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Moduli Spaces for Right Ideals of the Weyl Algebra

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1. INTRODUCTION

Recently, Cannings and Holland [7] showed that right ideals of the first Weyl algebra $A = A_1(\mathbb{C})$ come in families determined by certain "moduli." More precisely, for every $n \in \mathbb{N}$, any n-points on the affine line, $x_1, \dots, x_n$, and all n-tuple positive integers $m_1, \dots, m_n$ they construct a variety $X(x_1, \dots, x_n; m_1, \dots, m_n)$, the points of which parametrize the isoclasses of right A-ideals. Moreover, for each point they give generators for a representant.

Although this is in a sense the ultimate answer, there are a few unsatisfactory aspects. For example, given generators of a right ideal it seems to be rather hard to calculate the corresponding moduli. More important, there are too many moduli and the obtained varieties have no apparent connection with more traditional moduli problems.

The hope that another approach might be possible is based on an intriguing analogy noted by Stafford in [19, p. 625] between the study of right ideals of A and that of projective right ideals of a polynomial ring over a division algebra.

The special case of projective right ideals of $\mathbb{H}[x, y]$ (where $\mathbb{H}$ is the quaternion algebra) has been worked out extensively in a series of papers by Knus, Ojanguren, Parimala, and Sridharan; see for example [12] or [13]. Their approach is as follows. A projective (non-free) ideal P of $\mathbb{H}[x, y]$ is a free module of rank 2 over the subalgebra $\mathbb{C}[x, y]$. They show that P can be extended to a vector bundle $\mathscr{P}$ of rank 2 over the projective plane $\mathbb{P}^2(\mathbb{C})$ with first Chern number $c_1 = 0$ and even c_2. Using Beilinson's spectral sequence [17, Chap. II, Sect. 3] one can describe the moduli spaces of such bundles entirely by linear data; see for an example [10].

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Translating these results back to $\mathbb{H}[x, y]$ they obtain a "moduli space" description of the isoclasses of projective right ideals.

In view of these results one would expect some numerical invariants to be associated to a right ideal of A (replacing the role of Chern numbers above) to give a coarse classification of the isomorphism classes. Ideally, the corresponding moduli spaces would then be related to well understood moduli problems such as vector bundles on projective spaces [17].

In this paper we will show that these hopes are (at least partially) justified. One can mimick the Knus *et al.* approach by replacing the projective plane with Artin's quantum plane [1] associated to the homogenized Weyl algebra H. This quantum space has a scheme-like structure and one can extend a right ideal of A to a vector bundle over it and use a version of Beilinson's derived equivalence to study moduli spaces of such noncommutative vector bundles. For this approach the reader is referred to the preprint version of this paper [14].

In this revision we hope to give a more ring-theoretical treatment. If we homogenize a right ideal P of A with respect to the induced Bernstein filtration we obtain a graded reflexive right ideal of H which is an Auslander regular algebra of global dimension 3. We show that the E_2 term of the naturally associated spectral sequence has only two nonzero entries: the homogenization itself and a finite dimensional graded vectorspace V. The main point is that V contains enough information to reconstruct the right ideal and that a suitable shift of it is an isomorphism invariant.

In fact, we will prove that just two consecutive homogeneous components of V and the connecting multiplication maps suffice to reconstruct the right ideal. Hence, to a right ideal we can assign two integers m and n (the dimensions of these homogeneous components) and three $m \times n$ matrices (representing the action of the generators of H). The main result states that two right ideals are isomorphic if and only if the integers coincide and the triples of matrices are equivalent under the natural $GL_m \times GL_n$ action. This can then be used to construct moduli spaces $M(A; m, n)$.

In the last section we show that these numerical moduli separate the "canonical" right ideals $P_n = x^{n+1}A + (xy + n)A$ and that there is a rational map from the corresponding moduli space $M(A; n, n)$ to the moduli space $M_{\mathbb{P}^2}(n; 0, n)$ of stable vector bundles of rank n on $\mathbb{P}^2$ with Chern numbers $c_1 = 0$ and $c_2 = n$.

2. The Statement

This section will lead to the statement of the main result. The first Weyl algebra $A = A_1(\mathbb{C})$ is the algebra generated by x and y satisfying the canonical commutation relation $[x, y] = 1$. The Bernstein filtration by giving x and y degree 1 makes A into a positively filtered algebra $\mathbb{C} = A_0 \subset A_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} A_i$ with finite dimensional A_i and with associated commutative graded ring $\operatorname{gr}(A) = \mathbb{C}[x, y]$. Any right ideal P of A becomes a filtered A-module via the induced filtration; that is, $P_i = P \cap A_i$. The Bernstein filtration extends to a filtration on the Weyl skewfield $D = D_1(\mathbb{C})$.

