# DEPARTEMENT BEDRIJFSECONOMIE

## OPTIMAL STEADY-STATE REPLACEMENT POLICY IN AN ENVIRONMENT OF MULTIPLE PARALLEL MACHINES WITH VARIABLE INTENSITY OF UTILIZATION

by

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# Optimal steady-state replacement policy in an environment of multiple parallel machines with variable intensity of utilization

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**Abstract**

In this paper the concept of variable intensity of utilization is introduced in a multiple machine replacement model. Based on simple assumptions regarding the behavior of the cost of capacity utilization over the economic life of equipment, it is demonstrated that the intensity of utilization will fall as age increases. This proces is a formal representation of the concept of functional degradation.

# 1 Introduction

Optimal replacement policy is a subject that generated a lot of attention from economists, mathematicians as well as engineers. The work of Preinreich [18] is a standard reference in this area, but is was mainly Terborgh [22][23] who converted Preinreichs ideas into a manageable tool for business applications and gave the problem widespread attention. Later, the machine replacement problem was fine-tuned by Alchian [1] and Smith [21]. Soon, the problem also received attention from a purely mathematical point of view, since the replacement problem in many cases turns out to be a prototype of a dynamic programming problem (See among others the work of Bellman [5], Dreyfus [6] and Hastings [10]). Aside from a vast number publications on interesting practical applications of the theory, considerable attention went to further development of theoretical ideas on the subject, such as:

- the determinations of repair limits (See e.g. Hastings & Drinkwater [7][11]);
- the effect of technological progress (See e.g. Grinyer [9], Bethuyne [4]);
- the role of uncertainty (See e.g. Mauer & Ott [16], following the methodology of Dixit & Pindyck [8]).

The ideas developed for the machine replacement problem are also applied to related problems, such as the forestry problem and housing problem. The forestry problem is in fact a special case of a replacement problem, often encountered in agricultural economics. It involves determining the optimal time to harvest the wood of a growing tree and replace it with a new one. The problem was described in detail and solved by Hirshleifer [12, pp.82-92] and Samuelson [19]. The machine replacement problem is also related to housing problem. The housing problem refers to the decision of a landlord-builder who owns a house and tries to maximize his net revenue by maintaining the quality of the building at an optimal level. The relation with the machine replacement problem is obvious, since the landlord can also decide to demolish the building and replace it with a new one (See e.g. Arnott, Davidson & Pines [2][3]). Applications of the replacement problem in the housing market are of particular interest, because here other interesting variables such as maintenance and intensity of utilization are introduced in the model.

The importance of intensity of utilization in the single machine replacement model also received attention by the author [4] previously. It was demonstrated that the concept of a variable intensity of utilization is of considerable importance in the timing of replacement. It was shown that even in the case where operating costs do not depend on the past intensity of utilization (i.e. on the output produced in the past), the economic life of equipment is a function of the intensity of utilization future tasks impose on the equipment. This implies that a machine that is obsolete for one task, may still have economic value for another less demanding task - a concept that was already observed by Terborgh and named 'functional degradation'. Unfortunately, a further analysis of the dynamic evolution of the intensity of utilization over the economic life of equipment is extremely difficult in the framework of a single machine replacement model. Indeed, in single machine replacement analysis it is often assumed that production (and hence also the intensity of utilization) is constant over the economic life, in order not to confuse replacement investments and capacity investments. Of course, this rules out any serious analysis of the dynamic behavior of the intensity of utilization.

Generalizations of the problem to a situation where multiple parallel machines are operated remained scarce for a long time. The initial analysis of Preinreich mentioned parallel replacement, but his analysis remained at a very abstract level. Since then, important contributions came from Lutz & Lutz [13] (See also Massé for a description [15]) and mainly Malcomson [14] and Nickell [17]. The process described by Lutz & Lutz, which they call a 'synchronized process', contains in fact the bare essence of a multiple parallel-machine replacement model. However, it has been criticized strongly (among others by Samuelson [19]), for camouflaging the influence of the time-value of money on the timing of replacements. We will return to this point later.

Although the model fails to demonstrate clearly the effect of the interest rate, it has the advantage of relative simplicity compared to the real vintage models of Malcomson and Nickell. A brief description of this model will be presented in this paper as 'the simple static case'. The use of the term static is inspired by the fact that in this model, all exogenous variables are assumed constant. It will be demonstrated that even in these static surroundings where interest has little effect on the timing of replacements, the introduction of a variable intensity of utilization may cause interesting dynamic behavior in the use of the equipment.

The general outline of this paper is as follows. The definition and role of the intensity of utilization in single machine replacement analysis is examined in the second paragraph. It contains a generalization of a previous model of the author, in which only the direct effect of intensity on operating costs is taken into account. In the model presented here the model is extended with the concept of wear, which reflects age as well as the past intensity of utilization. In the third paragraph we develop the general structure and terminology of a multiple machine replacement. Necessary conditions for optimal replacement timing in steady-state are derived in the fourth paragraph. Finally, the multiple machine replacement model is extended with a variable intensity of utilization. Again, necessary conditions for optimal replacement are derived and interpreted. The interpretation allows for a clear graphical representation in two- and three-dimensional pictures. The paper is concluded with a summary of the main conclusions and some ideas for further research.

## 2 The intensity of utilization in the single machine model

For the single machine replacement problem, it has already been demonstrated that the intensity of utilization can have an important impact on economic life [4]. The model presented here can be considered as a generalization of this previous work. Whereas the earlier model only considered the direct effect of the intensity of utilization on operating costs, the present analysis considers both direct and indirect effects on operating costs and salvage value.

