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Types of Linkage of Quadratic Pfister Forms

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Abstract

Given a field F of positive characteristic p, $\theta \in H_p^{n-1}(F)$ and $\beta, \gamma \in F^\times$, we prove that if the symbols $\theta \wedge \frac{d\beta}{\beta}$ and $\theta \wedge \frac{d\gamma}{\gamma}$ in $H_p^n(F)$ share the same factors in $H_p^1(F)$ then the symbol $\theta \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$ in $H_p^{n+1}(F)$ is trivial. We conclude that when $p = 2$, every two totally separably $(n-1)$-linked n-fold quadratic Pfister forms are inseparably $(n-1)$-linked. We also describe how to construct non-isomorphic n-fold Pfister forms which are totally separably (or inseparably) $(n-1)$-linked, i.e. share all common $(n-1)$-fold quadratic (or bilinear) Pfister factors.

Keywords: Kato-Milne Cohomology, Fields of Positive Characteristic, Quadratic Forms, Pfister Forms, Quaternion Algebras, Linkage
2010 MSC: 11E81 (primary); 11E04, 16K20, 19D45 (secondary)

1. Introduction

Linkage of Pfister forms is a classical topic in quadratic form theory. We say that two n-fold Pfister forms over a field F are separably (inseparably, resp.) m-linked if there exists an m-fold quadratic (bilinear) Pfister form which is a common factor of both forms. When $\operatorname{char}(F) \neq 2$, there is no difference between quadratic and bilinear factors, so the terms coincide, and we simply say m-linked.

We say that two quadratic n-fold Pfister forms are totally separably (inseparably) m-linked if every quadratic (bilinear) m-fold Pfister factor of one of them is also a factor of the other. The following facts were proven in [4]:

- Two n-fold quadratic Pfister forms can be totally separably 1-linked, inseparably 1-linked, or even both, without being isometric. (The special case of quaternion algebras over fields of characteristic not 2 was covered in [9].)
- Total separable 1-linkage and total inseparable 1-linkage of n-fold quadraticPfister forms are independent properties, i.e. do not imply each other.

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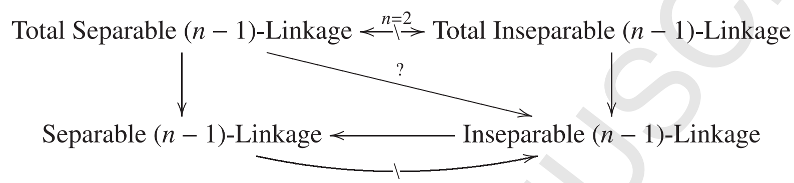
Clearly total separable (or inseparable) $(n-1)$-linkage implies (nontotal) separable (inseparable, resp.) $(n-1)$-linkage. It is known that inseparable $(n-1)$-linkage of quadratic n-fold Pfister forms implies separable $(n-1)$-linkage, but the converse is in general not true (see [12], [8], [5], [6], [3] and [1] for references).

Question 1.1. *Does total separable $(n-1)$-linkage of n-fold quadratic Pfister forms imply (nontotal) inseparable $(n-1)$-linkage?*



We answer this question in the positive in Section 3. We conclude it from deeper results on linkage of symbols in the Kato-Milne cohomology groups.

There are several other natural questions that arise in this setting:

Question 1.2. *Do there exist totally separably (inseparably) m-linked quadratic n-fold Pfister forms which are not isometric for a given $m \in \{1,\dots,n-1\}$?*

Question 1.3. *Over fields of characteristic 2, are total separable m-linkage and total inseparable m-linkage independent properties?*

Question 1.4. *Given $1 \leqslant \ell < m \leqslant n-1$, are there totally separably (or inseparably) ℓ-linked n-fold quadratic Pfister forms which are not totally separably (inseparably, resp.) m-linked?*

We answer Question 1.2 in full generality (in the positive), and Question 1.4 in the case of fields of characteristic not 2 and $m = n-1$ (see Section 4). Question 1.3 was answered in the negative in [4] for $m = 1$, but it remains open for arbitrary m.