With H we denote the homogenization of A with respect to a central element t. That is,

$$H = \bigoplus_{i=0}^{\infty} A_i t^i \subset A\left[t, t^{-1}\right].$$

H is a quadratic algebra on three generators $X = x \cdot t$, $Y = y \cdot t$ and $Z = 1 \cdot t$ with defining equations

$$\begin{aligned} XY - YX &= Z^2 \\ XZ - ZX &= 0 \\ YZ - ZY &= 0. \end{aligned} \tag{1}$$

Observe that $H/(Z - 1) \simeq A$ and that $H/(Z) \simeq \operatorname{gr}(A)$. This allows us to lift homological properties from A and $\operatorname{gr}(A)$ to H. We refer the reader to [16] and [15] for more details.

In particular, H will be an Auslander regular quadratic algebra of global dimension 3 which satisfies the Cohen–Macaulay property. We recall that this means that for every finitely generated one-sided H-module M, all i, and every finitely generated H-submodule N of $\operatorname{Ext}^i_H(M, H)$, one has $j_H(N) := \inf\{i\colon \operatorname{Ext}^i_H(N, H) \neq 0\} \cup \{\infty\} \geq i$. The Cohen–Macaulay property means that $GK\dim(M) + j_H(M) = GK\dim(H) = 3$ for all finitely generated H-modules M. In particular, Auslander regularity implies that for every finitely generated H-module M there is a convergent spectral sequence

$$E_2^{p,-q}(M) = \operatorname{Ext}^p_H(\operatorname{Ext}^q_H(M, H)) \Rightarrow H^{p-q}(M)$$

where $H^0(M) = M$ and $H^i(M) = 0$ for $i \neq 0$.

Although all this can be deduced from standard filtered techniques, it is also a direct consequence of the general results of [3] on graded Artin–Schelter regular algebras of global dimension 3. In their classification, H is a triple line example. Further, it follows from [2] that there is a resolution of the augmentation map of the form

$$0 \to H(-3) \to H(-2)^{\oplus 3} \to H(-1)^{\oplus 3} \to H \to \mathbb{C} \to 0$$

which makes H into a Gorenstein–Koszul algebra.

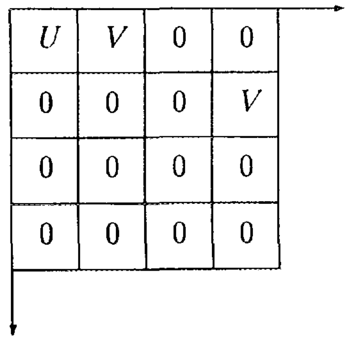

FIGURE 1

If P is a one-sided fractional ideal of A we denote its homogenization $\oplus P_i t^i \subset D[t, t^{-1}]$ by $h(P)$. If P is a right ideal of A, then $h(P)$ is a graded reflexive right ideal of H.

PROPOSITION 1. *Let $h(P)$ be the homogenization of a right A-ideal P. The E_2 term of the spectral sequence for $h(P)$ has the shape shown in Fig. 1, where $U = h(P)$ and V is a finite dimensional graded H-module.*

Proof. From Auslander regularity of H we know that the E_2 term of $h(P)$ looks like Fig. 2a below. As $E_2^{i,0} = E_\infty^{i,0}$ for $i = 2, 3$ we have $c = d = 0$. If we denote $h(P)^* = \mathrm{Hom}(h(P), H)$ this means that $\mathrm{Ext}^i(h(P)^*, H) = 0$ for $i = 2, 3$, or equivalently, that $pd(h(P)^*) \le 1$. As $h(P)$ is reflexive we can apply this argument to $h(P)^*$ and obtain that $pd(h(P)) \le 1$. This implies that $h = i = j = 0$.

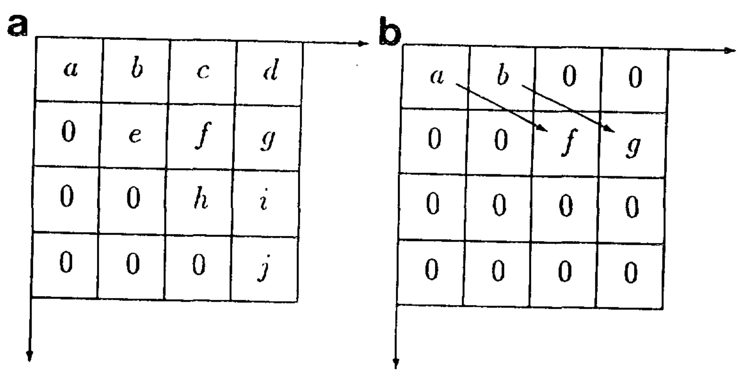

FIGURE 2

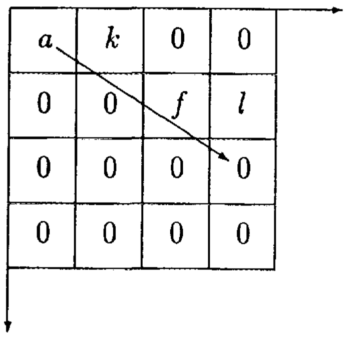

FIGURE 3

Combining this with [15, Theorem 2.2(b)] and the fact that $F^1(h(P)) = 0$ because H is 0-pure using the terminology of [15, 2.4], we see that $e = 0$. So, the E_2 term of $h(P)$ reduces to Fig. 2b with connecting morphisms α: $a \to f$ and β: $b \to g$. By reflexitivity $a = h(P)$ and as $E_3^{0,0} = E_\infty^{0,0} = h(P)$, α must be the zero map. If $k = \operatorname{Ker}(\beta)$ and $l = g/\operatorname{Im}(\beta)$, the E_3 term of $h(P)$ is shown in Fig. 3. As this is also the E_∞ term we get that $f = 0$ and that $k = l = 0$, so β is an isomorphism.

