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Simple Holonomic Modules over the Second Weyl Algebra A_2

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For simple generalized Weyl algebras Λ of Gelfand–Kirillov dimension 4, a class including the second Weyl algebra A_2, some simple factor algebras of the universal enveloping algebra of the Lie algebra $sl(2) \times sl(2)$ and of $U_q sl(2) \times U_q sl(2)$, etc., the simple holonomic Λ-modules are classified up to pairs of irreducible elements of certain noncommutative Euclidean ring. © 2000 Academic Press

1. INTRODUCTION

The results in this paper are valid for generalized Weyl algebras (GWA), but in this introduction we just phrase everything for the second Weyl algebra A_2 because it is a concrete way to introduce the ideas and results.

Let K be an algebraically closed uncountable field of characteristic zero. The nth *Weyl* algebra $A_n = A_n(K)$ over K is generated by $2n$ indeterminates $X_1, ..., X_n, \partial_1, ..., \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0, \qquad \text{if} \quad i \neq j, \quad [\partial_i, X_i] = 1, \quad i = 1, ..., n.$$

It is a simple Noetherian algebra of Gelfand–Kirillov dimension $2n$. The Weyl algebra A_n is isomorphic to the ring of differential operators $K[X_1, ..., X_n, \partial/\partial X_1, ..., \partial/\partial X_n]$ with polynomial coefficients. Bernstein

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has shown that $\mathrm{GK}(M) \geqslant n$ holds for any nonzero finitely generated A_n-module M [Be] (for a further generalization of this inequality for simple affine algebras and GWA resp., affine algebras see [Bav3, Bav4, Bav9], resp., [BL]). Then M is called *holonomic* if $\mathrm{GK}(M) = n$.

The problem of classifying the set $\hat{A}_n$ of isoclasses of simple A_n-modules and, in particular, the holonomic ones $\hat{A}_n$ (holonomic) is certainly a non-trivial one. The first step in this direction was made by Block who classified (up to irreducible elements) the simple A_1-modules [Bl1–Bl3], see also [Bav2] for an alternative approach (for a classification of the simple modules for other popular algebras the reader is refered to [Bav2, Bav5] and [BVO1]).

The simple A_1-modules are holonomic. The nth Weyl algebra is the tensor product of n copies of the first Weyl algebras, $A_n = A_1 \otimes \cdots \otimes A_1$. By tensoring n simple A_1-modules one obtains a simple holonomic A_n-module, i.e.,

$$\hat{A}_1 \otimes \cdots \otimes \hat{A}_1 \subseteq \hat{A}_n \text{ (holonomic)}.$$

T. Stafford first constructed an example of a simple non-holonomic A_n-module [St1]. Later Bernstein and Lunts [BeLu], resp., Lunts [Lu], showed for $n = 2$, resp., $n \geqslant 2$, that "almost each" element $x \in A_n$ generates a maximal left ideal, and as a consequence the simple module $A_n/A_n x$ is non-holonomic ($\mathrm{GK}(A_n/A_n x) = 2n - 1$).

The holonomic A_n-modules have nice properties; e.g., any holonomic module has finite length (see Björk's book for details [Bj]). Hence the structure of holonomic modules is defined by the simple holonomic modules and their extensions. In fact, there are not many known examples of simple holonomic A_n-modules. The most famous one is the polynomial ring $W_n = K[X_1, ..., X_n]$ ($\simeq A_n/A_n(\partial_1, ..., \partial_n)$) in n variables with evident action of the ring of differential operators.

The Weyl algebra A_n is equipped with the *Bernstein* filtration B, i.e., the standard filtration associated with canonical generators of A_n. The associated graded algebra $\mathrm{gr}_B A_n$ is a polynomial ring in $2n$ variables. Thus the growth of a finitely generated A_n-module M (w.r.t. B) is described by the *Hilbert* polynomial, the degree of which is $\mathrm{GK}(M)$, and the leading coefficient multiplied by $\mathrm{GK}(M)!$ is called the *multiplicity* $e(M)$ of M. The multiplicity is a natural number.

The next natural step in the classification of simple A_n-modules was to classify the "smallest" A_n-modules, in a sense that they have the smallest possible Gelfand–Kirillov dimension (i.e., n) and the smallest multiplicity (i.e., 1). All these modules are simple holonomic and W_n is a nice representative of them. Moreover, the others can be obtained from W_n in the following way.

THEOREM 1.1 [Bav5, Theorem 5]. *A finitely generated A_n-module M has Gelfand–Kirillov dimension* $\mathrm{GK}(M)=n$ *and multiplicity* $e(M)=1$ *iff* $M\simeq {}^{\tau}W_n$ *(a twisted module) for some algebra automorphism* $\tau\in \mathrm{Aut}_B A_n$ *which preserves the Bernstein filtration.*

The structure of this paper is as follows.

In Section 2 we recall basic facts about GWA.

In Section 3 the simple modules are related to those of a ring and its localization.

Section 4 contains the proofs of our main results (the classification of simple holonomic modules over GWA up to certain pairs of irreducible elements).

In Section 5 we specify results to the case of the second Weyl algebra A_2. Let us provide some more detail.

Let k be the Weyl skew field of A_1 (i.e., the full quotient ring of A_1). Then the second Weyl algebra $A_2=A_1\otimes A_1$ is a subalgebra of the first Weyl algebra $k\otimes A_1\equiv A_1(k)$ with coefficients from the skew Weyl field. Let M be a simple A_2-module. Then $\widetilde{M}:=A_1(k)\otimes_{A_2} M$ is an $A_1(k)$-module and a left k-vector space. One has three possibilities: $\dim_k \widetilde{M}=0$; finite; infinite. Correspondingly the set of simple A_2-modules is decomposed into 3 subsets (Proposition 5.1).

1. $\dim_k \widetilde{M}=0$ (i.e., $\widetilde{M}=0$) $\Leftrightarrow [M]\in \hat{A}_1\otimes \hat{A}_1$;
2. $1\leqslant \dim_k \widetilde{M}<\infty \Leftrightarrow [M]\in \hat{A}_2(\text{holonomic})\backslash \hat{A}_1\otimes\hat{A}_1$;
3. $\dim_k \widetilde{M}=\infty \Leftrightarrow [M]\in \hat{A}_2(\text{non-holonomic})$.

So, the simple A_2-module M is holonomic iff $\widetilde{M}$ is a finite dimensional $A_1(k)$-module. Note that existence of $\hat{A}_2(\text{holonomic})\backslash \hat{A}_1\otimes\hat{A}_1$ is a typical effect of "non-commutative geometry."

Next we classify (up to pairs of irreducible elements) the simple k-finite dimensional $A_1(k)$-modules (Proposition 5.3). We present the Weyl algebra $A_1(k)$ as a GWA

$$A_1(k)=\mathscr{D}(\sigma, a=H), \qquad \mathscr{D}=k[H], \qquad \sigma\in \mathrm{Aut}_k\,\mathscr{D}, \qquad \sigma(H)=H-1.$$

Then each simple k-finite dimensional $A_1(k)$-module is a semisimple $\mathscr{D}$-module (hence, $\mathscr{D}$-torsion) which contains finitely many isotypic components, say n; each isotypic component is a simple $\mathscr{D}$-module and any of them can be obtained from a fixed one by the action (twisting) of the group generated by σ. Thus

$$\hat{A}_1(k)(k\text{-fin.dim.})=\coprod_{n=1}^{\infty} \hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n).$$

So, the first irreducible element in the description of $\hat{A}_1(k)(k\text{-fin.dim.})$ is just an irreducible element of $\mathscr{D}$, say f, which defines a simple $\mathscr{D}$-submodule of a $A_1(k)$-module (i.e., $\mathscr{D}/\mathscr{D}f$).

The central result of Section 5 is Theorem 5.4 which provides a description of the "nontrivial" simple holonomic A_2-modules:

THEOREM 5.4. *The map*

$$\hat{A}_2(\text{holonomic})\backslash\hat{A}_1 \otimes \hat{A}_1 \to \coprod_{n=1}^{\infty} \hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n),$$

$$[N] \to [A_1(k) \otimes_{A_2} N],$$

is bijective with inverse $[M] \to [\operatorname{Soc}_{A_2} M]$, *where* $\operatorname{Soc}_{A_2} M$ *is the socle of* M *considered as an* A_2*-module.*

Corollary 5.5 allows us to represent an element of $\hat{A}_2(\text{holonomic})\backslash \hat{A}_1 \otimes \hat{A}_1$ as a factor module of A_2.

2. GENERALIZED WEYL ALGEBRAS

Let D be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ an n-tuple of commuting automorphisms of D, ($\sigma_i\sigma_j = \sigma_j\sigma_i$, for all i, j), and $a = (a_1, ..., a_n)$ an n-tuple of (non-zero) elements of the center $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$

The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly, GWA) of degree n with a *base* ring D is the ring generated by D and the $2n$ indeterminates $X_1^+, ..., X_n^+, X_1^-, ..., X_n^-$ subject to the defining relations [Bav1, Bav2]:

$$X_i^- X_i^+ = a_i, \qquad X_i^+ X_i^- = \sigma_i(a_i),$$

$$X_i^{\pm} \alpha = \sigma_i^{\pm 1}(\alpha)\, X_i^{\pm}, \qquad \forall \alpha \in D,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0, \qquad \forall i \neq j,$$

where $[x, y] = xy - yx$. We say that a and σ are the *defining* elements and automorphisms of A respectively. For a vector $k = (k_1, ..., k_n) \in \mathbf{Z}^n$ we put $v_k = v_{k_1}(1) \cdots v_{k_n}(n)$ where for $1 \leqslant i \leqslant n$ and $m \geqslant 0$: $v_{\pm m}(i) = (X_i^{\pm})^m$, $v_0(i) = 1$. In the case $n = 1$ we write v_m for $v_m(1)$ and $X = X_1^+$, $Y = X_1^-$. It follows from the definition of the GWA that

$$A = \bigoplus_{k \in \mathbf{Z}^n} A_k$$

is a $\mathbf{Z}^n$-graded algebra ($A_k A_e \subset A_{k+e}$, for all $k, e \in \mathbf{Z}^n$), where $A_k = Dv_k$.

The tensor product (over a base field) $A \otimes A'$ of generalized Weyl algebras of degree n and n' respectively is a GWA of degree $n+n'$:

$$A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a').$$

Let $A = D(\sigma, a)$ be a generalized Weyl algebra of degree 1, $a \in Z(D)$, $\sigma \in \mathrm{Aut}(D)$.

The ring A is generated by D and by two indeterminates $X = X_1^+$ and $Y = X_1^-$ subject to the defining relations:

$$X\alpha = \sigma(\alpha)\, X \qquad \text{and} \qquad Y\alpha = \sigma^{-1}(\alpha)\, Y, \qquad \forall \alpha \in D,$$

$$YX = a, \qquad \text{and} \qquad XY = \sigma(a).$$

The algebra $A = \oplus_{n \in \mathbf{Z}} A_n$ is $\mathbf{Z}$-graded, where $A_n = Dv_n$, $v_n = X^n$ $(n > 0)$, $v_n = Y^{-n}$ $(n < 0)$, $v_0 = 1$. It follows from the above relations that

$$v_n v_m = (n, m)\, v_{n+m} = v_{n+m} \langle n, m \rangle$$

for some $(n, m) \in D$. If $n > 0$ and $m > 0$ then

$$n \geqslant m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a),$$

$$(-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$

$$n \leqslant m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \qquad (-n, m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n, m) = 1$.

Let K be a field and $K[H]$ be the polynomial ring in one variable H over K. Note that up to a change of variable, any K-automorphism of $K[H]$ is either $H \to H - 1$ or $H \to \lambda H$ for some nonzero $\lambda \in K$.

Consider the following GWA of degree 1.

EXAMPLE 1. $A = K[H](\sigma, a \neq 0)$, $\sigma(H) = H - 1$. The algebra A is simple iff char $K = 0$ and if there is no irreducible polynomial $p \in \mathrm{Irr}\, K[H]$ such that both p and $\sigma^i(p)$ are multiples of a for some nonzero i [Jos, Bav2, and Ho1].

The first Weyl algebra $A_1 = K\langle X, Y \mid YX - XY = 1 \rangle$ is a GWA of this kind:

$$A_1 \simeq K[H](\sigma, H), \qquad X \leftrightarrow X, \qquad Y \leftrightarrow Y, \qquad YX \leftrightarrow H.$$

EXAMPLE 2. Let $A = K[H, H^{-1}](\sigma, a \neq 0)$, $\sigma(H) = \lambda H$, $\lambda \neq 0, 1 \in K$, where $K[H, H^{-1}]$ is a Laurent polynomial ring. Up to an algebra automorphism $X \leftrightarrow XH^i$, $Y \leftrightarrow Y$, $H \leftrightarrow H$ we shall assume that the defining element a is a polynomial $(a \in K[H])$ with nonzero constant coefficient

$(a(0)\neq 0)$. The algebra Λ is simple iff λ is not a root of unity and there is no irreducible polynomial $p\in \operatorname{Irr} K[H]$ such that both p and $\sigma^i(p)$ are multiples of a for some nonzero i [Bav3, Bav5].

In case $a=1$ we obtain a skew Laurent extension $\Lambda = K[H, H^{-1}][X, X^{-1}; \sigma]$. This ring is the localization of the *quantum plane* $K\langle X, Y \mid XY=\lambda YX\rangle \simeq K[H](\sigma, H)$ (resp., the *first quantum Weyl* algebra $K\langle X, Y \mid YX-\lambda^{-1}XY=1\rangle \simeq K[H](\tau, H)$, $\tau(H)=\lambda(H-1)$) at the powers of the element $H=YX$ (resp., $H=YX-\lambda(\lambda-1)^{-1}$).