2. Preliminaries

2.1. Quadratic Forms

For general reference on symmetric bilinear forms and quadratic forms see [7]. Throughout, let F be a field and V an F-vector space. A quadratic form over F is a map $\varphi : V \to F$ such that $\varphi(av) = a^2\varphi(v)$ for all $a \in F$ and $v \in V$ and the map defined by $B_\varphi(v,w) = \varphi(v+w)-\varphi(v)-\varphi(w)$ for all $v,w \in V$ is a bilinear form on V. The bilinear form B_φ is called the polar form of φ and is clearly symmetric. Two quadratic forms $\varphi : V \to F$ and $\psi : W \to F$ are isometric if there exists an isomorphism $M : V \to W$ such that $\varphi(v) = \psi(Mv)$ for all $v \in V$. We are interested in the isometry classes of quadratic forms, so when we write $\varphi = \psi$ we actually mean that they are isometric.

We say that φ is singular if B_φ is degenerate, and that φ is nonsingular if B_φ is nondegenerate. If F is of characteristic 2, every nonsingular form φ is even dimensional and can be written as

$$\varphi = [\alpha_1,\beta_1] \perp \cdots \perp [\alpha_n,\beta_n]$$

for some $\alpha_1, \ldots, \beta_n \in F$, where $[\alpha, \beta]$ denotes the two-dimensional quadratic form $\psi(x, y) = \alpha x^2 + xy + \beta y^2$ and $\perp$ denotes the orthogonal sum of quadratic forms. If the characteristic of F is different from 2, symmetric bilinear forms and quadratic forms are equivalent objects, and we do not distinguish between them in this case. The unique nonsingular two-dimensional isotropic quadratic form is $\mathbb{H} = [0, 0]$, called the hyperbolic plane. A hyperbolic form is an orthogonal sum of hyperbolic planes. Any quadratic form φ over F decomposes into an orthogonal sum of a uniquely determined anisotropic quadratic form and a number of hyperbolic planes. The number of hyperbolic planes appearing in this decomposition is called the Witt index and denoted $i_W(\varphi)$.

We denote by $\langle \alpha_1, \ldots, \alpha_n \rangle$ the diagonal bilinear form given by $(x, y) \mapsto \sum_{i=1}^n \alpha_i x_i y_i$. A bilinear n-fold Pfister form over F is a symmetric bilinear form isometric to a bilinear form

$$\langle 1, \alpha_1 \rangle \otimes \langle 1, \alpha_2 \rangle \otimes \cdots \otimes \langle 1, \alpha_n \rangle$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^\times$. We denote such a form by $\langle\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle\rangle$. By convention, the bilinear 0-fold Pfister form is $\langle 1 \rangle$. The 2-fold Pfister forms generate the fundamental ideal IF in the Witt ring of nondegenerate symmetric bilinear forms WF. Powers of IF are denoted by $I^n F$, and are generated by n-fold Pfister forms respectively.

Let $B : V \times V \to F$ be a symmetric bilinear form over F and $\varphi : W \to F$ be a quadratic form over F. We may define a quadratic form $B \otimes \varphi : V \otimes_F W \to F$ determined by the rule that $(B \otimes \varphi)(v \otimes w) = B(v, v) \cdot \varphi(w)$ for all $w \in W, v \in V$. We call this quadratic form the tensor product of B and φ. A quadratic n-fold Pfister form over F is a tensor product of a bilinear $(n-1)$-fold Pfister form $\langle\langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle\rangle$ and a two-dimensional quadratic form $[1, \beta]$ for some $\beta \in F$. We denote such a form by $\langle\langle \alpha_1, \ldots, \alpha_{n-1}, \beta]]$. Quadratic n-fold Pfister forms are isotropic if and only if they are hyperbolic (see [7, (9.10)]). The 2-fold quadratic Pfister forms generate the fundamental ideal, denoted $I_q F$ or $I_q^1 F$, of the Witt group of nonsingular quadratic forms. Let $I_q^n F$ denote the subgroup generated by scalar multiples of quadratic n-fold Pfister forms.

