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ISOMORPHISM PROBLEMS AND GROUPS OF AUTOMORPHISMS FOR GENERALIZED WEYL ALGEBRAS

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Abstract. We present solutions to isomorphism problems for various generalized Weyl algebras, including deformations of type-A Kleinian singularities and the algebras similar to $U(\mathfrak{sl}_2)$ introduced by S. P. Smith. For the former, we generalize results of Dixmier on the first Weyl algebra and the minimal primitive factors of $U(\mathfrak{sl}_2)$ by finding sets of generators for the group of automorphisms.

1. Introduction

Let k be an algebraically closed field of characteristic 0 and consider the Weyl algebra $A_1(k) = k\langle \partial, x : \partial x - x\partial = 1\rangle$. Dixmier [11] showed that the k-automorphism group of $A_1(k)$ is generated by the automorphisms $e^{\lambda \operatorname{ad} x^n}$ and $e^{\lambda \operatorname{ad} \partial^n}$ where $n \geq 1$ and $\lambda \in k$. Adapting his methods in [12], he found an analogous set of generators for the k-automorphism group of the infinite-dimensional primitive factor $B_\lambda := U(\mathfrak{sl}_2)/(C - \lambda)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, where C is the Casimir element and $\lambda \in k$. He also showed that $B_\lambda \simeq B_{\lambda'}$ if and only if $\lambda = \lambda'$.

Let $a \in k[h]$ and let $A(a)$ be the generalized Weyl algebra $k[h](\sigma, a)$, where σ is the k-automorphism of $k[h]$ such that $\sigma(h) = h - 1$. Thus $A(a)$ is the k-algebra generated by h, x and y subject to the relations

$$xh = (h-1)x,\ yh = (h+1)y,\ xy = a(h-1),\ yx = a(h).$$

Both $A_1(k)$ and B_λ are of the form $A(a)$ with $\deg(a) = 1$ and 2 respectively. Rings of the form $A(a)$, with a of arbitrary degree, were studied by the first author [3] and, under the name deformations of type-A Kleinian singularities, by Hodges [14]. The problem of when $A(a_1) \simeq A(a_2)$ was raised in [14, §5 (1)].

Smith [31] gave a substantial analysis of a class of algebras, similar to $U(\mathfrak{sl}_2)$, which can be interpreted as generalized Weyl algebras and which are closely related to the algebras $A(a)$. For $f \in K[H]$, let $R(f)$ denote the k-algebra generated by A, B and H subject to the relations

$$[H, A] = A,\ [H, B] = -B,\ [A, B] = f(H).$$

The isomorphism problem for these algebras was raised in [31, Remark (2)].

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We shall adapt the methods of Dixmier to find a set of generators for the automorphism group of $A(a)$ and to solve the isomorphism problems for $A(a)$ and $R(f)$. Loosely speaking, we shall see that $A(a_1) \simeq A(a_2)$ if and only if a_2 can be obtained from a_1 by a combination of non-zero scalar multiplication, translation $a(h) \mapsto a(h+\tau)$ and a "flip" $a(h) \mapsto a(-h)$ corresponding to an interchange of the roles of x and y. The situation for $R(f)$ is analogous. In some cases, including those of $A_1(k)$ and B_λ, the flip can be expressed in terms of translation and scalar multiplication and there is an automorphism Ω, well-known for both $A_1(k)$ and B_λ, interchanging the roles of x and y. We shall say that $a(h)$ is *reflective* if there exists $\rho \in k$ such that $a(\rho - h) = (-1)^d a(h)$, where $d = \deg a$. Polynomials of degree 1 and 2 must be reflective but cubics need not. When $a(h)$ is reflective, there is a k-automorphism Ω of $A(a)$ such that $\Omega(x) = y$, $\Omega(y) = (-1)^d x$ and $\Omega(h) = 1 + \rho - h$. In general, there is an isomorphism $\Lambda : A(a(h)) \to A(a(-h))$ such that $\Lambda(x) = y, \Lambda(y) = x$ and $\Lambda(h) = 1 - h$.

Let G denote the subgroup of the k-automorphism group $\operatorname{Aut}_k A(a)$ generated by the k-automorphisms $e^{\lambda \operatorname{ad} x^m}$ and $e^{\lambda \operatorname{ad} y^m}$, where $\lambda \in k$ and $m \geq 1$, and the k-automorphisms Θ_μ such that $\Theta_\mu(x) = \mu x, \Theta_\mu(y) = \mu^{-1} y$ and $\Theta_\mu(h) = h$, where $\mu \in k^*$. We shall prove:

- (Theorem 3.29) If a is reflective, then $\operatorname{Aut}_k A(a)$ is generated by G and Ω. If a is not reflective, then $\operatorname{Aut}_k A(a) = G$.
- (Theorem 3.28) For $a_1, a_2 \in k[h]$, $A(a_1) \simeq A(a_2)$ if and only if $a_2(h) = \eta a_1(\tau \pm h)$ for some $\eta, \tau \in k$ with $\eta \neq 0$.
- (Theorem 4.2) For $f_1, f_2 \in k[H]$, $R(f_1) \simeq R(f_2)$ if and only if $f_2(H) = \eta f_1(\tau \pm H)$ for some $\eta, \tau \in k$ with $\eta \neq 0$.

Theorem 4.2 is deduced from Theorem 3.28 using the fact that each of the algebras $R(f)$ has a distinguished central element C such that $R(f)/CR(f) \simeq A(a)$ for some a. Notice that when a_1 is reflective the condition for isomorphism in Theorem 3.28 becomes $a_2(h) = \eta a_1(h + \tau)$.

We shall also prove, in §5, an analogue of Theorem 4.2 for the algebras which are similar to the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ in the way that the algebras $R(f)$ are similar to $U(\mathfrak{sl}_2)$.

Another isomorphism problem concerns two deformations of $\mathfrak{sl}_2$ considered by Witten [33, (5.2) and (5.12)] and a third deformation due to Woronowicz [34]. In §6, we shall show that Witten's second deformation and Woronowicz's deformation are isomorphic. The background to this includes remarks in the literature of mathematical physics [33, 10] suggesting at least that these two algebras become isomorphic after localization or completion and a comment in [24], where a rigorous algebraic relationship between Witten's two deformations is explained, doubting the existence of a connection with the Woronowicz algebra.

2. Generalized Weyl algebras and ambiskew polynomial rings

2.1. Generalized Weyl algebras. Let D be a ring, let σ be an automorphism of D and let a be a central element of D. The *generalized Weyl algebra (of degree 1) $D(\sigma, a)$*, introduced by the first author [3, 4, 5, 7, 8], is the ring extension of D generated by two indeterminates x, y subject to the relations

$$xd = \sigma(d)x \text{ and } yd = \sigma^{-1}(d)y \text{ for all } d \in D, \tag{1}$$

$$xy = \sigma(a),\ yx = a. \tag{2}$$

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These algebras were also studied, without a name, by the second author [18], where D is assumed to be commutative, and, under the name *hyperbolic ring*, by Rosenberg [28]. Generalized Weyl algebras of higher degree were introduced in [4] and studied in [4, 7, 8, 9].

2.2. Ambiskew polynomial rings. The papers [3, 9, 15, 16, 17, 19, 20, 22] are concerned with a class of iterated skew polynomial rings, now called ambiskew polynomial rings [21], which can be viewed as important examples of generalized Weyl algebras.

Let B be a ring, let σ be an automorphism of B, let v be a central element of B and let p be a central unit in B. We denote by $R(B,\sigma,v,p)$ the iterated skew polynomial ring $B[x;\sigma][y;\sigma^{-1},\delta]$ where the automorphisms $\sigma^{\pm 1}$ are extended to $B[x;\sigma]$ by setting $\sigma^{\pm 1}(x) = p^{\mp 1}x$ and δ is the σ^{-1}-derivation of $B[x;\sigma]$ such that $\delta(B) = 0$ and $\delta(x) = v$. Thus $R(B,\sigma,v,p)$ is the ring extension of B generated by x and y subject to the relations

$$xb = \sigma(b)x \text{ and } yb = \sigma^{-1}(b)y \text{ for all } b \in B, \tag{3}$$

$$xy - pyx = v. \tag{4}$$

2.3. Example. The first Weyl algebra $A_1(k) = k\langle \partial, x \mid \partial x - x\partial = 1\rangle$ over a field k is a basic example for both constructions. To view $A_1(k)$ as a generalized Weyl algebra, one first adjoins $h = yx$ to obtain the polynomial ring $k[h]$. Then $A_1(k) = k[h](\sigma, h)$, where σ is the k-automorphism of $k[h]$ such that $\sigma(h) = h - 1$. Thus, in $k[h](\sigma,h)$, $xy = h - 1, yx = h$ and $yx - xy = 1$. When viewing $A_1(k)$ as an ambiskew polynomial ring, the base ring is k and $A_1(k) = R(k, \mathrm{id}, -1, 1)$. This illustrates the relationship between the two constructions.

2.4. Lemma. *Every ambiskew polynomial ring $R(B,\sigma,v,p)$ is a generalized Weyl algebra $B[w](\sigma,w)$, where $w = yx$ and σ is extended to $B[w]$ by setting $\sigma(w) = pw + v$.*

Proof. See [22, 2.6 Corollary] or [8, Lemma 1.2]. □

Every generalized Weyl algebra occurs as a homomorphic image of an ambiskew polynomial ring, obtained by factoring out a normal element z which exists under a certain condition on v. When this holds, the previous lemma can be sharpened by replacing w by z, which is an eigenvector for σ.

2.5. Lemma. *Let $R = R(B,\sigma,v,p)$. Suppose that v has the form $\sigma(u) - pu$, for some central element u of B, and let $z = yx - u = p^{-1}(xy - \sigma(u))$.*

(i) *z is normal in R, with $xz = pzx, yz = p^{-1}zy$ and $zb = bz$ for all $b \in B$, and R/zR is the generalized Weyl algebra $B(\sigma,u)$.*
(ii) *R is the generalized Weyl algebra $B[z](\sigma, z+u)$, where σ is extended to $B[z]$ by setting $\sigma(z) = pz$.*

2.6. $\mathbb{Z}$-grading. Every generalized Weyl algebra $A = D(\sigma,a)$ is $\mathbb{Z}$-graded, $A = \bigoplus_{n\in\mathbb{Z}} A_n$, where $A_n = Dx^n$ if $n \geq 0$ and $A_n = Dy^{-n}$ if $n < 0$. It follows from the defining relations that, for each positive integer n,

$$y^n x^n = a\sigma^{-1}(a)\cdots\sigma^{-(n-1)}(a) \text{ and } x^n y^n = \sigma(a)\sigma^2(a)\cdots\sigma^n(a). \tag{5}$$

The next lemma gives details of some automorphisms and isomorphisms for generalized Weyl algebras.

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2.7. Lemma. *Let $A = D(\sigma, a)$ be a generalized Weyl algebra and let τ be a ring automorphism of D.*

(i) $A \simeq D(\sigma^{-1}, \sigma(a))$ *with* $x \mapsto y, y \mapsto x$ *and* $d \mapsto d$ *for all* $d \in D$.

(ii) *If λ is a central unit in D, then* $A \simeq D(\sigma, a\lambda)$, *with* $x \mapsto x\lambda^{-1}, y \mapsto y$ *and* $d \mapsto d$ *for all* $d \in D$. *In particular, if a is a unit then* $D(\sigma, a) \simeq D[x^{\pm 1}; \sigma]$.

(iii) *With λ as in (ii), there is an automorphism Θ_λ of A with* $x \mapsto x\lambda$, $y \mapsto \lambda^{-1}y$ *and* $d \mapsto d$ *for all* $d \in D$.

(iv) *The automorphism τ extends to isomorphisms* $\tau^{e+} : A \to D(\tau\sigma\tau^{-1}, \tau(a))$, *with* $\tau^{e+}(x) = x$ *and* $\tau^{e+}(y) = y$, *and* $\tau^{e-} : A \to D(\tau\sigma^{-1}\tau^{-1}, \tau\sigma(a))$, *with* $\tau^{e-}(x) = y$ *and* $\tau^{e-}(y) = x$.

(v) *If* $\tau\sigma = \sigma\tau$, *then* $\tau^{e+} : A \to D(\sigma, \tau(a))$.

(vi) *If* $\tau\sigma = \sigma^{-1}\tau$, *then* $\tau^{e-} : A \to D(\sigma, \tau\sigma(a))$.

Proof. The proof is routine. In (iv), τ^{e-} is obtained from τ^{e+} by composition with the isomorphism $D(\tau\sigma\tau^{-1}, \tau(a)) \to D(\tau\sigma^{-1}\tau^{-1}, \tau\sigma(a))$ given by (i). □

The next lemma gives the details for the corresponding automorphisms and isomorphisms for ambiskew polynomial rings.