Let $\varphi$ be the intensity of utilization of a certain piece of equipment. This variable is defined as the rate of output production as a percentage of the maximum output rate: $0 \leq \varphi \leq 1$. Most types of equipment can be used with a variable intensity: locomotives, trucks, ... can perform variable mileage per day, or they can be used to haul heavy or light cargo, electrical power plants can vary the produced power, machines can produce at different speeds, etc. The age and past intensity of utilization are both reflected in what we call the wear of the machine, noted as $W$. The wear of new equipment is assumed to be zero, and wear changes in function of the intensity of utilization and age:

$$\begin{aligned} \frac{dW}{dl} &= f\left[\varphi\left(l\right), l\right] \geq 0 \\ W(0) &= 0 \end{aligned} \tag{1}$$

At zero intensity (i.e. when the equipment is not used), wear could be constant. For strict positive intensity of utilization, wear is assumed to be strictly increasing[1]. For simplicity, we will first deal with the case in which the intensity of utilization is constant over time. In that case, wear can be expressed as $W = W(\varphi, l)$ and the total cost of the equipment as:

$$k\left[W(\varphi, l), \varphi\right] \tag{2}$$

with $\frac{\partial k}{\partial W} > 0$ (costs are a strictly increasing function of wear) and $\frac{\partial^2 k}{\partial W^2} \geq 0$ (increasing marginal cost of wear). The utilization intensity influences the cost-function in a direct and indirect way. Indirectly, the utilization intensity determines the wear of the equipment and consequently also its operating cost and its value on the second-hand market. However, the operating cost of capital equipment is also directly related to intensity irrespective of wear (for instance through energy consumption, in the case of a machine).

The objective is to minimize total costs:

$$\min K = \frac{1}{1-e^{-iL}} \int_0^L k\left[W(\varphi, l), \varphi\right].e^{-il}\, dl \tag{3}$$

The term after the integral is the present value of all costs of one machine over its economic life $L$. The preceeding factor is the limit of an infinite geometric series with first term 1 and ratio $e^{-iL}$. This factor converts the integral (the cost of one machine) into the present cost of an infinite chain of identical machines, each lasting $L$ years. The necessary condition for optimal timing of replacement now takes the form:

$$k\left[W(\varphi, L), \varphi\right] = k_e\left[W(\varphi, L), \varphi\right] \tag{4}$$

in which:

$$k_e\left[W(\varphi, L), \varphi\right] = \frac{i}{1-e^{-iL}} \int_0^L k\left[W(\varphi, l), \varphi\right].e^{-il}\, dl \tag{5}$$

is the equivalent costflow, i.e. a perpetual constant costflow which with a present value equal to the objective function: $\frac{k_e}{i} = K$. Eq.(4) states that the optimal time of replacement will be a function of the intensity of utilization: $L(\varphi)$. The effect of $\varphi$ on the economic life can be clarified by taking the total differential of the optimality condition. Since $dk_e/dl = 0$ in the optimum, the effect of $\varphi$ on the economic life can be written as:

$$\frac{dL}{d\varphi} = \frac{\left(\frac{\partial k_e}{\partial W}\frac{\partial W}{\partial \varphi} + \frac{\partial k_e}{\partial \varphi}\right) - \left(\frac{\partial k}{\partial W}\frac{\partial W}{\partial \varphi} + \frac{\partial k}{\partial \varphi}\right)}{\frac{\partial k}{\partial W}\frac{\partial W}{\partial l}} \tag{6}$$

[1]A possible functional form for wear could be $W \propto \int_0^l \varphi(\theta)\, d\theta$, which reduces to $\varphi l$ in the case where the intensity of utilization is constant. Wear is then proportional to the cumulated output, expressed as a fraction of the potential output.

Since the second-order condition of the replacement problem implies that total costs have to be rising in the optimum, the sign of the derivative is solely determined by the sign of the numerator of (6). However, since $\frac{\partial^2 k}{\partial W^2} \geq 0$, we can state that:

$$\begin{aligned} & \frac{\partial\, k_e \left[W\left(\varphi, L\right), \varphi\right]}{\partial W} \\ & = \frac{i}{1-e^{-iL}} \int_0^L \frac{\partial k \left[W\left(\varphi, l\right), \varphi\right]}{\partial W} . e^{-il} dl \\ & \leq \frac{i}{1-e^{-iL}} \int_0^L \frac{\partial k \left[W\left(\varphi, L\right), \varphi\right]}{\partial W} . e^{-il} dl \\ & = \frac{\partial k \left[W\left(\varphi, L\right), \varphi\right]}{\partial W} \end{aligned} \tag{7}$$

Also, if we assume that $\frac{\partial^2 k}{\partial W \partial \varphi} \geq 0$ , then:

$$\begin{aligned} & \frac{\partial\, k_e \left[W\left(\varphi, L\right), \varphi\right]}{\partial \varphi} \\ & = \frac{i}{1-e^{-iL}} \int_0^L \frac{\partial k \left[W\left(\varphi, l\right), \varphi\right]}{\partial \varphi} e^{-il} dl \\ & \leq \frac{i}{1-e^{-iL}} \int_0^L \frac{\partial k \left[W\left(\varphi, L\right), \varphi\right]}{\partial \varphi} e^{-il} dl \\ & = \frac{\partial k \left[W\left(\varphi, L\right), \varphi\right]}{\partial \varphi} \end{aligned} \tag{8}$$

Hence the numerator of (6) is negative, meaning that a decrease in the intensity of utilization will increase the economic life of the equipment, based on the assumption we made about the sign of the second cross-partial derivative of the operational cost function. Of course we have to wonder if this assumption is reasonable. If we rewrite the derivative as $\partial \left(\frac{\partial k}{\partial \varphi}\right) / \partial W \geq 0$, we can see that our assumption implies that marginal cost of the intensity of utilization has to increase in function of wear. Provided that operational costs are proportional to the rate of intensity, the second cross-partial derivative will be positive. Indeed, if $k(W, \varphi) = f(\varphi) M(W)$, where $f(\varphi)$ is a positive monotonic function and $M(W)$ is the operational cost of a machine with wear $W$ when it is used at full intensity, the derivative will be positive since $M(W)$ is a positive function of wear.

