde M)). & \end{tikzcd} \qquad (**)^g$$

Since $\tilde\tau(n) > \tilde\sigma(n)$ it is clear that $\tilde\gamma^\sigma_\tau(n)$ is injective for each n. By the Mittag-Lefler conditions for all inverse systems involved it follows that $(\gamma^\sigma_\tau)^\mu$ is injective too. Note that $\tilde\tau > \tilde\sigma$ also implies that $\tilde\gamma^\sigma_\tau$ is injective.

3.14. Proposition. *With notation and assumptions as before*:

1. *$Q^g_{\tilde\kappa}(\tilde M)/(\tilde M/\tilde\kappa(\tilde M))$ is X-torsionfree. If $\tilde\tau > \tilde\sigma$ then also $Q^g_{\tilde\tau}(\tilde M)/\tilde\rho^\sigma_\tau(Q^g_{\tilde\sigma}(\tilde M))$ is X-torsionfree.*

2. *$Q^\mu_{\tilde\kappa}(\tilde M)/(\tilde M/\tilde\kappa(\tilde M))$ is X-torsionfree. If $\tilde\tau > \tilde\sigma$ then we also have that $Q^\mu_{\tilde\tau}(\tilde M)/(\rho^\sigma_\tau)^\mu(Q^\mu_{\tilde\sigma}(\tilde M))$ is X-torsionfree.*

3. *The map $\rho^\sigma_\tau: Q^\mu_\sigma(M) \to Q^\mu_\tau(M)$ is a strict filtered morphism and it is the unique strict filtered extension of the canonical strict filtered morphism $M/\sigma(M) \to M/\tau(M)$.*

Proof. 1. Suppose $a \in Q^g_{\tilde\kappa}(\tilde M)$ is such that $Xa \in \tilde M/\tilde\kappa(\tilde M)$. For some $\tilde I \in \mathscr{L}(\tilde\kappa)$ we have $\tilde I a \subset \tilde M/\tilde\kappa(\tilde M)$. If $\tilde I a \not\subset X(\tilde M/\tilde\kappa(\tilde M))$ then $0 = \dot\pi(\tilde I X a) = \pi(\tilde I)\,\dot\pi(Xa)$ with $\dot\pi(Xa) \neq 0$ in $\widetilde{G(M/K(M))} = G(\tilde M/\tilde\kappa(\tilde M))$, where $\pi: \tilde R \to \tilde R/X\tilde R$, $\dot\pi: \tilde M/\tilde\kappa(\tilde M) \to \widetilde{G(M/K(M))}$ are the canonical maps. Since $\pi(\tilde I) \in \mathscr{L}(\bar\kappa)$ and $\widetilde{G(M/\kappa(M))} = G(M)/\bar\kappa G(M)$ (Lemma 2.3(3)) is $\bar\kappa$-torsionfree, it follows that $\dot\pi(Xa) = 0$ or $Xa \in X(\tilde M/\tilde\kappa(\tilde M))$. Since $Q^g_{\tilde\kappa}(\tilde M)$ is itself X-torsionfree $a \in \tilde M/\tilde\kappa(\tilde M)$ follows. For the second statement, consider $b \in Q^g_{\tilde\tau}(\tilde M)$ such that $Xb \in \tilde\rho^\sigma_\tau(Q^g_{\tilde\sigma}(\tilde M))$. Pick $\tilde J \in \mathscr{L}(\tilde\sigma)$ such that $\tilde J X b \subset \tilde\rho^\sigma_\tau(\tilde M/\tilde\sigma(\tilde M)) = \tilde M/\tilde\tau(\tilde M)$. Since $\tilde J b \subset Q^g_{\tilde\tau}(\tilde M)$ the first statement yields that $\tilde J b \subset \tilde M/\tilde\tau(\tilde M)$. The perfectness of $Q^g_{\tilde\sigma}$ and the injectivity of $\tilde\gamma^\sigma_\tau$ yields that $Q^g_{\tilde\sigma}(\tilde R)\cdot \tilde J b = Q^g_{\tilde\sigma}(\tilde R) b \subset \tilde\gamma^\sigma_\tau Q^g_{\tilde\sigma}(\tilde M/\tilde\tau(\tilde M))$; hence $b \in \tilde\rho^\sigma_\tau(Q^g_{\tilde\sigma}(\tilde M))$.

2. Suppose that $a \in Q^\mu_{\tilde\kappa}(\tilde M)$ is such that $Xa \in \tilde M/\tilde\kappa(\tilde M)$. We represent a by $a_n \in Q^g_{\tilde\kappa(n)}(\tilde M/X^n\tilde M)$. Then $Xa \to Xa_n \in \bar{\tilde M}/\tilde\kappa(n)\cdot(\bar{\tilde M}) = (\tilde M/\tilde\kappa(\tilde M))/X^n(\tilde M/\tilde\kappa(\tilde M))$ (note that $\tilde M/\tilde\kappa\tilde M$ is indeed X-torsionfree). For some $\tilde I \in \mathscr{L}(\bar\kappa)$ we have $\tilde I \cdot a_n \subset \bar{\tilde M}/\kappa(n)(\bar{\tilde M})$ and $X\tilde I \cdot a_n \subset X(\bar{\tilde M}/\kappa(n)(\bar{\tilde M}))$. Calculating module X yields $\pi(\tilde I)\,\bar\pi_n(Xa_n) = 0$, where $\pi_n: \tilde M/X^n\tilde M \to \tilde M/X\tilde M$ and the induced $\bar\pi_n: \bar{\tilde M}/\tilde\kappa(n)\,\bar{\tilde M} \to G(M)/\bar\kappa G(M)$ are the canonical maps. Since $\pi(\tilde I) \in \mathscr{L}(\bar\kappa)$ it follows that $\bar\pi_n(Xa_n) = 0$ or $Xa_n \subset X(\bar{\tilde M}/\tilde\kappa(n)\,\bar{\tilde M})$; therefore either $a_n \in \bar{\tilde M}/\tilde\kappa(n)\,\bar{\tilde M}$ or $Xa_n = 0$ and then $a_n \subset X^{n-1}(\bar{\tilde M}/\tilde\kappa(n)\,\bar{\tilde M})$. In the latter case $a_{n-1} = 0$. If for some n we have $a_n \notin \bar{\tilde M}/\tilde\kappa(n)\,\bar{\tilde M}$ then this is true for all larger n too and so the chain representing a, $(\ldots, a_n, \ldots)$ has a tail of