2.8. Lemma. *Let $R = R(B, \sigma, v, p)$ be an ambiskew polynomial ring and let τ be a ring automorphism of B.*

(i) $R \simeq R(B, \sigma^{-1}, -p^{-1}v, p^{-1})$ *with* $x \mapsto y, y \mapsto x$ *and* $b \mapsto b$ *for all* $b \in B$.

(ii) *If λ is a central unit in B and* $\sigma(\lambda) = \lambda$, *then* $R \simeq R(B, \sigma, v\lambda, p)$, *with* $x \mapsto \lambda^{-1}x, y \mapsto y$ *and* $b \mapsto b$ *for all* $b \in B$.

(iii) *With λ as in (ii), there is an automorphism of R with* $x \mapsto \lambda x$, $y \mapsto \lambda^{-1}y$ *and* $b \mapsto b$ *for all* $b \in B$.

(iv) *There are isomorphisms* $\tau^{e+} : R \to R(B, \tau\sigma\tau^{-1}, \tau(v), \tau(p))$ *and* $\tau^{e-} : R \to R(B, \tau\sigma^{-1}\tau^{-1}, \tau(p^{-1}v), \tau(p^{-1})$, *extending τ, such that* $\tau^{e+}(x) = x$, $\tau^{e+}(y) = y$, $\tau^{e-}(x) = y$ *and* $\tau^{e-}(y) = -x$.

(v) *If* $\tau\sigma = \sigma\tau$, *then* $\tau^{e+} : R \to R(B, \sigma, \tau(v), \tau(p))$.

(vi) *If* $\tau\sigma = \sigma^{-1}\tau$, *then* $\tau^{e-} : R \to R(B, \sigma, \tau(p^{-1}v), \tau(p^{-1}))$.

Proof. The proof is again routine, though here the construction of τ^{e-} from τ^{e+} also involves (ii) with $\lambda = -1$. □

Two points to note, in comparing Lemmas 2.7 and 2.8, are the extra condition on λ in Lemma 2.8, which will be significant in §5, and the appearance of $\tau(p^{-1}v)$ rather than $\tau\sigma(v)$ in 2.8(vi).

2.9. Notation. In the remainder of the paper, k will denote an algebraically closed field and k^* will denote the multiplicative group of k.

3. Deformations of type-A Kleinian singularities

In this section we concentrate on the case where D is the polynomial ring $k[h]$ and $\sigma(h) = h - 1$. We assume throughout the section that $\operatorname{char} k = 0$. Let $a = a(h) \in k[h]$ and let $A(a) = k[h](\sigma, a)$. Thus $A(a)$ is the k-algebra generated by h, x and y subject to the relations

$$xh = (h-1)x, \ yh = (h+1)y, \ xy = a(h-1), \ yx = a(h).$$

Examples include the Weyl algebra $A_1(k)$ and, for $\lambda \in k$, the algebra $B_\lambda := U(\mathfrak{sl}_2)/(C - \lambda)$, where C is the Casimir element and $a = -h^2 - h - \frac{\lambda}{4}$; see 3.2 below.

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A lot is known about the algebras $A(a)$ through studies by the first author [3, 4, 5, 7, 9] and by Hodges [14]. The global dimension of $A(a)$ was computed in [3, 4, 14] generalizing a result of Stafford [32] for B_λ. The Krull dimension of $A(a)$ was determined in [4, 14] and the simple $A(a)$-modules were classified in [4, 7]. In [3, 4], it was shown that the spaces Ext^i and Tor_i for simple $A(a)$-modules are finite dimensional, generalizing a result of McConnell and Robson [25] on Ext^i in the case of $A_1(k)$. When the defining polynomial a has no multiple root, the Grothendieck group $K_0(A(a))$ was computed in [14]. The two-sided ideals for certain generalized Weyl algebras more general than $A(a)$ were classified in [5, 6]. In [4, Theorem 7], it was shown that if $a \notin k$ and $f \in A(a)\backslash k$, then the centralizer $C(f)$ is commutative and is a free $k[f]$-module of finite rank and this was applied to determine the structure of commutative subalgebras of $A(a)$, generalizing a result of Amitsur [1] for $A_1(k)$.

As observed in [14, Theorem 2.1(iii)], the algebra $A(a)$ has a filtration by finite-dimensional subspaces, with $\deg x = \deg y = d$ and $\deg h = 2$, such that $\mathrm{gr}\, A(a)$ is isomorphic to the commutative algebra $k[X, Y, Z]/(XY - Z^d)$. It follows from [26, 8.2.14(i) and 8.6.5] or [23, Theorem 4.5 and Proposition 6.6] that the Gelfand-Kirillov dimension $\mathrm{GK}(A(a)) = 2$.

The isomorphism problem for the algebras $A(a)$ was raised in [14, p. 289 (1)]. The solution to this will be given in Theorem 3.28 where it will be shown that the only situations where $A(a_1)$ and $A(a_2)$ are isomorphic arise when a_2 can be obtained from a_1 by a sequence of transformations of the three types specified in the following lemma.

3.1. Lemma. (i) *For all $\lambda \in k^*$, there is an isomorphism $\Gamma_\lambda : A(a) \to A(\lambda a)$ such that*

$$\Gamma_\lambda(x) = \lambda^{-1}x,\ \Gamma_\lambda(y) = y \text{ and } \Gamma_\lambda(h) = h.$$

(ii) *For all $\mu \in k$ there is an isomorphism $\mathrm{tr}_\mu : A(a(h)) \to A(a(h + \mu))$ such that*

$$\mathrm{tr}_\mu(x) = x,\ \mathrm{tr}_\mu(y) = y \text{ and } \mathrm{tr}_\mu(h) = h + \mu.$$

(iii) *There is an isomorphism $\Lambda : A(a(h)) \to A(a(-h))$ such that*

$$\Lambda(x) = y,\ \Lambda(y) = x \text{ and } \Lambda(h) = 1 - h.$$

Proof. Here (i) is a special case of Lemma 2.7(ii) while (ii) and (iii) follow from Lemma 2.7(v,vi) with $\mathrm{tr}_\mu = \tau^{\mathrm{e}+}$ and $\Lambda = \eta^{\mathrm{e}-}$, where $\tau(h) = h+\mu$, so that $\tau\sigma = \sigma\tau$, and $\eta(h) = 1 - h$, so that $\eta\sigma(h) = -h = \sigma^{-1}\eta(h)$. Note that $\Lambda^{-1} : A(-h) \to A(h)$ is defined by the same formulae as Λ: $\Lambda^{-1}(x) = y, \Lambda^{-1}(y) = x$, and $\Lambda^{-1}(h) = 1 - h$. □

3.2. Example ([14, Example 4.7]). Taking $a = -h^2 - h - \frac{\lambda}{4}$, where $\lambda \in k$, and making the change of variables $x \mapsto e, y \mapsto f, h \mapsto 2h$, we obtain the infinite-dimensional primitive factors B_λ of $U(\mathfrak{sl}_2)$ [12]. Since an arbitrary polynomial of degree 2 can be written in the form $\mu((h+\eta)^2 + (h+\eta) + \frac{\lambda}{4})$, it follows from Lemma 3.1 that if $\deg a = 2$, then $A(a) \simeq B_\lambda$ for some $\lambda \in k$. Similar calculations show that if $\deg a = 1$, then $A(a) \simeq A_1(k)$.

The fixed ring of $A_1(k)$ for the action of the cyclic group generated by the automorphism $y \mapsto \omega y$, $x \mapsto \omega^{-1}x$, where ω is a primitive nth root of unity, has the form $A(a)$ with $\deg a = n$. See [14, Example 4.8] for details.

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The solution of the isomorphism problem for the algebras B_λ was given by Dixmier [12] and our approach is based on his, which in turn was based on his earlier analysis of the Weyl algebra $A_1(k)$ [11].

3.3. Classification of elements. The following definitions and notation are from [11, 6.1] and [12, 3.1]. Let R be a k-algebra, let $w \in R$ and let $\lambda \in k$. Denote by $\operatorname{ad} w$ the inner derivation of R defined by w, that is $(\operatorname{ad} w)(z) = wz - zw$. Let $\mathcal{D}(w, \lambda) := \{z \in R : (\operatorname{ad} w)(z) = \lambda z\}$ and let $\mathcal{F}(w, \lambda) := \{z : (\operatorname{ad} w - \lambda)^i(z) = 0$ for sufficiently large $i\}$. Let $\mathcal{D}(w) := \bigoplus_{\lambda\in k} \mathcal{D}(w, \lambda)$ and $\mathcal{F}(w) := \bigoplus_{\lambda\in k} \mathcal{F}(w, \lambda)$. The elements z of $\mathcal{F}(w)$ are those for which the subspace $\sum_{n=0}^{\infty} k(\operatorname{ad} w)^n(z)$ is finite-dimensional. Let $\mathcal{N}(w) = \mathcal{F}(w, 0)$ and observe that $\mathcal{D}(w, 0)$ is the centralizer $\mathcal{C}(w)$ of w in R.

The element w is said to be *strictly semisimple* if $\mathcal{D}(w) = R$ in other words, if the derivation $\operatorname{ad} w$ is diagonalizable. In the generalized Weyl algebra $A(a)$, for each $n \in \mathbb{Z}$, $\mathcal{D}(h, n) = A_n$, the nth component of $A(a)$ in the $\mathbb{Z}$-grading of $A(a)$ in §2.6. Thus h is strictly semisimple. Note that $\mathcal{C}(h) = k[h]$ and it follows from the formula $xf(h) = f(h-1)x$ that the centre of $A(a)$ is k.

The element w is said to be *strictly nilpotent* (with respect to the adjoint action) if $\mathcal{N}(w) = A$, in other words, if $\operatorname{ad} w$ is locally nilpotent. If w is strictly nilpotent, then there is a well-defined k-automorphism $e^{\operatorname{ad} w}$ of R such that, for $r \in R$,

$$\mathrm{e}^{\operatorname{ad} w}(r) = \textstyle\sum_{i\geq 0} (\operatorname{ad} w)^i(r)/i!.$$

The inverse of $e^{\operatorname{ad} w}$ is $e^{\operatorname{ad}(-w)}$.

We now aim to show that x and y are strictly nilpotent in $A(a)$. For $m \in \mathbb{N}$, let Δ_m denote the linear map $\sigma^m - 1 : k[h] \to k[h]$, that is $\Delta_m(f) = \sigma^m(f) - f$ for all $f \in k[h]$. Then Δ_m is a σ^m-derivation of the algebra $k[h]$, that is $\Delta_m(fg) = \Delta_m(f)g + \sigma^m(f)\Delta_m(g)$, for all $f, g \in k[h]$. For $i \in \mathbb{N}$, $\ker((\Delta_m)^i) = \{f \in k[h] : \deg f < i\}$. Thus Δ_m is locally nilpotent.

Let $f \in k[h]$. For $m \in \mathbb{N}$, direct computations give

$$\operatorname{ad} x^m : x \mapsto 0,\ f \mapsto \Delta_m(f)x^m,\ h \mapsto -mx^m,\ y \mapsto \Delta_m(a)x^{m-1}. \tag{6}$$

For $i \geq 0$, $(\operatorname{ad} x^m)(\Delta_m^i(a)x^{im-1}) = \Delta_m^{(i+1)}(a)x^{(i+1)m-1}$. It follows by induction that, for $i \geq 1$,

$$(\operatorname{ad} x^m)^i(y) = \Delta_m^i(a)x^{im-1}. \tag{7}$$

Similarly,

$$\operatorname{ad} y^m : x \mapsto -y^{m-1}\Delta_m(a),\ f \mapsto -y^m\Delta_m(f),\ h \mapsto my^m,\ y \mapsto 0, \tag{8}$$

and, for $i \geq 1$,

$$(\operatorname{ad} y^m)^i(x) = (-1)^i y^{im-1}\Delta_m^i(a). \tag{9}$$

3.4. Lemma. *Let $d = \deg a(h)$, let $m \geq 0$ be an integer and let $\lambda \in k$. The inner derivations $\operatorname{ad} x^m$ and $\operatorname{ad} y^m$ of $A(a)$ are locally nilpotent. Hence there are k-automorphisms $\Psi_{m,\lambda} := e^{\lambda \operatorname{ad} x^m}$ and $\Phi_{m,\lambda} := e^{\lambda \operatorname{ad} y^m}$ of $A(a)$ such that*

$$\Psi_{m,\lambda} : x \mapsto x,\ h \mapsto h - m\lambda x^m,\ y \mapsto y + \sum_{i=1}^{d} \frac{\lambda^i}{i!}\Delta_m^i(a)x^{im-1} \tag{10}$$

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and

$$\Phi_{m,\lambda} : x \mapsto x + \sum_{i=1}^{d} \frac{(-\lambda)^i}{i!} y^{im-1}\Delta_m^i(a),\ h \mapsto h + m\lambda y^m,\ y \mapsto y. \tag{11}$$

Proof. As $A(a)$ is generated by x, y and h, this is immediate from (6)–(9) and the fact that, for $m \geq 1$, Δ_m is locally nilpotent. □

3.5. Automorphisms. We denote by G the subgroup of $\operatorname{Aut}_k A(a)$ generated by the k-automorphisms of the following three types:

(i) Θ_μ for $0 \neq \mu \in k$ (in the notation of Lemma 2.7(iii)),
(ii) $\Psi_{m,\lambda}$ for an integer $m \geq 0$ and $\lambda \in k$,
(iii) $\Phi_{m,\lambda}$ for an integer $m \geq 0$ and $\lambda \in k$.