Given a symmetric bilinear form B, we denote by $Q(B)$ the quadratic form given by the map $v \mapsto B(v, v)$.

Let π be an n-fold quadratic Pfister form over F. For $m \in \{1, \ldots, n\}$, we say an m-fold quadratic (resp. bilinear) Pfister form ψ (resp. B) is a factor of π if there exists an $(n-m)$-fold bilinear (reps. quadratic) Pfister form B' (resp. ψ') such that $\pi = B' \otimes \psi$ (resp. $\pi = B \otimes \psi'$).

Let ω be an n-fold quadratic Pfister form over F. We say π and ω are separably (resp. inseparably) m-linked if there exists an m-fold quadratic (resp. bilinear) Pfister form ψ such that ψ is a factor of both π and ω. We say π and ω are totally separably (resp. inseparably) m-linked if every quadratic (resp. bilinear) m-fold Pfister form is a factor of π if and only if it is a factor of ω. This terminology comes from the fact that in characteristic 2, the function fields of quadratic (resp. bilinear) Pfister forms are separable (resp. inseparable) extensions of the ground field.

2.2. Kato-Milne Cohomology

In this section, assume F is a field of characteristic $p > 0$. For $n > 0$, the Kato-Milne Cohomology group $H_p^{n+1}(F)$ is defined to be the cokernel of the Artin-Schreier map

$$\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$$

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto (\alpha^p - \alpha) \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n}.$$

We also fix $H_p^1(F)$ to be $F/\wp(F)$. The group $\nu_F(n)$ is defined to be the kernel of this map. By [2], $\nu_F(n) \cong K_nF/pK_nF$, with the isomorphism given by

$$\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \{\beta_1, \ldots, \beta_n\}.$$

It is known that $H_p^2(F) \cong {}_pBr(F)$ and for $p = 2$, $H_2^n(F) \cong I_q^n F / I_q^{n+1} F$. The first isomorphism is given by the map

$$\alpha \frac{d\beta}{\beta} \mapsto [\alpha, \beta)_{p,F}$$

where $[\alpha, \beta)_{p,F}$ stands for the symbol p-algebra

$$F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1\rangle.$$

The second isomorphism is given by the map

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \langle\langle \beta_1, \ldots, \beta_n, \alpha]].$$

We call the logarithmic differentials $\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}$ in $H_p^n(F)$ and $\frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}$ in $\nu_F(m)$ "symbols". There is a natural map

$$H_p^n(F) \times \nu_F(m) \to H_p^{m+n}(F)$$

defined by the wedge product

$$\left(\alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}, \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}\right) \mapsto \alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} \wedge \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}.$$

We define the linkage of symbols in an analogous manner to the linkage of Pfister forms. If a symbol ω in $H_p^{m+n}(F)$ is a wedge product $\theta \wedge \psi$ where θ is a symbol in $H_p^n(F)$ and ψ is a symbol in $\nu_F(m)$, then θ and ψ are called factors of ω. We say that two symbols π and ω are separably k-linked if they have a common factor in $H_p^k(F)$, and inseparably k-linked if they have a common factor in $\nu_F(k)$. We say that two symbols π and ω are totally separably k-linked if they share all factors in $H_p^k(F)$, and inseparably k-linked if they share all factor in $\nu_F(k)$.

3. Separably $(n-1)$-linked Symbols in $H_p^n(F)$

In this section, assume F is a field of characteristic $p > 0$. One of the main goals is to show that total separable $(n-1)$-linkage implies inseparable $(n-1)$-linkage for quadratic n-fold Pfister forms when $p = 2$.

Lemma 3.1. *For $\alpha \in F$ and $\beta \in F^\times$, let*

$$t = \alpha + \frac{(\alpha - \beta)}{\gamma}.$$

The symbol p-algebra $[\alpha, \gamma)_{p,F}$ contains the étale extension $F[x : x^p - x = t^p\gamma + \beta]$ of F.