Let $F_m=A_1^G$ be the fixed ring of the first Weyl algebra A_1 under the action of the cyclic group G of order m, acting by multiplication $\partial\to\omega\partial$, $X\to\omega^{-1}X$, where ω is a primitive mth root of unity. Then

$$F_m=K\langle \partial^m, \partial X, X^m\rangle \simeq K[H](\sigma, a=m^mH(H+1/m)\cdots(H+(m-1)/m),$$

is a simple algebra in case of char $K=0$, where $\sigma(H)=H-1$.

A simplicity criterion of GWA of degree 1 with commutative coefficients was obtained in [Jo1]; in the general situation and for arbitrary degree see [Bav3]. By Theorem 2.1 of [Bav2] and Proposition 9 of [Bav5] the algebras Λ in Examples 1 and 2 are with *restricted minimum condition*; this means that any proper left and right factor module of Λ has finite length. In the case of the first Weyl algebra this property is established in [Di1].

From now on let

$$\Lambda=\bigotimes_{i=1}^{n}\Lambda_i$$

be the tensor product of algebras Λ_i from Example 1 or 2 and let $k_1, ..., k_n$ be the degrees of the defining polynomials $a_1, ..., a_n$ of the algebras $\Lambda_1, ..., \Lambda_n$ respectively.

THEOREM 2.1 [Bav3, Corollary 4.8]. *Let $\Lambda=\bigotimes_{i=1}^n \Lambda_i$ be the tensor product of simple generalized Weyl algebras from Examples* 1 *and* 2 (*for example, the Weyl algebra $A_n=A_1^{\otimes n}$*). *Then*

1. *Λ is a central simple affine Noetherian algebra which is a domain, moreover, $\Lambda\otimes B$ is simple for every simple algebra B.*
2. *The Gelfand–Kirillov dimension* $\mathrm{GK}(\Lambda)$ *of Λ is $2n$.*
3. *The filter dimension* (*in the sense of* [Bav8]) *is* $\operatorname{fil.dim}\Lambda=1$.
4. $\mathrm{GK}(M)\geqslant n$ *for every non-zero finitely generated Λ-module M.*
5. (*See also* [Bav3].) *The Krull dimension* (*in the sense of Rentschler–Gabriel* [*RG*]) $\mathrm{K.dim}\,\Lambda=n$.
6. [Bav7]. *If K is an algebraically closed uncountable field and each defining polynomial has only simple roots, then the global dimension*

$\mathrm{gl.dim}\, \Lambda = n$; *moreover*, $\mathrm{gl.dim}\, \Lambda \otimes B = n + \mathrm{gl.dim}\, B$ *for every left Noetherian affine algebra B.*

7. [Bav6]. *Any left or right ideal of* Λ *can be generated by two elements.*

Remark. In the case of the nth Weyl algebra A_n (over the field of complex numbers) the Krull (resp., the global) dimension was computed in [RG] (resp. [Ri] in case A_1, and [Ro]). The fact that any left or right ideal of A_n has two generators was established in [St2].

A nonzero finitely generated module M over the algebra Λ as in Theorem 2.1 is called *holonomic* if it has (minimal possible) Gelfand–Kirillov dimension $\mathrm{GK}(M) = n$. We denote by $\hat{\Lambda}$(holonomic) the set of isoclasses of all simple holonomic modules.

Let A be a GWA from Examples 1 and 2 respectively and let k be the degree of the defining element (polynomial)

$$a = \alpha H^k + \cdots$$

of A where $0 \neq \alpha \in K$ is the leading coefficient of a. The algebra A is equipped with a finite filtration

$$B = \{B_m, m \geqslant 0\} : B_m = \sum \{KH^i v_j : 2\,|i| + k\,|j| \leqslant m\}$$

such that the associated graded algebra

$$\mathrm{gr}_B\, A \simeq K[H](\mathrm{id}, \alpha H^k),$$

resp.,

$$\mathrm{gr}_B\, A \simeq K[H, h](\sigma, \alpha H^k), \qquad \sigma(H) = \lambda H, \qquad \sigma(h) = \lambda^{-1} h, \qquad Hh = hH,$$

is a GWA of degree 1. In the first case the algebra $\mathrm{gr}_B\, A$ is commutative, generated by X, Y, and H subject to the relation

$$\mathrm{gr}_B\, A = K[X, Y, H]/(XY = \alpha H^k).$$

In the second case the elements H and h are normal. The factor algebras

$$\mathrm{gr}_B\, A/(h) \simeq K[H](\sigma, \alpha H^k), \qquad \sigma(H) = \lambda H,$$

and

$$\mathrm{gr}_B\, A/(H) \simeq K[h](\sigma, 0), \qquad \sigma(h) = \lambda^{-1} h,$$

are GWA. The elements H and h are normal in the factor algebras above and the factor algebra

$$\mathrm{gr}_B\, \Lambda/(H, h) \simeq K[X, Y]/(XY = YX = 0)$$

is a commutative algebra.

The algebra $\Lambda = \bigotimes_{i=1}^{n} \Lambda_i$ has a filtration, say $B = \bigotimes_{i=1}^{n} B_i$, which is the tensor product of the filtrations above. The associated graded algebra $\mathrm{gr}_B\, \Lambda$ is the tensor product $\bigotimes_{i=1}^{n} \mathrm{gr}_{B_i} \Lambda_i$ of the associated graded algebras. It is an affine Noetherian algebra. Note that in case of the Weyl algebra $A_n = A_1^{\otimes n}$ the filtration B coincides with the Bernstein filtration of A_n (the natural filtration associated with canonical generators). Now fix the filtration B.

A filtration $\Gamma = \{\Gamma_i, i \geqslant 0\}$ of a Λ-module $M = \bigcup_{i=0}^{\infty} \Gamma_i$ is called *good* if the associated graded $\mathrm{gr}_B\, \Lambda$-module $\mathrm{gr}_\Gamma M$ is finitely generated. A Λ-module M has a good filtration iff it is finitely generated, and if $\{\Gamma_i\}$ and $\{\Omega_i\}$ are two good filtrations on M, then there exists a natural number k such that $\Gamma_i \subseteq \Omega_{i+k}$ and $\Omega_i \subseteq \Gamma_{i+k}$ for all i. If a Λ-module M is finitely generated and M_0 is a finite-dimensional generating subspace of M, then the standard filtration $\{\Gamma_i = B_i M_0\}$ is good (see [Bj] or [LVO] for details).

LEMMA 2.2. *Let $\Lambda = \bigotimes_{i=1}^{n} \Lambda_i$ be as above, $k = \mathrm{l.c.m.}(k_1, ..., k_n, 2)$, and let M be a finitely generated Λ-module with good filtration $\Gamma = \{\Gamma_i\}$. Then*

1. *there exist k polynomials $\gamma_0, ..., \gamma_{k-1} \in \mathbf{Q}[t]$ with coefficients from $[k^{\mathrm{GK}(M)}\mathrm{GK}(M)!]^{-1}\, \mathbf{Z}$ such that*

$$\dim \Gamma_i = \gamma_j(i) \qquad \textit{for all} \quad i \gg 0 \qquad \textit{and} \qquad j \equiv i \pmod k;$$

2. *the polynomials γ_j have the same degree $\mathrm{GK}(M)$ and the same leading coefficients $e(M)/\mathrm{GK}(M)!$ where $e(M)$ is called the multiplicity of M. The multiplicity $e(M)$ does not depend on the choice of good filtration Γ.*

Proof. It follows from Theorem 3.2 of [Bav4] and the proof of Theorem 2.2 in [Bav4] using the fact that the elements $\{H, h\}$ (of the tensor multiples $\{\Lambda_i\}$) are normal in $\mathrm{gr}_B\, \Lambda$. ∎

Remark. In the case that all algebras Λ_i are from Example 1, Lemma 2.2 is proved in [Bav4, 4.6]. If all Λ_i belong to Example 2 and all $a_i = H_i$, the algebra Λ is (isomorphic to) a multiplicative analog of the nth Weyl algebra (see [MP] for details and refferences there), the existence of the Hilbert polynomial for M was established in [MP, Proposition 5.3].

Let $0 \to N \to M \to L \to 0$ be an exact sequence of finitely generated Λ-modules and let $\Gamma = \{\Gamma_i\}$ be a good filtration on M. Then

$\Gamma' = \{\Gamma'_i = \Gamma_i \cap N\}$ and $\Gamma'' = \{\Gamma''_i = (\Gamma_i + N)/N\}$ are filtrations on N and L respectively such that the sequence of $\mathrm{gr}_B \Lambda$-modules

$$0 \to \mathrm{gr}_{\Gamma'}(N) \to \mathrm{gr}_{\Gamma}(M) \to \mathrm{gr}_{\Gamma''}(L) \to 0$$

is exact. The ring $\mathrm{gr}_B \Lambda$ is Noetherian and the $\mathrm{gr}_B \Lambda$-module $\mathrm{gr}_\Gamma(M)$ is finitely generated, so the $\mathrm{gr}_B \Lambda$-modules $\mathrm{gr}_{\Gamma'}(N)$ and $\mathrm{gr}_{\Gamma''}(L)$ are finitely generated, i.e., the filtrations Γ' and Γ'' are good and we have

$$\dim \Gamma_i = \dim \Gamma'_i + \dim \Gamma''_i, \tag{2.1}$$

hence, by Lemma 2.2,

$$\mathrm{GK}(M) = \max\{\mathrm{GK}(N), \mathrm{GK}(L)\}, \tag{2.2}$$

and if $\mathrm{GK}(M) = \mathrm{GK}(N) = \mathrm{GK}(L)$, then

$$e(M) = e(N) + e(L). \tag{2.3}$$

Let R be an affine algebra over the algebraically closed and uncountable field K and let M be a simple R-module. Then the endomorphism ring $\mathrm{End}_R(M) = K$. Let S be an K-algebra and N be an S-module. Then the tensor product $M \otimes N$ of the modules M and N is an $R \otimes S$-module. By the density theorem any submodule of $M \otimes N$ is equal to $M \otimes N'$ for some S-submodule N' of N. In particular, $M \otimes N$ is a simple $R \otimes S$-module if ${}_S N$ is simple, i.e.,

$$\hat{R} \otimes \hat{S} \subseteq (R \otimes S)\hat{}. \tag{2.4}$$

COROLLARY 2.3. 1. *Suppose that the algebra Λ is simple. Then each holonomic Λ-module has finite length. If the field K is algebraically closed and uncountable, then*

$$\hat{\Lambda}_1 \otimes \cdots \otimes \hat{\Lambda}_n \subseteq \hat{\Lambda}(\text{holonomic}).$$

In case $n = 1$,

$$\hat{\Lambda} = \hat{\Lambda}(\text{holonomic}).$$

2. *Let M be a finitely generated Λ-module and $\varphi: M \to M$ be a monomorphism. Then* $\mathrm{GK}(M/\mathrm{Im}\,\varphi) \leqslant \mathrm{GK}(M) - 1$.

3. *Suppose that $n = 2$ and the algebra $\Lambda = \Lambda_1 \otimes \Lambda_2$ is simple. Then each simple non-holonomic Λ-module has Gelfand–Kirillov dimension* 3.

Proof. 1. It follows from Lemma 2.2(1) and (2.3) that the Λ-length $l(M)$ of a holonomic Λ-module M is less than or equal to $k^n e(M)$. In case

$n=1$, $\mathrm{GK}(\Lambda)=2$ (Theorem 2.1(2)). Let x be any nonzero element from Λ, then the map $\varphi\colon \Lambda\to\Lambda$, $u\to ux$, is a Λ-module monomorphism, so, by the Statement 2, $\mathrm{GK}(M)\leqslant 1$. The algebra Λ is simple infinite dimensional, thus it has no nonzero finite-dimensional modules. Hence each nonzero simple Λ-module has Gelfand–Kirillov dimension 1, and it is holonomic.

2. It follows from Lemma 2.2(2) and (2.1).

3. The algebra Λ is a domain. For any nonzero $l\in\Lambda$ the map $\Lambda\to\Lambda$, $x\to xl$, is a Λ-module monomorphism, so, by Statement 2, $\mathrm{GK}(\Lambda/\Lambda l)\leqslant 4-1=3$. The left Λ-module Λ is not simple, thus any simple Λ-module M is an epimorphic image of some $\Lambda/\Lambda l$, hence $\mathrm{GK}(M)\leqslant 3$. ∎

Let $U=Usl(2)$ be the universal enveloping algebra of the Lie algebra $sl(2)$ over a field K of characteristic zero and let C be the Casimir element of U. Then the factor algebra ($\lambda\in K$),

$$U(\lambda):=U/U(C-\lambda)\simeq K[H](\sigma, a=\lambda-H(H+1)),\qquad \sigma(H)=H-1,$$

[Bav1, Bav2], is simple iff $\lambda\notin\{(n^2-1)/4\mid n=1,2,...\}$ [St3, Bav2].

Let C be the Casimir element of $U_q=U_q sl(2)$; $0\notin q\in K$ is not a root of unity. Then for any $\lambda\in K$, the factor algebra

$$\begin{aligned} U_q(\lambda) &:= U_q/U_q(C-\lambda)\\ &\simeq K[H,H^{-1}](\sigma, a=\lambda+\{H^2/(q^2-1)-H^{-2}/(q^{-2}-1)\}/2h),\\ \sigma(H)&=qH,\end{aligned}$$

for some $h\in K$ (see [Bav2] for details). The algebra $U_q(\lambda)$ is simple iff $U_q(\lambda)$ has no simple finite dimensional module iff for each root, say μ, of the polynomial a no single scalar $q^i\mu$, $0\neq i\in\mathbf{Z}$, is a root of a (K is a.c.). For other examples the reader is refered to [Bav2–Bav7].