3.6. Notation. Let $0 \neq a \in k[h]$. For the remainder of this section, we write A for $A(a)$. By Lemma 3.1(i), we can assume that a is monic. We denote by d the degree of a, and by β the coefficient in a of h^{d-1}, thus

$$a = h^d + \beta h^{d-1} + \dots .$$

3.7. *Remark.* When the algebra B_λ from 3.2 is written in the form $A(a)$ with a monic, a has the form $h^2 + h + \mu$, so that $a(h-1) = a(-h)$. It follows that, in this case, the isomorphism Λ in Lemma 3.1(iii) is an automorphism. This automorphism is denoted by Ω in [12] where it is frequently used to justify symmetry of arguments involving x and y (e and f in [12]). In our more general situation, we can again appeal to symmetry using Λ. Technical results established for x and y will hold in $A(a(-h))$ and hence for y and x in A.

For some choices of a, there does exist an automorphism of $A(a)$ generalizing the above automorphism Ω. If there exists $\rho \in k$ such that $a(\rho - h) = (-1)^d a(h)$ we shall say that a is *reflective*. This definition is independent of the generator h. In particular, if $t = h+\tau$, where $\tau \in k$, and $a = a(h) = b(t)$, then $a(\rho-h) = (-1)^d a(h)$ if and only if $b(\rho+2\tau-t) = (-1)^d b(t)$. Let $\tau = \beta d$ so that the coefficient of t^{d-1} in $a = b(t)$ is 0. Then a is reflective with $a(\rho-h) = (-1)^d a(h)$ if and only if $\rho+2\tau = 0$ and $b(-t) = (-1)^d b(t)$. Consequently, if d is even, then a is reflective if and only if $a \in k[t^2]$ and, if d is odd, then a is reflective if and only if $a \in tk[t^2]$. In particular, all quadratics are reflective but a monic cubic $h^3 + \mu h^2 + \lambda h + \xi$ is reflective if and only if $27\xi = 9\mu\lambda - 2\mu^3$.

3.8. Lemma. *If a is reflective with $a(\rho - h) = (-1)^d a(h)$, then there exists an automorphism Ω of $A(a)$ with $\Omega(x) = y$, $\Omega(y) = (-1)^d x$ and $\Omega(h) = 1 + \rho - h$.*

Proof. In the notation of Lemma 3.1,

$$\Omega = \Gamma_{(-1)^d}\, \Lambda\, \operatorname{tr}_\rho : A(a) \to A((-1)^d a(\rho - h)) = A(a).$$

□

3.9. Filtrations. Let $\mathcal{P}(d)$ denote the set of all ordered pairs (i, j) where i and j are non-negative integers such that $i \equiv j \bmod d$. Let $f_0(h) = 1$ and, for $n \geq 1$, let

$$f_n(h) = h^n + \gamma_n h^{n-1},\ \text{where } \gamma_n := \frac{n(n + 2\beta - d)}{2d}.$$

Note that $\{f_n(h) : n \geq 0\}$ is a k-basis for $k[h]$. It can be easily checked that, for all $i, j \geq 0$,

$$\gamma_{i+j} = \gamma_i + \gamma_j + ij/d. \tag{12}$$

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If $j > i$, with $j = i + md$ and $m > 0$, then set $U_{(i,j)} = f_i(h)x^m$ and if $j \leq i$, with $i = j + md$ and $m \geq 0$, then set $U_{(i,j)} = y^m f_j(h)$. It follows from §2.6 that $\{U_p : p \in \mathcal{P}(d)\}$ is a k-basis for A.

Let $\sigma, \rho \in \mathbb{R}$ be such that $\rho + \sigma > 0$ and let $H_{\rho,\sigma}$ be the additive submonoid of $\mathbb{R}$ generated by σ and ρ. For $p = (i,j) \in \mathcal{P}(d)$, let $v_{\rho,\sigma}(U_p) = \rho i + \sigma j \in H_{\rho,\sigma}$. For $h \in H_{\rho,\sigma}$, let A_h be the k-subspace of A spanned by $\{U_p : v_{\rho,\sigma}(p) \leq h\}$, and let $A_{<h} = \bigcup_{j<h} A_j$. We shall see that $\{A_h\}_{h \in H_{\rho,\sigma}}$ is a filtration of A by the ordered monoid $H_{\rho,\sigma}$; that is,

(i) $A = \bigcup_{h \in H_{\rho,\sigma}} A_h$,
(ii) $A_h \subset A_k$ for all $h < k \in H_{\rho,\sigma}$,
(iii) $A_h A_k \subseteq A_{h+k}$ for all $h, k \in A_h$,
(iv) $\bigcap_{h \in H_{\rho,\sigma}} A_h = 0$.

This coincides with the definition in [29, 1.2.13] except for the reversal of $>$ in (ii). In the language of [27], where only filtrations by $\mathbb{Z}$ are considered, (i) and (iv) would be omitted from the definition of filtration and would say that the filtration is exhaustive and separated. The associated $H_{\rho,\sigma}$-graded ring, $\operatorname{gr} A := \bigoplus_{h \in H_{\rho,\sigma}} A_h/A_{<h}$, of A with respect to this filtration can be constructed as for filtrations by $\mathbb{N}$ or $\mathbb{Z}$, [26, 27, 29]. For $r \in A_h$ and $s \in A_k$, $(r + A_{<h})(s + A_{<k}) := rs + A_{<h+k}$. We shall use the following notation adapted from that used in [26] for filtrations by $\mathbb{N}$: for $w \in A$, $\overline{w} := w + A_{<h}$, where $h = v_{\rho,\sigma}(w) := \min\{h \in H : w \in A_h\}$. Below we specify a commutative $H_{\rho,\sigma}$-graded algebra which will turn out to be isomorphic to $\operatorname{gr} A$.

3.10. Associated graded rings. Let $\mathcal{P}(d), \sigma, \rho$ and $H_{\rho,\sigma}$ be as in §3.9. Let $C^{(d)}$ denote the k-subalgebra of the commutative polynomial algebra $k[T,S]$ generated by T^d, S^d and TS. Then $C^{(d)}$ has a k-basis consisting of monomials of the form T^iS^j where $(i,j) \in \mathcal{P}(d)$. For a monomial $T^iS^j \in C^{(d)}$, let $v_{\rho,\sigma}(T^iS^j) = \rho i + \sigma j$ and, for $0 \neq q = \sum \alpha_{ij} T^iS^j \in C^{(d)}$, let $v_{\rho,\sigma}(q) = \sup_{\alpha_{ij} \neq 0} v_{\rho,\sigma}(T^iS^j)$. Let $v_{\rho,\sigma}(0) = -\infty$. Then there is an $H_{\rho,\sigma}$-grading on $C^{(d)}$ such that, for $h \in H_{\rho,\sigma}$, $C^{(d)}_h$ is spanned by those monomials T^iS^j with $v_{\rho,\sigma}(q) = h$. As indicated above, this will be seen to be the associated $H_{\rho,\sigma}$-graded ring of A for a filtration of A by $H_{\rho,\sigma}$. Furthermore, we shall see that commutation in A induces a Poisson bracket on $C^{(d)}$.

3.11. Poisson bracket. Let $C = C^{(1)} = k[T,S]$. The *Poisson bracket* on C, $\{,\} : C \times C \to C$, is given by

$$\{b,c\} = \frac{\partial b}{\partial T}\frac{\partial c}{\partial S} - \frac{\partial b}{\partial S}\frac{\partial c}{\partial T}.$$

Thus

$$\{T^iS^j, T^pS^q\} = (iq - jp)T^{i+p-1}S^{j+q-1}, \tag{13}$$

from which it follows that, for $d \geq 1$, the subalgebra $C^{(d)}$ is invariant under $\{,\}$ and there is an induced Poisson bracket on $C^{(d)}$. It will be convenient to normalize this bracket and write, for $b, c \in C^{(d)}$,

$$\{b,c\}_d = \frac{1}{d}\left(\frac{\partial b}{\partial T}\frac{\partial c}{\partial S} - \frac{\partial b}{\partial S}\frac{\partial c}{\partial T}\right).$$

For $c \in C^{(d)}$, we denote by $\operatorname{pad} c$ the Poisson-inner derivation of $C^{(d)}$ given by $(\operatorname{pad} c)(b) = \{c,b\}_d$. Thus $\operatorname{pad} c = \frac{1}{d}(\frac{\partial c}{\partial T}\frac{\partial}{\partial S} - \frac{\partial c}{\partial S}\frac{\partial}{\partial T})$.

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3.12. Lemma. *In the notation of 3.9, let* $(i,j),(p,q)\in\mathcal{P}(d)$. *Then*

$$U_{(i,j)}U_{(p,q)}\equiv U_{(i+p,j+q)}-\frac{jp}{d}U_{(i+p-1,j+q-1)} \bmod A_{\rho(i+p-2)+\sigma(j+q-2)}, \tag{14}$$

and

$$[U_{(i,j)},U_{(p,q)}]\equiv \frac{iq-jp}{d}U_{(i+p-1,j+q-1)} \bmod A_{\rho(i+p-2)+\sigma(j+q-2)}. \tag{15}$$

Proof. As (15) is an immediate consequence of (14), it suffices to establish (14) and this requires consideration of four cases, two of which split into subcases.

Case 1: $j\le i$ and $q>p$, with $i=j+dm$ and $q=p+dr$.

Subcase 1A: $r>m$. Here $j+q=(i+p)+d(r-m)$, $U_{(i+p,j+q)}=f_{i+p}(h)x^{r-m}$ and

$$\begin{aligned}U_{(i,j)}U_{(p,q)}&=y^mf_j(h)f_p(h)x^r\\&=y^mx^mf_j(h+m)f_p(h+m)x^{r-m}\\&=\left(\prod_{s=0}^{m-1}a(h+s)\right)f_j(h+m)f_p(h+m)x^{r-m}\qquad\text{(by (5))}\\&=g(h)x^{r-m},\end{aligned}$$

where $g(h)=(\prod_{s=0}^{m-1}a(h+s))f_j(h+m)f_p(h+m)\in k[h]$ is monic of degree $dm+j+p=i+p$. To establish (14) in this case, it suffices to show that the coefficient of h^{i+p-1} in $g(h)$ is $\gamma_{i+p}-jp/d$. The coefficient of h^{i+p-1} in $g(h)$ is

$$\begin{aligned}&\left(\sum_{s=0}^{m-1}(\beta+ds)\right)+(jm+\gamma_j)+(pm+\gamma_p)\\&=m\beta+dm(m-1)/2+m(j+p)+\gamma_{j+p}-jp/d\qquad\text{(by (12))}\\&=\gamma_{dm}+dm(j+p)/d+\gamma_{j+p}-jp/d\\&=\gamma_{dm+j+p}-jp/d\qquad\text{(by (12))}\\&=\gamma_{i+p}-jp/d.\end{aligned}$$

Subcase 1B: $r\le m$. Here $U_{(i+p,j+q)}=y^{m-r}f_{j+q}(h)$ and

$$\begin{aligned}U_{(i,j)}U_{(p,q)}&=y^mf_j(h)f_p(h)x^r\\&=y^{m-r}f_j(h+r)f_p(h+r)y^rx^r\\&=y^{m-r}\left(\prod_{s=0}^{r-1}a(h+s)\right)f_j(h+r)f_p(h+r)\\&=y^{m-r}g(h),\end{aligned}$$

where $g(h)=(\prod_{s=0}^{r-1}a(h+s))f_j(h+r)f_p(h+r)\in k[h]$ is monic of degree $dr+j+p=j+q$. The same calculation as in 1A, with m replaced by r, shows that (14) again holds.

Case 2: $j\le i$ and $q\le p$, with $i=j+dm$ and $p=q+dr$. Here $U_{(i+p,j+q)}=y^{m+r}f_{j+q}(h)$ and

$$\begin{aligned}U_{(i,j)}U_{(p,q)}&=y^mf_j(h)y^rf_q(h)\\&=y^{m+r}f_j(h-r)f_q(h)\\&=y^{m+r}g(h),\end{aligned}$$

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where $g(h)=f_j(h-r)f_q(h)\in k[h]$ is monic of degree $j+q$. The coefficient of h^{j+q-1} in $g(h)$ is $-jr+\gamma_j+\gamma_q=\gamma_{j+q}-(jp)/d$ by (12). Hence (14) holds in this case.