zeros on the left or else all $a_n \in \bar{\tilde{M}}/\tilde{\kappa}(n)\,\bar{\tilde{M}} = (\tilde{M}/\kappa\ (\tilde{M}))/X^n(\tilde{M}/\tilde{\kappa}(\tilde{M}))$ or $a \in (\tilde{M}/\tilde{\kappa}(\tilde{M}))^{\wedge X}$. Since $\tilde{M}/\tilde{\kappa}(\tilde{M})$ is X-closed in $(\tilde{M}/\tilde{\kappa}(\tilde{M}))^{\wedge X}$ it follows that $a \in \tilde{M}/\tilde{\kappa}(\tilde{M})$.

For the second statement take $b \in Q^{\mu}_{\tilde{\tau}}(\tilde{M})$ such that $Xb \in (\rho^{\sigma}_{\tau})^{\mu}\,(Q^{\mu}_{\tilde{\sigma}}(\tilde{M}))$. Represent b by b_n in $Q^{g}_{\tau(n)}(\tilde{M}/X^n\tilde{M})$ such that $Xb_n \in \tilde{\rho}^{\sigma}_{\tau}(n)(Q^{g}_{\tilde{\sigma}(n)}(\tilde{M}/X^n\tilde{M}))$. For some $\tilde{I} \in \mathscr{L}(\tilde{\sigma})$ we obtain: $X\tilde{I}\cdot b_n \subset \tilde{\rho}^{\sigma}_{\tau}(n)(\bar{\tilde{M}}/\tilde{\sigma}(n)\,\bar{\tilde{M}}) = \bar{\tilde{M}}/\tilde{\tau}(n)\,\bar{\tilde{M}}$.

Then either $\tilde{I}\cdot b_n \subset \bar{\tilde{M}}/\tilde{\tau}(n)\,\bar{\tilde{M}}$ or else $\tilde{I}\cdot b_n \subset X^{n-1}(\bar{\tilde{M}}/\tilde{\tau}(n)\,\bar{\tilde{M}})$, i.e., $\tilde{I}\cdot b_{n-1} = 0$ in $(\tilde{M}/X^{n-1}\tilde{M})/\tilde{\tau}(n-1)(\tilde{M}/X^{n-1}\tilde{M})$ or $b_{n-1} = 0$. In case all $b_n \in \tilde{\rho}^{\sigma}_{\tau}(n)(Q^{g}_{\tilde{\sigma}(n)}(\bar{\tilde{M}}))$ we obtain $b \in (\tilde{\rho}^{\sigma}_{\tau})^{\wedge}\,(Q^{g}_{\tilde{\sigma}}(\bar{\tilde{M}}))^{\wedge X}$ but the latter is $(\rho^{\sigma}_{\tau})^{\mu}\,(Q^{\mu}_{\tilde{\sigma}}(\tilde{M}))$ in view of Theorem 3.12. In case some $b_n \notin \tilde{\rho}_{\tau}(n)(Q^{g}_{\tilde{\sigma}(n)}(\bar{\tilde{M}}))$ then this is also true for all bigger n so $b_m = 0$ for all $m \geqslant n$ for some given n; hence $b_n = 0$ for all n and $b = 0$ in this case.

3. Obvious from 2. ∎

3.15. *Remark.* Using $Q^{g}_{\tilde{\kappa}}(\tilde{M})^{\wedge X} = Q^{\mu}_{\tilde{\kappa}}(\tilde{M})$ from the beginning of the proof of Proposition 3.14(2) one may reduce the proof to 1 plus the observation that the morphism $\tilde{\rho}^{\sigma}_{\tau}$ is an $\mathscr{F}_X$-morphism such that $X^nQ^{g}_{\tilde{\tau}}(\tilde{M}) \cap \tilde{\rho}^{\sigma}_{\tau}(Q^{g}_{\tilde{\sigma}}(\tilde{M})) = X^n\tilde{\rho}^{\sigma}_{\tau}(Q^{g}_{\tilde{\sigma}}(\tilde{M}))$; hence $\tilde{\rho}^{\sigma}_{\tau}$ is strict filtered in the X-adic topologies of the modules. The composition $Q^{g}_{\tilde{\sigma}}(\tilde{M}) \xrightarrow[\tilde{\rho}^{\sigma}_{\tau}]{} Q^{g}_{\tilde{\tau}}(\tilde{M}) \to (Q^{g}_{\tilde{\tau}}(\tilde{M}))^{\wedge X}$ is strict for the X-adic topologies involved; hence it factorizes through a strict $(Q^{g}_{\tilde{\sigma}}(\tilde{M}))^{\wedge X} \xrightarrow[(\rho^{\sigma}_{\tau})^{\mu}]{} (Q^{g}_{\tilde{\tau}}(\tilde{M}))^{\wedge X}$.