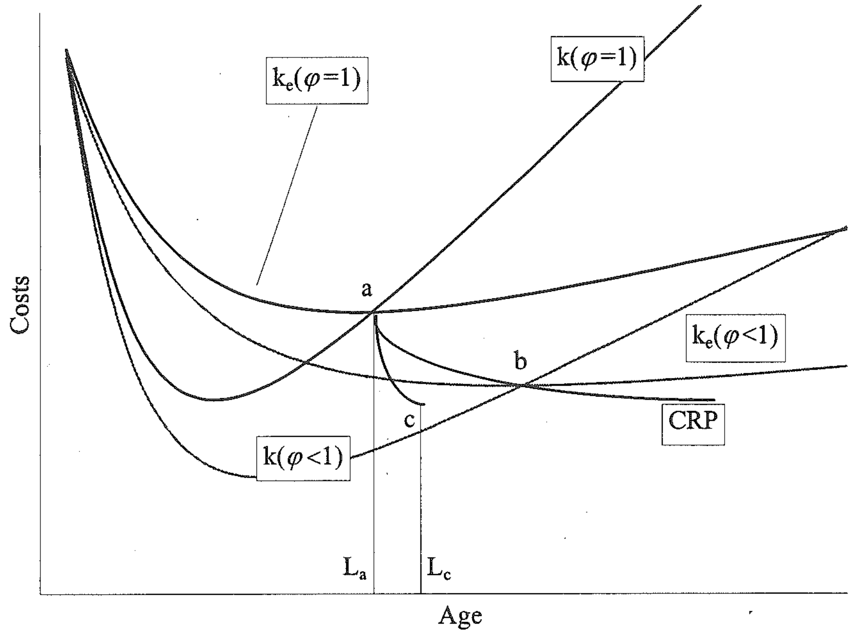


Figure 1: The single-machine model

The problem can be visualized as in figure 1. Suppose that in the initial situation, the intensity is 100%. Optimality implies equal total costs and equivalent costflow (point a). If the equipment is operated over its entire economic life at a reduced intensity, the economic life will increase and costs will be reduced, i.e. the minimum of the equivalent costflow will shift down and to the right as the intensity falls. We will call the path along the optimality condition when $\varphi$ is falling the cost-reduction path (CRP); it shows how optimal life and minimal costs evolve as intensity of utilization falls. A three-dimensional representation of the problem (as in [4]) is an easy exercise.

We can now also address the issue of different intensities of utilization over economic life. Assume for instance that an operator used his equipment at full intensity over its economic life $L_a$. It would be suboptimal to continue using this equipment at the same intensity, but it may still be optimal to continue using it at a lower intensity. However, by doing so the intersection of $k$ and $k_e$ (point $a$ in figure 1) will not shift along the cost-reduction path, since this would imply that the reduction of the utilization intensity in the future (i.e. after $L_a$) would also reduce the wear, caused by the use of the machine before $L_a$. Of course such retro-active effect of a reduction of the intensity never occurs. To describe the actual movement of the optimal point $a$, we reconsider the total differential of the optimality condition (6). This time however, since we only change the future values of $\varphi$, the wear at $L_a$ is unaffected and the indirect effect of $\varphi$ through $W$ will vanish: $\frac{\partial W}{\partial \varphi} = 0$. Hence:

$$\left(\frac{\partial k_e}{\partial W} - \frac{\partial k}{\partial W}\right)\frac{\partial W}{\partial \varphi} = 0 \tag{9}$$

In that case a negative term in de numerator of (6) disappears and:

$$\left.\frac{dL}{d\varphi}\right|_{\frac{\partial W}{\partial \varphi}=0} \geq \frac{dL}{d\varphi} \tag{10}$$

The costs will now change along the path $ac$. Notice that the path $ac$ is local to the original optimum $a$ since the assumption $\frac{\partial W}{\partial \varphi} = 0$ can not be maintained over non-infinitesimal periods of time. If so, for longer periods the optimum will be situated somewhere in the region between de cost reduction path and the path $ac$.

There are important advantages of this model-specification. First, it resolves the debate about the correct measure to determine the time of replacement. In the model, time is chosen as the main variable, but the solution to the problem is contingent upon the intensity of utilization. This new variable captures the influence that the rate of production exercises on the timing of replacement. The model can also be applied to illustrate the principle of functional degradation. This last point requires some further explanation. Suppose a new machine were acquired which can be used at different degrees of intensity. If the new equipment is used at full intensity, its economic life will end at $L_a$ (see figure 1). In the basic replacement model without variable intensity of utilization, the equipment was necessarily abandoned at this age. However, in this model there is an alternative, since the equipment can be maintained for lighter duty. For instance, if the equipment is used with an intensity of $\varphi < 1$, the economic life can be prolonged to $L_c$. This principle is often observed in real-life production applications and has been verbally described by Terborgh [22], as mentioned before. In formal models however, this principle seems to be neglected.