It only remains to show that $V = b \simeq g$ is finite dimensional. From the second row in the E_2 term we get that $j(\operatorname{Ext}^1(h(P), H) = 3$. We could have applied the same argument to $h(P^*)$ so also $j(b) = j(\operatorname{Ext}^1(h(P)^*, H)$ $= 3$. From the Cohen-Macaulay property we get that $GK\dim(b) = 0$, finishing the proof. ▮

COROLLARY 1. *With notations as above,*

1. *There exist integers $n_i, m_j \in \mathbb{Z}$ and an exact sequence*

$$0 \to \bigoplus_i H(n_i) \to \bigoplus_j H(m_j) \to h(P) \to 0$$

where $H(k)$ is the graded free rank-one module with $H(k)_i = H_{k+i}$.

2. *The finite dimensional vector space $V = \operatorname{Ext}^1(h(P)^*, H)$ is nonzero iff P is not principal.*

Proof. (1) Use the fact that $pd(h(P)) = 1$ and that every finitely generated graded projective H-module is free and hence a direct sum of $H(k)$'s.

(2) $V = 0$ iff $h(P)^*$ is graded projective, whence free. But then, P^* and hence P must be principal. ▮

As $V = \mathrm{Ext}^1_H(h(P)^*, H)$ is a graded finite dimensional H-module, there exist $k, l \in \mathbb{Z}$ such that $V = \oplus_{j=k}^{l} V_j$. Hence V can be represented by the finite dimensional vector spaces V_j $(k \leq j \leq l)$ and matrices X_j, Y_j, Z_j: $V_j \to V_{j+1}$ for $k \leq j \leq l$ satisfying the relations

$$\begin{aligned} X_j Y_{j+1} - Y_j X_{j+1} &= Z_j Z_{j+1} \\ X_j Z_{j+1} - Z_j X_{j+1} &= 0 \\ Y_j Z_{j+1} - Z_j Y_{j+1} &= 0. \end{aligned} \tag{2}$$

The main result of this paper will be that a suitable shift of V, namely $V(d)$ (where d is the minimal filtration degree of an element of P), is an isomorphism invariant for the nonprincipal right ideal P of A. In fact, we will prove a stronger result:

THEOREM 1. *Let P be a nonprincipal right ideal of A with minimal filtration degree $d = d(P)$. Let $V = \mathrm{Ext}^1_H(h(P)^*, H)$ which is a finite dimensional graded right H-module. Then, P is determined by the Kronecker module*

$$V_{d-3} \overset{\cdot X}{\underset{\cdot Z}{\overset{\cdot Y}{\rightrightarrows}}} V_{d-2}.$$

That is, a right ideal is determined by two integers $m(P) = \dim(V_{d-3})$ and $n(P) = \dim(V_{d-2})$ and three $m(P) \times n(P)$ matrices $(X(P), Y(P), Z(P))$.

If P' is another right ideal of A, then P is isomorphic to P' if and only if $m = m(P) = m(P')$, $n = n(P) = n(P')$, and $(X(P), Y(P), Z(P)) \sim (X(P'), Y(P'), Z(P'))$ under the natural $GL_m \times GL_n$ action.

We will end this section with a few remarks on the calculation of these invariants. This is an application of the theory of stairs due to Galligo [9].

An element $f \in A$ can be written uniquely as a finite sum $f = \sum_{(m,n) \in \mathbb{N}^2} f_{(m,n)} x^m y^n$. Define a total order of $\mathbb{N}^2$ by $(m, n) > (m', n')$ iff $m + n > m' + n'$ or if $m + n = m' + n'$ then $m > n$. With respect to this order we define $\exp(f) = \max\{(m, n): f_{(m,n)} \neq 0\}$.

Given a right ideal P of A we define its stairs to be $\mathrm{Stairs}(P) = \{\exp(f): f \in P\}$ which can be depicted as shown in Fig. 4.