4. A Microstructure Sheaf

In this section we assume that R is a Zariski filtered ring such that $G(R)$ *is a commutative Noetherian domain* and we will also assume that $G(R)$ *is a positively graded ring.* For some of the results one may consider a much more general setting, but for the applications we know it is enough to consider this classical situation. Note that the assumptions on $G(R)$ alone already imply that FR is Zariskian. If one drops the assumption that $G(R)$ is positively graded the results remain true but at the price of replacing $\mathrm{Proj}(G(R))$ by the graded spectrum $\mathrm{Spec}^{g}(G(R))$ which is a less commonly used geometric space. $\mathrm{Proj}(G(R))$ with its Zariski topology has two structure sheaves defined on it, the graded one and the ungraded one obtained by taking parts of degree zero in the graded one. It is the latter one that is regarded as the structure sheaf of Proj. We write $X = \mathrm{Proj}(G(R))$, and $X(I) \subset \mathrm{Proj}(G(R)) = \{P \in \mathrm{Spec}^{g}(G(R)),\ P \not\supset G(R)_+ = \bigoplus_{n>0} G(R)_n\}$ is a Zariski open set associated to a graded ideal of I of $G(R)$ given by $X(I) = \{P \in \mathrm{Proj}(G(R)),\ P \not\supset I\}$. The graded structure sheaf $\mathbf{O}^{g}_{X}$ is obtained by taking for the sections over $X(I)$ the graded ring $Q^{g}_{I}(G(R))$. The structure sheaf $\mathbf{O}_X$ is then obtained by associating $(Q^{g}_{I}(G(R)))_0$ to the Zariski open set $X(I)$ and this is indeed the classical projective structure sheaf (but we

used the description of all sections by localizations in the philosophy of D. Murdoch and F. Van Oystaeyen, cf. [12], that also reappeared in the graded context, cf. [20]). If $J \supset I$ are graded ideals of $G(R)$ then $X(I) \subset X(J)$ and $\bar{\kappa}_I \geqslant \bar{\kappa}_J$ or $\mathscr{L}(\bar{\kappa}_I) \subset \mathscr{L}(\bar{\kappa}_J)$. Restricting to ideals of the form $I = G(R)\bar{r}$ where $\bar{r}$ is homogeneous in $G(R)$, we see that $\mathrm{Proj}(G(R))$ has a topology basis such that the associated $\bar{\kappa}_I$ are saturated; indeed $\tilde{\kappa}_I$ will then be associated to the Ore set $\pi^{-1}(\{1, \bar{r}, \bar{r}^2, \ldots\})$ in $\tilde{R}$. For properties of sheaves we may almost always restrict to a basis of the topology so we may fix from now on *a basis $\mathscr{B}$ for the Zariski topology of X consisting of sets $X(I)$ for which $\bar{\kappa}_I$ is saturated and $\tilde{\kappa}_I$ is perfect* (i.e., it is enough that $\mathscr{B}$ contains the $X(I)$ for graded principal ideals of $G(R)$) and we also write $\mathscr{B}$ for the set of graded kernel functors corresponding to the $X(I) \in \mathscr{B}$; that is, we will write $\bar{\kappa} \in \mathscr{B}$ if $\bar{\kappa} = \bar{\kappa}_I$ for some $X(I) \in B$. To a $\bar{\kappa} \in \mathscr{B}$ we associate $Q^{\mu}_{\bar{\kappa}}(\tilde{R})$ and $Q^{\mu}_{\kappa}(R)$ as defined in the foregoing section. From Proposition 3.14(3) it follows that we have defined presheaves $\tilde{\mathbf{O}}^{\mu}_X$ and $\mathbf{O}^{\mu}_X$, respectively, on X. The stalk of $\mathbf{O}^{g}_X$ at $P \in \mathrm{Proj}(G(R))$ is $Q^{g}_P(G(R))$ and for $\mathbf{O}_X$ we obtain $(Q^{g}_P(G(R)))_O$. The stalk of $\tilde{\mathbf{O}}^{\mu}_X$ may easily be calculated.

4.1. THEOREM. *For $P \in X$ the stalk $\mathbf{O}^{\mu}_P$ is a Zariski ring and moreover $(\mathbf{O}^{\mu}_P)^{\wedge} = Q^{\mu}_P(R)$ is just the microlocalization at $P \in \mathrm{Spec}^{g}(G(R))$, where $\wedge$ denotes the completion with respect to the Zariskian filtration of $\mathbf{O}^{\mu}_P$.*

Proof. Put $S_p = \mathbf{O}^{\mu}_P = \varinjlim_{f \in G(R) - p} Q^{\mu}_f(R)$, f homogeneous. If $x \in F_{-1}S_p$, $a \in F_0 S_p$ then for some homogeneous $f \in G(R) - p$ we have $x \in F_{-1}Q^{\mu}_f(R)$, $a \in F_0 Q^{\mu}_f(R)$. Since $Q^{\mu}_f(R)$ is a Zariski ring we have that $(1 - ax)^{-1} \in F_0 Q^{\mu}_f(R)$; hence $(1 - ax)^{-1} \in F_0 S_p$ and this leads to $F_{-1}S_p \subset J(F_0 S_p)$. It remains to check whether $\tilde{S}_p$ is Noetherian. Take a left ideal $\tilde{L} \subset \tilde{S}_p$ (for proving the left Zariskian condition, a similar argument will hold for right ideals). For a fixed homogeneous $f \in G(R) - P$ we have that $Q^{\mu}_f(R)$ is a Zariski ring and $(Q_f(R))^{\wedge} = Q^{\mu}_f(R)$ where the completion is flat because $Q_f(R)^{\sim}$ is Noetherian (cf. [10]). We have $\tilde{L}(f) = \tilde{L} \cap Q^{\mu}_f(R)^{\sim}$ is finitely generated. For another homogeneous $g \in G(R) - P$ it is clear that

$$\tilde{L}(fg) \cap Q_{fg}(R)^{\sim} = Q_{fg}(R)^{\sim} (\tilde{L}(f) \cap Q_f(R))^{\sim}$$

because if $z \in \tilde{L}(fg) \cap Q_{fg}(R)^{\sim}$ then $I^{\sim} z \subset \tilde{L}(f) \cap Q_f(R)^{\sim}$ for some $\tilde{I} \in \mathscr{L}(\tilde{\kappa}_{fg})$ and $Q_{fg}(R)^{\sim} I^{\sim} = Q_{fg}(R)^{\sim}$ by perfectness of $\tilde{\kappa}_{fg}$. The flatness of $Q_{fg}(R)^{\wedge}$ as a right $Q_{fg}(R)$-module, or rather the corresponding property on the Rees level, yields that

$$\begin{aligned}
\tilde{L}(fg) &= (Q_{fg}(R)^{\sim})^{\wedge\,\sim} (\tilde{L}(fg) \cap Q_{fg}(R)^{\sim}) \\
&= (Q_{fg}(R)^{\sim})^{\wedge\,\sim} (\tilde{L}(f) \cap Q_f(R)^{\sim}) \\
&= (Q^{\mu}_{fg}(R))^{\sim} \tilde{L}(f)
\end{aligned}$$