Case 3: $j>i$ and $q>p$, with $j=i+dm$ and $q=p+dr$. Here $U_{(i+p,j+q)}=f_{i+p}(h)x^{m+r}$ and

$$\begin{aligned} U_{(i,j)}U_{(p,q)} &= f_i(h)x^m f_p(h)x^r \\ &= f_i(h)f_p(h-r)x^{m+r} \\ &= g(h)x^{m+r}, \end{aligned}$$

where $g(h)=f_i(h)f_p(h-r)\in k[h]$ is monic of degree $i+p$. The same calculation as in Case 2, with the interchanges $j\leftrightarrow p$ and $i\leftrightarrow q$, shows that (14) holds again.

Case 4: $j>i$ and $q\le p$, with $j=i+dm$ and $p=q+dr$.

Subcase 4A: $m>r$. Here $j+q=(i+p)+d(m-r)$, $U_{(i+p,j+q)}=f_{i+p}(h)x^{m-r}$ and

$$\begin{aligned} U_{(i,j)}U_{(p,q)} &= f_i(h)x^m y^r f_q(h) \\ &= f_i(h)\left(\prod_{s=1}^{r} a(h-s-m+r)\right) f_q(h-m+r)x^{m-r} \qquad \text{(by (5))} \\ &= g(h)x^{m-r}, \end{aligned}$$

where $g(h)=f_i(h)(\prod_{s=1}^{r} a(h-s-m+r))f_q(h-m+r)\in k[h]$ is monic of degree $i+dr+q=i+p$. The coefficient of h^{i+p-1} in $g(h)$ is

$$\begin{aligned} &\left(\sum_{s=0}^{r-1}(d(s-m)+\beta)\right)+\gamma_i+\gamma_q+q(r-m) \\ &= r(-dm+\beta)+dr(r-1)/2+\gamma_{i+q}-iq/d+q(r-m) \qquad \text{(by (12))} \\ &= \gamma_{dr}-rdm+(\gamma_{i+q}-iq/d)+q(r-m) \\ &= \gamma_{dr+i+q}-dr(i+q)/d-rdm-iq/d+q(r-m) \qquad \text{(by (12))} \\ &= \gamma_{i+p}-rj-qj/d \\ &= \gamma_{i+p}-jp/d, \end{aligned}$$

whence (14) holds.

Subcase 4B: $m\le r$. Here $i+p=(j+q)+d(r-m)$, $U_{(i+p,j+q)}=y^{m-r}f_{j+q}(h)$ and

$$\begin{aligned} U_{(i,j)}U_{(p,q)} &= f_i(h)x^m y^r f_q(h) \\ &= y^{r-m}f_i(h-r+m)\left(\prod_{s=1}^{r} a(h-s-r+m)\right) f_q(h) \\ &= y^{r-m}g(h), \end{aligned}$$

where $g(h)=f_i(h-r+m)(\prod_{s=1}^{r} a(h-s-r+m))f_q(h)\in k[h]$ is monic of degree $i+dr+q=i+p$. The same calculation as in 4A, with the interchanges $m\leftrightarrow r$, $i\leftrightarrow q$ and $j\leftrightarrow p$, then shows that (14) holds again. □

3.13. *Remark.* The proof of Lemma 3.12 shows that

$$U_{(i,j)}U_{(p,q)}=U_{(i+p,j+q)}-\frac{jp}{d}U_{(i+p-1,j+q-1)}+\sum_{k\ge 2}\lambda_k U_{(i+p-k,j+q-k)}$$

for some scalars $\lambda_k=\lambda_k((i,j),(p,q))$.

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3.14. Theorem. *In the notation of 3.9,*

(i) $\{A_h\}_{h\in H_{\rho,\sigma}}$ *is a filtration of* A *by* $H_{\rho,\sigma}$.

(ii) *For this filtration,* $\operatorname{gr} A$ *is isomorphic to* $C^{(d)}$*, with the* $H_{\rho,\sigma}$*-grading on* $C^{(d)}$ *specified in 3.10. An isomorphism, which we shall use to identify* $\operatorname{gr} A$ *with* $C^{(d)}$*, is given by*

$$\sum_{(i,j)\in\mathcal{P}(d)} \alpha_{(i,j)}\overline{U_{(i,j)}} \leftrightarrow \sum_{(i,j)\in\mathcal{P}(d)} \alpha_{(i,j)}T^iS^j.$$

(iii) *Let* $w_1, w_2 \in A$ *and let* $z = [w_1, w_2]$*. For* $i = 1, 2$*, let* $v_i = v_{\rho,\sigma}(w_i)$ *and let* $q_i = \overline{w_i}$*. Then* $v_{\rho,\sigma}(w_1w_2) = v_1+v_2$ *and* $v_{\rho,\sigma}(z) \le v_1+v_2-(\rho+\sigma)$*. Moreover,* $v_{\rho,\sigma}(z) = v_1 + v_2 - (\rho + \sigma) \Leftrightarrow \{q_1, q_2\}_d \neq 0$*, in which case* $\overline{z} = \{q_1, q_2\}_d$.

Proof. This all follows directly from Lemma 3.12 and the fact that $\{U_p : p \in \mathcal{P}(d)\}$ is a k-basis for A. □

The next few results, culminating in Lemma 3.19, are aimed at extending key results, [11, Proposition 7.4] and [12, Proposition 3.3], for the cases $d = 1, 2$ to higher degree and in particular to the case $d = 4$ which turns out to require special attention.

3.15. Lemma. *Let* $c = \lambda T^d + \mu S^d \in C^{(d)}$*, where* $\lambda, \mu \in k^*$*, and let* n *be a positive integer. Let* $b \in C^{(d)}$*. If* $\{c^n, b\}_d = 0$*, then* $b \in k[c]$.

Proof. We use the $H_{1,1}$-grading on $C^{(d)}$. Suppose that $\{c, b\}_d = 0$ for some $b \in C^{(d)}\backslash k[c]$ and choose such an element b with $f := v_{1,1}(b)$ minimal. Let $I = \{i \in \mathbb{Z} : 0 \le i \le f, (f - i, i) \in \mathcal{P}(d)\}$ and let u be the leading term of b. Then

$$u = \sum_{i\in I} \alpha_i T^{f-i}S^i,$$

where $\alpha_i \in k$ for each $i \in I$, and $\{c, u\}_d = 0$. Thus

$$0 = \sum_{i\in I} \alpha_i(i\lambda T^{f+d-i-1}S^{i-1} - (f - i)\mu T^{f-i-1}S^{i+d-1}). \tag{16}$$

Let $i \in I$. Comparing coefficients of $T^{f+d-i-1}S^{i-1}$ in (16), if $i - d \notin I$, then $i\alpha_i\lambda = 0$ and if $i - d \in I$, then

$$i\lambda\alpha_i = (f + d - i)\mu\alpha_{i-d}. \tag{17}$$

It follows that if i is minimal such that $i \in I$ and $\alpha_i \neq 0$, then $i = 0$ and that if $j \not\equiv 0 \bmod d$, then $\alpha_j = 0$. Hence $f \equiv 0 \bmod d$ and we can assume that $\alpha_0 = \lambda^m$, where $m = f/d$. By induction on r, it now follows from (17) that, for $0 \le r \le m$,

$$\alpha_{rd} = \binom{m}{r}\lambda^{m-r}\mu^r.$$

Thus u is the leading term of c^m. Then $\{b - c^m, c\}_d = 0$, contradicting the minimality of f. This completes the proof for the case $n = 1$.

For the general case, suppose that $\{c^n, b\}_d = 0$ and let δ be the derivation $\operatorname{pad} b$ of $C^{(d)}$. Then $nc^{n-1}\delta(c) = \delta(c^n) = 0$, so, as $C^{(d)}$ is a domain and $\operatorname{char} k = 0$, $\delta(c) = 0$ and, by the case $n = 1$, $b \in k[c]$. □

3.16. Lemma. *Let* $c = \lambda T^2 + \mu S^2 \in C^{(2)}$*, where* $\lambda, \mu \in k^*$*, and let* $n > 0$ *and* $m \ge 0$ *be integers. There does not exist* $b \in C^{(2)}$ *such that* $\{c^n, b\}_2 = \rho c^m$ *for some non-zero* $\rho \in k$.

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Proof. Again we use the $H_{1,1}$-grading on $C^{(2)}$. Suppose that $\{c^n, b\}_2 = \rho c^m$ for some non-zero $\rho \in k$. As c^m and c^n are homogeneous with $v_{1,1}(c^n) = 2n$ and $v_{1,1}(c^m) = 2m$, we can assume that b is homogeneous with $v_{1,1}(b) = 2(m-n) + 2 \geq 2$. Thus $m \geq n$. With $\delta = -\operatorname{pad} b$, $\rho c^m = \delta(c^n) = nc^{n-1}\delta(c)$, whence $\frac{\rho}{n}c^{m-n+1} = \delta(c) = \{c, b\}_2$ and we may assume that $n = 1$. Thus $\{c, b\}_2 = \rho c^m$ for some non-zero $\rho \in k$ and b is homogeneous with $v_{1,1}(b) = 2m$. The ring $C^{(2)}$ has a $\mathbb{Z}_2$-grading in which the even part, C_+, is spanned by the monomials T^iS^j with i and j both even and the odd part, C_- is spanned by the monomials T^iS^j with i and j both odd. Now $c, c^m \in C_+$, $\{c, C_+\}_2 \subseteq C_-$ and $\{c, C_-\}_2 \subseteq C_+$, so we can assume that $b \in C_-$, that is b has the form

$$TS\left(\sum_{i=0}^{m-1} \alpha_i T^{2(m-i-1)} S^{2i}\right),\ \alpha_i \in k. \tag{18}$$

Then

$$\begin{aligned}
\{c, b\}_2 =& \lambda \sum_{i=0}^{m-1} \alpha_i (2i+1) T^{2(m-i)} S^{2i} \\
&- \mu \sum_{i=0}^{m-1} \alpha_i (2(m-i-1)+1) T^{2(m-i-1)} S^{2(i+1)} \\
=& \lambda\alpha_0 T^{2m} - \mu\alpha_{m-1} S^{2m} \\
&+ \sum_{i=1}^{m-1} \left(\lambda\alpha_i(1+2i) - \mu\alpha_{i-1}(1+2(m-i)\right) T^{2(m-i)} S^{2i}.
\end{aligned}$$

Setting this equal to $\rho c^m = \rho \sum_{i=0}^{m} \binom{m}{i} (\lambda T^2)^{m-i} (\mu S^2)^i$, we see that

(i) $\alpha_0 = \rho\lambda^{m-1}$;
(ii) for $1 \leq i \leq m-1$, $\lambda\alpha_i(1+2i) = \mu\alpha_{i-1}(1+2(m-i)) + \rho\binom{m}{i}\lambda^{m-i}\mu^i$;
(iii) $\alpha_{m-1} = -\rho\mu^{m-1}$.

From (i) and (ii), it follows inductively that, for $0 \leq i \leq m-1$, there exists $q_i \in \mathbb{Q}$, with $q_i > 0$, such that $\alpha_i = \rho q_i \lambda^{m-i-1}\mu^i$. Here $q_0 = 1$ and, for $i > 0$,

$$q_i = \frac{1 + 2(m-i)q_{i-1} + \binom{m}{i}}{1+2i} > 0.$$

In particular, $\alpha_{m-1} = \rho q_{m-1}\mu^{m-1}$, where $q_{m-1} > 0$, contradicting (iii). □

3.17. Lemma. *Let $c = \lambda T^2 + \mu S^2 \in C^{(2)}$, where $\lambda, \mu \in k^*$, and let $n > 0$ and $m \geq 0$ be even integers.*

(i) *For $b \in C^{(4)}$, if $\{c^n, b\}_4 = 0$, then $b \in k[c^2]$.*
(ii) *There does not exist $b \in C^{(4)}$ such that $\{c^n, b\}_4 = \rho c^m$ for some non-zero $\rho \in k$.*

Proof. Note that $C^{(4)} \subset C^{(2)}$ and $\{c^n, b\}_2 = 2\{c^n, b\}_4$.

(i) Let $b \in C^{(4)}$ be such that $\{c^n, b\}_4 = 0$. Then $\{c^n, b\}_2 = 0$ and, by Lemma 3.15, $b \in k[c] \cap C^{(4)} = k[c^2]$.