The tensor product $\Lambda=\bigotimes_{i=1}^n \Lambda_i$ of algebras $\Lambda_i=U(\lambda_i)$ or $\Lambda_i=U_q(\lambda_i)$ is an example of the algebra Λ. In particular, an algebra $\bigotimes_{i=1}^n U(\lambda_i)$ is isomorphic to the factor algebra

$$\bigotimes_{i=1}^{n} Usl(2)/(C_i-\lambda_i)\simeq U(sl(2)\times\cdots\times sl(2))/(C_1-\lambda_1, ..., C_n-\lambda_n).$$

3. SIMPLE MODULES OF A RING AND ITS LOCALIZATION

Let A be a ring and let $B=S^{-1}A$ be the left (Ore) localization of the ring A at an Ore set $S\ni 1$ of A. We have the natural ring homomorphism $A\to B$, $a\to a/1$, which, in general, is not a monomorphism. For a left ideal

$\mathbf{m}$ of B we denote by $A \cap \mathbf{m}$ the inverse image of $\mathbf{m}$ in A. The localization defines the localization functor

$$S^{-1}\colon A\text{-mod} \to B\text{-mod}, \qquad M \to S^{-1}M = B \bigotimes_A M,$$

from the category of A-modules to the category of B-modules. An A-module M contains the S-*torsion* submodule

$$\operatorname{tor}_S(M) = \{m \in M :\ sm = 0 \text{ for some } s = s(m) \in S\}.$$

If the A-module M is simple, then its localization $S^{-1}M$ is either zero ($\Leftrightarrow M = \operatorname{tor}_S(M)$) or not ($\Leftrightarrow \operatorname{tor}_S(M) = 0$); in the last case $S^{-1}M$ is a simple B-module. Correspondingly we say that a simple A-module is either S-*torsion* or S-*torsionfree*, i.e.,

$$\hat{A} = \hat{A}(S\text{-torsion}) \cup \hat{A}(S\text{-torsionfree}). \tag{3.1}$$

The sum of all simple submodules of an A-module M is called the *socle* $\operatorname{Soc}_A M$ of M. It is the largest semisimple submodule of M. A B-module N is called A-*socle* (or, *socle*, for short) provided $\operatorname{Soc}_A N \neq 0$. Denote by $\hat{B}(A\text{-socle})$ the set of isoclasses of simple A-socle B-modules. A submodule M' of M is called *essential* if it intersects each nonzero submodule of M nontrivially.

LEMMA 3.1. 1. *The canonical map*

$$S^{-1}\colon \hat{A}(S\text{-}torsionfree) \to \hat{B}(A\text{-}socle), \qquad [M] \to [S^{-1}M],$$

is a bijection with inverse Soc: $[N] \to [\operatorname{Soc}_A(N)]$.

2. *Each simple S-torsionfree A-module has the form*

$$M_{\mathbf{m}} := A/A \cap \mathbf{m} \tag{3.2}$$

for some left maximal ideal $\mathbf{m}$ *of the ring B. Two such modules are isomorphic, $M_{\mathbf{m}} \simeq M_{\mathbf{n}}$, iff the B-modules $B/\mathbf{m}$ and $B/\mathbf{n}$ are isomorphic.*

Proof. 1. It is easy to see that the maps S^{-1} and Soc are well defined. If $[M] \in \hat{A}(S\text{-torsionfree})$, then M is a simple essential A-submodule of $S^{-1}M$, so $\operatorname{Soc}_A(S^{-1}M) = M$. Conversely, if $[N] \in \hat{B}(A\text{-socle})$, then $\operatorname{Soc}_A(N)$ is nonzero, thus $S^{-1}(\operatorname{Soc}_A(N)) = N$ (as a nonzero submodule of a simple module).

2. Evident. ∎

Write LMAX(B) for the set of all left maximal ideals of B. A maximal left ideal $\mathbf{m}$ of the ring B is called *socle*, resp., *convenient*, provided

$\mathrm{Soc}_A\, M_{\mathbf{m}} \neq 0$, resp. $M_{\mathbf{m}}$ is a simple A-module and the sets of all such ideals are denoted by LMAX.soc(B) and LMAX.con(B). Clearly, LMAX.con(B) $\subseteq$ LMAX.soc(B). In general, not every left maximal (resp. socle) ideal is socle (resp., convenient).

For a socle maximal left ideal $\mathbf{m}$ of B let $J(\mathbf{m})$ be the smallest of the left ideals of A strictly containing $A \cap \mathbf{m}$, then

$$J(\mathbf{m})/A \cap \mathbf{m} = \mathrm{Soc}_A\, M_{\mathbf{m}}.$$

Since $S^{-1}(J(\mathbf{m})/A \cap \mathbf{m}) = S^{-1}\,\mathrm{Soc}_A\, M_{\mathbf{m}} = B/\mathbf{m}$, the set

$$\mathbf{a}(\mathbf{m}) := J(\mathbf{m}) \cap S \tag{3.3}$$

is not empty.

LEMMA 3.2. *Let* $\mathbf{m} \in \mathrm{LMAX.soc}(B)$ *and* $\alpha \in S$. *The following are equivalent.*

1. $\alpha \in \mathbf{a}(\mathbf{m})$;
2. $J(\mathbf{m}) = A\alpha + A \cap \mathbf{m}$;
3. $M_{\mathbf{m}\alpha^{-1}}$ *is a simple* A-*module*;
4. $\mathbf{m}\alpha^{-1} \in \mathrm{LMAX.con}(B)$.

Proof. (1⇒2) Put $I(\alpha) = A\alpha + A \cap \mathbf{m}$, then $I(\alpha) \subseteq J(\mathbf{m})$ and $S^{-1}I(\alpha) = B$, so $I(\alpha) \neq A \cap \mathbf{m}$, hence $I(\alpha) = J(\mathbf{m})$.

(2 ⇒ 1) $\alpha \in J(\mathbf{m}) \cap S = \mathbf{a}(\mathbf{m})$.

(2 ⇒ 3) Follows from

$$\begin{aligned}\mathrm{Soc}_A(M_{\mathbf{m}}) &= J(\mathbf{m})/A \cap \mathbf{m} \\ &= (\bar{A}\alpha + \bar{A} \cap \mathbf{m})/\bar{A} \cap \mathbf{m} \simeq \bar{A}\alpha/\bar{A}\alpha \cap \mathbf{m} \simeq \bar{A}/\bar{A} \cap \mathbf{m}\alpha^{-1} \simeq M_{\mathbf{m}\alpha^{-1}},\end{aligned}$$

where $\bar{A}$ is the image of A in B under the map $A \to B$, $a \to a/1$.

(3 ⇔ 4) Evident.

(3 ⇒ 2) As we have seen above,

$$M_{\mathbf{m}\alpha^{-1}} \simeq (\bar{A}\alpha + \bar{A} \cap \mathbf{m})/\bar{A} \cap \mathbf{m} \simeq (A\alpha + A \cap \mathbf{m})/A \cap \mathbf{m},$$

hence $J(\mathbf{m}) = A\alpha + A \cap \mathbf{m}$. ∎

LEMMA 3.3. *Let* $\mathbf{m}$ *be a maximal left ideal of* B. *Then* $A \cap \mathbf{m}$ *is a maximal left ideal of* $A \Leftrightarrow A = A\alpha + A \cap \mathbf{m}$ *for all* $\alpha \in S$.

Proof. ($\Rightarrow$) Set $I(\alpha) = A\alpha + A \cap \mathbf{m}$. Then $S^{-1}I(\alpha) = B$ and $S^{-1}(A \cap \mathbf{m}) = \mathbf{m}$, so $I(\alpha) \neq A \cap \mathbf{m}$ and, by maximality of $A \cap \mathbf{m}$, $I(\alpha) = A$.

($\Leftarrow$) Suppose that $A \cap \mathbf{m}$ is not a maximal left ideal of A and let J be a maximal left ideal of A strictly containing $A \cap \mathbf{m}$. We see that $S^{-1}J = B$ (otherwise, $S^{-1}J = \mathbf{m}$, then $J \subseteq A \cap S^{-1}J = A \cap \mathbf{m}$, a contradiction), thus $J \cap S \neq \varnothing$. If $\alpha \in J \cap S$, then $A = A\alpha + A \cap \mathbf{m} \subseteq J$, a contradiction. ∎

LEMMA 3.4. *Let* $\mathbf{m}$ *be a maximal left ideal of* B.

1. *The* A*-module* $M_{\mathbf{m}}$ *is simple* $\Leftrightarrow \operatorname{Hom}_A(M_{\mathbf{m}}, N) = 0$ *for all simple* S*-torsion* A*-modules* N.

2. *If the left ideal* $A \cap \mathbf{m}$ *of* A *contains a nonzero element* v *that acts injectively on every simple* S*-torsion* A*-module, then the* A*-module* $M_{\mathbf{m}}$ *is simple.*

Proof. 1. ($\Rightarrow$) Evident.

($\Leftarrow$) Suppose that $M_{\mathbf{m}}$ is a nonsimple A-module. Then $A \cap \mathbf{m}$ is strictly contained in some left maximal ideal J of the ring A and $S^{-1}J = B$ (otherwise, $S^{-1}J = \mathbf{m}$ and $J \subseteq A \cap S^{-1}J = A \cap \mathbf{m}$, a contradiction). So, $J \cap S \neq \varnothing$ and $N = A/J$ is a simple S-torsion A-module. Then $\operatorname{Hom}_R(M_{\mathbf{m}}, N) \neq 0$, since there exists a nonzero homomorphism $M_{\mathbf{m}} \to N$, $u + A \cap \mathbf{m} \to u + J$.

2. Let φ be a homomorphism from $M_{\mathbf{m}}$ to a simple S-torsion A-module N. Then $0 = \varphi(v) = v\varphi(1 + A \cap \mathbf{m})$, and, by the choice of v, $\varphi(1 + A \cap \mathbf{m}) = 0$, hence $\varphi = 0$. Now the result follows from Statement 1. ∎

4. SIMPLE HOLONOMIC Λ-MODULES

Let K be an algebraically closed uncountable field and let the algebra

$$\Lambda = C \otimes A$$

be the tensor product of GWA's from Example 1 or 2. From now on we will assume that the algebra Λ is *simple*. The main example of such an algebra is the second Weyl algebra $A_2 = A_1 \otimes A_1$. We fix the notation

$$A = D(\sigma, a) = \bigoplus_{i \in \mathbf{Z}} Dv_i, \qquad X \equiv X_2^+ \equiv v_1, \qquad Y \equiv X_2^- \equiv v_{-1},$$

where either

$$D = K[H], \qquad \sigma: H \to H-1 \qquad \text{or} \qquad D = K[H, H^{-1}], \qquad \sigma: H \to \lambda H.$$

Denote by k the skew field which is the full quotient ring

$$k = C_*^{-1} C, \qquad C_* = C \backslash \{0\},$$

of the algebra C. Hence k is either the *skew Weyl* field or the skew field of the *quantum plane*. This follows immediately from the fact that the localization of the GWA $K[H](\sigma, a \neq 0)$ at $K[H]\backslash\{0\}$ is the skew Laurent extension $K(H)[X, X^{-1}; \sigma]$ with coefficients from the field $K(H)$ of rational functions. Note that in both cases the center of the skew field k is the base field,

$$Z(k) = K.$$

The ring

$$\mathscr{A} := C_*^{-1} \Lambda = k \otimes A$$

is the GWA $\mathscr{A} = \mathscr{D}(\sigma, a)$ of degree 1 with base ring

$$\mathscr{D} = k \otimes K[H] = k[H] \qquad \text{or} \qquad \mathscr{D} = k \otimes K[H, H^{-1}] = k[H, H^{-1}],$$

and with the defining automorphism $\sigma \in \mathrm{Aut}_k \mathscr{D}$ acting trivially on k.

Let M be a nonzero simple Λ-module; the localization

$$C_*^{-1} M = \mathscr{A} \bigotimes_{\Lambda} M$$

of the module M at C_* is either zero or not. In the last case it is a simple $\mathscr{A}$-module. With respect to these two possibilities we say that the module M is either C-*torsion* or C-*torsionfree*, so

$$\hat{\Lambda} = \hat{\Lambda}(C\text{-torsion}) \cup \hat{\Lambda}(C\text{-torsionfree}). \tag{4.1}$$

PROPOSITION 4.1.

$$\hat{\Lambda}(C\text{-torsion}) = \hat{C} \otimes \hat{A} \subseteq \hat{\Lambda}(\text{holonomic}).$$

Proof. The inclusion $\subseteq$ was established in Corollary 2.3.

The algebras C and A are affine Noetherian (but not Artinian) integral domains, so the inclusion $\supseteq$ follows from (2.4).

On the other hand, let M be a nonzero simple C-torsion Λ-module. The algebra C satisfies the restricted minimum condition, so M contains a simple C-submodule, say N. Then the Λ-module M is an epimorphic image of the A-module $N \otimes A$ but, as we have seen at the end of Section 2, any submodule of $N \otimes A$ is equal to $N \otimes J$ for some left ideal J of A. Hence, $M \simeq N \otimes A/J$, where the A-module A/J is simple. ∎

The simple modules over the first Weyl algebra A_1 were classified in [Bl1–Bl3]. For the algebras from Examples 1 and 2 this was done in [Bav2, Bav5]. The reader is refered to these papers for detail. For certain generalized crossed products including the algebras mentioned, the simple modules were classified in [BVO1].

Denote by $k(H)$ the skew field $\mathscr{D}_*^{-1}\mathscr{D}$ where $\mathscr{D}_* = \mathscr{D}\backslash\{0\}$ and by

$$\mathscr{B} := \mathscr{D}_*^{-1}\mathscr{A} = k(H)[X, X^{-1}; \sigma] \tag{4.2}$$

the localization of the algebra $\mathscr{A}$ at $\mathscr{D}_*$. The ring $\mathscr{B}$ is a (left and right) principal ideal domain. We identify $\mathscr{A}$ with its image in $\mathscr{B}$ via the algebra monomorphism $\mathscr{A} \to \mathscr{B}$, $x \to x/1$. So, we have

$$\hat{\mathscr{A}} = \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}) \cup \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsionfree}). \tag{4.3}$$

A $\mathscr{D}$-module V is simple iff $V \simeq \mathscr{D}/\mathscr{D}f$ for some irreducible polynomial $f \in \operatorname{Irr} \mathscr{D}$.