($Q_{fg}(R)^\wedge$ is also flat as a $Q_f(R)$-module). Hence the finite set of generators, $\tilde{\lambda}_1, ..., \tilde{\lambda}_n$, say, for $\tilde{L}(f)$ over $Q_f^\mu(R)^\sim$ is still generating $\tilde{L}(fg)$ over $Q_{fg}^\mu(R)^\sim$. Since $\tilde{L} = \bigcup_{f \in h(G(R)\ \ P)} \tilde{L}(f)$ it follows that $\tilde{L}$ is generated by $\{\tilde{\lambda}_1, ..., \tilde{\lambda}_n\}$ over $\tilde{S}_p$. Hence $\mathbf{O}_P^\mu$ is a Zariski ring. (Note that an easy extension of the idea of the proof of Proposition 3.11 together with Proposition 3.14(3) or Remark 3.15 allows one to see that a graded left ideal of $Q_{fg}^\mu(R)^\sim$ is generated as a $Q_{fg}^\mu(R)^\sim$-module by its intersection with $Q_f^\mu(R)^\sim$.) Now since $\mathscr{L}(\bar{\kappa}_p) = \bigcup\{\mathscr{L}(\bar{\kappa}),\ \bar{\kappa} \in \mathscr{B}$ and $P \notin \mathscr{L}(\bar{\kappa})\}$ and $\bar{\kappa}_p$ as well as all the $\bar{\kappa}$ involved here are saturated and such that $\bar{\kappa}_p$ as well as the $\bar{\kappa}$ are perfect it follows that for every $\tilde{I} \in \mathscr{L}(\tilde{\kappa})$ we obtain $(Q_p^\mu(R)^\sim)\, I^\sim = Q_P^\mu(R)^\sim$ and also $Q_p^g(G(R))\, \pi(\tilde{I}) = Q_p^g(G(R))$; hence in view of Proposition 3.4 it follows that $\tilde{S}_p \subset Q_p^\mu(R)^\sim$. Since $Q_p(R)^\sim \subset \tilde{S}_p$ follows from $\tilde{S}_p = \varinjlim_{\tilde{\kappa}} \{Q_\kappa^\mu(R)^\sim,\ P \notin \mathscr{L}(\bar{\kappa})\} \supset \varinjlim_{\tilde{\kappa}} \{Q_\kappa(R)^\sim,\ \ P \notin \mathscr{L}(\bar{\kappa})\} = Q_p(R)^\sim$ and $Q_p^\mu(R)^\sim = (Q_p(R)^\sim)^{\wedge X}$, we arrive at $(\tilde{S}_p)^{\wedge X} = Q_p^\mu(R)^\sim$ or $\tilde{S}_p = Q_p^\mu(R)$.

Note. Since we have assumed that $G(R)$ is a domain, all the maps in the directed system were injective, so in fact $\varinjlim_{\tilde{\kappa}}$ is just the union of the $Q_{\tilde{\kappa}}^\mu$ viewed as subrings of the total microlocalization $Q_G^\mu(R)^\sim$ obtained by microlocalizing at $hG(R)^*$; the latter microlocalization is not a gr-skewfield (all homogeneous elements are invertible) because X is not inverted in $Q_{G(R)^*}^\mu(R)^\sim$. In fact $\Delta = Q_{G(R)^*}^\mu(R)$ is a skewfield and its filtration is the filtration of a discrete valuation (given by $v(x) = \deg \sigma(x)$ for $x \in \Delta$) (see also [11]) so its Rees ring is a gr-valuation ring.

4.2. THEOREM. *With notation as before*: $\tilde{\mathbf{O}}_X^\mu$, $\mathbf{O}_X^\mu$ *are sheaves.*

Proof. Consider a covering $X(I) = \bigcup_\alpha X(I_\alpha)$. We use the notation of Proposition 3.14 and the remarks preceding it, so let $(\rho_{I_\alpha}^I)^\mu = \rho_\alpha^\mu$ and $\rho_{I_\alpha}^I = \rho_\alpha$ be the restriction morphisms from $X(I)$ to $X(I_\alpha)$ in $\tilde{\mathbf{O}}_X^\mu$, resp. $\mathbf{O}_X^\mu$. If for some $g \in Q_{\tilde{\kappa}(I)}^\mu(\tilde{R})$ we have that $\rho_\alpha^\mu(g) = 0$ for every α then we have to establish that $g = 0$. In our situation here the $(\rho_{I_\alpha}^I)^\mu$ and $(\rho_{I_\alpha}^I)$ are injective so $\tilde{O}_X^\mu$, and similarly $\mathbf{O}_X^\mu$, is indeed a separated presheaf. Now consider a covering $X(I) = \bigcup_\alpha X(I_\alpha)$ and given elements $g_\alpha \in Q_{\tilde{\kappa}_\alpha}^\mu(\tilde{R})$ such that $(\rho_{\alpha\beta}^\alpha)^\mu(g_\alpha) = (\rho_{\alpha\beta}^\beta)^\mu(g_\beta)$ for all α and β, then we have to construct $g \in Q_{\tilde{\kappa}_I}^\mu(\tilde{R})$ such that for every α, $(\rho_\alpha^I)^\mu(g) = g_\alpha$.