(ii) If $\{c^n, b\}_4 = \rho c^m$, then $\{c^n, b\}_2 = 2\rho c^m$, which is impossible by Lemma 3.16. □

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3.18. Lemma. *Let $d = 4$ and let $u = c^{2j}$ for some $j \geq 1$, where $c = \lambda T^2 + \mu S^2$ and $\lambda, \mu \in k^*$. Let $b \in C^{(4)} \backslash k[c^2]$ be homogeneous in the $H_{1,1}$-grading. Then $(\operatorname{pad} u)^i(b) \neq 0$ for all positive integers i.*

Proof. Suppose not and choose the least positive integer i such that $(\operatorname{pad} u)^i(b) = 0$. By 3.17(i), $i > 1$. Let $b_1 = (\operatorname{pad} u)^{i-1}(b)$ and $b_2 = (\operatorname{pad} u)^{i-2}(b)$, which are non-zero and homogeneous. Then $\{c^{2j}, b_1\}_4 = 0$ so, by 3.17(i) and homogeneity, $\{c^{2j}, b_2\}_4 = b_1 = \rho c^m$ for some even positive integer m. By 3.17(ii), this is impossible. □

3.19. Lemma. *Let $d = 4$ and let $c = \lambda T^2 + \mu S^2$, where $\lambda, \mu \in k^*$. Let $w \in A$ and let $u = \overline{w} \in \operatorname{gr} A = C^{(4)}$. If $u = c^n$ for some even positive integer n, then $\mathcal{F}(w) \neq A$.*

Proof. We use the $H_{1,1}$-grading on $C^{(4)}$. Let $z \in A$ be such that, in $\operatorname{gr} A$, $\overline{z} \notin k[c^2]$, let $b = \overline{z}$ and let $v = v_{1,1}(z)$. For $i \geq 0$, let $z_i = (\operatorname{ad} w)^i(z)$ and $b_i = (\operatorname{pad} u)^i(b)$. By Lemma 3.18, $b_i \neq 0$ for all $i \geq 0$. Hence, by Theorem 3.14(iii), for each i, $b_i = \overline{z_i}$ and $v_{1,1}(z_i) = v + i(2n-2) \to \infty$ as $i \to \infty$. Thus $z \notin \mathcal{F}(w)$. □

3.20. Lemma. *Let σ and ρ be positive integers. Let $w \in A$, let $v = v_{\rho,\sigma}(w)$ and let $q = q(T, S) = \overline{w} \in \operatorname{gr} A = C^{(d)}$. Suppose that $v > \rho + \sigma$, that q is not a monomial and that $\mathcal{F}(w) = A$. Then one of the following holds:*

(i) *$\sigma > \rho$, σ is a multiple of ρ, and $q = \xi T^\alpha (T^{\sigma/\rho} + \mu S)^\beta$ for some $\xi, \mu \in k^*$ and some integers $\alpha \geq 0$ and $\beta > 0$.*
(ii) *$\sigma < \rho$, ρ is a multiple of σ, and $q = \xi S^\alpha (S^{\rho/\sigma} + \mu T)^\beta$ for some $\xi, \mu \in k^*$ and some integers $\alpha \geq 0$ and $\beta > 0$.*
(iii) *$d = 1$, $\sigma = \rho$ and*

$$q = \xi(\mu T + \nu S)^\alpha (\mu' T + \nu' S)^\beta \tag{19}$$

for some $\xi, \mu, \nu, \mu', \nu' \in k$, with ξ and at least three of μ, ν, μ', ν' non-zero, and some integers $\alpha, \beta \geq 0$, with $\alpha + \beta > 0$.
(iv) *$d = 2$, $\sigma = \rho$ and q has the form in (19) with $\alpha + \beta$ even.*

Proof. As was observed in §3.3, the centre of A is k so $\mathcal{F}(w) \neq \mathcal{C}(w)$. By the proofs of [11, Lemme 7.3 and Proposition 7.4], either (i) holds or (ii) holds or $\rho = \sigma$ and q is as in (19). Note that the non-zero conditions on the parameters are consequences of the fact that $q \neq 0$ and is not a monomial. An element of the form (19) is in $C^{(2)}$ if and only if $\alpha + \beta$ is even. Thus if $d \leq 2$, then (iii) holds or (iv) holds. Therefore we can assume that $d > 2$. To show that if $\rho = \sigma$, then either (iii) or (iv) holds, it suffices to show that if $q \in C^{(d)}$ has the form (19), then $d = 4$, α is even and $q = \gamma(\lambda T^2 + \mu S^2)^\alpha$ for some $\gamma, \lambda, \mu \in k$, with $\gamma \neq 0$ and, as q is not a monomial, $\lambda \neq 0 \neq \mu$. Then, by Lemma 3.19, $\mathcal{F}(w) \neq A$.

We can assume that $\xi = 1$. If all four of μ, ν, μ', ν' are non-zero, we can assume that $\mu = 1 = \mu'$. In this case the coefficients in q of $T^{\alpha+\beta-1}S$ and $TS^{\alpha+\beta-1}$ are $\alpha\nu + \beta\nu'$ and $\alpha\nu^{\alpha-1}\nu'^\beta + \beta\nu^\alpha\nu'^{\beta-1}$ respectively, and as $d > 2$ and $q \in C^{(d)}$, so that d divides $\alpha + \beta$ but not $\alpha + \beta - 2$, these must be 0. Thus

$$\alpha\nu + \beta\nu' = 0 = \alpha\nu' + \beta\nu.$$

As α and β cannot both be 0 and neither is negative, it follows that $\nu = -\nu'$ and $\beta = \alpha$. Thus $q = (T^2 - \nu^2 S^2)^\alpha$, which has non-zero coefficients of $T^{2\alpha}$ and $T^{2\alpha-2}S^2$. Hence d divides 2α and $2\alpha - 4$. As $d > 2$ it follows that $d = 4$ and α is even. If only three of μ, ν, μ', ν' are non-zero, easy calculations show that either

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the coefficients of $T^{\alpha+\beta}$ and $T^{\alpha+\beta-1}S$ are non-zero or the coefficients of $S^{\alpha+\beta}$ and $TS^{\alpha+\beta-1}$ are not zero and hence, in this case, that $q \notin C^{(d)}$ if $d > 2$. □

3.21. Automorphisms of $\operatorname{gr} A$. Let n be a positive integer and, for $\mu \in k$, consider the automorphism $\Phi_{n,\mu}$. By (11), this is a filtered automorphism for the filtration of A by $H_{1,dn-1}$, that is $\Phi_{n,\mu}(A_h) = A_h$ for all $h \in H_{1,dn-1}$. Consequently, it induces an automorphism $\overline{\Phi_{n,\mu}}$ of the associated $H_{1,dn-1}$-graded ring $C^{(d)}$. As the leading term (in the usual grading of $k[h]$) of $\frac{1}{i!}\Delta_n^i(a)$ is $(-n)^i\binom{d}{i}$, it follows from (11) that $\overline{\Phi_{n,\mu}}$ acts as the restriction to $C^{(d)}$ of the automorphism of $k[T, S]$ such that

$$T \mapsto T,\ S \mapsto S + n\mu T^{dn-1}. \tag{20}$$

3.22. Lemma. *Let $w \in A$ be of the form*

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_r x^r + \alpha h$$

with $0 \neq \alpha \in k$ and each $\alpha_j \in k$. There exist $\Phi \in G$ and $\beta, \beta_0 \in k$ such that $\Phi(w) = \beta_0 + \beta h$.

Proof. The proof is as for [12, Lemme 5.1]. For an appropriate value of μ, $\Phi_{r,\mu}(w) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{r-1}x^{r-1} + \alpha h$ and the result follows by induction. □

3.23. Lemma. *Let $w = \sum \alpha_{i,j}U_{(i,j)} \in A$. Let r be the least non-negative integer such that $\alpha_{i,0} = 0$ whenever $i > r$. Let s be the least non-negative integer such that $\alpha_{0,j} = 0$ whenever $j > s$. Suppose that there exist integers i_1, j_1 such that:*

(1) $\alpha_{i_1,j_1} \neq 0$;
(2) $(i_1, j_1) \neq (1, 1)$;
(3) either $si_1 + rj_1 > rs$ or $r = s = 0$ and $(i_1, j_1) \neq (0, 0)$. Then $\mathcal{F}(w) \neq A$.

Proof. The proof is an adaptation of the proof of [12, Lemme 5.2], which in turn was adapted from the proof of [11, Lemme 8.7]. As in [12], by hypothesis (3), $i_1 > 0, j_1 > 0$ and there exist positive real numbers ρ, σ, linearly independent over $\mathbb{Q}$, such that $\sigma i_1 + \rho j_1 > \rho s$ and $\sigma i_1 + \rho j_1 > \sigma r$. There exist $i_2, j_2 \geq 0$ such that $\alpha_{i_2,j_2} \neq 0$, $v_{\rho,\sigma}(w) = \sigma i_2 + \rho j_2$ and, in $\operatorname{gr} A$ for the filtration of A by $H_{\rho,\sigma}$, $\overline{w} = \alpha_{i_2,j_2}T^{i_2}S^{j_2}$. As in [12], $i_2 > 1$ or $j_2 > 1$. By symmetry (see Remark 3.7), we can suppose that $i_2 \leq j_2$.

For $n \geq 0$, let $x_n = \delta^n(x)$, where $\delta = \operatorname{ad} w$. As in [12], one shows by induction on n that $\overline{x_n} = \beta_n T^{n(i_2-1)}S^{d+n(j_2-1)}$, for some $0 \neq \beta_n \in k$. This is true for $n = 0$ as $\overline{x} = S^d$. Supposing it to be true for some $n \geq 0$ and applying Theorem 3.14(iii) to w and x_n, one finds that $\overline{x_{n+1}} = \frac{1}{d}(di_2 + nj_2 - ni_2)\alpha_{i_2,j_2}\beta_n T^{(n+1)(i_2-1)}S^{d+(n+1)(j_2-1)}$. It follows that $v_{\rho,\sigma}(x_n) = \sigma n(i_2 - 1) + \rho(d + n(j_2 - 1)) \to \infty$ as $n \to \infty$, whence $x \notin \mathcal{F}(w)$ and $\mathcal{F}(w) \neq A$. □

3.24. *Remark.* In the proofs of the next two lemmas, we can assume that $d > 2$ for otherwise, as observed in §3.2, either $A \simeq B_\lambda$ for some λ, in which case [12, Lemme 5.3 and Lemme 5.4] are applicable, or $A \simeq A_1(k)$, in which case we can apply [11, Lemme 8.8, Théorème 9.2 and proof of Lemme 8.4].

3.25. Lemma. *Let $w \in A$ be such that $\mathcal{F}(w) = A$. There exists $\Phi \in G$ such that $\Phi(w) \in k[x]$ or $\Phi(w) \in k[y]$ or $\Phi(w) = \lambda x + \eta h + \mu y + \tau$ for some $\lambda, \eta, \mu, \tau \in k$.*

Proof. Again this is adapted from the proofs of [11, Lemme 8.8] and [12, Lemme 5.3]. Let $r = r(w)$ and $s = s(w)$ be as in Lemma 3.23. Write $w = \sum \alpha_{i,j}U_{(i,j)}$.

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Applying Lemmas 3.22 and 3.23, the result is true if $r = 0$ or $s = 0$ so, by symmetry (see Remark 3.7), it suffices to consider the case where $r \geq s \geq d$. Let $n = r + s$. By induction, we can assume that the result is true whenever $r(w) + s(w) < n$. As $r \geq s > 2$, we have $rs > r + s$.

We work with the filtration of A by $H_{s,r}$. By Lemma 3.23, if $\alpha_{i,j} \neq 0$, then either $i = j = 1$, in which case $v_{s,r}(U_{(i,j)}) = s + r < rs$, or $v_{s,r}(U_{(i,j)}) = is + jr \leq rs$. As $\alpha_{r,0} \neq 0$ and $\alpha_{0,s} \neq 0$, $v_{s,r}(w) = rs$, and $\overline{w}$ has the form

$$q = \beta_0 T^r + \ldots + \beta_s S^s \text{ where } \beta_0 \neq 0, \beta_s \neq 0. \tag{21}$$

As $q \in C^{(d)}$, $r = jd$ and $s = td$ for some positive integers j, t.