There is a minimal set $\mathrm{St}(P) = \{\alpha_1, \ldots, \alpha_m\}$ such that $\mathrm{Stairs}(P) = \bigcup_{i=1}^{m} (\alpha_i + \mathbb{N}^2)$. A set $\{f_1, \ldots, f_m\}$ of elements of P is called a standard basis for P iff $\exp(f_i) = \alpha_i$. Such a set defines a partition of $\mathbb{N}^2$ as follows: $\Delta_1 = \alpha_1 + \mathbb{N}^2$, $\Delta_2 = (\alpha_2 + \mathbb{N}^2) - \Delta_1, \ldots, \Delta_m = (\alpha_m + \mathbb{N}^2) - (\Delta_1 \cup \cdots \cup \Delta_{m-1})$, and $\Delta = \mathbb{N}^2 - (\Delta_1 \cup \cdots \cup \Delta_m)$. The relevant property of a standard basis is that for every $g \in A$ there exist uniquely determined

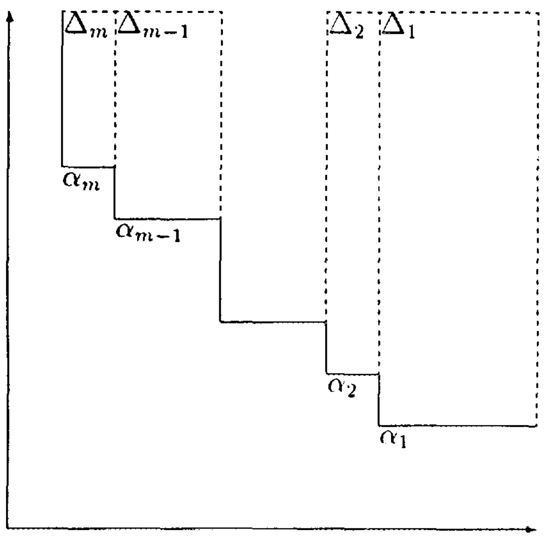

FIGURE 4

elements $g_1, \ldots, g_m, r \in A$, such that

$$g = \sum_i f_i g_i + r,$$

with $\exp(f_i g_i) \in \Delta_i$ and $\exp(r) \in \Delta$. In particular, $g \in P$ iff $r = 0$. In [9, Sect. 2] an algorithm is given to find the stairs and a standard basis for P given generators for P.

If P is a right ideal of A with standard basis $\{f_1, \ldots, f_m\}$ and $\exp(f_i) = (m_i, n_i)$ such that $m_1 > m_2 > \cdots > m_m$ then there is an exact sequence

$$0 \to \bigoplus_{i=1}^{m-1} H(-m_i - n_{i+1}) \to \bigoplus_{i=1}^{m} H(-m_i - n_i) \to h(P) \to 0$$

where the rightmost map is given by multiplication with the homogenizations of the f_i and the leftmost map by the homogenizations of the canonical sums giving $f_{i+1}x^{m_{i+1}-m_i}$. Similarly, one can find a resolution of the graded left H-module $h(P)^*$, and dualizing it gives the finite dimensional graded vector space V.

3. The Proof

In this section we will prove the main theorem. Instead of working in the category $\mathrm{gr}(H)$ of all finitely generated graded right H-modules we work in the quotient category $\mathrm{Coh}(\mathbb{P}_q)$ modulo the Serre subcategory of all H_+-torsion modules (that is, the finite dimensional graded H-modules) with degree preserving maps.

In analogy with the commutative case, one can view $\mathrm{Coh}(\mathbb{P}_q)$ as the category of coherent sheaves on a quantum plane P_q, see [1]. Ring theoretically this can be interpreted as follows. Let $\mathscr{O}, \mathscr{M}$ (resp. $\mathscr{O}(n), \mathscr{M}(n)$) be the objects in $\mathrm{Coh}(\mathbb{P}_q)$ corresponding to H, M (resp. to $H(n), M(n)$). Let us denote

$$H^i(\mathbb{P}_q, \mathscr{M}) = \mathrm{Ext}^i_{\mathrm{Coh}(\mathbb{P}_q)}(\mathscr{O}, \mathscr{M}).$$

Then for every graded right H-module M we have

$$\bigoplus_{n \in Z} H^0(\mathbb{P}_q, \mathscr{M}(n)) = Q^g_{\kappa_+}(M)$$

where $Q^g_{\kappa_+}$ is graded localization at the kernel functor determined by the filter of graded ideals with basis $(H_+)^k$ $(k \in \mathbb{N})$.

In particular, if M is κ_+-closed (such as $H, h(P)$, and $h(P^*)$) we get that $M = \oplus_{n \in \mathbb{Z}} H^0(\mathbb{P}_q, \mathscr{M}(n))$.

The Koszul resolution becomes in $\mathrm{Coh}(\mathbb{P}_q)$ the exact sequence

$$0 \to \mathscr{O}(-3) \to \mathscr{O}(-2)^{\oplus 3} \to \mathscr{O}(-1)^{\oplus 3} \to \mathscr{O} \to 0.$$

Hence $H^2(\mathbb{P}_q, \mathscr{O}(-3)) \neq 0$ and, more precisely, one can translate the Gorenstein property of H into the statements $H^1(\mathbb{P}_q, \mathscr{O}(k)) = 0$ for all $k \in \mathbb{Z}$ and $H^2(\mathbb{P}_q, \mathscr{O}(k)) \simeq H^*_{-k-3}$ as $\mathbb{C}$-vector spaces as in the commutative case.