Proof. Let i and j be a pair of generators of $[\alpha, \gamma)_{p,F}$ with $i^p - i = \alpha$, $j^p = \gamma$ and $jij^{-1} = i + 1$. Take $x = i + tj + ij$ in $[\alpha, \gamma)_{p,F}$. Then $x^p - x$ is equal to

$$\gamma\alpha^p + \gamma^{1-p}\alpha^p - \gamma^{1-p}\beta^p + \beta = t^p\gamma + \beta$$

by [3, Lemma 3.1]. Hence the subalgebra $F[x]$ of $[\alpha, \gamma)_{p,F}$ is as required. □

Proposition 3.2. *Consider two separably $(n-1)$-linked symbols*

$$\pi = \alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \quad \text{and} \quad \omega = \alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma}$$

in $H_p^n(F)$ and let $t = \alpha + \frac{(\alpha-\beta)}{\gamma}$. If $t^p\gamma + \beta$ is a factor in $H_p^1(F)$ of π, then the class of $\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$ in $H_p^{n+1}(F)$ is trivial.

Proof. Note first that if $t^p\gamma + \beta = 0$ then $d\gamma \wedge d\beta = 0$ and the result holds. Assume otherwise. The class of $t^p\gamma + \beta$ in $H_p^1(F)$ is a factor of ω by Lemma 3.1, so it is a common factor of π and ω. We have

$$\alpha\frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = \alpha\frac{d(\beta\gamma^{-1})}{\beta\gamma^{-1}} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta}$$

(see [6, Lemma 5.1, (e)]). Now,

$$\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} =$$

$$\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta\gamma^{-1}}{\beta\gamma^{-1}} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} =$$

$$\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} - \alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta}.$$

Since $t^p\gamma + \beta$ is a factor in $H_p^1(F)$ of $\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta}$, we have

$$\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} = (t^p\gamma + \beta)\tau \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta}$$

for some $\tau \in \nu_F(n-1)$. As $(t^p\gamma + \beta)\frac{d(t^p\gamma+\beta)}{t^p\gamma+\beta} = d(t^p\gamma + \beta)$, it is trivial in $H_p^2(F)$. Hence $\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p\gamma+\beta)}{t^p\gamma+\beta} = 0$. Similarly, since $t^p\gamma + \beta$ is a factor in $H_p^1(F)$ of $\alpha\frac{d\gamma}{\gamma}$, we have $\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma} \wedge \frac{d(t^p\gamma+\beta)}{t^p\gamma+\beta} = 0$. Therefore $\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$ as required. □

Corollary 3.3. *Let*

$$\pi = \alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \quad \text{and} \quad \omega = \alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma}$$

be two separably $(n-1)$-linked symbols in $H_p^n(F)$. If π and ω are totally separably 1-linked then the class of

$$\alpha\frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$$

in $H_p^{n+1}(F)$ is trivial.

When $p = 2$, by the identification of the symbols with quadratic n-fold Pfister forms, we obtain the following results:

Proposition 3.4. *Assume $p = 2$. Let*

$$\pi = \langle\langle \beta, \delta_{n-2}, \dots, \delta_1, \alpha]] \quad \text{and} \quad \omega = \langle\langle \gamma, \delta_{n-2}, \dots, \delta_1, \alpha]]$$

be two separably $(n-1)$-linked n-fold quadratic Pfister forms over F and let $t = \alpha + \frac{(\alpha-\beta)}{\gamma}$. If the 1-fold Pfister form $\langle\langle t^p\gamma + \beta]]$ is a factor of π, then $\langle\langle \beta, \gamma, \delta_{n-2}, \dots, \delta_1, \alpha]]$ is trivial. In particular, π and ω are inseparably $(n-1)$-linked.