First observe that it suffices to prove the claim for finite coverings (since $G(R)$ is Noetherian every $X(I)$ is compact so there is a finite subcovering of $X(I) = \bigcup_\alpha X(I_\alpha)$ and by the classical trick one verifies that it is indeed sufficient to establish the claim for this finite subcovering), so we assume from here on that $X(I) = \bigcup_\alpha X(I_\alpha)$ is a finite covering. Write $\tilde{\kappa} = \tilde{\kappa}_I$, $\tilde{\kappa}_\alpha = \tilde{\kappa}_{I_\alpha}$. Since the localizations at $\tilde{\kappa}$ and $\tilde{\kappa}(n)$ as well as each $\tilde{\kappa}_\alpha$ and $\tilde{\kappa}_\alpha(n)$ coincide on $\tilde{R}/X^n\tilde{R}$-modules for each n we write $\tilde{\kappa}$, $\tilde{\kappa}_\alpha$ for $\tilde{\kappa}(n)$, $\tilde{\kappa}_\alpha(n)$ resp. when we are concerned with localization of $\tilde{R}/X^n\tilde{R}$-modules. We write $\tilde{\kappa}_{\alpha\beta}$

for the $\tilde{\kappa}_{I_\alpha I_\beta}$ which is associated to $X(I_\alpha) \cap X(I_\beta)$. For every $n \in \mathbb{N}$ we have the following commutative diagram:

$$\begin{tikzcd}
Q^g_{\tilde{\kappa}_{\alpha\beta}}(\tilde{R})/X^{n+1}Q^g_{\tilde{\kappa}_{\alpha\beta}}(\tilde{R}) \arrow[rr] & & Q^g_{\tilde{\kappa}_{\alpha\beta}}(\tilde{R})/X^nQ^g_{\tilde{\kappa}_{\alpha\beta}}(\tilde{R}) \\
& Q^g_{\tilde{\kappa}_\alpha}(\tilde{R})/X^{n+1}Q^g_{\tilde{\kappa}_\alpha}(\tilde{R}) \arrow[ul, "\tilde{\rho}^\alpha_{\alpha\beta}(n)"'] \arrow[rr] & & Q^g_{\tilde{\kappa}_\alpha}(\tilde{R})/X^nQ^g_{\tilde{\kappa}_\alpha}(\tilde{R}) \arrow[ul, "\tilde{\rho}^\alpha_{\alpha\beta}(n)"'] \\
Q^g_{\tilde{\kappa}}(\tilde{R})/X^{n+1}Q^g_{\tilde{\kappa}}(\tilde{R}) \arrow[uul, "\tilde{\rho}^\kappa_{\kappa\alpha\beta}(n+1)"] \arrow[ur, "\rho^\alpha_{\kappa_\alpha}(n+1)"] \arrow[rr, "\psi^{n+1}_n"] & & Q^g_{\tilde{\kappa}}(\tilde{R})/X^nQ^g_{\tilde{\kappa}}(\tilde{R}). \arrow[uul, "\tilde{\rho}^\kappa_{\kappa\alpha\beta}(n)"'] \arrow[ur, "\tilde{\rho}^\kappa_\alpha(n)"']
\end{tikzcd}$$

Now $g_\alpha \in Q^\mu_{\tilde{\kappa}_\alpha}(\tilde{R})$ determines $g_\alpha(n) \in Q^g_{\tilde{\kappa}_\alpha}(\tilde{R})/X^nQ^g_{\tilde{\kappa}_\alpha}(\tilde{R})$, for every n, such that $\tilde{\rho}^\alpha_{\alpha\beta}(n)\, g_\alpha(n) = \tilde{\rho}^\beta_{\alpha\beta}(n)\, g_\beta(n)$ for all α, β and for all n. Fix n for a moment. Then there exists a $g(n) \in Q^g_{\tilde{\kappa}}(\tilde{R})/X^nQ^g_{\tilde{\kappa}}(\tilde{R})$ such that $\tilde{\rho}^\kappa_{\kappa_\alpha}(n)(g(n)) = g_\alpha(n)$ for every α because the $Q^g_{\tilde{\kappa}}(\tilde{R}/X^n\tilde{R})$ do determine a sheaf over X. Indeed if $g_i\colon \tilde{H}_i \to \tilde{R}/X^n\tilde{R}$ represents an element of $Q^g_{\tilde{\kappa}_i}(\tilde{R}/X^n\tilde{R})$, for $i = 1, 2$, such that g_1 and g_2 coincide on some $L \in \mathscr{L}(\tilde{\kappa}_1 \vee \tilde{\kappa}_2)$, $L \subset \tilde{H}_1 \cap \tilde{H}_2$ then g_1 and g_2 coincide on $\tilde{H}_1 \cap \tilde{H}_2$ (because $\tilde{H}_1 \cap \tilde{H}_2/L$ is $\tilde{\kappa}_1 \vee \tilde{\kappa}_2$-torsion and $\tilde{R}/X^n\tilde{R}$ is $\tilde{\kappa}_1 \vee \tilde{\kappa}_2$-torsion free) and we may define $g \in Q^g_{\tilde{\kappa}_1 \vee \tilde{\kappa}_2}(\tilde{R}/X^n\tilde{R})$, $g\colon \tilde{H}_1 + \tilde{H}_2 \to \tilde{R}/X^n\tilde{R}$ by $g \mid \tilde{H}_1 = g_1$, $g \mid \tilde{H}_2 = g_2$, such that g restricts to g_1 and g_2 as desired (then extend the argument to finite coverings in the obvious way). Now $\psi^{n+1}_n(g(n+1)) - g(n)$ is mapped to zero in each $Q^g_{\tilde{\kappa}_\alpha}(\tilde{R})/X^nQ^g_{\tilde{\kappa}_\alpha}(\tilde{R})$; hence $\psi^{n+1}_n(g(n+1)) = g(n)$ follows from the separatedness of the sheaf at level n. Consequently $(g(n),\ n \in \mathbb{N})$ determines $g \in Q^\mu_{\tilde{\kappa}}(\tilde{R})$ such that $(\rho^\kappa_{\kappa_\alpha})^\mu(g) = g_\alpha$, because for every n we have $g_\alpha(n) = \tilde{\rho}^\kappa_{\kappa_\alpha}(n)$, as desired. The statement about $\mathbf{O}^\mu_X$ follows immediately from the foregoing. ∎