In particular, $\mathscr{E} = \mathscr{O} \oplus \mathscr{O}(1) \oplus \mathscr{O}(2)$ is an exceptional (or tilting) object of $\mathrm{Coh}(\mathbb{P}_q)$, as in [6], [4] and hence gives rise to a derived equivalence.

Let $B = \mathrm{End}_{\mathrm{Coh}(\mathbb{P}_q)}(\mathscr{E})$; then B is the incidence algebra of the quiver

$$\bullet \; \overset{X_1}{\underset{Z_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{Y_1}}}}} \; \bullet \; \overset{X_2}{\underset{Z_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{Y_2}}}}} \; \bullet$$

with relations

$$\begin{aligned} X_1Y_2 - Y_1X_2 &= Z_1Z_2 \\ X_1Z_2 - Z_1X_2 &= 0 \\ Y_1Z_2 - Z_1Y_2 &= 0. \end{aligned} \tag{3}$$

Then we have following version of Beilinson's derived equivalence [5]:

PROPOSITION 2. *The functors*

$$F = \mathrm{Hom}_{\mathrm{Coh}(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), -) : \mathrm{Coh}(\mathbb{P}_q^2) \to \mathrm{mod}(B)$$

$$G = - \otimes_B (\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) : \mathrm{mod}(B) \to \mathbb{P}_q^2$$

induces an equivalence of derived categories

$$\mathcal{D}^b(\mathrm{Coh}(\mathbb{P}_q)) \simeq \mathcal{D}^b(\mathrm{mod}(B)).$$

Proof. This is just [6, Theorem 6.2] adapted to our situation. The required conditions follow from the Koszul sequence. ∎

Of course, we would prefer to be able to assign to an object in $\mathrm{Coh}(\mathbb{P}_q)$ a right B-module rather than a bounded complex of such modules. This can be achieved for certain subclasses of objects.

With $\mathcal{X}_i$ we denote the set of all $\mathcal{M} \in \mathrm{Coh}(\mathbb{P}_q)$ such that

$$\mathrm{Ext}^j_{\mathrm{Coh}(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), \mathcal{M}) = 0 \quad \text{for all } j \neq i.$$

Likewise, with $\mathcal{Y}_i$ we denote the set of all $M \in \mathrm{mod}(B)$ such that

$$\mathrm{Tor}^B_j(M, \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) = 0 \qquad \text{for all } j \neq i.$$

Then one deduces precisely as in e.g. [4, Sect. 3.2].

COROLLARY 2. *The functors*

$$F^i = \mathrm{Ext}^i_{\mathrm{Coh}(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), -)$$

$$G_i = \mathrm{Tor}^B_i(-, \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$$

establish an equivalence

$$X_i \simeq Y_i.$$

Hence, an object $\mathcal{M} \in \mathcal{X}_i$ is uniquely determined by a B-module and hence by linear data. In particular, $\mathcal{M} \in \mathcal{X}_1$ if the $H^j(\mathbb{P}_q, \mathcal{M}(i))$ have the shape shown in Fig. 5. Hence, if $\mathcal{M} \in \mathcal{X}_1$, then $\mathcal{M}$ is completely determined by the B-module

$$V_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; V_2 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; V_3,$$

				j	
?	0	0	0	0	
?	*	*	*	?	
0	0	0	0	?	
-3	-2	-1	0	1	i

FIGURE 5

where $V_i = H^1(\mathbb{P}_q, \mathscr{M}(-3+i))$ and where the maps are induced by multiplication with X (resp. Y and Z). Sometimes, one can even do better and show that $\mathscr{M}$ is determined by the three rightmost maps, hence by a Kronecker module. To do this, we can repeat the argument of Baer given in [4, Corollary 7.2]. Then we get that $\mathscr{M} \in \mathrm{Coh}(\mathbb{P}_q)$ is uniquely determined by a Kronecker module if both $\mathscr{M}$ and $\mathscr{M}(1)$ belong to $\mathscr{X}_1$. That is, the cohomology groups $H^j(\mathbb{P}_q, \mathscr{M}(i))$ have the shape shown in Fig. 6.

Using these general results we are now in a position to prove the main theorem. We will need a result on the minimal filtration degree of elements of right ideals in A. For a fractional onesided A-ideal F let $d(F)$ denote the minimal filtration degree of a nonzero element of F.