Proof. By [5], the $(n+1)$-fold Pfister form $\langle\langle \beta, \gamma, \delta_{n-1}, \dots, \delta_2, \alpha]]$ is trivial if and only if π and ω are inseparably $(n-1)$-linked. □

Corollary 3.5. *Assume $p = 2$. If a pair of separably $(n-1)$-linked n-fold quadratic Pfister forms over F are totally separably 1-linked then they are also inseparably $(n-1)$-linked. In particular, if a pair of n-fold quadratic Pfister forms over F are totally separably $(n-1)$-linked then they are inseparably $(n-1)$-linked.*

Remark 3.6. A similar result to Corollary 3.3 holds more straight-forwardly for Milnor K-groups. Let p be a prime integer, n a positive integer and F an arbitrary field. If $p = 2$ then further assume that $\sqrt{-1} \in F$. Then the following is trivial:

- If $\{\alpha\} \cup \theta \in K_nF/pK_nF$ has $\{\beta\}$ as a factor in K_1F/pK_1F then $\{\alpha, \beta\} \cup \theta = 0$ in $K_{n+1}F/pK_{n+1}F$.
- Therefore, if $\{\alpha\}\cup\theta$ and $\{\beta\}\cup\theta$ in K_nF/pK_nF are totally 1-linked then $\{\alpha, \beta\}\cup\theta = 0$ in $K_{n+1}F/pK_{n+1}F$.

Remark 3.7. The result analogous to Corollary 3.3 for inseparable linkage is also straight-forward. For fields F of characteristic $p > 0$ and positive integer n,

- If $\omega \wedge \frac{d\beta}{\beta} \in H_p^nF$ has $\frac{d\gamma}{\gamma}$ in $\nu_F(1)$ as a factor then $\omega \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$.
- Therefore, if $\omega \wedge \frac{d\beta}{\beta}$ and $\omega \wedge \frac{d\gamma}{\gamma}$ in H_p^nF are totally inseparably 1-linked then $\omega \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} = 0$ in $H_p^{n+1}(F)$.

4. Totally Linked Quadratic Pfister Forms

In [4] we considered whether total 1-linkage of Pfister forms implied isometry. In general it does not. In this section, we consider whether total m-linkage implies isometry.

Lemma 4.1. *Let n be an integer $\geqslant 2$ and $m \in \{1, \dots, n-1\}$. Let φ be an n-fold quadratic Pfister form, π an m-fold quadratic Pfister form and θ an $(m-1)$-fold quadratic Pfister form. Assume θ is a common factor of φ and π. Then $i_W(\varphi \perp -\pi) = 2^m$ if and only if π is a factor of φ. Otherwise, $i_W(\varphi \perp -\pi) = 2^{m-1}$.*

Proof. By [7, Corollary 24.3], $i_W(\varphi \perp -\pi)$ must be a power of 2. Since θ is a factor of φ, $i_W(\varphi \perp -\pi) \geqslant 2^{m-1}$. Therefore, the only other possible value is 2^m, in which case π is a subform of φ and therefore a factor of φ. □

Lemma 4.2. *Let n be an integer $\geqslant 2$ and $m \in \{1, \dots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$-linked and totally separably m-linked n-fold quadratic Pfister forms over F. Let π be an $(m+1)$-fold quadratic Pfister form such that π is a factor of φ but not ψ. Then there exists a field extension L such that π_L is a factor of ψ_L and φ_L and ψ_L are neither isometric nor hyperbolic.*

Proof. Since φ and ψ are separably $(n-1)$-linked, the form $\varphi \perp \psi$ is congruent mod $I_q^{n+1}F$ to some anisotropic n-fold Pfister form ϕ. Let π_0 be an m-fold quadratic Pfister factor of π. Since φ and ψ are totally separably m-linked, π_0 is a common factor of both forms.