By Lemma 3.20, the possibility that $r = s$ only arises when either $d = 1$ or $d = 2$. Thus we may suppose that $r > s$. By Lemma 3.20, $s|r$ and $q = \lambda(T^m + \mu S)^s$ for some $\lambda, \mu \in k^*$, where $m = \frac{r}{s}$. As $q \in C^{(d)}$, $d|s$ and, as the coefficient of $T^m S^{s-1}$ is non-zero, $d|s - 1 - m$. Thus $d|m + 1$ and there exist positive integers n, t such that $s = dt$, $m = dn - 1$ and $r = (dn - 1)s = (dn - 1)dt$. Thus $q = \lambda(T^{dn-1} + \mu S)^{dt}$ has the form

$$q = \sum_{i=0}^{dt} \beta_i T^{(dn-1)(dt-i)} S^i. \tag{22}$$

Let $\nu \in k$. By (20), $v_{1,dn-1}(\overline{\Phi_{n,\nu}}(q)) \leq v_{1,dn-1}(q) = dt(dn - 1)$ and the coefficients of S^{dt} and $T^{dt(dn-1)}$ in $\overline{\Phi_{n,\nu}}(q)$ are β_{dt} and

$$\sum_{i=0}^{dt} \beta_i n^i \nu^i \tag{23}$$

respectively. We can choose ν so that (23) is 0. Then $r(\Phi_{n,\nu}(w)) < r(w)$ and $s(\Phi_{n,\nu}(w)) = s(w)$ so, by induction there exists $\Phi_1 \in G$ such that $\Phi_1(\Phi_{n,\nu}(w)) \in k[y]$ or $\Phi_1(\Phi_{n,\nu}(w)) = \lambda x + \eta h + \mu y + \tau$ for some $\lambda, \gamma, \mu, \tau \in k$. The result follows on setting $\Phi = \Phi_1 \Phi_{n,\nu} \in G$. □

3.26. Proposition. *Let $w \in A$. Then w is strictly semisimple if and only if there exists $\Phi \in G$ such that $\Phi(w) = \gamma h + \tau$, for some $\gamma, \tau \in k$ with $\gamma \neq 0$.*

Proof. Suppose that w is strictly semisimple. Then w is not strictly nilpotent so $w \notin k[y]$ and, by Lemma 3.25, there exists an automorphism $\Phi_1 \in G$ such that $w_1 := \Phi_1(w) = \lambda x + \gamma h + \mu y + \tau$ for some $\lambda, \gamma, \mu, \tau \in k$. As $d > 2$, $\overline{w_1} = \lambda T^d + \mu S^d$. By Lemma 3.20 with $\rho = \sigma = 1$, this must be a monomial so either $\lambda = 0$ or $\mu = 0$.

Without loss of generality, suppose $\lambda = 0$. Thus $\Phi_1(w) = \gamma h + \mu y + \tau$ and, as y is strictly nilpotent, $\gamma \neq 0$. Now let $\Phi = \Phi_{1,\rho} \Phi_1 \in G$, where $\rho = -\gamma^{-1}\mu$. By (10), $\Phi_{1,\rho}(y) = y$ and $\Phi_{1,\rho}(h) = h + \rho y$, so $\Phi(w) = \gamma h + \tau$.

Conversely, if $w = \Phi^{-1}(\gamma h + \tau)$, for some $\gamma \in k^*, \tau \in k$ and $\Phi \in G$, then, as h is strictly semisimple, so too is w. □

3.27. Theorem. *Let $x_1, y_1, h_1 \in A$. Suppose that x_1, y_1 and h_1 generate A and satisfy the relations*

$$x_1 h_1 = (h_1 - 1)x_1, y_1 h_1 = (h_1 + 1)y_1, x_1 y_1 = a_1(h_1 - 1), y_1 x_1 = a_1(h_1)$$

for some monic $a_1 \in k[h]$. There exists $\Phi \in G$ such that, for some $\tau \in k$, either $\Phi(x_1) = x, \Phi(y_1) = y$, $\Phi(h_1) = h + \tau$ and $a_1(h + \tau) = a(h)$, or $\Phi(x_1) = y, \Phi(y_1) = (-1)^d x$, $\Phi(h_1) = \tau - h$ and $(-1)^d a_1(\tau - h - 1) = a(h)$.

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Proof. There is a surjection $A(a_1) \twoheadrightarrow A(a)$ so, as $\mathrm{GK}(A(a_1)) = 2 = \mathrm{GK}(A(a))$, it follows from [23, Proposition 3.15] that $A(a_1) \simeq A(a)$.

As h_1 is strongly semisimple, by Lemma 3.26, there exists $\Phi \in G$ such that $\Phi(h_1) = \gamma h + \tau$, for some $\gamma, \tau \in k$, with $\gamma \neq 0$. Let $x_2 = \Phi(x_1)$ and $y_2 = \Phi(y_1)$. The set of eigenvalues for $\operatorname{ad} h_1$ is $\mathbb{Z}$ and the set of eigenvalues for $\operatorname{ad}(\gamma h + \tau)$ is $\gamma\mathbb{Z}$ so $\gamma = \pm 1$.

Suppose that $\gamma = 1$, that is $\Phi(h_1) = h + \tau$. In this case $[h, x_2] = \Phi([h_1, x_1]) = \Phi(x_1) = x_2$ and similarly $[h, y_2] = -y_2$. The eigenspace for $\operatorname{ad} h$ for the eigenvalues 1 and -1 are $k[h]x$ and $yk[h]$ respectively. Hence $x_2 = b(h)x$ and $y_2 = yc(h)$ for some $b, c \in k[h]$. Then

$$x_2 y_2 = b(h)xyc(h) = b(h)a(h-1)c(h) \tag{24}$$

and

$$y_2 x_2 = yc(h)b(h)x = c(h+1)yxb(h+1) = c(h+1)a(h)b(h+1). \tag{25}$$

But

$$x_2 y_2 = \Phi(x_1 y_1) = \Phi(a_1(h-1)) = a_1(h+\tau-1) \tag{26}$$

and

$$y_2 x_2 = \Phi(y_1 x_1) = \Phi(a_1(h)) = a_1(h+\tau). \tag{27}$$

Therefore $a_1(h+\tau) = c(h+1)a(h)b(h+1)$. Hence $\deg a_1 \geq \deg a$ and, by symmetry, $\deg a \geq \deg a_1$, $A(a_1)$ being isomorphic to $A(a)$. Thus $\deg a_1 = \deg a$ and $\deg c = \deg b = 0$. By monicity, $a_1(h+\tau) = a(h)$ and $bc = 1$. Applying an automorphism from §3.5(i), we can assume that $c = b = 1$, whence $\Phi(x_1) = x$ and $\Phi(y_1) = y$.

Now suppose that $\gamma = -1$, that is $\Phi(h_1) = \tau - h$. Here $[h, x_2] = -x_2$ and $[h, y_2] = y_2$ so $x_2 = yb(h)$ and $y_2 = c(h)x$ for some $b, c \in k[h]$. Then

$$x_2 y_2 = yb(h)c(h)x = b(h+1)a(h)c(h+1) \tag{28}$$

and

$$y_2 x_2 = c(h)xyb(h) = c(h)a(h-1)b(h). \tag{29}$$

But $x_2 y_2 = a_1(\tau - h - 1)$ and $y_2 x_2 = a_1(\tau - h)$. Therefore $a_1(\tau - h - 1) = b(h+1)a(h)c(h+1)$ and $\deg a_1 \geq \deg a$. By symmetry, $\deg a_1 = \deg a = d$ and $\deg c = \deg b = 0$. By monicity, $a_1(\tau - h - 1) = (-1)^d a(h)$ and $bc = (-1)^d$. We can assume that $b = 1$ and $c = (-1)^d$, whence $\Phi(x_1) = y$ and $\Phi(y_1) = (-1)^d x$. □

3.28. Theorem. *For $a_1, a_2 \in k[h]$, $A(a_1) \simeq A(a_2)$ if and only if $a_2(h) = \eta a_1(\tau \pm h)$ for some $\eta, \tau \in k$ with $\eta \neq 0$.*

Proof. If a_1 and a_2 are monic, then this is immediate from Theorem 3.27. The general case reduces to the monic case on application of isomorphisms of the form given in Lemma 3.1(i). □

3.29. Theorem. *If a is reflective, then $\operatorname{Aut}_k A(a)$ is generated by G and Ω. If a is not reflective, then $\operatorname{Aut}_k A(a) = G$.*

Proof. Let $\Gamma \in \operatorname{Aut}_k A(a)$. By Theorem 3.27, with $x_1 = \Gamma(x), y_1 = \Gamma(y), h_1 = \Gamma(h)$ and $a_1 = a$, there exist $\Phi \in G$ and $\tau \in k$ such that either $x = \Phi\Gamma(x), y = \Phi\Gamma(y)$, $h + \tau = \Phi\Gamma(h)$ and $a(h+\tau) = a(h)$ or $\Phi\Gamma(x) = y, \Phi\Gamma(y) = (-1)^d x$, $\Phi\Gamma(h) = \tau - h$ and $(-1)^d a(\tau - h - 1) = a(h)$. In the latter case a is reflective, with $\rho = \tau - 1$, and $\Gamma = \Phi^{-1}\Omega \in \langle G, \Omega \rangle$. In the former case, which must hold if a is not reflective, τ must be 0 and $\Gamma = \Phi^{-1} \in G$. □

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3.30. *Remark.* When $d = 2$, $\Omega \in G$; see [12, 4.4]. It would be interesting to know when $\Omega \in G$.

3.31. Proposition. *Let $w \in A$. Then w is strictly nilpotent if and only if there exists $\Phi \in G$ such that $\Phi(w) \in k[x]$ or $\Phi(w) \in k[y]$.*

Proof. This result holds when $d \leq 2$ by [11, Théorème 9.1] and [12, Théorème 6.2], so we may assume that $d > 2$. By Lemma 3.4, x and y are strictly nilpotent and it follows, as in [12, 4.1], that if there exists an automorphism Φ as stated, then w is strictly nilpotent.

Suppose that w is strictly nilpotent. By Lemma 3.25, there exists $\Phi \in G$ such that $\Phi(w) \in k[x]$ or $\Phi(w) \in k[y]$ or $\Phi(w) = \lambda x + \eta h + \mu y + \tau$ for some $\lambda, \eta, \mu, \tau \in k$. It suffices to show, in the third possibility, that $\eta = 0$ and that $\lambda = 0$ or $\mu = 0$. Let $z = \Phi(w)$. In the $H_{1,1}$-grading, $\overline{z} = \lambda T^d + \mu Y^d$ so it follows from Lemma 3.20 that this is a monomial, that is $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then y is an eigenvector for $\operatorname{ad} z$ with eigenvalue $-\eta$ and if $\mu = 0$, then x is an eigenvector for $\operatorname{ad} z$ with eigenvalue η. As z is strictly nilpotent, $\eta = 0$ as required. □

3.32. *Remark.* When a is reflective $\Omega(x) = y$ so the criterion for w to be strictly nilpotent becomes the existence of $\Phi \in \operatorname{Aut}_k A$ such that $\Phi(w) \in k[x]$.

4. Algebras similar to $U(\mathfrak{sl}_2)$

Smith [31] considers a class of algebras similar to the enveloping algebra of $\mathfrak{sl}_2$. Let $f = f(H) \in k[H]$. We denote by $R(f)$ the k-algebra generated by A, B and H subject to the relations

$$[H, A] = A,\ [H, B] = -B,\ [A, B] = f(H).$$

As an ambiskew polynomial ring, $R(f) = R(k[H], \sigma, f(H), 1)$, where σ is the k-automorphism of $k[H]$ such that $\sigma(H) = H - 1$. By Lemma 2.4 $R(f) = k[H, W](\sigma, W)$, where $\sigma(H) = H - 1$ and $\sigma(W) = W + f(H)$.

We continue to assume that $\operatorname{char} k = 0$. In [31] $k = \mathbb{C}$, but the proofs of those results we quote from [31] are valid more generally. By [31, Proposition 1.5], the centre of $R(f)$ is generated by the Casimir element

$$C := AB + BA + \tfrac{1}{2}(u(H+1) - u(H)) = 2AB + u(H) = 2BA + u(H+1),$$

where $u \in k[H]$ is such that $f(H) = u(H+1) - u(H)$ and is unique up to addition of scalars. Let x, y and h denote the images in $R(f)/CR(f)$ of A, B and H respectively. Then the k-algebra $R(f)/CR(f)$ is generated by x, y, h subject to the relations

$$xh = (h-1)x,\ yh = (h+1)y,\ xy = -\tfrac{1}{2}u(h),\ yx = -\tfrac{1}{2}u(h+1).$$

Thus $R(f)/CR(f) = A(a)$, where $a(h) = -\frac{1}{2}u(h+1)$. More generally, for $\gamma, \delta \in k$, with $\gamma \neq 0$,

$$R(f)/(\gamma C - \delta)R(f) = A(\tfrac{\delta}{2\gamma} - \tfrac{1}{2}u(h+1)). \tag{30}$$

In [31, Lemma 6.1 and Remark 2], Smith notes the first two parts of the following lemma and remarks that he believes it will be very difficult to understand precisely when $R(f_1) \simeq R(f_2)$.