The subgroup G of $\operatorname{Aut}_k \mathscr{D}$ generated by σ acts by twisting on the set $\hat{\mathscr{D}}$: ${}^{\sigma}[V] = [{}^{\sigma}V]$, where ${}^{\sigma}V$ is the *twisted* $\mathscr{D}$-module. As an abelian group ${}^{\sigma}V$ coincides with V but the (twisted) action of $\mathscr{D}$ is defined as follows: $d * v = \sigma(d)\, v$, where $d \in \mathscr{D}$, $v \in V$, and by $*$ we denote the new action.

Denote by $\mathscr{O}(V) \equiv \mathscr{O}([V])$ the orbit in $\hat{\mathscr{D}}$ of the isoclass $[V]$. We say that an orbit $\mathscr{O}(V)$ is *cyclic* of length n (resp. *linear*) provided it contains finitely (resp. infinitely) many elements $n = |\mathscr{O}(V)|$. By Cyc (Lin) we denote the set of all cyclic (linear) orbits, i.e.,

$$\hat{\mathscr{D}}/G = \mathrm{Cyc} \cup \mathrm{Lin}.$$

Let Cyc_n, $n \geqslant 1$, be the set of all cyclic orbits of length n, $\mathrm{Cyc} = \bigcup_{n \geqslant 1} \mathrm{Cyc}_n$. Denote by $\hat{\mathscr{D}}(\mathrm{Cyc}_n)$ the set of isoclasses of simple $\mathscr{D}$-modules $[V]$ with $\mathscr{O}(V) \in \mathrm{Cyc}_n$. Then set

$$\hat{\mathscr{D}}(\mathrm{Cyc}) = \bigcup_{n=1}^{\infty} \hat{\mathscr{D}}(\mathrm{Cyc}_n).$$

An $\mathscr{A}$-module M is called *weight* if ${}_{\mathscr{D}}M$ is semisimple. The *support* $\mathrm{Supp}(M)$ of a weight module M is the set of isoclasses of the simple $\mathscr{D}$-submodules of M. The weight module M is the sum of its *isotypic* components

$$M = \sum_{[V] \in \hat{\mathscr{D}}} M_{[V]},$$

where $M_{[V]}$ is the sum of all submodules of M isomorphic to V. The module M is decomposed into the direct sum of $\mathscr{A}$-submodules

$$M = \sum \{M_{\mathscr{O}} \mid \mathscr{O} \text{ is an orbit}\}, \tag{4.4}$$

where $M_{\mathscr{O}} = \sum \{M_{[V]} \mid [V] \in \mathscr{O}\}$.

Let V be a simple $\mathscr{D}$-module. Then the induced $\mathscr{A}$-module

$$\mathscr{A}(V) := \mathscr{A} \bigotimes_{\mathscr{D}} V = \bigoplus_{i \in \mathbf{Z}} v_i \otimes V \tag{4.5}$$

is a $\mathbf{Z}$-graded weight module.

$$\mathscr{A}(V) = \bigoplus_{i \in \mathbf{Z}} \mathscr{A}(V)_i, \qquad \mathscr{A}(V)_i = v_i \otimes V,$$

with respect to the $\mathbf{Z}$-grading of the GWA

$$\mathscr{A} = \bigoplus_{i \in \mathbf{Z}} \mathscr{A}_i, \qquad \mathscr{A}_i = \mathscr{D} v_i,$$

and for each i the map

$$v_i \otimes V \to {}^{\sigma^{-i}}V, \qquad v_i \otimes v \to v$$

is a $\mathscr{D}$-module isomorphism.

If $V = \mathscr{D}/\mathscr{D}f$ for some irreducible polynomial $f \in \operatorname{Irr} \mathscr{D}$, then the map

$$\mathscr{A}/\mathscr{A}f \to \mathscr{A}(V), \qquad 1 + \mathscr{A}f \to 1 \otimes (1 + \mathscr{D}f),$$

is an $\mathscr{A}$-module isomorphism. Denote by $\hat{\mathscr{A}}$ (weight) the set of isoclasses of the simple weight $\mathscr{A}$-modules. It follows from (4.4) that any simple weight $\mathscr{A}$-module M is a homomorphic image of some $\mathscr{A}(V)$ and the support $\operatorname{Supp} M$ of M belongs to the orbit of $[V]$. The ring $\mathscr{D}$ has the restricted minimum condition, so

$$\hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}) = \hat{\mathscr{A}}\,(\text{weight}) \tag{4.6}$$

and

$$\hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}) = \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, linear}) \cup \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, cyclic}), \tag{4.7}$$

i.e., a simple $\mathscr{D}$-torsion $\mathscr{A}$-module with support from a linear (resp. cyclic) orbit belongs to the first set (resp. to the second). Moreover, $\hat{\mathscr{A}}$ ($\mathscr{D}$-torsion, cyclic) $= \bigsqcup_{n=1}^{\infty} \hat{\mathscr{A}}$ ($\mathscr{D}$-torsion, $\mathrm{Cyc}n_n$) where $[M] \in \hat{\mathscr{A}}$ ($\mathscr{D}$-torsion, $\mathrm{Cyc}n_n$) iff M is a simple $\mathscr{D}$-torsion $\mathscr{A}$-module with support from a finite orbit containing n elements.

Let $\lambda_1, \ldots, \lambda_t$ be all the distinct roots of the defining polynomial a of the GWA A. The algebra A is simple, so the simple $\mathcal{D}$-modules $V_i := \mathcal{D}/\mathcal{D}(H-\lambda_i)$ belong to the distinct linear orbits $\mathcal{O}_i = \mathcal{O}(V_i)$, $i = 1, \ldots, t$, which are called *degenerate*. For each i set

$$\Gamma_i^- = \{[{}^{\sigma^j}V_i], j \geqslant 0\} \qquad \text{and} \qquad \Gamma_i^+ = \{[{}^{\sigma^j}V_i], j < 0\}.$$

Denote by $\hat{\mathcal{D}}$(linear), resp. $\hat{\mathcal{D}}$(cyclic), the set of isoclasses of simple $\mathcal{D}$-modules which belong to the linear, resp. cyclic, orbits. We say that two isoclasses from $\hat{\mathcal{D}}$(linear) are *equivalent*, $[V] \sim [W]$, if they both belong either to one of $\Gamma_i^{\pm}$, $i = 1, \ldots, t$, or to a non-degenerate orbit.

PROPOSITION 4.2. *The map*

$$\hat{\mathcal{D}}(\text{linear})/\sim \;\to \hat{\mathcal{A}}\,(\mathcal{D}\text{-torsion, linear}), \qquad [\Gamma] \to [L(\Gamma)],$$

is bijective with inverse $[M] \to \operatorname{Supp} M$, *where*

1. *if* $\Gamma = \mathcal{O}(V)$ *is a non-degenerate orbit, then* $L(\Gamma) = \mathcal{A}(V)$;
2. *if* $\Gamma = \Gamma_i^-$ *for some* $i = 1, \ldots, t$, *then* $L(\Gamma) = \mathcal{A}(V_i)/\sum_{j>0} v_j \otimes V_i$;
3. *if* $\Gamma = \Gamma_i^+$ *for some* $i = 1, \ldots, t$, *then* $L(\Gamma) = \mathcal{A}({}^{\sigma^{-1}}V_i)/\sum_{j<0} v_j \otimes {}^{\sigma^{-1}}V_i$.

All the modules $L(\Gamma)$ *are infinite-dimensional left k-vector spaces.*

Proof. It follows immediately from the fact that $\mathcal{O}(V)$ is linear and the $\mathcal{D}$-module $\mathcal{A}(V)$ is semisimple with non-isomorphic simple isotypic components $\{v_i \otimes V\}$. ▮

PROPOSITION 4.3. *Any* $[M] \in \hat{\mathcal{A}}\,(\mathcal{D}$-torsion, linear) *has* $\operatorname{Soc}_\Lambda M = 0$.

Proof. Each simple $\mathcal{A}$-module $L = L(\Gamma)$ from Proposition 4.2 is a semisimple $\mathcal{D}$-module with non-isomorphic simple isotypic components $\{L_i \simeq v_i \otimes V\}$, $i \in \mathbf{Z}$, $L = \oplus L_i$. If N is a nonzero Λ-submodule of L, then $N^{\text{hom}} = \oplus\,(N_i := N \cap L_i)$ is a nonzero Λ-submodule of N and each component N_i is a $\mathcal{C}$-module, where $\mathcal{C}$ equals either $C \otimes K[H]$ or $C \otimes K[H, H^{-1}]$ with respect to two cases. If the socle $N = \operatorname{Soc}_\Lambda L$ exists, then it is a simple Λ-module, so $N = N^{\text{hom}}$, and each nonzero component N_i is a simple $\mathcal{C}$-module. The center $Z(\mathcal{C})$ of the ring $\mathcal{C}$ is equal to either $Z(C) \otimes K[H] = K[H]$ or $Z(C) \otimes K[H, H^{-1}] = K[H, H^{-1}]$ and the field K is algebraically closed and uncountable, hence there exists $\mu \in K$ such that $(H-\mu) N_i = 0$, i.e., N_i is a simple C-module ($C \simeq \mathcal{C}/\mathcal{C}(H-\mu)$), thus $\mathcal{A} \otimes_\Lambda N = 0$, a contradiction. ▮

LEMMA 4.4. *Let $k[H]$ (resp., $k[H, H^{-1}]$) be a (resp., Laurent) polynomial ring with coefficients in a skew field k. Then a polynomial $f \in k[H]$ (resp., $f \in k[H, H^{-1}]$) is normal iff $f = \lambda g$ (resp., $f = \lambda H^i g$) for some $\lambda \in k$ (resp., $i \in \mathbf{Z}$) and $g \in Z(k)[H]$.*

Proof. The second statement follows from the first. So, we prove the first.

($\Rightarrow$) Let $f = \sum \lambda_i H^i$ for some $\lambda_i \in k$. The ring $k[H]$ is an integral domain, so there is an automorphism, say τ, of the ring $k[H]$ such that

$$fd = \tau(d) f \qquad \text{for} \quad d \in k[H].$$

A degree-argument shows that $\tau(k) = k$, so the condition above for $d = \lambda \in k$ is equivalent to $\tau(\lambda) = \lambda_i \lambda \lambda_i^{-1} = \lambda_j \lambda \lambda_j^{-1}$ for all $\lambda \in k$ and all nonzero λ_i and λ_j. Hence $\lambda_j = \mu_j \lambda_i$ for some $\mu_j \in Z(k)$. Now $f = \lambda_i (\sum \mu_j H^j)$.

($\Leftarrow$) Obvious. ∎

LEMMA 4.5. 1. *Let V be a simple $\mathscr{D}$-module with the nonzero annihilator $\mathbf{a} = \operatorname{ann}_{\mathscr{D}} V$. Then $\mathbf{a} = \mathscr{D}(H - \lambda)$ for some $\lambda \in K$ and $V \simeq \mathscr{D}/\mathscr{D}(H - \lambda) \simeq k$.*

2. *Let V be a simple $\mathscr{D}$-module such that $\mathscr{O}(V)$ is cyclic. Then* $\operatorname{ann}_{\mathscr{D}} V = 0$.

Proof. 1. Any nonzero left or right ideal of $\mathscr{D}$ is principal and is generated by a nonzero element of minimal degree. Thus $\mathbf{a} = \mathscr{D}f = f\mathscr{D}$ for some (normal) element f which, by Lemma 4.4, may be chosen from $K[H]$. The $\mathscr{D}$-module V is simple, so $\mathbf{a}$ is a prime ideal of $\mathscr{D}$, hence f is an irreducible polynomial in $K[H]$ or in $K[H, H^{-1}]$, so $f = H - \lambda$ for some $\lambda \in K$ ($\lambda \neq 0$ in the case of $K[H, H^{-1}]$) and the result follows.

2. The algebra Λ is simple, so each simple $\mathscr{D}$-module $\mathscr{D}/\mathscr{D}(H - \lambda)$ from 1 belongs to a linear orbit. ∎

Put $S = K[H] \backslash \{0\}$ and let $K(H) = S^{-1}K[H]$ be the field of rational functions. Denote by $B := S^{-1}\mathscr{A}$ the localization of the ring $\mathscr{A}$ at S. The ring

$$B = k \otimes K(H)[X, X^{-1}; \sigma]$$

is the skew Laurent extension with coefficients in the tensor product $k \otimes K(H)$ of fields. By the ring monomorphism

$$\mathscr{A} \to B, \qquad X \to X, \qquad Y \to aX^{-1}, \qquad d \to d, \qquad d \in \mathscr{D},$$

the ring $\mathscr{A}$ may be identified with its image in B. For $n \geqslant 1$, denote by $\mathscr{A}_{[n]}$ the subring of $\mathscr{A}$ generated by $\mathscr{D}$, v_n, and v_{-n}; it is a GWA:

$$\mathscr{A}_{[n]} = \mathscr{D}(\sigma^n, (-n, n) = \sigma^{-n+1}(a) \cdots \sigma(a)\, a) = \bigoplus_{i \in \mathbf{Z}} \mathscr{D} v_{in}.$$

The localization $B_{[n]} = S^{-1} \mathscr{A}_{[n]}$ is the skew Laurent extension

$$B_{[n]} = k \otimes K(H)[X^n, X^{-n}; \sigma^n] \tag{4.8}$$

which is a subring of B. The ring

$$\mathscr{A} = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} \mathscr{A}_{\mathbf{i}}, \qquad \text{resp.} \qquad B = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} B_{\mathbf{i}},$$

is a $(\mathbf{Z}_n = \mathbf{Z}/\mathbf{Z}n)$-graded ring, where

$$\mathscr{A}_{\mathbf{i}} = \bigoplus_{j \in \mathbf{Z}} \mathscr{D} v_{i+jn}, \qquad \mathbf{i} = i + n\mathbf{Z}, \qquad \mathscr{A}_{\mathbf{0}} = \mathscr{A}_{[n]},$$

resp.

$$B_{\mathbf{i}} = \bigoplus_{j \in \mathbf{Z}} k \otimes K(H)\, X^{i+jn}, \qquad B_{\mathbf{0}} = B_{[n]}.$$

For each $\mathbf{i} \in \mathbf{Z}_n$: $\mathscr{A}_{\mathbf{i}} \subseteq B_{\mathbf{i}}$.