By Lemma 4.1, $\psi \perp -\pi$ is Witt equivalent to some anisotropic 2^n-dimensional form θ. Write $L = F(\theta)$ for the function field of θ over F. If one of the forms φ_L, ψ_L and ϕ_L were hyperbolic, then θ would be similar to a subform of the form by [7, Corollary 22.5]. However, since the forms are of the same dimension, this would imply that θ is similar to an n-fold Pfister form. This is impossible because the mth cohomological invariant of θ is nontrivial. It follows that ψ_L and φ_L are not isometric as ϕ_L is not hyperbolic. □

Theorem 4.3. *Let n be an integer $\geqslant 2$ and $m \in \{1, \dots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$-linked and totally separably m-linked quadratic n-fold Pfister forms over F. Then there exists a field extension K of F such that φ_K and ψ_K are totally separably $(m+1)$-linked but not isometric nor hyperbolic.*

Proof. Let S be the set of $(m+1)$-fold quadratic Pfister forms π over F such that π is a factor of φ but not of ψ. Then as in Lemma 4.2, for each $\pi \in S$, there exists a 2^n-dimensional quadratic form θ such that θ is Witt equivalent to $\pi \perp \psi$. Let F_0 be the compositum of the function fields of all such θ.

Similarly, let T be the set of $(m+1)$-fold quadratic Pfister forms π' over F such that π is a factor of ψ but not of φ. Again, as in Lemma 4.2, for each $\pi' \in T$, there exists a 2^n-dimensional quadratic form θ' such that θ' is Witt equivalent to $\pi' \perp \varphi$. Let F'_0 be the field compositum of the function fields of all such θ'.

Let K_0 be the compositum of F_0 and F'_0. Then by Lemma 4.2, φ_{K_0} and ψ_{K_0} are not isometric nor hyperbolic, and every $(m+1)$-fold Pfister form over K_0 defined over F is a factor of φ_{K_0} if and only if it is a factor of ψ_{K_0}. Using this construction inductively, we obtain the required field extension K/F. □

Lemma 4.4. *Assume* char$(F) = 2$. *Let θ be an anisotropic n-fold bilinear Pfister form over F and let θ' be an $(n-1)$-fold bilinear Pfister form factor of θ. Let β be an element represented by θ but not by θ'. Then $Q(\theta) = Q(\langle\langle\beta\rangle\rangle \otimes \theta')$.*

Proof. This follows easily from [7, Proposition 10.4]. □

Lemma 4.5. *Assume* char$(F) = 2$. *Let n be an integer $\geqslant 2$ and $m \in \{1, \ldots, n-1\}$. Let φ be an anisotropic n-fold quadratic Pfister form, π an anisotropic m-fold bilinear Pfister form and θ an $(m-1)$-fold bilinear Pfister form. Assume θ is a common factor of φ and π. Then $i_W(\varphi \perp Q(\pi)) = 2^m$ if and only if π is a factor of φ. Otherwise, $i_W(\varphi \perp Q(\pi)) = 2^{m-1}$.*

Proof. Since θ is a factor of φ, there exists a $2^{n-m} - 1$ dimensional bilinear form b and a quadratic 1-fold Pfister form ρ such that

$$\varphi = \theta \otimes (\langle 1\rangle \perp b) \otimes \rho\,.$$

Set $\tau = \theta \otimes b \otimes \rho$ (note that $b \otimes \rho$ is the so-called pure part of the quadratic Pfister form $(\langle 1\rangle \perp b) \otimes \rho$, see [7, p.66]). Then we have that

$$\varphi \perp Q(\theta) = 2^{m-1} \otimes \mathbb{H} \perp \tau \perp Q(\theta)\,.$$

Note that $\tau \perp Q(\theta)$ is anisotropic as φ is anisotropic.