LEMMA 1. *Let P be a non-principal right ideal of A, then $d(P^*) > -d(P) + 1$.*

Proof. Observe first that the statements are preserved under isomorphism. Hence we can take a representant in the isomorphism class such that $P \cap \mathbb{C}[x] \neq 0$ (this can be done by [19]). Now look at Fig. 7. Here, the top right corner region (marked 1) is the stairs of P. By assumption,

				j	
?	0	0	0	0	0
?	*	*	*	*	?
0	0	0	0	0	?
-3	-2	-1	0	1	i

FIGURE 6

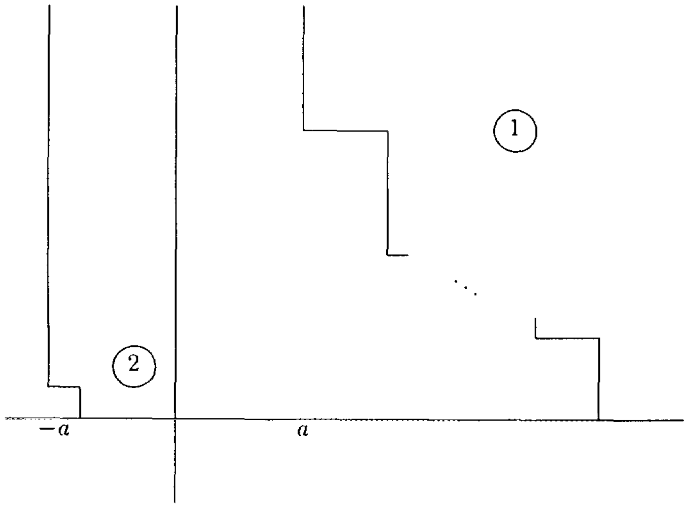

FIGURE 7

this region is bounded at the bottom right by the horizontal axis and at the top left by the line $n = a$. Therefore, the stairs of P^* lie in the region marked 2 which is bounded on the left by the line $n = -a$.

As P (and equivalently P^*) is nonprincipal, $(a, 0)$ (resp. $(-a, 0)$) does not lie in the stairs of P (resp. P^*). Therefore, $d(P) \geq a + 1$ and $d(P^*) \geq -a + 1$ from which the result follows. ∎

Proof of Theorem 1. As $H, h(P)$, and $h(P)^*$ are κ_+-closed, we have for the objects $\mathscr{O}$, $\mathscr{P}$, and $\mathscr{P}^*$ which they represent in $\mathrm{Coh}(\mathbb{P}_q)$ that

$$\bigoplus_i H^0(\mathbb{P}_q, \mathscr{O}) = H, \ \bigoplus_i H^0(\mathbb{P}_q, \mathscr{P}) = h(P), \ \bigoplus_i H^0(\mathbb{P}_q, \mathscr{P}^*) = h(P)^*.$$

Therefore, in order to prove Theorem 1 we have to verify that $\mathscr{P}(d-2)$ and $\mathscr{P}(d-1)$ belong to $\mathscr{X}_1$ where $d = d(P)$. For this we have to show that

$$H^2\big(\mathbb{P}_q, \mathscr{P}(d-4)\big) = 0.$$

The starting point is the resolution of graded left H-modules

$$0 \to \bigoplus_i H(-u_i) \to \bigoplus_j H(-v_j) \to h(P)^* \to 0$$

where for every j we have that $v_j \geq d(P^*)$. As $\mathrm{Ext}^1(h(P)^*, H)$ is finite dimensional we get after dualizing this sequence an exact sequence in $\mathrm{Coh}(\mathbb{P}_q)$,

$$0 \to \mathscr{P}(k) \to \bigoplus_j \mathscr{O}(k + v_j) \to \bigoplus_i \mathscr{O}(k + u_i) \to 0$$

for all $k \in \mathbb{Z}$. Consider the sequence with $k = d - 4$ and assume that $H^2(\mathbb{P}_q, \mathscr{P}(d-4)) \neq 0$; then by the long exact cohomology sequence there must be a j such that $H^2(\mathbb{P}_q, \mathscr{O}(d - 4 + v_j)) \neq 0$ or equivalently that $d - 4 + v_j \leq -3$ or $v_j \leq -d + 1$. As $d(P^*) \leq v_j$ this contradicts Lemma 1, finishing the proof.

To a right ideal P of A we can therefore assign two numerical isomorphism invariants $m = m(P) = \dim(V_{d-3})$ and $n = n(P) = \dim(V_{d-2})$ where $V = \mathrm{Ext}^1(h(P)^*, H)$. For given integers m, n we can construct a moduli space $M(A; m, n)$ whose points represent isomorphism classes of right A-ideals with these numerical invariants.

$M(A; m, n)$ can be constructed as follows: consider the subvariety of triples of $m \times n$ matrices (A, B, C) which can be extended to a B-module of dimension vector $(2m - n - 1, m, n)$ and such that this extension is unique up to B-isomorphism. In this variety consider the closed subvariety of B-modules M such that $\mathrm{Tor}_j^B(M, \mathscr{O} \oplus \mathscr{O}(1) \oplus \mathscr{O}(2)) = 0$ for $j \neq 1$. The moduli space $M(A; m, n)$ is then the quotient of this subvariety under the natural $GL_m \times GL_n$ action on the matrices. In the next section we will see that these moduli spaces sometimes have connections with classical moduli problems.

4. The Example

In this section we will show that the numerical invariants introduced above separate the "canonical" nonprincipal right ideals

$$P_n = x^{n+1}A + (xy + n)A.$$

We have seen how Galligo stairs can be used to get resolutions of the homogenized ideals. Here we will see that for the right ideals P_n everything can be seen immediately from eigenspace calculations as in [8, Sect. 3; 11, Sect. 1; or 18].