4.3. *Remark.* If $M \in R$-filt is such that $G(M)$ is (absolutely) torsion free (i.e., $\overline{rm} = 0$ for $\bar{r} \in G(R)$, $\bar{m} \in G(M)$ yields $\bar{m} = 0$) then the foregoing proof carries over to $\tilde{\mathbf{O}}^\mu_M$ and $\mathbf{O}^\mu_M$ given by $Q^\mu_{\tilde{\kappa}}(\tilde{M})$ and $Q^\mu_\kappa(M)$ resp. over the Zariski open set corresponding to $\bar{\kappa} \in \beta$. We do not restrict to the torsion free case usually so we write $\tilde{\mathbf{O}}^\mu_M$, resp. $\mathbf{O}^\mu_M$, for the sheaf associated to the presheaf determined by $Q^\mu_{\tilde{\kappa}}(\tilde{M})$, resp. $Q^\mu_\kappa(M)$, over the Zariski open set corresponding to $\bar{\kappa} \in \mathscr{B}$. The proof of Lemma 4.1 carries over to this situation and therefore we have $(\tilde{\mathbf{O}}^\mu_M)_p = Q^\mu_{\tilde{\kappa}_p}(\tilde{M})$, $(\mathbf{O}^\mu_M)_p = Q^\mu_{\kappa_p}(M)$ for $P \in X$. The sheaf $\mathbf{O}^\mu_M$ is a sheaf of filtered rings and we can say that $\mathbf{O}^\mu_X$ is a sheaf of Zariski rings since on a basis of the topology $\mathscr{B}$ the sections $Q^\mu_\kappa(R)$ are Zariski rings.

Define a filtered Ring to be a sheaf of rings $\mathbf{R}$ endowed with a family of subsheaves of groups, $\mathbf{R}_n$, $n \in \mathbb{Z}$, such that $\mathbf{R}_n \subset \mathbf{R}_{n+1}$, $\mathbf{R}_n \cdot \mathbf{R}_m \subset \mathbf{R}_{n+m}$ for $n, m \in \mathbb{Z}$, the unit section $\mathbf{1}$ is in R_0 and $R = \bigcup_n \mathbf{R}_n$ where the term on the right stands for the sheaf associated to the presheaf $X(I) \to \bigcup_n \mathbf{R}_n(X(I))$. It

is clear how to define a filtered Module over a filtered Ring and one can continue to introduce "good filtrations" on Modules and obtain sheaf versions of many of the ring theoretical properties that have been obtained, e.g., in [2, 9] .... It is not difficult to prove that a Zariski ring $\mathbf{R}$ (i.e., a sheaf of Zariski rings) such that $\mathbf{G}(\mathbf{R})$ is coherent, is itself coherent; moreover if $\mathcal{M}$ is an $\mathcal{R}$-module which is locally of finite type then $\mathcal{M}$ is coherent if and only if $\mathbf{G}(\mathcal{M})$ is coherent. Also, if $\mathcal{M}$ is coherent and $F\mathcal{M}$ is good then every coherent submodule $\mathcal{N}$ of $\mathcal{M}$ has an induced filtration $F\mathcal{N} = \mathcal{N} \cap \mathbf{F}\mathcal{M}$ and this filtration is again a good filtration of the sheaf $\mathcal{N}$. We do not go deeper into the theory of coherent filtered sheaves here.

The sheaf $\mathbf{O}_X^\mu$ is a filtered sheaf in the sense introduced above; let us write $\mathcal{R} = \mathbf{O}_X^\mu$. Then $\tilde{\mathcal{R}} = \tilde{\mathbf{O}}_X^\mu$ is a graded sheaf and $G(\mathcal{R}) = \tilde{\mathcal{R}}/\mathbf{X}\tilde{\mathcal{R}}$ is also a graded sheaf (where $\mathbf{X}$ stands for the global section of $\tilde{\mathcal{R}}$ determined by $X_{Q_{\bar{\kappa}}^\mu(\bar{R})}$ for each $\bar{\kappa} \in \mathcal{B}$). More explicitly: $\mathcal{F}_n\mathcal{R}(X(I)) = F_n Q_{\kappa_I}^\mu(R)$ where $\bar{\kappa}_I \in \mathcal{B}$ and $G(\mathcal{R})$ is the sheaf associated to the presheaf $\bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n/\mathcal{F}_{n-1}$. In this case $\mathcal{R} = \mathbf{O}_X^\mu$ we see that $G(\mathcal{R})$ is nothing but the graded structure sheaf $\mathbf{Q}_X^g$ of $\mathrm{Proj}(G(R))$. Similarly, to a filtered R-module M with good filtration FM there corresponds the microstructure sheaf $\mathbf{O}_M^\mu$ that is a filtered module over $\mathbf{O}_X^\mu$ such that $G(\mathbf{O}_M^\mu)$ is the graded structure sheaf of $G(M)$ over $\mathrm{Proj}(G(R))$.

So $\mathcal{F}_0\mathcal{R}$ is a Ring with $\mathcal{F}_0\mathcal{R}(X(I)) = F_0 Q_{\kappa_I}^\mu(R)$ if $\bar{\kappa}_I \in \mathcal{B}$ and this defines a subring of $\mathcal{R}$ isomorphic to the part of degree zero $(\tilde{\mathcal{R}})_0$ of the graded sheaf $\tilde{\mathcal{R}}$ and the image of $(\tilde{\mathcal{R}})_0$ in the graded structure sheaf $\mathbf{O}_X^g$ is the classical structure sheaf $\mathbf{O}_X$ of $\mathrm{Proj}(G(R))$.