The model presented here also implies that if two separate tasks with different intensity were to be performed by two machines with different age, but otherwise identical, it is optimal to perform the least intensive task by the oldest machine. This can be demonstrated as follows. Suppose the intensity of the first task is $\varphi$, the one of the second task is $\varphi + \Delta\varphi$. Assume also that the age of the first machine is $l$, and that the second machine is $\Delta l$ older. If in that case the newest machine performed the task with the lowest intensity and vice versa, the total cost of operation would be: $k(l, \varphi) + k(l + \Delta l, \varphi + \Delta\varphi)$. A second-order approximation of this cost is:

$$2.k(l,\varphi)+\frac{\partial k}{\partial l}\Delta l+\frac{\partial k}{\partial \varphi}\Delta\varphi+\frac{1}{2}\frac{\partial^2 k}{\partial l^2}(\Delta l)^2+\frac{1}{2}\frac{\partial^2 k}{\partial \varphi^2}(\Delta\varphi)^2+\frac{\partial^2 k}{\partial l \partial \varphi}(\Delta l)(\Delta\varphi) \quad (11)$$

If on the other hand the newest machine performed the most intensive task and vice versa, the total cost of operation would be: $k(l + \Delta l, \varphi) + k(l, \varphi + \Delta\varphi)$. This can be approximated as:

$$2.k(l,\varphi)+\frac{\partial k}{\partial l}\Delta l+\frac{\partial k}{\partial \varphi}\Delta\varphi+\frac{1}{2}\frac{\partial^2 k}{\partial l^2}(\Delta l)^2+\frac{1}{2}\frac{\partial^2 k}{\partial \varphi^2}(\Delta\varphi)^2 \quad (12)$$

The cost-difference between both approaches is the term $\frac{\partial^2 k}{\partial l \partial \varphi}(\Delta l)(\Delta\varphi)$. This term vanishes only in the case where the difference in age and in intensity are marginal. Since both differences in this expression are positive and non-marginal, the sign of the cost difference depends solely on the cross-partial derivative, which is assumption positive since $\frac{\partial^2 k}{\partial l \partial \varphi} = \frac{\partial^2 k}{\partial W \partial \varphi}\frac{\partial W}{\partial l}$ and we assumed $\frac{\partial^2 k}{\partial W \partial \varphi} \geq 0$ and $\frac{\partial W}{\partial l} \geq 0$. Hence producers will use the most recent equipment for the most intensive tasks, which again is compatible with the concept of functional degradation.

An important consequence is that the economic life depends on the specific task a piece of equipment has to perform. Machines that are obsolete for one task may still have economic value to perform other tasks. Of course, this is hard to demonstrate in a situation where only one machine performs a task with a specific and fixed intensity of utilization. To make the principle work in a single-machine production environment, we have to assume that another operator is willing to buy the machine after its first economic life, to use it for lighter duty. Apparently, the introduction of a variable intensity of utilization calls for a replacement model in a multiple machine environment. In what follows, we will concentrate on the direct effect of the intensity of utilization on operating costs and omit indirect effect through wear ($\frac{\partial W}{\partial \varphi} = 0$). If wear is not affected by the past intensity of utilization, the path $ac$ will coincide with the cost-reduction path. This specification still allows for functional degradation to appear and will allow us to find optimality conditions for a parallel replacement process, using simple static optimization techniques.

## 3 General multiple-machine model

Before addressing the problem of finding optimality conditions in a multiple machine environment, some basic relations need to be clarified. Let $m(l)$ be the operating cost of a machine of age $l$. For the moment, we do not include the intensity of utilization explicitly as a variable. Also, let $n(l,t)$ be the number of machines with age $l$ at time $t$. In that case, if $T(t)$ is the age of the oldest equipment in service at time $t$, the total operating cost of all equipment in a time interval $dt$ is:

$$c(t) = \int_0^{T(t)} n(l,t).m(l)\,dl \tag{13}$$

The present value of total operating costs over an infinite period of time is therefore:

$$K_1 = \int_0^{\infty} c(t).e^{-it}dt = \int_0^{\infty} \left\{ \int_0^{T(t)} n(l,t).m(l)\,dl \right\} .e^{-it}dt \tag{14}$$

The acquisition of new equipment and the scrapping of old equipment generates additional costs $K_2$. Let $a(t)$ the number of new machines installed at time $t$ and $s(t)$ the number of old machines scrapped at time $t$. Let $V(l)$ be the salvage value of a machine with age $l$, the initial cost of a new machine is $V(0)$. Then $K_2$ can be expressed as follows:

$$K_2 = \int_0^{\infty} \{a(t).V(0) - s(t).V\,[T(t)]\}\,.e^{-it}dt \tag{15}$$

Notice that we assume that only new equipment will be purchased.

The objective will be to minimize $K = K_1 + K_2$, subject to the constraint that the productive capacity at time $t$ has to allow a given amount of production $q(t)$. This can be expressed as:

$$\int_0^{T(t)} n(l,t)\,dl \geq q(t) \tag{16}$$

We assume that only the oldest equipment is scrapped, and that it is always decided to scrap an entire vintage, i.e. all machines of the same age are scrapped at the same time. Finally we also assume that once a machine is scrapped, it is definitively lost and can not be put back in operation at a later time. Consequently, the number of production units of age $l$ at time $t$ ($n(l,t)$) equals the number of machines acquired at time $(t-l)$, for $0 \leq l < T(t)$. Formally:

$$n(l,t) = \begin{cases} a(t-l) \text{ if } 0 \leq l \leq T(t) \\ 0 \text{ if } l > T(t) \end{cases} \tag{17}$$

Furthermore, the number of machines scrapped at time $t$ can also be expressed as:

$$s(t) = n\,[T(t),t] = a\,[t - T(t)] \tag{18}$$

Taking all these considerations into account, the objective becomes to minimize:

$$K = \int_0^\infty \left\{ \int_0^{T(t)} a(t-l).m(l)\,dl + a(t).V(0) - a\,[t-T(t)]\,.V\,[T(t)] \right\} .e^{-it}dt \tag{19}$$

subject to the constraint:

$$\int_0^{T(t)} a(t-l)\,dl \geq q(t) \tag{20}$$

In what follows we will examine the characteristics of the steady-state equilibrium of this dynamic model.