For $t \in \mathbb{Z}$ define $A(t) = \{f \in A | [xy, f] = tf\}$. Then, $A = \oplus_{t=-\infty}^{\infty} A(t)$ with $A(t)$ equal to

$$\begin{aligned} y^t\mathbb{C}[xy] &= \mathbb{C}[xy]y^t && \text{for } t \geq 0 \\ x^{-t}\mathbb{C}[xy] &= \mathbb{C}[xy]x^{-t} && \text{for } t < 0. \end{aligned} \tag{4}$$

LEMMA 2. *If* $P_n = x^{n+1}A + (xy+n)A$ *then* $P_n(t) = x^{n+1}A(t+n+1) + (xy+n)A(t)$ *is equal to*

$$\begin{aligned} &(xy+n)\mathbb{C}[xy]y^t && \text{for } t \geq 0 \\ &(xy+n)\mathbb{C}[xy]x^{-t} && \text{for } -n \leq t < 0 \\ &\mathbb{C}[xy]x^{-t} && \text{for } t < -n. \end{aligned} \tag{5}$$

Proof. Let $t = -1$, then $P_n(-1)$ is $x^{n+1}\mathbb{C}[xy]y^n + (xy+n)\mathbb{C}[xy]x$ which, using $x^{n+1}y^{n+1} = xy(xy+1)\cdots(xy+n)$, is equal to

$$xy(xy+1)\cdots(xy+n)\mathbb{C}[xy]y^{-1} + (xy+n)\mathbb{C}[xy]x.$$

The first factor is $(xy+1)\cdots(xy+n)\mathbb{C}[xy]x$ giving the desired result. All other calculations are similar. ∎

PROPOSITION 3. *We have an exact sequence of graded right H-modules*

$$0 \to H(-n-2) \to H(-2) \oplus H(-n-1) \to h(P_n) \to 0.$$

Proof. It is clear that $h(P_n)$ is generated by X^{n+1} and $XY + nZ^2$ and that there is a relation between these two generators, namely $X^{n+1}Y = (XY + nT^2)X^n$. This gives the required sequence. In order to verify that it is exact we have to compute the Hilbert series of $h(P_n)$. Using the foregoing lemma we see that it is equal to

$$\frac{1}{(1-s)}\left(\frac{s^2}{(1-s)(1-s^2)} + \frac{s^2\sum_{i=1}^{n-1}s^i}{(1-s^2)} + \frac{s^{n+1}}{(1-s)(1-s^2)}\right),$$

which simplifies to $(s^2 + s^{n+1} - s^{n+2})/(1-s)^3$ which fits with exactness of the sequence. ∎

Next, we perform similar computations for the dual module.

LEMMA 3. *The left A-module $P_n^* = Ax^{-n-1} \cap A(xy+n)^{-1}$ has eigenspace decomposition $P_n^*(t) = A(t-n-1)x^{-n-1} \cap A(t)(xy+n)^{-1}$ which is equal to*

$$\begin{array}{rl} y^t\mathbb{C}[xy](xy+n)^{-1} & \text{for } t \geq n+1 \\ y^t\mathbb{C}[xy] & \text{for } 0 \leq t \leq n \\ x^{-t}\mathbb{C}[xy] & \text{for } t < 0. \end{array} \tag{6}$$

Proof. Let us illustrate the calculations with the case $t = n$. Then $P_n^*(t) = x\mathbb{C}[xy]x^{-n-1} \cap y^n\mathbb{C}[xy](xy+n)^{-1}$ which equals

$$y^n\mathbb{C}[xy](xy+n-1)^{-1} \cdots (xy)^{-1} \cap y^n\mathbb{C}[xy](xy+n)^{-1},$$

giving the desired result. The other computations are similar. ∎

PROPOSITION 4. *There is an exact sequence of graded left H-modules*

$$0 \to H(-n) \to H \oplus H(-n+1) \to h(P_n^*) \to 0.$$

Proof. Again, it is easy to see the generators (1 and $Y^{n+1}(XY + nZ^2)^{-1}$) and the relation between them. Therefore it is sufficient to show that the Hilbert series of $h(P_n^*)$ is of the required shape. Using the above lemma the series is

$$\frac{1}{(1-s)}\left(\frac{s^{n-1}}{(1-s)(1-s^2)} + \frac{\sum_{i=0}^n s^i}{(1-s^2)} + \frac{2}{(1-s)(1-s^2)}\right),$$

which simplifies (as required) to $(1 + s^{n-1} - s^n)/(1-s)^3$. ∎

Observe that $h(P_n)$ is not projective as $h(P_n^*)h(P_n)$ doe not have elements of degree 0, so it cannot be equal to H.

As the minimal filtration degree for elements of P_n is 2 we have to calculate the dimensions of

$$V_1 = H^1(\mathbb{P}_q, \mathscr{P}(-1)) \quad \text{and} \quad V_2 = H^1(\mathbb{P}_q, \mathscr{P})$$

and the three maps between them induced by multiplication with the variables.