4.4. THEOREM. *The presheaf defined by associating $F_0 Q_{\kappa_I}^\mu(R)$ to $X(I)$, where $\bar{\kappa}_I \in \mathcal{B}$, is a Noetherian and coherent sheaf; the stalk at $P \in X$ is $F_0 Q_{\kappa_P}^\mu(R)$. The ideal $\mathcal{F}_{-1}\mathcal{R}$ is coherent and $\mathcal{F}_0\mathcal{R}/\mathcal{F}_{-1}\mathcal{R} = \mathbf{O}_X$, the structure sheaf of $X = \mathrm{Proj}(G(R))$.*

Proof. All claims follow from the foregoing observations, up to the following argument. Viewing $\mathcal{F}_0\mathcal{R}$ with the filtration $\mathcal{F}$ given by $\mathcal{F}_{-n}\mathcal{R}$, $\mathcal{F}_{-n}\mathcal{R}(X(I)) = F_{-n} Q_{\kappa_I}^\mu(R)$ for $n \geqslant 0$, we obtain $G_{\mathcal{F}}(\mathcal{F}_0\mathcal{R}) = (\mathbf{O}_X^g) = \bigoplus_{n \leqslant 0} (\mathbf{O}_X^g)_n$ which is Noetherian and coherent. Since each graded ring $Q_{\kappa_I}^g(G(R))$, is Noetherian and $\mathcal{F}$ is Zariskian on the ring $\mathcal{F}_0\mathcal{R}$ it follows that each $\mathcal{F}_{-n}\mathcal{R}$ is locally of finite type as an $\mathcal{F}_0\mathcal{R}$-module (this is just the phrasing in terms of sheaves of corresponding results in [2, 11]). Therefore the coherent and the Noetherian property of $G_{\mathcal{F}}(\mathcal{F}_0\mathcal{R})$ lift to $\mathcal{F}_0\mathcal{R}$ and in a similar way one establishes that $\mathcal{F}_{-1}\mathcal{R}$ is a Noetherian coherent ideal. For Noetherian sheaves see also [15].

We call $\mathcal{F}_0\mathcal{R}$ the *sheaf of quantum-sections* of $\mathbf{O}_X$ (inspired by terminology mentioned in [6]).

4.5. COROLLARY. *Let $M \in R$-filt have good filtration FM. Then we have a filtered sheaf $\mathcal{M} = \mathbf{O}^{\mu}_{M}$ of $\mathcal{R}$-modules, with $G(\mathcal{M})$ equal to the graded structure sheaf $\mathbf{O}^{g}_{G(M)}$. Since the latter is coherent and $\mathcal{M}$ is locally finite it follows that $\mathcal{M}$ is coherent. We obtain a coherent $\mathcal{F}_0\mathcal{M}$ with an $\mathcal{F}_0\mathcal{R}$-submodule $\mathcal{F}_{-1}\mathcal{M}$ such that $\mathcal{F}_0\mathcal{M}/\mathcal{F}_{-1}\mathcal{M} = \mathbf{O}_{G(M)}$ the usual structure sheaf of $G(M)$ over* Proj$(G(R))$. *Again, we call $\mathcal{F}_0\mathcal{M}$ the sheaf of quantum-sections of $\mathbf{O}_{G(M)}$.*

4.6. PROPOSITION. *If $M \xrightarrow[f]{} N$ is a strict morphism in R-filt then f induces a strict morphism on the sheaves of quantum-sections $\mathcal{F}_0\mathcal{M} \xrightarrow[\varphi]{} \mathcal{F}_0\mathcal{N}$. If FM is good then $\mathcal{F}_{\cdot}$ is good on $\mathcal{F}_0\mathcal{M}$.*

Proof. If f is strict then $M \xrightarrow[f]{} N \to L \to 0$, with $L = \operatorname{Coker} f$, is strict exact and so $\tilde{M} \xrightarrow[\tilde{f}]{} \tilde{N} \to \tilde{L} \to 0$ is exact with $\tilde{L}$ being X-torsionfree. Take a $\bar{\kappa} \in \mathcal{B}$. By the exactness of $Q^{\mu}_{\bar{\kappa}}(-)$ we obtain the following exact sequence of graded $Q^{\mu}_{\bar{\kappa}}(\tilde{R})$-modules:

$$Q^{\mu}_{\bar{\kappa}}(\tilde{M}) \to Q^{\mu}_{\bar{\kappa}}(\tilde{N}) \to Q^{\mu}_{\bar{\kappa}}(\tilde{L}) \to 0.$$

Now since $\tilde{L}/\tilde{\kappa}(\tilde{L}) = (L/\kappa(L))^{\sim}$ (Lemma 2.3(1)) it is clear that $\tilde{L}/\tilde{\kappa}(\tilde{L})$ is X-torsionfree. From Proposition 3.14(1) we retain that $Q^{\mu}_{\bar{\kappa}}(\tilde{L})/(\tilde{L}/\tilde{\kappa}(\tilde{L}))$ is X-torsionfree and hence it follows from both observations combined that $Q^{\mu}_{\bar{\kappa}}(\tilde{L})$ is X-torsionfree. Therefore $X^n Q^{\mu}_{\bar{\kappa}}(\tilde{N}) \cap Q^{\mu}_{\bar{\kappa}}(\tilde{f})\, Q^{\mu}_{\bar{\kappa}}(\tilde{M}) = X^n Q^{\mu}_{\bar{\kappa}}(\tilde{f})$ $(Q^{\mu}_{\bar{\kappa}}(\tilde{M}))$ and thus the graded morphism $Q^{\mu}_{\bar{\kappa}}(\tilde{f})$ is strict in the respective X-adic topologies of the modules considered. Therefore we have a strict exact sequence:

$$Q^{\mu}_{\bar{\kappa}}(M) \xrightarrow[Q^{\mu}_{\bar{\kappa}}(f)]{} Q^{\mu}_{\bar{\kappa}}(N) \to Q^{\mu}_{\bar{\kappa}}(L) \to 0.$$