4.1. Lemma. *Let $f_1, f_2 \in k[H]$.*

(i) *If $f_2 = \gamma f_1$ for some non-zero $\gamma \in k$, then $R(f_1) \simeq R(f_2)$ via $A \mapsto \gamma^{-1}A, B \mapsto B$ and $H \mapsto H$.*

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(ii) *If* $f_2(H) = f_1(H+\tau)$ *for some* $\tau \in k$*, then* $R(f_1) \simeq R(f_2)$ *via* $A \mapsto A, B \mapsto B$ *and* $H \mapsto H+\tau$*.*

(iii) *If* $f_2(H) = f_1(-H)$*, then* $R(f_1) \simeq R(f_2)$ *via* $A \mapsto B, B \mapsto -A$*, and* $H \mapsto -H$*.*

Proof. (i), (ii) and (iii) are special cases of Lemma 2.8(i), (v) and (vi), respectively. In (iii), one applies 2.8 with $p = 1$ and $\tau(H) = -H$. □

Combining the three parts of Lemma 4.1, $R(f_1) \simeq R(f_2)$ if $f_2(H) = \gamma f_1(\tau \pm H)$ for some $\gamma, \tau \in k$ with $\gamma \neq 0$. We shall now show that this condition is also necessary for $R(f_1) \simeq R(f_2)$.

4.2. Theorem. *For* $f_1(H), f_2(H) \in k[H]$*,* $R(f_1) \simeq R(f_2)$ *if and only if* $f_2(H) = \eta f_1(\tau \pm H)$ *for some* $\eta, \tau \in k$ *with* $\eta \neq 0$*.*

Proof. Lemma 4.1 establishes that the given condition is sufficient for $R(f_1) \simeq R(f_2)$. For necessity, suppose that there is an isomorphism $\theta : R(f_1) \to R(f_2)$. For $i = 1, 2$, let $C_i = 2AB + u_i(H) = 2BA + u_i(H+1) \in R(f_i)$, where $u_i \in k[H]$ is such that $f_i(H) = u_i(H+1) - u_i(H)$. The centre of R_i is $k[C_i]$ and so there exist $\gamma, \delta \in k$, with $\gamma \neq 0$, such that $\theta(C_1) = \gamma C_2 - \delta$. There is an induced isomorphism $R(f_1)/C_1R(f_1) \simeq R(f_2)/(\gamma C_2 - \delta)R(f_2)$ so, by (30),

$$A(-\tfrac{1}{2}v_1) \simeq A(\sigma, \tfrac{\delta}{2\gamma} - \tfrac{1}{2}v_2),$$

where each $v_i(h) = u_i(h+1)$. By Lemma 3.1(i),

$$A(v_1) \simeq A(\rho + v_2),$$

where $\rho = -\frac{\delta}{\gamma} \in k$. By Theorem 3.28, there exist $\eta, \tau \in k$, with $\eta \neq 0$, such that $\rho + v_2(h) = \eta v_1(\tau \pm h)$. Then

$$\begin{aligned} f_2(H) &= u_2(H+1) - u_2(H) \\ &= v_2(H) - v_2(H-1) \\ &= (v_2(H) + \rho) - (v_2(H-1) + \rho) \\ &= \eta v_1(\tau \pm H) - \eta v_1(\tau \pm (H-1)) \\ &= \eta(u_1(\tau \pm H + 1) - u_1(\tau \pm H \mp 1 + 1)) \\ &= \eta f_1(\tau + H) \text{ or } -\eta f_1(1 + \tau - H). \end{aligned}$$

□

5. Algebras similar to the primitive factors of $U_q(\mathfrak{sl}_2)$

In this section we concentrate on the case where D is the Laurent polynomial ring $k[h^{\pm 1}]$ and $\sigma(h) = qh$, where $q \in k^*$ is not a root of unity. Let $a = a(h) \in k[h^{\pm 1}]$ and let $W = k[h^{\pm 1}](\sigma, a)$. Thus W is the k-algebra generated by h, h^{-1}, x and y subject to the relations

$$xh = qhx,\ yh = q^{-1}hy,\ xy = a(qh),\ yx = a(h).$$

Examples include the minimal primitive factors of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, for which the details will be given in Example 5.3. These algebras may be viewed as quantizations of those considered in Section 3 but the isomorphism problem is influenced by the existence of the unit h. The following lemma lists some routine consequences of the definitions, Lemma 2.7(ii) and the fact that $k[h^{\pm 1}]$ is a domain. Here $U(W)$ denotes the group of units of W.

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5.1. Lemma. (i) *If $a \in k[h^{\pm 1}]$ is a non-zero non-unit, then $U(W)/k^*$ is cyclic, generated by h.*

(ii) *If $a = \lambda h^n$ is a unit in $k[h^{\pm 1}]$, then $U(W)/k^*$ is free abelian of rank two, generated by h and x, and W is isomorphic to the skew Laurent polynomial ring $k[h^{\pm 1}][x^{\pm 1};\sigma]$.*

(iii) *For each positive integer n, $\{w \in W : h^{-1}wh = q^n w\} = k[h^{\pm 1}]x^n$ and $\{w \in W : h^{-1}wh = q^{-n}w\} = y^n k[h^{\pm 1}]$.*

5.2. Theorem. *Let $0 \neq a_1, a_2 \in k[h^{\pm 1}]$, let $q \in k^*$ and let $\sigma \in \mathrm{Aut}_k\, k[h^{\pm 1}]$ be such that $\sigma(h) = qh$. Then $k[h^{\pm 1}](\sigma, a_1) \simeq k[h^{\pm 1}](\sigma, a_2)$ if and only if $a_2(h) = \eta h^m a_1(\mu h^{\pm 1})$ for some $\eta, \mu \in k^*$ and some integer m.*

Proof. For $i = 1, 2$, let $W_i = k[h^{\pm 1}](\sigma, a_i)$. By Lemma 5.1, we can assume that a_1 and a_2 are non-units; if both are units, then the stated condition holds and, by 5.1(ii), $W_1 \simeq k[h^{\pm 1}][x^{\pm 1};\sigma] \simeq W_2$, while if only one is a unit, then the stated condition fails and, by 5.1(i) and (ii), $W_1 \not\simeq W_2$.

If $a_2(h) = \eta h^m a_1(\mu h)$, then, by Lemma 2.7(ii), with $\lambda = \eta h^m$, and Lemma 2.7(v) with $\tau(h) = \mu h$, $W_1 \simeq W_2$ with

$$x \mapsto x\eta^{-1}h^{-m},\ y \mapsto y \text{ and } h \mapsto \mu h.$$

If $a_2(h) = \eta h^m a_1(\mu h^{-1})$, then, by Lemma 2.7(ii) and Lemma 2.7(vi), with $\tau(h) = q^{-1}\mu h^{-1}$ so that $\tau\sigma(h) = \mu h^{-1} = \sigma^{-1}\tau(h)$, $W_1 \simeq W_2$ with

$$x \mapsto y,\ y \mapsto x\eta^{-1}h^{-m}, \text{ and } h \mapsto q^{-1}\mu h^{-1}.$$

For the converse, suppose that there is an isomorphism $\Gamma : W_1 \to W_2$. By Lemma 5.1(i), there exists $\tau \in k^*$ such that $\Gamma(h) = \tau h^{\pm 1}$. Suppose first that $\Gamma(h) = \tau h$. By Lemma 5.1(iii), there exist $f_1, g_1 \in k[h^{\pm 1}]$ such that $\Gamma(x) = f_1(h)x$ and $\Gamma(y) = yg_1(h)$. In W_2,

$$\begin{aligned} a_1(\tau h) &= \Gamma(a_1(h)) = \Gamma(yx) = yg_1(h)f_1(h)x \\ &= g_1(q^{-1}h)f_1(q^{-1}h)yx = g_1(q^{-1}h)f_1(q^{-1}h)a_2(h). \end{aligned}$$

Hence

$$a_1(h) = g_1(\tau^{-1}q^{-1}h)f_1(\tau^{-1}q^{-1}h)a_2(\tau^{-1}h). \tag{31}$$

By symmetry, there exist $f_2, g_2 \in k[h^{\pm 1}]$ such that

$$a_2(h) = g_2(\tau q^{-1}h)f_2(\tau q^{-1}h)a_1(\tau h). \tag{32}$$

Using (32) to substitute in (31) for $a_2(\tau^{-1}h)$, we obtain

$$a_1(h) = g_1(\tau^{-1}q^{-1}h)f_1(\tau^{-1}q^{-1}h)g_2(q^{-1}h)f_2(q^{-1}h)a_1(h).$$

From this it follows that f_1, f_2, g_1 and g_2 are all units in $k[h^{\pm 1}]$ and hence, by (32), that $a_2(h) = \eta h^m a_1(\tau h)$ for some $\eta \in k^*$ and some integer m.

Now suppose that $\Gamma(h) = \tau h^{-1}$. By Lemma 5.1(iii), there exist $f_1, g_1 \in k[h^{\pm 1}]$ such that $\Gamma(y) = f_1(h)x$ and $\Gamma(x) = yg_1(h)$. In W_2,

$$\begin{aligned} a_1(\tau h^{-1}) &= \Gamma(a_1(h)) = \Gamma(yx) \\ &= f_1(h)xyg_1(h) = f_1(h)a_2(qh^{-1})g_1(h). \end{aligned}$$

Hence

$$a_1(h) = f_1(\tau^{-1}h^{-1})g_1(\tau^{-1}h^{-1})a_2(q\tau^{-1}h^{-1}). \tag{33}$$

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By symmetry, there exist $f_2, g_2 \in k[h^{\pm 1}]$ such that

$$a_2(h) = f_2(\tau^{-1}h^{-1})g_2(\tau^{-1}h^{-1})a_1(q\tau^{-1}h^{-1}). \tag{34}$$

Using (32) to substitute in (31) for $a_2(q\tau^{-1}h^{-1})$, we obtain

$$a_1(h) = f_1(\tau^{-1}h^{-1})g_1(\tau^{-1}h^{-1})f_2(q^{-1}h)g_2(q^{-1}h)a_1(h).$$

It follows that f_1, f_2, g_1 and g_2 are all units in $k[h^{\pm 1}]$ and hence, by (34), that $a_2(h) = \eta h^m a_1(q\tau^{-1}h^{-1})$ for some $\eta \in k^*$ and some integer m. □

5.3. Example. When q is not a root of unity, the minimal primitive ideals of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ are generalized Weyl algebras of the form $C_\mu := k[t^{\pm 1}](\sigma, u - \mu)$, where $\sigma(t) = q^2 t$, $\mu \in k$ and $u = -\frac{q^{-2}t^2 + q^2 t^{-2}}{(q^2 - q^{-2})^2}$; see [16, 7.9]. A simple calculation based on Theorem 5.2 shows that, for $\lambda, \mu \in k$, $C_\lambda \simeq C_\mu$ if and only if $\mu = \pm\lambda$. There are isomorphisms $\Gamma_1, \Gamma_2 : C_\lambda \to C_{-\lambda}$ such that

$$\Gamma_1(x) = -x,\ \Gamma_1(y) = y \text{ and } \Gamma_1(t) = it$$

and

$$\Gamma_2(x) = -y,\ \Gamma_2(y) = x \text{ and } \Gamma_2(t) = iq^2 t^{-1}.$$

The first of these was noted in [16, 7.9] for two particular values of λ, namely those for which C_λ has infinite global dimension.

Because of the term h^m, Theorem 5.2 cannot be lifted in the way that Theorem 3.28 was lifted modulo a central element to yield Theorem 4.2. In the analogue of Theorem 4.2, the term h^m disappears.

5.4. Theorem. *For $i = 1, 2$, let R_i be the k-algebra generated by $H^{\pm 1}, X$ and Y subject to the relations*

$$XH = qHX,\ YH = q^{-1}HY,\ XY - YX = v_i(H),$$

where $q \in k^*$ is not a root of unity and $0 \neq v_i(H) \in k[H^{\pm 1}]$. Suppose that both v_1 and v_2 have zero constant term. Then $R_1 \simeq R_2$ if and only if $v_2(H) = \eta v_1(\mu H^{\pm 1})$ for some $\eta, \mu \in k^*$.

Proof. Each R_i is an ambiskew polynomial ring, $R(k[H^{\pm 1}], \sigma, v_i, 1)$, where $\sigma(H) = qH$. By the condition on constant terms, there exist $u_1, u_2 \in k[H^{\pm 1}]$ such that each $v_i = u_i(qH) - u_i(H)$. For $i = 1, 2$, let $Z_i = YX - u_i(H) = XY - u_i(qH)$. By [17, 2.1(ii)], for $i = 1, 2$, $k[Z_i]$ is the centre of R_i. As a generalized Weyl algebra, $R_i = k[H^{\pm 1}, Z](\sigma, Z + u_i)$, where $\sigma(H) = qH$ and $\sigma(Z) = Z$.

Suppose that there is a k-isomorphism $\Gamma : R_1 \to R_2$. It is easy to see that the units of R_i have the form λH^m, where $\lambda \in k^*$ and $m \in \mathbb{Z}$. Hence there exists $\lambda \in k^*$ such that $\Gamma(H) = \lambda H^{\pm 1}$.