# 4 The simple static case

In its simplest form (i.e. without variable intensity of utilization), the multiple machine replacement model can be defined as a synchronized process, as in Lutz & Lutz [13]. If the quantity $q(t)$ to be produced is constant and that there is an equal number of machines of each age $l \leq T(t)$ at time $t$, $n(l,t)$ does not depend on $t$. This implies constant acquisitions $a(t) = a$ and an invariable scrapping age $T(t) = L$. In these static conditions, optimal values for $a$ and $L$ can be found solving:

$$\begin{aligned} \min \quad & \frac{a}{i} \left\{ \int_0^L m(l)\,dl + V(0) - V\,(L) \right\} \\ \text{sub} \quad & a.L = q \end{aligned} \tag{21}$$

The constraint has been altered to an equality constraint, since it will never be optimal in a steady-state to maintain machines in operation above the number strictly necessary for production. Hence, we can substitute $a$ in the objective function by $\frac{q}{L}$, to obtain first-order conditions for the optimal time of replacement:

$$\frac{q}{iL}\left[m(L) - v(L)\right] - \frac{q}{iL^2}\left\{\int_0^L m(l)\,dl + V(0) - V(L)\right\} = 0 \qquad (22)$$

where $v(l) = \frac{dV}{dl}$. Rearranging gives us the first-order condition as:

$$m(L) - v(L) = \frac{\int_0^L m(l)\,dl + V(0) - V(L)}{L} \qquad (23)$$

In words, the optimality condition states the equipment has to be replaced at an age, when the marginal cost of operating the equipment one extra period of time equals the average cost of operating the equipment per unit of time. Notice that the interest-rate has no influence at all on the optimal replacement age, a result that was strongly criticized by Samuelson, since it seems to suggest that the time-value of money is of no interest in the replacement problem.

However, such a conclusion would be overhasty. First of all, the fact that the timing of the replacement is not influenced by the interest-rate is purely a consequence of the steady-state nature of the solution. Since all costs are evenly smoothed over time and changes in the optimal age of replacement cause no variations of the distribution of costs over time, the time-value of money is not relevant for optimal replacement timing. However, when we leave the steady-state conditions, it will no longer be possible to cancel out interest, as demonstrated by Malcomson [14] and Nickell [17]. The absence of the interest-rate in the optimality condition seems to be an exception, rather then the rule. Furthermore, the minimum value of the cost function still depends on the interest-rate, even in steady-state. Although the outcome of the model may lead the inattentive researcher astray, the model is fundamentally correct. Since this paper does not primarily deal with the effects of the interest-rate on the optimal timing of replacement, we see no harm in the further use of this model.

# 5 The static case with variable intensity of utilization

## 5.1 First-order conditions

The synchronized process as described before is in many ways an oversimplification of any realistic situation. One crucial element is the implicit assumption that each machine produces constant output over its entire economic life. Since it seems logical to assume that it becomes increasingly more difficult (or better, more expensive) to maintain the same intensity of utilization as a machine becomes older, it is necessary to allow for variability in the intensity of utilization in the multiple machine model, as it was demonstrated before in a single machine environment. Therefore, suppose that the equipment can be operated at different intensities and let intensity vary with age: $0 \leq \varphi(l) \leq 1$. Notice that in this model, the intensity for a given age does not depend on chronological time. The operating cost will now be a function of the age of the equipment as well as the intensity the equipment is used with: $m\,[l, \varphi(l)]$. We consider only the direct effect of the intensity of utilization on the operating cost, without using the concept of wear.

Strictly speaking, the indirect effect of intensity of utilization on operating cost through wear could also be introduced in the model. However, such generalization would force us to use dynamic optimization techniques to determine necessary conditions for optimality. At this stage of the exposition it is preferred to rely only on non-linear programming techniques. Not only will this provide us with very simple and understandable optimality-conditions, but it will also demonstrate the possible use of 'static' non-linear programming techniques for modeling dynamic behavior. Indeed, it will be demonstrated that although the model and the optimization technique are both essentially static, distinct dynamic behavior may still result from the analysis.

The objective is now:

$$MIN \quad K = \frac{a}{i}\left\{\int_0^L m\,[l, \varphi(l)]\,dl + V(0) - V(L)\right\} \tag{24}$$

$$SUB \quad q = a\int_0^L \varphi(l)\,dl$$

First-order conditions can be found setting the first derivatives of the Lagrangian $\ell$ equal to zero ($\lambda$ is the Lagrange-multiplier):

$$\frac{\partial \ell}{\partial L} = \frac{a}{i}\{m[L,\varphi(L)] - v(L)\} - \lambda.a.\varphi(L) = 0 \quad (25)$$

$$\frac{\partial \ell}{\partial a} = \frac{1}{i}\left\{\int_0^L m[l,\varphi(l)]\,dl + V(0) - V(L)\right\} - \lambda\int_0^L \varphi(l)\,dl = 0 \quad (26)$$

$$\frac{\partial \ell}{\partial \varphi(l)} = \frac{a}{i}\frac{\partial m[l,\varphi(l)]}{\partial \varphi(l)} - \lambda a = 0 \quad (27)$$

$$\frac{\partial \ell}{\partial \lambda} = a\int_0^L \varphi(l)\,dl - q = 0 \quad (28)$$