For each $[V] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$, the $\mathscr{A}$-module $\mathscr{A}(V)$ is $\mathbf{Z}_n$-graded:

$$\mathscr{A}(V) = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} \mathscr{A}(V)_{\mathbf{i}}, \qquad \text{where} \qquad \mathscr{A}(V)_{\mathbf{i}} = \bigoplus_{j \in \mathbf{Z}} v_{i+jn} \otimes V \tag{4.9}$$

is an $\mathscr{A}_{[n]}$-module and it is the isotypic component of the $\mathscr{D}$-module $\mathscr{A}(V)$ which corresponds to the simple $\mathscr{D}$-module ${}^{\sigma^{-i}}V$. The zero component $\mathscr{A}(V)_{\mathbf{0}}$ of $\mathscr{A}(V)$, as an $\mathscr{A}_{[n]}$-module, is (naturally) isomorphic to

$$\mathscr{A}(V)_{\mathbf{0}} \simeq \mathscr{A}_{[n]}(V) := \mathscr{A}_{[n]} \bigotimes_{\mathscr{D}} V. \tag{4.10}$$

An element $u \in \mathscr{A}_{[n]}(V)$ can be written uniquely as a sum of homogeneous (graded) components

$$u = v_{ns} \otimes u_s + v_{n(s+1)} \otimes u_{s+1} + \cdots + v_{nt} \otimes u_t,$$

where all $u_i \in V$, $u_s \neq 0$, $u_t \neq 0$. The number

$$l_{[n]}(u) \equiv l(u) := t - s$$

is called the $([n])$-*length of* u.

LEMMA 4.6. *Let* $[V] \in \hat{\mathcal{D}}(\mathrm{Cyc}_n)$. *Then for any nonzero* $\alpha \in K[H]$ *and any* $i \in \mathbf{Z}$ *the maps from* $\mathcal{A}(V)$ *to* $\mathcal{A}(V)$ *defined as* $\hat{\alpha}: u \to \alpha u$ *and* $\hat{v}_i: u \to v_i u$ *are bijective.*

Proof. The $\mathcal{D}$-module $\mathcal{A}(V)$ is semisimple and $\alpha \in Z(\mathcal{D})$. Now the bijectivity of $\hat{\alpha}$ follows from Lemma 4.5(2). For each natural number m,

$$Y^m X^m = \sigma^{-m+1}(a) \cdots \sigma^{-1}(a)\, a \qquad \text{and} \qquad X^m Y^m = \sigma^m(a) \cdots \sigma(a) \tag{4.11}$$

are elements of $K[H]$, so the second statement follows from the first. ∎

LEMMA 4.7. 1 (A Division Algorithm with Remainder). *Let* $[V] \in \hat{\mathcal{D}}(\mathrm{Cyc}_n)$ *and let* $0 \neq u$, $u' \in \mathcal{A}_{[n]}(V)$ *with* $l(u') \geqslant l(u)$. *Then there exist* $\alpha \in \mathcal{A}_{[n]}$ *and* $u'' \in \mathcal{A}_{[n]}(V)$ *such that* $u' = \alpha u + u''$ *and* $l(u'') < l(u)$.

2. *Any nonzero* $\mathcal{A}_{[n]}$*-submodule of* $\mathcal{A}_{[n]}(V)$ *is* 1*-generated and it is generated by any nonzero element of minimal length in the submodule.*

Proof. 1. Straightforward (using Lemma 4.6).

2. It follows from Statement 1. ∎

DEFINITION. Let $[V] \in \hat{\mathcal{D}}(\mathrm{Cyc}_n)$. An element $u \in \mathcal{A}_{[n]}(V)$ is called *V-irreducible* if $\mathcal{A}_{[n]} u$ is a maximal submodule of $\mathcal{A}_{[n]}(V)$ and has minimal length in $\mathcal{A}_{[n]} u$.

LEMMA 4.8. *Let* $[V] \in \hat{\mathcal{D}}(\mathrm{Cyc}_n)$ *and let* $u = \sum v_{in} \otimes u_i \in \mathcal{A}_{[n]}(V)$ *be a V-irreducible element. Then*

1. $\mathrm{ann}_{\mathcal{D}}\, u = \mathrm{ann}_{\mathcal{D}}\, (v_{in} \otimes u_i)$ *for each* $0 \neq u_i \in V$.
2. *The* $\mathcal{A}_{[n]}$*-module homomorphism*

$$\mathcal{A}_{[n]}(V) \to \mathcal{A}_{[n]}(V), \qquad \alpha(v_{in} \otimes u_i) \to \alpha u, \qquad \alpha \in \mathcal{A}_{[n]},$$

is a monomorphism for any $u_i \neq 0$.

3. *Let* $V = \mathcal{D}/\mathcal{D}f$ *for some irreducible polynomial* $f \in \mathcal{D}$ *and suppose that* $0 \neq v \in V$ *satisfies* $\sigma^n(f)\, v = 0$ (*it exists since* ${}^{\sigma^n}V \simeq V$). *Then the element* $u = Y^n \otimes v + 1 \otimes \bar{1} \in \mathcal{A}_{[n]}(V)$ *is V-irreducible, where* $\bar{1} = 1 + \mathcal{D}f \in V$.

Proof. 1. It follows immediately from Lemma 4.7.

2. It follows easily from Statement 1 and the decomposition $\mathcal{A}_{[n]}(V) = \bigoplus_{i \in \mathbf{Z}} v_{in} \otimes V$.

3. Since $\mathrm{ann}_{\mathcal{D}}\, u = \mathcal{D}f$ is a maximal left ideal of $\mathcal{D}$ and $l_{[n]}(u) = 1$, the element u is V-irreducible. ∎

Remark. Lemma 4.8(3) shows that for each $[V] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$ the $\mathscr{A}_{[n]}$-module $\mathscr{A}_{[n]}(V)$ is not simple, so each V-irreducible element is nonzero.

Let $[V = \mathscr{D}/\mathscr{D}f] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$ for some irreducible polynomial $f \in \mathrm{Irr}\,\mathscr{D}$ and let $u \in \mathscr{A}_{[n]}(V)$ be a V-irreducibe element. Then the $\mathscr{A}_{[n]}$-module $\mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u$ is simple. Set

$$M(V, u) := \mathscr{A} \bigotimes_{\mathscr{A}_{[n]}} \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u. \tag{4.12}$$

By Lemma 4.6 there is the decomposition

$$M(V, u) = \bigoplus_{i=0}^{n-1} X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u = \bigoplus_{i=0}^{n-1} Y^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u, \tag{4.13}$$

where each component $M(V, u)_{\mathbf{i}} = X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u$ is a simple $\mathscr{A}_{[n]}$-module and it is the isotypic component of the semisimple $\mathscr{D}$-module $M(V, u)$ which corresponds to $[{}^{\sigma^{-i}}V]$. Lemma 4.6 and (4.11) show that the action of $0 \neq \alpha \in K[H]$ and v_i, $i \in \mathbf{Z}$, on $M(V, u)$ provides the bijective maps $\hat{\alpha}, \hat{v}_i \colon M(V, u) \to M(V, u)$. Now it is clear that

$$[M(V, u)] \in \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}, \mathrm{Cyc}n_n).$$

On the other hand, let $[M] \in \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}, \mathrm{Cyc}n_n)$. Then M is an epimorphic image of an $\mathscr{A}$-module $\mathscr{A}(V)$ and it follows from its semisimplicity as a $\mathscr{D}$-module that

$$M = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} M_{\mathbf{i}} \tag{4.14}$$

is a $\mathbf{Z}_n$-graded module, each component $M_{\mathbf{i}}$ is a nonzero simple $\mathscr{A}_{[n]}$-module. Thus M is an epimorphic image of some $M(V, u)$, since the last is a simple $\mathscr{A}$-module, $M \simeq M(V, u)$. So, we proved that

- $[M] \in \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion}, \mathrm{Cyc}n_n) \Leftrightarrow M$ is isomorphic to some $M(V, u)$ as above.

In view of Lemma 4.6 and (4.11), the k-linear map

$$1 \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]} \to X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}, \qquad 1 \otimes u \to X^i \otimes u,$$

is an isomorphism of left k-vector spaces with inverse $X^i \otimes u \to ((-i, i)^{-1})^{\wedge}\, Y^i(X^i \otimes u)$, thus all components in (4.13) have the same left k-dimension

$$\dim_k X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]} = l_{[n]}(u) \dim_k V = l_{[n]}(u) \deg f. \tag{4.15}$$

The map

$$\mathscr{A}_{[n]}(V=\mathscr{D}/\mathscr{D}f)=\bigoplus_{i\in\mathbf{Z}} v_{ni}\otimes\mathscr{D}/\mathscr{D}f\to\mathscr{A}_{[n]}/\mathscr{A}_{[n]}f,$$
$$v_{ni}\otimes(d+\mathscr{D}f)\to v_{ni}d+\mathscr{A}_{[n]}f,\qquad d\in\mathscr{D},\tag{4.16}$$

is a canonical isomorphism of $\mathscr{A}_{[n]}$-modules. We will identify the above modules via this isomorphism. Let $\tilde{u}$ be any element of $\mathscr{A}_{[n]}$ which maps onto u under the natural epimorphism

$$\mathscr{A}_{[n]}\to\mathscr{A}_{[n]}/\mathscr{A}_{[n]}f,\qquad x\to x+\mathscr{A}_{[n]}f.$$

Now, from composition of the natural isomorphisms of $\mathscr{A}_{[n]}$-modules

$$\mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u\simeq(\mathscr{A}_{[n]}/\mathscr{A}_{[n]}f)/(\mathscr{A}_{[n]}(f,\tilde{u})/\mathscr{A}_{[n]}f)\simeq\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f,\tilde{u}),$$

we obtain a natural isomorphism of $\mathscr{A}_{[n]}$-modules:

$$\begin{aligned}\mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u&\to\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f,\tilde{u}),\\ v_{ni}\otimes(d+\mathscr{D}f)+\mathscr{A}_{[n]}u&\to v_{ni}d+\mathscr{A}_{[n]}(f,\tilde{u}).\end{aligned}\tag{4.17}$$

We identify these $\mathscr{A}_{[n]}$-modules by (4.17).

$$M(V,u)=\bigoplus_{i=0}^{n-1}X^i\otimes\mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u=\bigoplus_{i=0}^{n-1}X^i\otimes\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f,\tilde{u}).\tag{4.18}$$

Using Lemma 4.6, we may always assume that the element u from $\mathscr{A}_{[n]}(V)$ (resp., $\tilde{u}$ from $\mathscr{A}_{[n]}$) can be chosen from $\sum_{i\geqslant 0}X^{ni}\otimes V$ (resp., from $\mathscr{D}[X^n;\sigma^n]:=\sum_{i\geqslant 0}\mathscr{D}X^{ni}$). The k-automorphism σ of the ring $\mathscr{D}$ can be lifted to a k-automorphism of the skew polynomial ring $\mathscr{D}[X^n;\sigma^n]$ by setting $\sigma(X^n)=X^n$.

Let us show that the $\mathscr{A}_{[n]}$-module epimorphism

$$\begin{aligned}\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(\sigma^i(f),\sigma^i(\tilde{u}))&\to X^i\otimes\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f,\tilde{u}),\\ \overline{1}=1+\mathscr{A}_{[n]}(\sigma^i(f),\sigma^i(\tilde{u}))&\to X^i\otimes\overline{1},\end{aligned}\tag{4.19}$$

is an isomorphism, *where* $\overline{1}=1+\mathscr{A}_{[n]}(f,\tilde{u})$.

Since

$$\operatorname{ann}_{\mathscr{A}_{[n]}}(X^i\otimes\overline{1})\supseteq\mathscr{A}_{[n]}(\sigma^i(f),\sigma^i(\tilde{u}))$$

the map is well-defined.

The automorphism σ of $\mathscr{D}$ is k-linear, so the map (4.19) is k-linear and, under the identification (4.16), the length $l_{[n]}(\sigma^i(u))$ of the element

$\sigma^i(u) := \sigma^i(\tilde{u}) + \mathscr{A}_{[n]}\sigma^i(f) \in \mathscr{A}_{[n]}/\mathscr{A}_{[n]}\sigma^i(f)$ is equal to the length $l_{[n]}(u)$, and $\deg \sigma^i(f) = \deg f$. By (4.15) the dimensions of the left k-vector spaces in (4.19) coincide,

$$\begin{aligned}\dim_k \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(\sigma^i(f), \sigma^i(\tilde{u}))\\ &= l_{[n]}(\sigma^i(u)) \deg \sigma^i(f) = l_{[n]}(u) \deg f\\ &= \dim_k X^i \otimes \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u}),\end{aligned} \tag{4.20}$$

consequently the map (4.19) is bijective. Let us show that the $\mathscr{A}$-module epimorphism

$$\begin{aligned}\mathscr{A}/\mathscr{A}(f, \tilde{u}) \to M(V, u) = \bigoplus_{i=0}^{n-1} X^i \otimes \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u}),\\ \overline{1} = 1 + \mathscr{A}(f, \tilde{u}) \to 1 \otimes (1 + \mathscr{A}_{[n]}(f, \tilde{u})),\end{aligned} \tag{4.21}$$

is an *isomorphism*. Set $L = \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u})$. It follows from $v_{ni}L = L$, $i \in \mathbf{Z}$, and $\mathscr{A} = \sum_{-n+1 \leqslant i \leqslant n-1} v_i \mathscr{A}_{[n]}$ that

$$\mathscr{A}/\mathscr{A}(f, \tilde{u}) = \sum_{-n+1 \leqslant i \leqslant n-1} v_i L = \sum_{0 \leqslant i \leqslant n-1} X^i L,$$

hence,

$$\dim_k \mathscr{A}/\mathscr{A}(f, \tilde{u}) \leqslant n \dim_k L = \dim_k M(V, u),$$

and (4.21) is an isomorphism.