Suppose $\tau \perp Q(\pi)$ is isotropic. Since $\tau \perp Q(\theta)$ is anisotropic, there exists an element β represented by τ and $Q(\pi)$ but not by $Q(\theta)$. As β is represented by $Q(\pi)$ but not by $Q(\theta)$, it follows from Lemma 4.4 that $Q(\langle\langle\beta\rangle\rangle \otimes \theta) = Q(\pi)$. As β is represented by $Q(\tau)$ but not by $Q(\theta)$, it follows from [7, Proposition 15.7] that $\langle\langle\beta\rangle\rangle \otimes \theta$ is a factor of φ. In particular, φ becomes isotropic over the function field of π. Hence π is a factor of φ by [11, (1.4)] and repeated use of [7, (15.6)]. □

Theorem 4.6. *Assume* char$(F) = 2$ *and n be an integer $\geqslant 2$ and $m \in \{1, \ldots, n-1\}$. Let φ and ψ be two non-hyperbolic, separably $(n-1)$-linked and totally inseparably m-linked n-fold quadratic Pfister forms over F. Then there exists a field extension K of F such that φ_K and ψ_K are totally inseparably $(m+1)$-linked but neither isometric nor hyperbolic.*

Proof. The result follows from Lemma 4.5 in a similar way to Theorem 4.3. □

Using Theorems 4.3 and 4.6, one can construct examples of non-isometric pairs of n-fold quadratic Pfister forms which are totally separably (or inseparably, or both) m-linked for any $m \in \{1, \ldots, n-1\}$.

Example 4.7. Start with the field $F = \mathbb{F}_2(\!(x_1)\!) \ldots (\!(x_{n+1})\!)$ of iterated Laurent series in $n+1$ indeterminates over $\mathbb{F}_2$. The forms

$$\varphi = \langle\langle x_1, \ldots x_{n-1}, x_1 \cdot \ldots \cdot x_n]] \quad \text{and} \quad \psi = \langle\langle x_2, \ldots x_n, x_2 \cdot \ldots \cdot x_{n+1}]]$$

over F are $(n-1)$-linked but not totally 1-linked (see [5, Sections 9&10]). By iterating Theorem 4.3 (or 4.6) m times, we end up with a field K over which φ_K and ψ_K are totally m-linked, but neither isometric nor hyperbolic.

Question 4.8. *When* $\mathrm{char}(F) = p$*, does there exist a similar process that extends two non-equal, nontrivial separably* $(n-1)$ *symbols in* $H_p^n(F)$ *to two non-equal, nontrivial totally separably (or inseparably)* $(n-1)$*-linked symbols?*

Theorem 4.9. *Assume* $\mathrm{char}(F) \neq 2$ *and* n *be an integer* $\geqslant 3$ *and* $m \in \{1, \dots, n-3\}$*. Let* φ *and* ψ *be two non-hyperbolic* $(n-1)$*-linked and totally* m*-linked* n*-fold quadratic Pfister forms over* F*. Assume there exists an* $(n-1)$*-fold quadratic Pfister form* ω *such that*

(a) ω *is a factor of* φ*,*
(b) ω *is not a factor of* ψ*,*
(c) there exists an $(n-2)$*-fold quadratic Pfister form that is a factor of both* ω *and* ψ*.*

Then there exists a field extension K *of* F *such that* φ_K *and* ψ_K *are totally* $(m+1)$*-linked but not totally* $(n-1)$*-linked nor hyperbolic.*

Proof. This is essentially the same proof as in Theorem 4.3. Here we just need to note the following: The form $\psi \perp -\omega$ is Witt equivalent to some anisotropic 2^n dimensional form. The latter remains anisotropic under scalar extension to L by [10, Theorem 5.4]. Therefore ω_L is not a subform of ψ_L, and that completes the proof. □

Example 4.10. Start with the field $F = \mathbb{C}(\!(x_1)\!) \dots (\!(x_{n+1})\!)$ of iterated Laurent series in $n+1$ indeterminates over $\mathbb{C}$. The forms $\varphi = \langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ and $\psi = \langle\!\langle x_2, \dots, x_{n+1} \rangle\!\rangle$ over F are $(n-1)$-linked but not totally 1-linked. By iterating Theorem 4.3 m times, we end up with a field K over which φ_K and ψ_K are totally m-linked, but neither hyperbolic nor isometric. If $m \leqslant n-2$ then by Theorem 4.9, φ_K and ψ_K are also not totally $(n-1)$-linked.

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