Notice that eq.(27) is in fact a set of equations for all possible values of $l \in [0, L]$. The marginal cost caused by an extra unit of output $q$ can be found as follows:

$$dK = \frac{\partial K}{\partial L}\frac{\partial L}{\partial q}dq_L + \frac{\partial K}{\partial a}\frac{\partial a}{\partial q}dq_a + \int_0^L \left[\frac{\partial K}{\partial \varphi(l)}\frac{\partial \varphi(l)}{\partial q}dq_{\varphi(l)}\right] dl \quad (29)$$

Where $dq_j$ is the change in output caused by a change in $j \in \{a, L, \varphi(l)\}$. It follows that:

$$\frac{dK}{dq} = \frac{\frac{\partial K}{\partial L}}{\frac{\partial q}{\partial L}} f_L + \frac{\frac{\partial K}{\partial a}}{\frac{\partial q}{\partial a}} f_a + \int_0^L \left[\frac{\frac{\partial K}{\partial \varphi(l)}}{\frac{\partial q}{\partial \varphi(l)}} f_\varphi\right] dl \quad (30)$$

Where $f_j = \frac{dq_j}{dq}$, i.e. the fraction of the extra capacity, caused by an increase in $j \in \{a, L, \varphi(l)\}$, so that $f_L + f_a + \int_0^L f_{\varphi(l)}dl = 1$. A marginal unit of capacity can be provided in three different ways: by increasing the economic life of the equipment, by increasing the number of parallel machines or by increasing the intensity of utilization at different stages of the economic life. The marginal cost of such extra capacity can then be expressed as the sum of the marginal costs of each of the previous capacity expansions. Now, we can derive that:

$$\frac{\frac{\partial K}{\partial L}}{\frac{\partial q}{\partial L}} = \frac{1}{i}.\frac{m[L,\varphi(L)] - v(L)}{\varphi(L)} = \lambda \quad (31)$$

$$\frac{\frac{\partial K}{\partial a}}{\frac{\partial q}{\partial a}} = \frac{1}{i} \cdot \frac{\int_0^L m\left[l, \varphi(l)\right] dl + V(0) - V(L)}{\int_0^L \varphi(l)\, dl} = \lambda \tag{32}$$

$$\frac{\frac{\partial K}{\partial \varphi(l)}}{\frac{\partial q}{\partial \varphi(l)}} = \frac{1}{i} \cdot \frac{\partial m\left[l, \varphi(l)\right]}{\partial \varphi(l)} = \lambda \tag{33}$$

The first equality in each equation follows directly from differentiating $K$ and $q$ with respect to the appropriate variables. The second equality follows from the first-order conditions (25) to (27).

Before engaging in a rigorous interpretation of all previous equations, it is useful to elaborate on the true meaning of $\frac{dK}{dq}$. Although time plays an important role in virtually every replacement model, it is essential to understand that the present model is basically static. After all, we assumed that all parameters were constants. Consequently, the optimal values of all variables found in this analysis can be considered as optimal steady-state values of an underlying dynamic model. It would be wrong to pry some interpretation about the dynamic behavior of the variables considered out of the optimality conditions. More in particular, it would be wrong to assign a dynamic interpretation to $\frac{dK}{dq}$. The marginal cost function considered here does not give any information about the dynamic behavior of costs when confronted with variations in $q$. The analysis is limited to the comparison of the steady-state before and after the adjustment and is thus simply an exercise in comparative static analysis. The evolution of costs and costflows under dynamic behavior of parameters like $q$ needs to be examined in a true dynamic model. Before engaging in such effort, a full comprehension of the steady-state will prove useful.

From eq.(30) and conditions (31) to (33) it follows immediately that $\frac{dK}{dq} = \lambda$, the usual interpretation of the Lagrange-multiplier. The left-hand side of eq.(31) can be interpreted as the marginal cost (in steady-state) of an extra unit of capacity, caused by an increase of the economic life of existing equipment. In the numerator we find the marginal cost per unit of increment in $L$. This cost consists of the extra operating cost in the marginal period of operation and the extra depreciation caused by maintaining the equipment in operation for an extra period of time. The denominator measures the marginal capacity of a unit of increment in $L$. The preceding factor $\frac{1}{i}$ simply converts constant flows in their present value.

The left-hand side of eq.(32) is the marginal cost (again in steady-state) of an extra unit of capacity if the existing equipment were expanded with one new extra unit. An extra unit of capacity would cause extra operating costs (the first term of the numerator) and its acquisition would cost a net price of $V(0) - V(L)$ in steady-state. The denominator is equals the total capacity added by one extra unit of equipment. Finally, the left-hand side of eq.(33) determines the optimal intensity of utilization of equipment of age $l$. It states that the marginal effect of an increase of the intensity of utilization on costs has to be equal, irrespective of the age of the equipment.

In the optimum, these terms are all equal to $\lambda$. This can be explained as follows. Suppose the outputflow $q$ increased with one unit over the entire time-horizon. In that case, the extra capacity could be found in three different ways: the existing equipment could be held in operation for a longer period of time, new equipment could be installed or finally the intensity of utilization of the existing equipment of age $l$ could be increased. The extra costs of each of these three measures can be found in eq.(31) to (33) respectively. The fact that the three equations share the same right-hand side simply means that in optimum, no extra cost-reductions can be made by reducing the capacity in one way and compensating in another way. In the optimum, the operator of the equipment is indifferent between changes of capacity caused by prolonged lifetime, additional equipment or increased intensity of utilization.