Denote by $\hat{\mathscr{A}}$ (k-fin.dim.) the set of isoclasses of simple $\mathscr{A}$-modules which are finite-dimensional left k-vector spaces. Any module from $\hat{\mathscr{A}}$ ($\mathscr{D}$-torsionfree) contains a free $\mathscr{D}$-submodule of rank 1 and $\dim_k \mathscr{D} = \infty$. So, by (4.3), (4.20), and by Proposition 4.2, we have

$$\hat{\mathscr{A}}\,(k\text{-fin.dim.}) = \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, cyclic}). \tag{4.22}$$

THEOREM 4.9. 1. $[M] \in \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, Cyc}n_n) \Leftrightarrow M \simeq M(V, u)$ *for some* $[V] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$ *and some V-irreducible element* $u \in \sum_{i \geqslant 0} X^{ni} \otimes V$;

$$\dim_k M(V, u) = n\, l_{[n]}(u) \dim_k V < \infty.$$

2. *Two modules from* $\hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, Cyc}n_n)$ *are isomorphic,* $M(V, u) \simeq M(V', u')$, *iff the* $\mathscr{A}_{[n]}$*-modules* $\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u})$ *and* $\mathscr{A}_{[n]}/\mathscr{A}_{[n]}(\sigma^i(f'), \sigma^i(\tilde{u}'))$ *are isomorphic for some* $i \in \{0, ..., n-1\}$. *In particular,* $\mathscr{O}(V) = \mathscr{O}(V')$.

Proof. The first statement has been proved above. It is evident that the isoclass $[M(V, u)]$ is uniquely determined by the isoclass of any of its $\mathbf{Z}_n$-graded components $[X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u]$. Now, the statement follows from (4.18) and (4.19). ∎

We identify the ring $\mathscr{A}_{[n]}$ with its image in the localization $B_{[n]} = S^{-1}\mathscr{A}_{[n]} = k \otimes K(H)[X^n, X^{-n}; \sigma^n]$ or, more precisely, via the map

$$X^n \to X^n, \qquad Y^n \to (-n, n)\, X^{-n}, \qquad \mathscr{D} \ni d \to d. \tag{4.23}$$

The ring $B = B_{[1]}$ admits an inner automorphism

$$\omega\colon b \to XbX^{-1}, \qquad b \in B,$$

which preserves all subrings $B_{[n]}$ ($\omega(B_{[n]}) = B_{[n]}$) but, in general, does not leave $\mathscr{A}_{[n]}$ invariant, as can be seen from:

$$\omega(\mathscr{A}_{[n]}) = \mathscr{D}\langle X^n, \sigma((-n, n))\, X^{-n}\rangle.$$

Denote by $\hat{\mathscr{A}}_{[n]}(\mathscr{D}\text{-torsion}, \mathrm{Cyc}n_n)$ the set of isoclasses of the simple $\hat{\mathscr{A}}_{[n]}$-modules

$$L = \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u = \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u}) \tag{4.24}$$

as in Theorem 4.9. It follows from Lemma 4.6 that

- any $\mathscr{A}_{[n]}$-module L is also a simple $B_{[n]}$-module,

$$L = \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(f, \tilde{u}) = B_{[n]}/B_{[n]}(f, \tilde{u}). \tag{4.24.a}$$

- Two such $\mathscr{A}_{[n]}$-modules are isomorphic iff they are isomorphic as $B_{[n]}$-modules.

Hence, for any $i \in \mathbf{Z}$, the twisted $B_{[n]}$-module ${}^{\omega^i}L$ is simple. Moreover, it follows from (4.23) that ${}^{\omega^i}L$ is a simple $\mathscr{A}_{[n]}$-module and, as can be easily deduced from (4.18) that

$${}^{\omega^{-i}}L \simeq \mathscr{A}_{[n]}/\mathscr{A}_{[n]}(\sigma^i(f), \sigma^i(\tilde{u})), \tag{4.24.b}$$

the ith component of the $\mathscr{A}$-module $M(V = \mathscr{D}/\mathscr{D}f, u)$. By Theorem 4.9, ${}^{\omega^n}L \simeq L$ as $\mathscr{A}_{[n]}$-modules. Thus we have an action of the group $\mathbf{Z}_n \simeq \langle\omega\rangle/\langle\omega^n\rangle$ on the set $\hat{\mathscr{A}}_{[n]}(\mathscr{D}\text{-torsion}, \mathrm{Cyc}n_n)$ defined by twisting. The orbit $\mathscr{O}([L])$ contains precisely n distinct elements $[L], ..., [{}^{\omega^{n-1}}L]$

and it coincides with the $\hat{\mathscr{A}}_{[n]}$-support $\mathrm{Supp}_{\mathscr{A}_{[n]}} M(V,u)$ of the $\mathscr{A}$-module $M(V,u)$ (which is a semisimple $\mathscr{A}_{[n]}$-module).

COROLLARY 4.10. *The map*

$$\hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, Cyc}n_n) \to \hat{\mathscr{A}}_{[n]}(\mathscr{D}\text{-torsion, Cyc}n_n)/\mathbf{Z}_n,$$

$$[M] \to \mathrm{Supp}_{\mathscr{A}_{[n]}} M$$

is bijective with inverse, $\mathscr{O}([L]) \to [\mathscr{A} \otimes_{\mathscr{A}_{[n]}} L]$.

LEMMA 4.11. *Let Λ be an affine subalgebra of an affine algebra A such that A is a finitely generated left Λ-module. Let M be a finitely generated A-module. Then* $\mathrm{GK}({}_A M) = \mathrm{GK}({}_\Lambda M)$.

Proof. Let $A = \Lambda W$ for some finite dimensional subspace W of A. Choose finite-dimensional subspaces of algebra generators $A_1 \ni 1$ and $\Lambda_1 \ni 1$ of the algebras A and Λ respectively, such that $A_1 \supseteq W$, $A_1 \supseteq \Lambda_1$. Then $A_1 A_1 \subseteq \Lambda_1^c W$ for some natural number c and, by induction,

$$A_1^n \subseteq \Lambda_1^{(n-1)c} W \qquad \text{for all} \quad n \geqslant 2.$$

$(A_1^n = A_1^{n-1} A_1 \subseteq \Lambda_1^{(n-2)c} W A_1 \subseteq \Lambda_1^{(n-2)c} A_1 A_1 \subseteq \Lambda_1^{(n-1)c} W)$. The A-module M is finitely generated, so we may consider a finite-dimensional generating subspace M_0 of ${}_A M$. The A-module M has the standard filtration $M = AM_0 = \bigcup_{i=0}^{\infty} A_1^i M_0$. It follows from $M = AM_0 = \Lambda W M_0$ that the Λ-module M is finitely generated and is equipped with the standard filtration $M = \bigcup_{i=0}^{\infty} \Lambda_1^i W M_0$. For $i \geqslant 1$,

$$\Lambda_1^i W M_0 \subseteq A_1^{i+1} M_0 \subseteq \Lambda_1^{ic} W M_0,$$

hence, $\mathrm{GK}({}_A M) = \mathrm{GK}({}_\Lambda M)$. ∎

For $n \geqslant 1$, the GWA A contains the subrings $A_{[n]}$, $n \geqslant 1$, generated by $K[H]$, v_n, and v_{-n}. The algebra $A_{[n]}$ is a GWA of degree 1,

$$A_{[n]} = K[H](\sigma^n, (-n, n) = \sigma^{-n+1}(a) \cdots a),$$

which is isomorphic to GWA from Example 1 (for set $h = H/n$, then $\sigma^n(h) = h - 1$). It is a central simple Noetherian affine algebra which is an integral domain. The algebra A is a left and right finitely generated $A_{[n]}$-module:

$$A = \sum_{-n+1 \leqslant i \leqslant n-1} A_{[n]} v_i = \sum_{-n+1 \leqslant i \leqslant n-1} v_i A_{[n]}. \tag{4.25}$$

Then $\Lambda = C \otimes A$ is a left and right finitely generated, module over the subalgebra

$$\Lambda_{[n]} := C \otimes A_{[n]}.$$

Note that algebras $\Lambda_{[n]}$ are of the type Λ.

THEOREM 4.12. *The map*

$$\hat{\Lambda}(\text{holonomic}) \backslash \hat{C} \otimes \hat{A} \to \bigsqcup_{n=1}^{\infty} \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, Cyc}n_n), \quad [N] \to [\mathscr{A} \bigotimes_{\Lambda} N],$$

is bijective with inverse $[M] \to [\mathrm{Soc}_\Lambda M]$.

Proof. If M is a simple holonomic Λ-module, then $\mathscr{A} \otimes_\Lambda M$ is a simple $\mathscr{D}$-torsion $\mathscr{A}$-module. Since otherwise, the module M contains a free $C \otimes K[H]$-submodule of rank 1, so

$$2 = \mathrm{GK}(M) \geqslant \mathrm{GK}(C \otimes K[H]) = 3,$$

a contradiction.

In view of Proposition 4.3 the only fact we have to prove is that each $[M] \in \hat{\mathscr{A}}\,(\mathscr{D}\text{-torsion, Cyc}n_n)$ has nonzero socle $\mathrm{Soc}_\Lambda M$ (then it is automatically a nonzero simple Λ-module). It follows from the $\mathbf{Z}_n$-graded structure of the $\mathscr{A}$-module M, (4.14) or (4.13), and from the simplicity of $\mathrm{Soc}_\Lambda M$ that

$$\mathrm{Soc}_\Lambda M = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} (\mathrm{Soc}_\Lambda M)_{\mathbf{i}}, \qquad \text{where} \qquad (\mathrm{Soc}_\Lambda M)_{\mathbf{i}} = \mathrm{Soc}_\Lambda M \cap M_{\mathbf{i}},$$

and each $(\mathrm{Soc}_\Lambda M)_{\mathbf{i}}$ is a simple $\Lambda_{[n]}$-module; moreover,

$$(\mathrm{Soc}_\Lambda M)_{\mathbf{i}} = \mathrm{Soc}_{\Lambda_{[n]}}(M_{\mathbf{i}}) \qquad \text{and} \qquad M_{\mathbf{i}} = (k \otimes A_{[n]}) \bigotimes_{\Lambda_{[n]}} \mathrm{Soc}_{\Lambda_{[n]}}(M_{\mathbf{i}}).$$

By Lemma 4.6, $\mathrm{Soc}_\Lambda M \neq 0$ iff some (all) $(\mathrm{Soc}_\Lambda M)_{\mathbf{i}} \neq 0$. If $(\mathrm{Soc}_\Lambda M)_{\mathbf{i}} \neq 0$, then it is the unique simple $\Lambda_{[n]}$-submodule of $M_{\mathbf{i}}$ which is contained in any nonzero submodule of $M_{\mathbf{i}}$. Thus it remains to show that each component $M_{\mathbf{i}} = X^i \otimes \mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u$, $i = 0, ..., n-1$, from (4.18), contains a simple $\Lambda_{[n]}$-submodule.

The *idea of the proof* is to find in $M_{\mathbf{i}}$ a holonomic $\Lambda_{[n]}$-submodule. But any holonomic $\Lambda_{[n]}$-module, by Corollary 2.3(1), has finite length, so it contains a simple $\Lambda_{[n]}$-submodule. We prove the statement in the case $\mathbf{i} = \mathbf{0}$; in the remaining cases arguments may be repeated word for word.