PROPOSITION 5. *The right ideal P_n is determined by the Kronecker module with dimension vector (n, n) and where the linear maps correspond-*

ing to multiplication with X (resp. Y and Z) can be represented by the matrices

$$\cdot X = \cdot \begin{pmatrix} 0 & -n+1 & 0 & & \\ & 0 & -n+2 & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & -1 \\ & & & & 0 \end{pmatrix}$$

$$\cdot Y = \cdot \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{pmatrix}$$

$$\cdot Z = \cdot I_n \cdot$$

Proof. The starting point is the resolution

$$0 \to H(-n) \xrightarrow[\cdot(Y^n,\, -X)]{} H \oplus H(-n+1) \xrightarrow[\cdot(Y^{n+1}(XY+nZ^2)^{-1}]{} h(P_n^*) \to 0.$$

As $V \to \operatorname{Ext}^1(h(P^*), H)$ is finite dimensional, dualizing this sequence gives an exact sequence in $\operatorname{Coh}(\mathbb{P}_q)$,

$$0 \to \mathcal{P}_n \to \mathcal{O} \oplus \mathcal{O}(n-1) \to \mathcal{O}(n) \to 0.$$

The long exact cohomology sequence gives the diagram

$$\begin{array}{ccccccc} H(-1) \oplus H(n-2) & \xrightarrow{(Y^n,\,-X)\cdot} & H(n-1) & \longrightarrow & H^1(\mathcal{P}_n(-1)) & \longrightarrow & 0 \\ H(0) \oplus H(n-1) & \xrightarrow[(Y^n,\,-X)\cdot]{} & H(n) & \to & H^1(\mathcal{P}_n) & \longrightarrow & 0 \end{array}$$

So we see that a basis for $H^1(\mathbb{P}_q, \mathcal{P}_n(-1))$ is given by the images of

$$Y^{n-1}, Y^{n-2}Z, \ldots, YZ^{n-2}, Z^{n-1}$$

and a basis for $H^1(\mathbb{P}_q, \mathcal{P}_n)$ is given by the images of

$$Y^{n-1}Z, Y^{n-2}Z^2, \ldots, YZ^{n-1}, Z^n.$$

Hence $m(P_n) = m(P_n) = n$ and with respect to these bases the maps are given by multiplication with the given matrices. ∎

Recall from [10] that one can associate to a triple of $n \times n$ matrices (M_1, M_2, M_3) a stable vector bundle on the commutative projective plane $\mathbb{P}^2(\mathbb{C})$ provided $\dim_{\mathbb{C}}(\mathbb{C} \cdot M_1 \cdot v + \mathbb{C} \cdot M_2 \cdot v + \mathbb{C} \cdot M_3 \cdot v) \geq 2$ for all $0 \neq v \in \mathbb{C}^n$. If one of these matrices is the identity matrix, the rank of the vector bundle is equal to the rank of the commutator matrix of the other two. Applying these facts to the above computations we see that there is a stable rank n vector bundle on $\mathbb{P}_2(\mathbb{C})$ with Chern numbers $c_1 = 0$ and $c_2 = n$ associated to the right ideal P_n.

In a neighborhood of P_n in $M(A; n, n)$ this gives a rational map $M(A; n, n) \to M_{\mathbb{P}^2}(n; 0, n)$. We will show that its image is a small subvariety of $M(n; 0, n)$. For in a neighborhood of P_n we may assume the Z-matrix to remain invertible and so by a base change we may assume it to be the identity matrix. As the Kronecker module determines uniquely the corresponding B-module with dimension vector $(n-1, n, n)$,

$$\mathbb{C}^{n-1} \;\begin{array}{c}\xrightarrow{X_1}\\ \xrightarrow{Y_1}\\ \xrightarrow{Z_1}\end{array}\; \mathbb{C}^{n} \;\begin{array}{c}\xrightarrow{X_2}\\ \xrightarrow{Y_2}\\ \xrightarrow{Z_2}\end{array}\; \mathbb{C}^{n},$$

we deduce from the defining relations

$$\begin{aligned} X_1 I_n &= Z_1 X_2 \\ Y_1 I_n &= Z_1 Y_2 \\ X_1 Y_2 - Y_1 X_2 &= Z_1 I_n \end{aligned} \tag{7}$$

that

$$Z_1(X_2Y_2 - Y_2X_2 - I_n) = 0,$$

and by unicity of the extended B-module we must have that $rk(Z_1) = n - 1$ or equivalently that $rk([X_2, Y_2] - I_n) = 1$. Therefore the image must lie in the subvariety of $M(n; 0, n)$ determined by a couple of $n \times n$ matrices (X, Y) such that $rk([X, Y] - I_n) = 1$. Recall that $M(n; 0, n) = (M_n \times M_n)/PGL_n$ so the image is a rather small subvariety.

It would be interesting to know whether all moduli spaces $M(A; m, n)$ connect with the usual moduli problems.

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