It is also important to notice that $a$ does not appear in the optimality conditions (31) to (33). This means that the marginal cost of a capacity expansion does not depend on the number of machines already in operation. This conclusion is of particular importance, since it implies that $\frac{d\lambda}{dq} = \frac{d\lambda}{da} = 0$. Since $\lambda$ is independent of the rate of production or the number of machines in operation, it follows that economic life $L$ and the intensity-function $\varphi(l)$ are also independent of the production-rate and the number of machines in operation. This is an important observation, because it says that the characteristics of operation do not depend on the size of the production system. In fact, such a conclusion follows from the implicit assumption that there are no (dis)economies of scale associated with parallel production. There are indications that economic life is prolonged in periods of capacity-expansion (See e.g. Smith [20, p.32]), but this kind of behavior is of a typically dynamic nature and does not correspond with the steady-state conditions employed here.

Using some further assumptions on the nature of the cost-function $m$ allows us to draw some interesting conclusions about the progress of $\varphi$ over time. Assume the general outline of the relation between $m$ and $\varphi$ is as depicted in figure 2. Operating costs equal zero when there is no production but increase rapidly when intensity rises. The rate of increment first

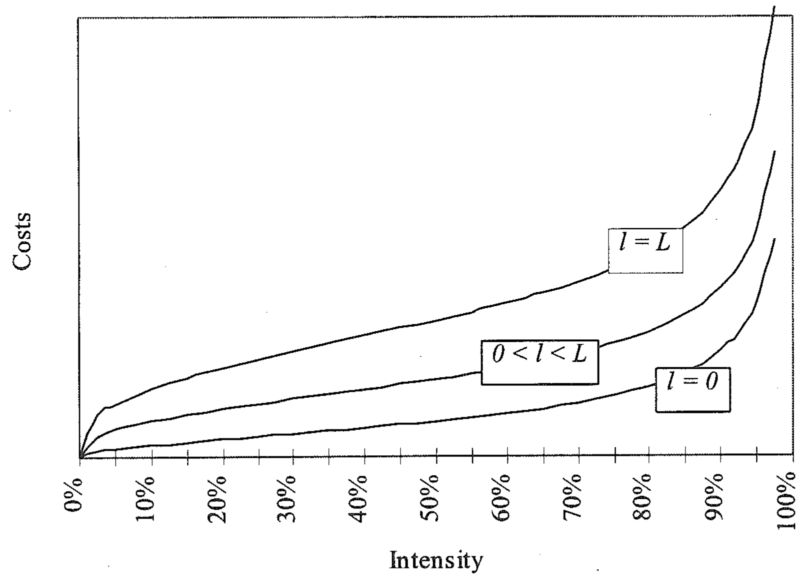


Figure 2: Operating costs at different levels of age and intensity of utilization

diminishes as $\varphi$ increases, but as the intensity approaches unity, costs rise quickly and eventually approach infinity. The idea behind this is, that it is nearly impossible to operate a machine at full capacity for 100% of the time. The cost of operating a machine at a certain level of intensity is also assumed to rise with age. In figure 2 three cost functions for different ages are represented. For the interpretation of the first-order conditions we will use the corresponding marginal and average cost-functions, as depicted in figure 3. To facilitate the graphical interpretation, lets assume $v(L) = 0$, i.e. suppose the equipment has lost all its value above the scrap-value before the age of replacement. Such an assumption can be well motivated, since the operator of the equipment can (and will) reduce the intensity of utilization in function of age, in order to prolong the economic life. Hence there is no need for a second-hand market to find an other operator who is willing to use the equipment at lower intensity, as it was the case in the single machine model. Since the operator will be able to reduce the intensity until it has reached a point where economic life cannot be prolonged any more, depreciation of the equipment will be complete and the remaining value will be the scrap-value.

In that case, bringing together eq.(31) and (33) gives us:

$$\frac{m\left[L,\varphi(L)\right]}{\varphi(L)}=\frac{\partial\, m\left[L,\varphi(L)\right]}{\partial\,\varphi(L)} \tag{34}$$

Hence, if $L$ is the optimal age of replacement, the average cost of capacity has to equal the marginal cost. This means that $[L,\varphi(L)]$ will be found at the minimum of an average cost-function. Let the average and marginal cost-curves at the time of replacement be represented by the highest curves in figure 3. In that case, the minimum intensity of utilization (i.e. at the moment of replacement) is $f_L$ in figure 3. Since the marginal cost of intensity has to be equal at all ages, the intensity of utilization at each specific age will be found at the intersection of line $AB$ and the marginal cost curve at age $l$. From this we can conclude that the initial intensity of utilization at age 0 in figure 3 will be $f_0$. Since the marginal cost of intensity rises with age, the intensity of utilization will fall as age increases from $f_0$ to $f_L$.