So, let $\mathbf{i}=\mathbf{0}$ and $M_{\mathbf{0}}=\mathscr{A}_{[n]}(V)/\mathscr{A}_{[n]}u$, where $V=\mathscr{D}/\mathscr{D}f$ for some irreducible polynomial $f\in k[H]$ (which can be chosen from $C\otimes K[H]$) and for some V-irreducible element

$$u=\sum_{i\geqslant 0} X^{ni}\otimes u_i, \qquad u_0=\overline{1},$$

such that all $cu_i\in C\otimes K[H]\,\overline{1}\subseteq\mathscr{D}/\mathscr{D}f$, for some $c\in C_*$ where $\overline{1}=1+\mathscr{D}f$. Consider a $\Lambda_{[n]}$-submodule

$$L:=\Lambda_{[n]}(1\otimes\overline{1})\simeq\Lambda_{[n]}\tilde{1}$$

of $\mathscr{A}_{[n]}(V)\simeq\mathscr{A}_{[n]}/\mathscr{A}_{[n]}f$ (by (4.16)), where $1\otimes\overline{1}$ and $\tilde{1}=1+\mathscr{A}_{[n]}f$ are generators of the $\mathscr{A}_{[n]}$-modules $\mathscr{A}_{[n]}(V)$ and $\mathscr{A}_{[n]}/\mathscr{A}_{[n]}f$, respectively. The ring $\Lambda_{[n]}$ is an integral domain and $0\neq df\in\Lambda_{[n]}$, for some $d\in C_*$, so the map

$$\Lambda_{[n]}\to\Lambda_{[n]}, \qquad x\to x\,df,$$

is a $\Lambda_{[n]}$-module monomorphism. By Corollary 2.3(2),

$$\mathrm{GK}(\Lambda_{[n]}/\Lambda_{[n]}\,df)\leqslant\mathrm{GK}(\Lambda_{[n]})-1=4-1=3. \tag{4.26}$$

The $\Lambda_{[n]}$-module L is an epimorphic image of $\Lambda_{[n]}/\Lambda_{[n]}\,df$ ($\Lambda_{[n]}/\Lambda_{[n]}\,df\to L$, $1+\Lambda_{[n]}\,df\to\tilde{1}$), hence

$$3\leqslant\mathrm{GK}(L)\leqslant\mathrm{GK}(\Lambda_{[n]}/\Lambda_{[n]}\,df)\leqslant 3. \tag{4.27}$$

Lemma 4.8(2) provides the $\mathscr{A}_{[n]}$-module monomorphism

$$\mathscr{A}_{[n]}(V)\to\mathscr{A}_{[n]}(V), \qquad 1\otimes\overline{1}\to u. \tag{4.28}$$

For every $s\in C_*$, $\mathscr{D}$-modules $\mathscr{D}/\mathscr{D}f$ and $\mathscr{D}/\mathscr{D}fs^{-1}$ are isomorphic, so the $\Lambda_{[n]}$-submodule L_s of L generated by $s\cdot 1\otimes\overline{1}$ is isomorphic to $\Lambda_{[n]}/\Lambda_{[n]}\wedge\mathscr{A}_{[n]}fs^{-1}$, so $GK(L_s)=GK(L)=3$. Fix s such that L_s has minimal possible multiplicity, say e. Changing (if necessary) f for fs^{-1}, we can assume that all submodules $L_b, b\in C_*$, have the multiplicity e. Now all $\Lambda_{[n]}$-modules

$$L\supseteq L\cap\mathscr{A}_{[n]}u\supseteq L\wedge\tilde{L}\supseteq\varphi(L_c)=\Lambda_{[n]}cu$$

have Gelfand–Kirillov dimension 3 and multiplicity e. The restriction of the map above to L defines a $\Lambda_{[n]}$-module isomorphism

$$\varphi\colon L\to\tilde{L}:=\Lambda_{[n]}u, \qquad 1\otimes\overline{1}\to u.$$

Applying Lemma 2.2, we may conclude that

$$\mathrm{GK}(P:=L/\Lambda_{[n]}cu)\leqslant\mathrm{GK}(L)-1\leqslant 3-1=2. \tag{4.29}$$

The $\Lambda_{[n]}$-submodule

$$Q := \Lambda_{[n]}(1 \otimes \bar{1})/\Lambda_{[n]}(1 \otimes \bar{1}) \cap \mathscr{A}_{[n]} u$$

of $M_{\mathbf{0}}$ is nonzero, since $\mathscr{A}_{[n]} \otimes_{\Lambda_{[n]}} Q \simeq M_{\mathbf{0}}$. It follows from

$$\Lambda_{[n]} cu \subseteq \Lambda_{[n]}(1 \otimes \bar{1}) \cap \mathscr{A}_{[n]} u$$

that we have a natural epimorphism of $\Lambda_{[n]}$-modules:

$$\begin{aligned} P = \Lambda_{[n]}(1 \otimes \bar{1})/\Lambda_{[n]} cu &\to Q, \\ 1 \otimes \bar{1} + \Lambda_{[n]} cu &\to 1 \otimes \bar{1} + \Lambda_{[n]}(1 \otimes \bar{1}) \cap \mathscr{A}_{[n]} u. \end{aligned}$$

By (4.29), $\mathrm{GK}(Q) \leqslant \mathrm{GK}(P) \leqslant 2$, and, by Theorem 2.1(4), $\mathrm{GK}(Q) = 2$, i.e., Q is a holonomic $\Lambda_{[n]}$-module. Hence, by Corollary 2.3(1), it contains a simple $\Lambda_{[n]}$-submodule. ∎

Definition. The element $f \in \mathscr{D}$ is called n-minimal if $[\mathscr{D}/\mathscr{D}f] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$ and all modules $\Lambda_{[n]}/\Lambda_{[n]} \wedge \mathscr{A}_{[n]} fs^{-1}$, $s \in C_*$, have the same multiplicity.

It was proved above that, for every $f \in \mathscr{D}$ such that $[\mathscr{D}/\mathscr{D}f] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$, there exists $s \in C_*$ such that fs^{-1} is n-minimal. Clearly, if f is n-minimal then so is fs^{-1} for every $s \in C_*$.

Corollary 4.13. *Let M be a nonzero simple Λ-module and let $\tilde{M} = \mathscr{A} \otimes_\Lambda M$. Then*

1. $\tilde{M} = 0 \Leftrightarrow [M] \in \hat{C} \otimes \hat{A}$;
2. $1 \leqslant \dim_k \tilde{M} < \infty \Leftrightarrow [M] \in \hat{\Lambda}(\text{holonomic}) \backslash \hat{C} \otimes \hat{A}$;
3. $\dim_k \tilde{M} = \infty \Leftrightarrow [M] \in \hat{\Lambda}(\text{non-holonomic})$.

Hence, M is holonomic (resp., non-holonomic) iff $\dim_k \tilde{M} < \infty$ (*resp., M contains a free $C \otimes K[H]$-module of rank* 1).

Corollary 4.14. *Let $[V = \mathscr{D}/\mathscr{D}f] \in \hat{\mathscr{D}}(\mathrm{Cyc}_n)$ for some $f \in C \otimes K[H]$ that is irreducible in $\mathscr{D}$ and let $\tilde{u} \in \Lambda_{[n]}$ be such that $u = \tilde{u} + \mathscr{A}_{[n]} f$ is a V-irreducible element. Then*

1. *the $\Lambda_{[n]}$-modules $P = \Lambda_{[n]}/(\Lambda_{[n]} \cap \mathscr{A}_{[n]} f + \Lambda_{[n]} \tilde{u})$ and $Q = \Lambda_{[n]}/\Lambda_{[n]} \cap \mathscr{A}_{[n]}(f, \tilde{u})$ are holonomic provided f is n-minimal*;
2. *the Λ-module*

$$M(f, \tilde{u}) := \Lambda/\Lambda \cap \mathscr{A}(f, \tilde{u})$$

is holonomic provided $\sigma^i(f)$ is n-minimal for $-n+1 \leqslant i \leqslant n-1$ and $\mathrm{Soc}_\Lambda M(f, \tilde{u}) \simeq \mathrm{Soc}_\Lambda M(V, u)$;

3. *let J be a left ideal of Λ containing $\Lambda \cap \mathscr{A}(f, \tilde{u})$ and $J/\Lambda \cap \mathscr{A}(f, \tilde{u}) = \mathrm{Soc}_\Lambda\, M(f, \tilde{u})$. Then the left ideal $\mathbf{a} = J \cap C$ of C is nonzero and for any nonzero element $c \in \mathbf{a}$*

$$\mathrm{Soc}_\Lambda\, M(f, \tilde{u}) \simeq \Lambda/\Lambda \cap \mathscr{A}(f, \tilde{u})\, c^{-1},$$

and hence

$$[\Lambda/\Lambda \cap \mathscr{A}(f, \tilde{u})\, c^{-1}] \in \hat{\Lambda}(\text{holonomic}) \backslash \hat{C} \otimes \hat{A}.$$

Consequently, any element of $\hat{\Lambda}(\text{holonomic}) \backslash \hat{C} \otimes \hat{A}$ is an isoclass of some Λ-module $\Lambda/\Lambda \cap \mathscr{A}(f, \tilde{u})\, c^{-1}$ (for some f, $\tilde{u}$, c, and n as above).

Proof. 1. The $\Lambda_{[n]}$-modules P and Q above are isomorphic to the holonomic $\Lambda_{[n]}$-modules P and Q from the proof of Theorem 4.12, respectively.

2. The element f is homogeneous with respect to the $\mathbf{Z}_n$-grading of the rings Λ and $\mathscr{A}$. Without loss of generality we may assume that the element $\tilde{u}$ is also homogeneous, so the left ideals $\mathscr{A}(f, \tilde{u})$ and $\Lambda(f, \tilde{u})$ are homogeneous. By (4.25), the Λ-module

$$M(f, \tilde{u}) = \bigoplus_{\mathbf{i} \in \mathbf{Z}_n} M_{\mathbf{i}}$$

is $\mathbf{Z}_n$-graded, where

$$\begin{aligned} M_{\mathbf{i}} = {} & (\Lambda_{[n]} X^i + \Lambda_{[n]} Y^{n-i})/(\Lambda_{[n]} X^i + \Lambda_{[n]} Y^{n-i}) \\ & \cap (\mathscr{A}_{[n]} X^i + \mathscr{A}_{[n]} Y^{n-i})(f, \tilde{u}). \end{aligned}$$

The $\Lambda_{[n]}$-module $M_{\mathbf{i}}$ is a natural epimorphic image of the direct sum of two $\Lambda_{[n]}$-modules

$$L_{\mathbf{i}} = \Lambda_{[n]} X^i/\Lambda_{[n]} X^i \cap \mathscr{A}_{[n]} X^i(f, \tilde{u})$$

and

$$N_{\mathbf{i}} = \Lambda_{[n]} Y^{n-i}/\Lambda_{[n]} Y^{n-i} \cap \mathscr{A}_{[n]} Y^{n-i}(f, \tilde{u}).$$

Let us show that these modules are holonomic. Now we use the freedom in the choice of the V-irreducible element u: for any $0 \neq \alpha \in K[H]$ and any $i \in \mathbf{Z}$, the element $\alpha v_i u$ is also V-irreducible and $\mathscr{A}_{[n]}(f, \tilde{u}) = \mathscr{A}_{[n]}(f, \alpha v_i \tilde{u})$. In the expression of $L_{\mathbf{i}}$ (resp. $N_{\mathbf{i}}$) we may suppose that the element $\tilde{u}$ is from $\sum C \otimes K[H]\, v_i$, where i runs through $i \geqslant 0$ (resp. $i \leqslant 0$) because of

$$\mathscr{A}_{[n]} X^i + \mathscr{A}_{[n]} Y^{n-i} = X^i \mathscr{A}_{[n]} + Y^{n-i} \mathscr{A}_{[n]}.$$

The ring $\Lambda_{[n]}$ is an integral domain, so the $\Lambda_{[n]}$-module

$$\begin{aligned} L_{\mathbf{i}} & = \Lambda_{[n]} X^i/\Lambda_{[n]} X^i \cap \mathscr{A}_{[n]}(\sigma^i(f), \sigma^i(\tilde{u}))\, X^i \\ & \simeq \Lambda_{[n]}/\Lambda_{[n]} \cap \mathscr{A}_{[n]}(\sigma^i(f), \sigma^i(\tilde{u})) \end{aligned}$$

is holonomic by Statement 1. The same argument (or symmetric version) shows that $N_{\mathbf{i}}$ is a holonomic $\Lambda_{[n]}$-module.

Now, each component $M_{\mathbf{i}}$ of M is a holonomic $\Lambda_{[n]}$-module, hence M is a holonomic $\Lambda_{[n]}$-module. The affine algebra Λ is a finitely generated left module over the affine subalgebra $\Lambda_{[n]}$, thus, by Lemma 4.11, the Λ-module M is holonomic.

Since $\mathscr{A} \otimes_{\Lambda} M(f, \tilde{u}) \simeq \mathscr{A}/\mathscr{A}(f, \tilde{u}) \simeq M(V, u)$ (by (4.21)), we conclude that $\mathrm{Soc}_{\Lambda} M(f, \tilde{u}) \simeq \mathrm{Soc}_{\Lambda} M(V, u)$.

3. Immediates from Lemma 3.2. ∎

Let $f = \lambda_m H^m + \lambda_{m+1} H^{m+1} + \cdots + \lambda_n H^n \in K[H, H^{-1}]$, all $\lambda_i \in k$ and $\lambda_m, \lambda_n \neq 0$. The *length* $l_H(f)$ of f is defined as $l_H(f) = n - m$.

THEOREM 4.15. *Let Λ/J be a simple holonomic Λ-module for a maximal left ideal J of Λ. Then the minimal number of generators of the left ideal J is* 2.

Proof. In view of Theorem 2.3(2) it remains to be established that J is not 1-generated. Suppose the contrary, that $J = \Lambda w$ for some $w \in \Lambda$.

If $[\Lambda/J] \in \hat{C} \otimes \hat{A}$, then $J = V \otimes A + C \otimes U$ for some nonzero maximal left ideals V of C and U of A. Choose nonzero elements $v \in V$ and $u \in U$. Then $v = gw$ for some $g \in \Lambda$, but considering this equality in $k \otimes A$ where v is a unit, we see that w is a unit, so $w \in C$. By symmetry, $w \in A$, hence $w \in C \cap A = K$, a contradiction.

Let $[\Lambda/J] \in \hat{\Lambda}(\text{holonomic}) \backslash \hat{C} \otimes \hat{A}$. Then the left ideal J contains nonzero elements $f \in C \otimes D$, where $D = K[H]$ (resp., $D = K[H, H^{-1}]$) and $\tilde{u} = \sum_{i \geqslant 0} X^{ni} \tilde{u}_i$, where $\tilde{u}_i \in C \otimes D$ and $\deg_H \tilde{u}_i < \deg_H f$ (resp., $l_H(\tilde{u}_i) < l_H(f)$). Recall that f is an irreducible polynomial in $\mathscr{D}$. Since $f \in \Lambda w$, there exists $g \in \Lambda$ such that

$$f = gw. \tag{4.30}$$

Considering this equality in $\mathscr{B} = k(H)[X, X^{-1}; \sigma]$, where f is a unit, we conclude that both $g, w \in C \otimes D \subseteq \mathscr{D}$. The element f is irreducible in $\mathscr{D}$, so, by (4.30), either $w \in C$ or $\deg_H w = \deg_H f$, if $D = K[H]$ (resp., $l_H(w) = l_H(f)$, if $D = K[H, H^{-1}]$). The first case, i.e., $w \in C$, is impossible, since $[\Lambda/J] \notin \hat{C} \otimes \hat{A}$.