The restriction of $Q^{\mu}_{\bar{\kappa}}(f) = \varphi_{\bar{\kappa}}$ to the quantum-sections over $X(\bar{\kappa})$ is again strict $\varphi_{\bar{\kappa}}: F_0 Q^{\mu}_{\bar{\kappa}}(M) \to F_0 Q^{\mu}_{\bar{\kappa}}(N)$ and so we define a sheaf map $\varphi: \mathcal{F}_0\mathcal{M} \to \mathcal{F}_0\mathcal{N}$; $\varphi(X(\bar{\kappa})) = \varphi_{\bar{\kappa}}$, which is strict in the $\mathcal{F}_{\cdot}$-filtration of the sheaves. For the second condition we consider $\tilde{M}$ which is a finitely generated graded $\tilde{R}$-module. By the perfectness of $\bar{\kappa}$ we obtain then that $Q^{\mu}_{\bar{\kappa}}(\tilde{M})$ is a finitely generated $Q^{\mu}_{\bar{\kappa}}(\tilde{R})$-module, say $Q^{\mu}_{\bar{\kappa}}(\tilde{M}) = Q^{\mu}_{\bar{\kappa}}(\tilde{R})\, e_1 + \cdots + Q^{\mu}_{\bar{\kappa}}(\tilde{R})\, e_s$ with e_i homogeneous in $Q^{\mu}_{\bar{\kappa}}(\tilde{M})$. Then $(Q^{\mu}_{\bar{\kappa}}(\tilde{M}))_{\cdot} = Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\leqslant d_1}\, e_1 + \cdots + Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\geqslant d_s}$ where $\deg e_i = -d_i$, $i = 1, \ldots, s$.

If $d_i \leqslant 0$ then $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{d_i}\, e_i$ is finitely generated as a $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\cdot}$-module since $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\leqslant d_i}$ is finitely generated in $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\cdot}$. If $d_i > 0$ then $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\leqslant d_i} \supset Q^{\mu}_{\bar{\kappa}}(\tilde{R})$ but again $Q^{\mu}_{\bar{\kappa}}(\tilde{R})_{\leqslant d_i}$ is finitely generated as a $Q^{\mu}_{\bar{\kappa}}(R)$ module (see also the reference in the proof of Theorem 4.4 where it is stated that each $\mathcal{F}$ is locally finite as an $\mathcal{F}_0\mathcal{R}$-module). So it follows that $F_0 Q^{\mu}_{\bar{\kappa}}(M)$, having $Q^{\mu}_{\bar{\kappa}}(\tilde{M})$ for $(F_0 Q^{\mu}_{\bar{\kappa}}(M))^{\sim}$ with respect to its $\mathcal{F}_{\cdot}$-filtration, is a good

filtered module. Since the latter statement holds for all $\bar{\kappa}$ in the basis $\mathscr{B}$ for the topology on X, it follows that $\mathscr{F}_0.\mathscr{M}$ is a good filtration.

If we look at an arbitrary $M \in R$-filt with good filtration FM then some microlocalizations trivialize because $G(M)$ may be $\bar{\kappa}$-torsion; in order to avoid these trivializations one may restrict attention to the characteristic variety, of M (compare [5]) in $\mathrm{Proj}(G(R))$. Let $A = \mathrm{Ann}_{G(R)} G(M)$ be the annihilator of $G(M)$ in $G(R)$ and let $V(A) \subset \mathrm{Proj}(G(R))$ be the closed set determined by the (graded) ideal A. Any $\bar{\kappa}$ appearing as $\bigwedge\{\bar{\kappa}_P$, some $P \in V(A)\}$ does not contain A in $\mathscr{L}(\bar{\kappa})$.

4.7. Lemma. *If $\bar{\kappa} = \bigwedge\{\bar{\kappa}_P\}$ and some $P \in V(A)$, i.e., $\mathscr{L}(\bar{\kappa}) = \bigcap\{\mathscr{L}(\bar{\kappa}_P)\}$ and some $P \in V(A)$ then $\bar{\kappa}G(M) \neq G(M)$, or $Q^g_\kappa(G(M)) \neq 0$.*

Proof. If $\bar{\kappa}G(M) = G(M)$ then since $G(M)$ is a finitely generated $G(R)$-module and since $G(R)$ is commutative, we have $JG(M) = 0$ for some $J \in \mathscr{L}(\bar{\kappa})$. Thus $J \subset A$ but then $A \in \mathscr{L}(\bar{\kappa})$ meaning $A \not\subset P$ for some $P \in V(A)$, a contradiction.

For an $M \in R$-filt with good filtration FM we define the *quantum-sections over the characteristic variety* by the restricted sheaf: $\mathscr{F}_0.\mathscr{M}|V(A)$, associating $F_0 Q^\mu_{\kappa_I}(M)$ to $X(I) \cap V(A)$. When $X(I) \cap V(A)$ is not empty then $Q^\mu_{\kappa_I}(M)$ is not zero in view of the lemma. When the graded ideal I of $G(R)$ is not generated by elements in $G(R)_0$ (and usually $G(R)_0$ is a field in most applications) then the perfectness of $\bar{\kappa}_I \in \mathscr{B}$ entails that $1 = \sum q_\alpha i_\alpha$ for some homogeneous $i_\alpha \in I$, $q_\alpha \in Q^g_{\kappa_I}(G(R))$; hence there is a nontrivial negative part in $Q^g_{\kappa_I}(G(R)) = G(Q^\mu_{\kappa_I}(R))$, consequently $F_0 Q^\mu_{\kappa_I}(M) \neq 0$. Consequently the sheaf $\mathscr{F}_0.\mathscr{M}|V(A)$ has nonvanishing sections everywhere. The sheaf of quantum-sections over the characteristic variety should prove to be useful in the study of holonomic or pure modules over certain rings of differential operators; this is the topic of work in progress. Quantum sections provided the tool for the introduction of generalised gauge algebras in [21].

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