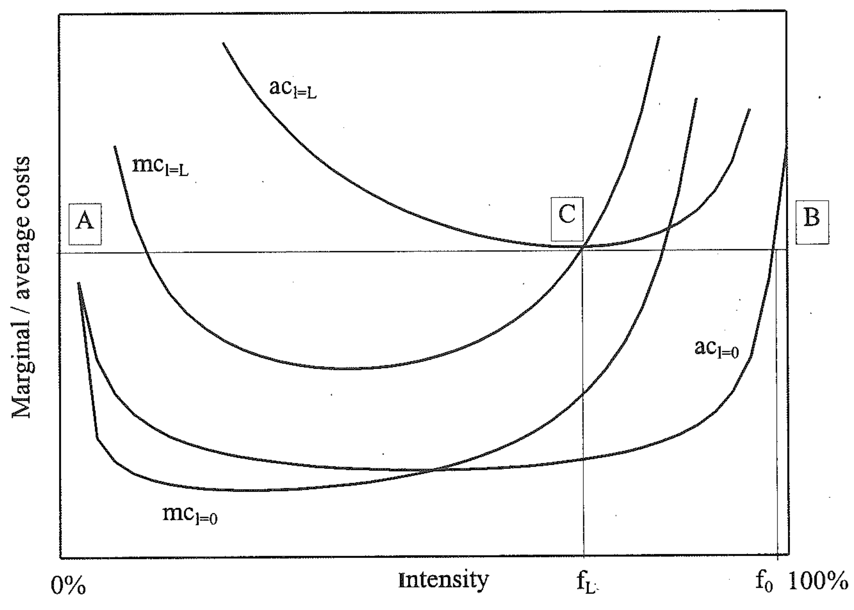


Figure 3: Marginal and average cost of intensity at different levels of age

The information contained in figure 3 can also be represented in three dimensions. This can give a better insight about the intensity of utilization of a piece of equipment over time. Such a representation can be found in figure 4. Intensity is represented on one of the horizontal axes, the average and marginal cost of intensity is on the vertical axis. In order not to overload the figure, only the maximum of average and marginal cost is represented. To the left of line $DE$ the marginal cost is smaller than the average cost, so the average cost is represented. To the right, the opposite. Hence $DE$ connects all minima of the average cost function. Now, consider the relation between $\varphi$ and $l$. We know that at the time of replacement $L$, the intensity of utilization must satisfy eq.(34), so the average cost at $[L, \varphi(L)]$ has to be located on line $AB$. Suppose for instance that $C$ represents the optimal average and marginal cost at the time of replacement. Since the marginal cost of each piece of equipment has to be equal, irrespective of age, all optimal combinations of $\varphi$ and $l$ will be located at the contour line of the marginal cost through $C$. Consequently the evolution of the intensity of utilization over time will start at a point near $B$ and follow the contour until it reaches $C$. Of course, the exact location of the ending-point $C$ and the corresponding contour $AB$ can only be determined with knowledge of the exact functional form of the underlying cost-function.

Finally, figure 5 gives a two-dimensional view of figure 4 from above. Line $AB$, connecting all minima of the average cost-function, and line $CD$, representing the relation between intensity of utilization and age, can again be recognized. Notice the apparent fall in intensity of utilization as age increases, especially near the end of the economic life of the equipment.

## 5.2 Second-order conditions

The bordered Hessian of the minimization problem is:

$$\begin{bmatrix} 0 & a.\varphi(L) & \int_0^L \varphi(l)dl & a \\ a.\varphi(L) & \frac{a}{i}\left[\frac{\partial m(L)}{\partial L} - \frac{\partial v(L)}{\partial L}\right] & \frac{m(L)-v(L)}{i} - \lambda\varphi(L) & 0 \\ \int_0^L \varphi(l)dl & \frac{m(L)-v(L)}{i} - \lambda\varphi(L) & 0 & \frac{1}{i}\frac{\partial m(l)}{\partial \varphi(l)} - \lambda \\ a & 0 & \frac{1}{i}\frac{\partial m(l)}{\partial \varphi(l)} - \lambda & \frac{a}{i}\frac{\partial^2 m(l)}{\partial \varphi(l)^2} \end{bmatrix}.$$

For brevity, $m[l, \varphi(l)]$ is noted as $m(l)$. Substituting the first-order conditions in this matrix simplifies it to:

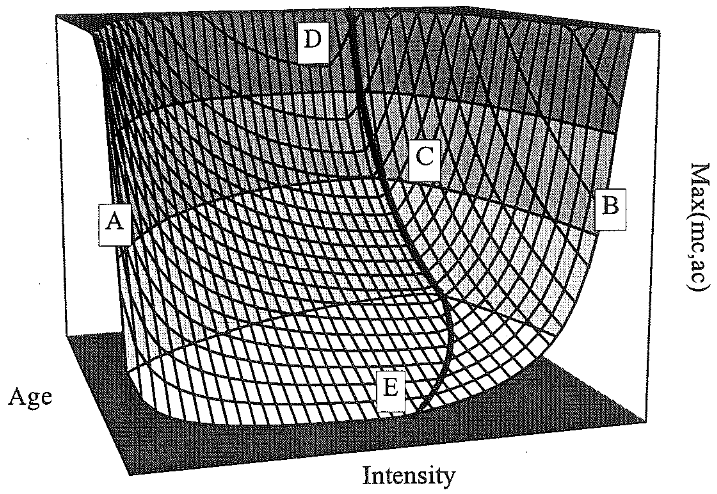


Figure 4: Maximum of average and marginal cost of intensity

$$\begin{bmatrix} 0 & a.\varphi(L) & \int_0^L \varphi(l)dl & a \\ a.\varphi(L) & \frac{a}{i}\left(\frac{\partial m}{\partial L} - \frac{\partial v}{\partial L}\right) & 0 & 0 \\ \int_0^L \varphi(l)dl & 0 & 0 & 0 \\ a & 0 & 0 & \frac{a}{i}\frac{\partial^2 m[l,\varphi(l)]}{\partial \varphi(l)^2} \end{bmatrix}$$

Second-order conditions require that all border-preserving minors are negative. For most of these this follows directly from the assumptions about the characteristics of $m\,[l, \varphi(l)]$ and $V(t)$. However, one of them deserves further attention:

dynamic behavior of wear.

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