The element $\tilde{u} \in \Lambda w$, so

$$\tilde{u} = hw$$

for some $h \in \Lambda$. Then $\deg_H \tilde{u}_i \geqslant \deg_H w = \deg_H f$ (resp., $l_H(\tilde{u}_i) \geqslant l_H(w) = l_H(f)$) for any nonzero $\tilde{u}_i$, a contradiction. ∎

THEOREM 4.16. *Let $\Lambda = \bigotimes_{i=1}^{n} \Lambda_i$ be a simple algebra, $n \geqslant 2$, and let $0 \neq x \in \Lambda$ be a non-unit. Then*

$$\mathrm{GK}(\Lambda/\Lambda x) = \mathrm{GK}(\Lambda) - 1 = 2n - 1.$$

Proof. By Corollary 2.3(2), $\mathrm{GK}(\Lambda/\Lambda x) \leqslant \mathrm{GK}(\Lambda) - 1 = 2n - 1$.
Using the $\mathbf{Z}^n$-grading of the algebra Λ, for every non-unit $0 \neq x \in \Lambda$, we can find a subalgebra $E(x)$ of Gelfand–Kirillov dimension $2n-1$ such that $E(x) \cap \Lambda x = 0$, hence $GK(\Lambda/\Lambda x) = 2n - 1$. ∎

COROLLARY 4.17. *Let $\Lambda = \bigotimes_{i=1}^{n} \Lambda_i$ be a simple algebra, $n \geqslant 2$.*

1. *Let J be a nonzero left (right) ideal of Λ such that $\mathrm{GK}(\Lambda/J) < 2n - 1$. Then the minimal number of generators of J as left (right) ideals is 2.*

2. *Let Λ/J be a holonomic Λ-module. Then the minimal number of generators of J as left (right) ideals is 2.*

Proof. 1. It follows immediately from Theorems 2.1(2) and 4.16 and [Bav 6].

2. It follows directly from 1. ∎

The algebra $\mathscr{B} = k(H)[X, X^{-1}; \sigma]$, defined by (4.2), is the localization $\mathscr{B} = S^{-1}\Lambda$ of Λ at $S := C \otimes K[H] \backslash \{0\}$. We identify the algebra Λ with its image in $\mathscr{B}$ under the map $\Lambda \to \mathscr{B}$, $\lambda \to \lambda/1$ (in particular, $X \to X$).

PROPOSITION 4.18. 1. *The canonical map*

$$S^{-1}\colon \hat{\Lambda}(\textit{non-holonomic}) \to \hat{\mathscr{B}}(\Lambda\textit{-socle}), \qquad [M] \to [S^{-1}M], \tag{4.31}$$

is a bijection with inverse Soc: $[N] \to [\mathrm{Soc}_\Lambda(N)]$.

2. *Each simple non-holonomic Λ-module has the form*

$$M_b := \Lambda/\Lambda \cap \mathscr{B}b \tag{4.32}$$

for some irreducible element b of the ring $\mathscr{B}$. Two such modules are isomorphic, $M_b \simeq M_c$, iff the $\mathscr{B}$-modules $\mathscr{B}/\mathscr{B}b$ and $\mathscr{B}/\mathscr{B}c$ are isomorphic.

Proof. By Corollary 4.13 the map (4.31) is well-defined. The rest follows from Lemma 3.1 and the fact that any left ideal of $\mathscr{B}$ is 1-generated. ∎

5. THE SIMPLE HOLONOMIC A_2-MODULES

We stick to the notation of the previous sections. Let K be an algebraically closed uncountable field of characteristic zero. The first Weyl algebra A_1 is a GWA (Example 1),

$$A_1 = K[H](\sigma, a = H), \qquad \sigma(H) = H - 1.$$

We denote by $k = (A_1)_*^{-1} A_1$ its full quotient ring, the Weyl skew field. The second Weyl algebra

$$A_2 = A_1 \otimes A_1$$

is the tensor product (over K) of two first Weyl algebras. The set of isoclasses of simple A_2-modules is partitioned as

$$\hat{A}_2 = \hat{A}_2(\text{holonomic}) \cup \hat{A}_2(\text{non-holonomic}),$$

i.e., those having Gelfand–Kirillov dimension 2 and 3, respectively (Corollary 2.3(3)). Each simple A_1-module has Gelfand–Kirillov dimension 1, so

$$\hat{A}_1 \otimes \hat{A}_1 \subseteq \hat{A}_2(\text{holonomic}).$$

Now, setting $\Lambda = A_2$, $A = C = A_1$, we introduce the localization $\mathscr{A}$ of A_2 at $(A_1)_*$. From Section 4 we recall that

$$\mathscr{A} = (A_1)_*^{-1} A_2 = k \otimes A_1 = A_1(k)$$

can be identified with the first Weyl algebra $A_1(k)$ with coefficients in the Weyl skew field. It is a GWA of degree 1,

$$A_1(k) = \mathscr{D}(\sigma, a = H), \qquad \mathscr{D} = k \otimes K[H] = k[H],$$

$$\sigma \in \operatorname{Aut}_k \mathscr{D}, \qquad \sigma(H) = H - 1.$$

PROPOSITION 5.1 (Corollary 4.13). *Let M be a nonzero simple A_2-module and let $\tilde{M} = A_1(k) \otimes_{A_2} M$. Then*

1. $\tilde{M} = 0$ (*i.e.*, $\dim_k \tilde{M} = 0$) $\Leftrightarrow [M] \in \hat{A}_1 \otimes \hat{A}_1$;
2. $1 \leqslant \dim_k \tilde{M} < \infty \Leftrightarrow [M] \in \hat{A}_2(\text{holonomic}) \backslash \hat{A}_1 \otimes \hat{A}_1$;
3. $\dim_k \tilde{M} = \infty \Leftrightarrow [M] \in \hat{A}_2(\text{non-holonomic})$.

Hence, M is holonomic (resp., non-holonomic) iff $\dim_k \tilde{M} < \infty$ (resp., M contains a free $A_1 \otimes K[H]$-module of rank 1).

The localization $\mathscr{B}$ of $A_1(k)$ at $\mathscr{D}_*$,

$$\mathscr{B} = \mathscr{D}_*^{-1} A_1(k) = k(H)[X, X^{-1}; \sigma], \qquad \sigma \in \mathrm{Aut}_k\, k(H), \qquad \sigma(H) = H - 1,$$

is the skew Laurent polynomial ring with coefficients in the skew field $k(H) = \mathscr{D}_*^{-1}\mathscr{D}$. Now

$$\hat{A}_1(k) = \hat{A}_1(k)(\mathscr{D}\text{-torsion}) \cup \hat{A}_1(k)(\mathscr{D}\text{-torsionfree}).$$

We aim to describe $\hat{A}_1(k)(\mathscr{D}\text{-torsion})$. The action of the group $G = \langle \sigma \rangle$ on this set yields the partition

$$\hat{A}_1(k)(\mathscr{D}\text{-torsion}) = \hat{A}_1(k)(\mathscr{D}\text{-torsion, linear}) \cup \hat{A}_1(k)(\mathscr{D}\text{-torsion, cyclic}).$$

The defining polynomial $a = H$ has the unique root 0 and $V_0 := \mathscr{D}/\mathscr{D}H \simeq k$ is the corresponding simple $\mathscr{D}$-module. So, there is the unique degenerate orbit $\mathscr{O}(V_0)$ with subsets

$$\Gamma^- = \{[{}^{\sigma^j}V_0],\ j \geqslant 0\} \qquad \text{and} \qquad \Gamma^+ = \{[{}^{\sigma^j}V_0],\ j < 0\}.$$

Two isoclasses from $\hat{\mathscr{D}}(\text{linear})$ are equivalent, $[V] \sim [W]$, if they both belong either to one of $\Gamma^{\pm}$ or to a non-degenerate orbit.

Proposition 5.2 (Proposition 4.2). *The map*

$$\hat{\mathscr{D}}(\text{linear})/\sim\ \to \hat{A}_1(k)(\mathscr{D}\text{-torsion, linear}), \qquad [\Gamma] \to [L(\Gamma)],$$

is bijective with inverse $[M] \to \mathrm{Supp}\, M$, *where*

1. *if* $\Gamma = \mathscr{O}(V)$ *is a non-degenerate orbit, then* $L(\Gamma) = A_1(k)(V) := A_1(k) \otimes_{\mathscr{D}} V$;
2. *if* $\Gamma = \Gamma^-$, *then* $L(\Gamma) = A_1(k)(V_0)/\sum_{j>0} v_j \otimes V_0$;
3. *if* $\Gamma = \Gamma^+$, *then* $L(\Gamma) = A_1(k)({}^{\sigma^{-1}}V_0)/\sum_{j<0} v_j \otimes {}^{\sigma^{-1}}V_0$.

All modules $L(\Gamma)$ *are infinite-dimensional left* k*-vector spaces.*

As we have seen,

$$\hat{A}_1(k)(\mathscr{D}\text{-torsion, linear}) = \hat{A}_1(k)(\mathscr{D}\text{-torsion, } k\text{-inf.dim.}),$$

and

$$\hat{A}_1(k)(\mathscr{D}\text{-torsion, cyclic}) = \hat{A}_1(k)(k\text{-fin.dim.}).$$

Moreover,

$$\hat{A}_1(k)(\mathscr{D}\text{-torsion, cyclic}) = \bigsqcup_{n=1}^{\infty} \hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n).$$

The subalgebra ($n \geqslant 1$),

$$\begin{aligned} A_1(k)_{[n]} &= k[H](\sigma^n, (-n, n) = \sigma^{-n+1}(H) \cdots H \\ &= (H+n-1)(H+n-2) \cdots H), \qquad \sigma^n(H) = H-n, \end{aligned}$$

is a GWA of degree 1.

PROPOSITION 5.3 (Theorem 4.9). 1.

$$\begin{aligned} &[M] \in \hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n) \\ &\quad \Leftrightarrow M \simeq M(V, u) := A_1(k) \bigotimes_{A_1(k)_{[n]}} A_1(k)_{[n]}(V)/A_1(k)_{[n]} u \\ &\quad \simeq A_1(k)/A_1(k)(f, \tilde{u}), \end{aligned}$$

for some $[V = \mathscr{D}/\mathscr{D}f] \in \hat{\mathscr{D}}(\text{Cyc}n_n)$ *and some* V*-irreducible element* $u \in \sum_{i \geqslant 0} X^{ni} \otimes V$;

$$\dim_k M(V, u) = nl_{[n]}(u) \dim_k V = nl_{[n]}(u) \deg_H f < \infty.$$

2. *Two modules from* $\hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n)$ *are isomorphic,* $M(V, u) \simeq M(V', u')$, *iff the* $A_1(k)_{[n]}$*-modules* $A_1(k)_{[n]}/A_1(k)_{[n]}(f, \tilde{u})$ *and* $A_1(k)_{[n]}/A_1(k)_{[n]}(\sigma^i(f'), \sigma^i(\tilde{u}'))$ *are isomorphic for some* $i \in \{0, ..., n-1\}$. *In particular,* $\mathscr{O}(V) = \mathscr{O}(V')$.

The next theorem describes the simple holonomic A_2-modules which are not the tensor product of two simple A_1-modules.

THEOREM 5.4 (Theorem 4.12). *The map*

$$\hat{A}_2(\text{holonomic}) \backslash \hat{A}_1 \otimes \hat{A}_1 \to \bigsqcup_{n=1}^{\infty} \hat{A}_1(k)(\mathscr{D}\text{-torsion, Cyc}n_n),$$

$$[N] \to [A_1(k) \bigotimes_{A_2} N]$$

is bijective with inverse $[M] \to [\mathrm{Soc}_{A_2} M]$.

In the next corollary we obtain the precise form of the simple holonomic A_2-modules from Theorem 5.4.

COROLLARY 5.5 (Corollary 4.14). *Let* $[V=\mathscr{D}/\mathscr{D}f]\in\hat{\mathscr{D}}(\mathrm{Cyc}_n)$ *for some element* $f\in A_1\otimes K[H]$ *irreducible in* $\mathscr{D}$ *and let* $\tilde{u}\in A_1(k)_{[n]}$ *be such that* $u=\tilde{u}+A_1(k)_{[n]}f$ *is a V-irreducible element. Then*

1. *The* A_2*-module*

$$M(f,\tilde{u}):=A_2/A_2\cap A_1(k)_{[n]}(f,\tilde{u})$$

is holonomic with socle $\mathrm{Soc}_{A_2}M(f,\tilde{u})\simeq\mathrm{Soc}_{A_2}M(V,u)$ *provided* $\sigma^i(f)$ *is n-minimal for* $-n+1\leqslant i\leqslant n+1$.

2. *Let* J *be a left ideal of* A_2 *containing* $A_2\cap A_1(k)_{[n]}(f,\tilde{u})$ *and* $J/A_2\cap A_1(k)_{[n]}(f,\tilde{u})=\mathrm{Soc}_{A_2}M(f,\tilde{u})$. *Then the left ideal* $\mathbf{a}=J\cap A_1$ *of* $A_1=A_1\otimes 1$ *is nonzero and for any nonzero element* $c\in\mathbf{a}$

$$[A_2/A_2\cap A_1(k)_{[n]}(f,\tilde{u})\,c^{-1}]\in\hat{A}_2(\mathrm{holonomic})\backslash\hat{A}_1\otimes\hat{A}_1.$$

So, any element of $\hat{A}_2(\mathrm{holonomic})\backslash\hat{A}_1\otimes\hat{A}_1$ *is an isoclass of some* A_2*-module* $A_2/A_2\cap A_1(k)_{[n]}(f,\tilde{u})\,c^{-1}$ (*for some* f, $\tilde{u}$, c, *and* n *as above*).

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