There exist $\beta \in k^*$ and $\gamma \in k$ such that $\Gamma(Z_1) = \beta Z_2 + \gamma$. Then

$$\Gamma(YX) = \Gamma(Z_1 + u_1(H)) = \beta Z_2 + \gamma + u_1(\lambda H^{\pm 1}). \tag{35}$$

It follows from consideration of degrees in Y and X in the iterated skew polynomial ring $R_2 = k[H^{\pm 1}][X; \sigma][Y; \sigma^{-1}, \delta]$ that one of $\Gamma(X)$ and $\Gamma(Y)$ has the form $f_1(H)Y + g_1(H)$, $f_1, g_1 \in k[H^{\pm 1}]$, and the other has the form $Xf_2(H) + g_2(H)$, $f_2, g_2 \in k[H^{\pm 1}]$.

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Suppose that $\Gamma(H) = \lambda H$. As $H^{-1}XH = qX$ and $H^{-1}YH = q^{-1}Y$, we must have $\Gamma(X) = Xf_2(H)$ and $\Gamma(Y) = f_1(H)Y$. By (35),

$$f_1(H)f_2(H)YX = \Gamma(YX) = \beta(YX - u_2(H)) + \gamma + u_1(\lambda H),$$

whence $u_2(H) = \beta^{-1}(u_1(\lambda H) + \gamma)$ and $v_2(H) = \beta^{-1}v_1(\lambda H)$.

Now suppose that $\Gamma(H) = \lambda H^{-1}$. In this case we must have $\Gamma(X) = f_1(H)Y$ and $\Gamma(Y) = Xf_2(H)$. By (35),

$$f_2(qH)f_1(qH)XY = p^{-1}\beta(XY - u_2(qH)) + \gamma + u_1(\lambda H^{-1}),$$

whence $u_2(H) = p\beta^{-1}(u_1(\lambda H^{-1}) + \gamma)$ and $v_2(H) = -p\beta^{-1}v_1(\lambda q^{-1}H^{-1})$. In both cases, $v_2(H) = \eta v_1(\mu H^{\pm 1})$ for some $\eta, \mu \in k^*$.

For the converse, if $v_2(H) = \eta v_1(\mu H)$ for some $\eta, \mu \in k^*$, then $R_1 \simeq R_2$ by Lemma 2.8(i) with $\lambda = \eta$ and Lemma 2.8(v) with $\tau(H) = \mu H$. If $v_2(H) = \eta v_1(\mu H^{-1})$ for some $\eta, \mu \in k^*$, then $R_1 \simeq R_2$ by Lemma 2.8(i) with $\lambda = \eta$ and Lemma 2.8(vi) with $\tau(H) = \mu H^{-1}$. □

6. The deformations of Witten and Woronowicz

6.1. Notation. In this section we consider a class of algebras including two algebras named in [10, 35] as *Woronowicz's deformation* and *Witten's second deformation*. We offer two proofs that these two algebras are isomorphic. One is based on an analysis of this class of algebras and the second is a direct application of Lemma 2.7 based on the identification of Woronowicz's deformation and Witten's second deformation as generalized Weyl algebras in [8].

We consider ambiskew polynomial rings of the form $R(k[t], \sigma, v, p)$, where, for some $q \in k^*$ and $c \in k$, σ is the k-automorphism of $k[t]$ such that $\sigma(t) = qt + c$ and $v = dt + e \in k[t]$ has degree 1. Such a ring A is the k-algebra generated by x, y and t subject to the relations

$$xt - qtx = cx,\ yt - q^{-1}ty = -q^{-1}cy,\ xy - pyx = dt + e.$$

By Lemma 2.4, A is the generalized Weyl algebra $k[t, w](\sigma, w)$, where σ is extended to the polynomial ring $k[t, w]$ by setting $\sigma(w) = pw + v$. If $p \neq q$ and either $e = 0$ or $p \neq 1$, then there exists $u = ft + g \in k[t]$ of degree 1 such that $v = \sigma(u) - pu$, whence, by Lemma 2.5(ii), $A = k[t, z](\sigma, ft + z + g)$, where $\sigma(z) = pz$.

6.2. Example. Taking $q = r^2, c = -r, d = 1, e = 0$ and $p = s^2$, where $r, s \in k^*$, one obtains a class of algebras discussed by Fairlie [13]. The defining relations are

$$rtx - r^{-1}xt = x,\ ryt - r^{-1}ty = y,\ s^{-1}xy - syx = t.$$

In [13], the generators x, y, t are written as W_+, W_- and W_0 respectively. We shall denote the algebra with these generators and relations by $F_{s,r}$. For Woronowicz's deformation, introduced in [34, p. 150] and discussed in the survey article [30], take $r = \nu^2$ and $s = \nu \in k^*$. In [34, 30], $k = \mathbb{C}$ and $\nu \in [-1, 1]$ is real. The generators ∇_0, ∇_1 and ∇_2 given in [34, 30] are related to ours by the formulae

$$\nabla_1 = (1 + \nu^2)t, \nabla_0 = (1 + \nu^2)x, \nabla_2 = -y.$$

For Witten's second deformation, introduced in [33, 5.13], take $r = q^{\frac{1}{2}}$ and $s = q \in k^*$. In [33], the generators x, y, t are written as t_+, t_- and t_0 respectively.

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6.3. Notation. For a given pair of parameters p and q with $q \neq 1$, the algebras specified in §6.1 fall into two isomorphism classes and there is a simple procedure, outlined in the proof of the next theorem, to determine which isomorphism class a given algebra is in. Let σ be the k-automorphism of $k[t]$ such that $\sigma(t) = qt$. Let $W_{p,q} = R(k[t], \sigma, t+1, p)$ and $V_{p,q} = R(k[t], \sigma, t, p)$. If $p \neq q$, then by Lemma 2.5(ii), $V_{p,q} = k[t, z](\sigma, z+t/(q-p))$ and, if also $p \neq 1$, $W_{p,q} = k[t, z](\sigma, z+t/(q-p)+1/(1-p))$, where $\sigma(z) = pz$. By Lemma 2.4, $W_{1,q} = k[t, w](\sigma, w)$, where $\sigma(w) = w+t+1$.

6.4. Theorem. *Let A be one of the rings specified in §6.1. If $q \neq 1$, then either $A \simeq W_{p,q}$ or $A \simeq V_{p,q}$.*

Proof. Let $t' = t + \frac{c}{q-1}$. Then $\sigma(t') = qt'$ and $A = R(k[t'], \sigma, dt' + e', p)$ where $e' = e - \frac{dc}{q-1}$.

If $e' = 0$, then there is an isomorphism from A to $V_{p,q}$ given by $t' \mapsto d^{-1}t, x \mapsto x$ and $y \mapsto y$.

If $e' \neq 0$, then there is an isomorphism from A to $W_{p,q}$ given by $t' \mapsto e'd^{-1}t, x \mapsto e'x$ and $y \mapsto y$. □

The next lemma indicates symmetry in the roles of the pairs (t, q) and (z, p) in $W_{p,q}$ and $V_{p,q}$.

6.5. Lemma. (i) *$V_{q,p} \simeq V_{p,q}$ and, if $q \neq 1$ and $p \neq 1$, $W_{q,p} \simeq W_{p,q}$.*
(ii) *$W_{p,q} \simeq W_{p^{-1},q^{-1}}$ and $V_{p,q} \simeq V_{p^{-1},q^{-1}}$.*

Proof. (i) We may assume that $p \neq q$. We have observed that $V_{p,q} = k[t, z](\sigma, z + t/(q-p))$ and, if $p \neq 1$, $W_{p,q} = k[t, z](\sigma, z + t/(q-p) + 1/(1-p))$, where $\sigma(t) = qt$ and $\sigma(z) = pz$. Applying Lemma 2.7(iv) with $\tau(z) = t/(q-p)$ and $\tau(t) = (q-p)z$, so that $\tau\sigma\tau^{-1}(z) = qz$, $\tau\sigma\tau^{-1}(t) = pt$ and $\tau(z + t/(q-p)) = z + t/(q-p)$, we obtain $V_{p,q} \simeq V_{q,p}$ and, if $p \neq 1$ and $q \neq 1$, $W_{p,q} \simeq W_{q,p}$.

(ii) follows easily from Lemma 2.8(i). □

6.6. *Remark.* In 6.1, if $q = 1$, then $A \simeq R(k[t], \sigma, t, p)$, where $\sigma = \mathrm{id}$ or $\sigma(t) = t-1$, depending on whether $c = 0$. In particular, $W_{p,1} \simeq V_{p,1}$ which, by Lemma 6.5(i), is isomorphic to $V_{1,p}$. However, if $p \neq 1$, $W_{1,p}$ and $V_{1,p}$ are not isomorphic. It can be deduced from [22, 3.1] that, if $p \neq 1$, $W_{1,p}$ has a unique 1-dimensional simple module whereas $V_{1,p}$ has infinitely many, because in this case $k[t, z]$ has a maximal ideal invariant under σ.

6.7. Example. We apply the algorithm from the proof of Theorem 6.4 to the algebras $F_{s,r}$ in Example 6.2, assuming that $r^2 \neq 1$. In the notation of that proof, $t' = t - \frac{r}{r^2-1}$ and $e' = \frac{r}{r^2-1} \neq 0$. Consequently, $F_{s,r} \simeq W_{s^2,r^2}$.

6.8. Corollary. *Let $q = \nu^2 \in k^*$. If $q^2 \neq 1$, then Woronowicz's deformation and Witten's second deformation are isomorphic algebras.*

Proof. From the discussions in Examples 6.2 and 6.7, Woronowicz's deformation F_{ν,ν^2} is isomorphic to W_{ν^2,ν^4} while Witten's second deformation $F_{q,q^{1/2}}$ is isomorphic to $W_{q^2,q} = W_{\nu^4,\nu^2}$. By Lemma 6.5(i), the two are isomorphic.

An alternative proof is based upon the identifications of the two deformations as generalized Weyl algebras given by the first author [8]. Let D be the commutative polynomial algebra $k[C, H]$ and let σ be the k-automorphism of D such that $\sigma(C) = r^4C$ and $\sigma(H) = r^2H$. From [8], Woronowicz's deformation is $D(\sigma, a)$, where $a = C-(H-r/(1-r^2))(H-r^3/(1-r^2))/r^2(r+r^{-1})$, and Witten's second deformation

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is $D(\sigma, a_1)$, where $a_1 = H + \mu C + \rho$ for some $\mu, \rho \in k^*$. By straightforward changes of generators, we can assume that $a = C - \lambda H^2 + H + 1$ for some $\lambda \in k^*$ and that $a_1 = C + H + 1$.

Let τ be the k-automorphism of D such that $\tau(C) = C - \lambda H^2$ and $\tau(H) = H$. Then $\sigma\tau = \tau\sigma$ and $\tau(a) = a_1$ so, by Lemma 2.7(v), $D(\sigma, a) \simeq D(\sigma, a_1)$. □

6.9. *Remark.* The algebra known as *Witten's first deformation* [10, 35], which was introduced in [33, 5.2], is of the form $R(k[t], \sigma, v, p) = k[t, w](\sigma, w)$, with $p = 1$, specified in §6.1 except that v is quadratic in t. The algebraic relationship between Witten's two deformations was explained in rigorous terms by Le Bruyn [24]. Below we apply his method more generally, using a change of variables of the form used in the proof of Theorem 6.4, and observe that the drop in degree from 2 to 1 in the degree of the element v in the passage from Witten's first deformation to his second is, in a sense, singular.

Let A be a k-algebra of the form $R(k[t], \sigma, v, p)$, where $p, q \in k^*$, $q \neq 1$, $\sigma(t) = qt + c$ and $v \in k[t]$ has degree 2. As in the proof of Theorem 6.4, a change of variables allows us to assume that $c = 0$. Thus A is the k-algebra generated by x, y and t subject to the relations

$$xt = qtx,\ yt = q^{-1}ty,\ xy - pyx = ft^2 + dt + e, \tag{36}$$

for some $f, d, e \in k$ with $f \neq 0$. The *homogenization* B, say, of A is the graded k-algebra generated by x, y, t and z subject to the relations

$$xt = qtx,\ yt = q^{-1}ty,\ xy - pyx = ft^2 + dtz + ez^2,$$
$$zx = xz,\ zy = yz,\ zt = tz.$$

The element t is normal in B and the automorphism τ of B for which $tb = \tau(b)t$ for all $b \in B$ is given by

$$x \mapsto q^{-1}x,\ y \mapsto qy,\ t \mapsto t,\ z \mapsto z.$$

Following [2, §8], the *twist* B_τ of B by τ is the graded k-algebra which is isomorphic to B as a graded abelian group, with an isomorphism written as $a \mapsto a_\tau$, and has multiplication determined by the rule $a_\tau b_\tau = (\tau^d(a)b)_\tau$, where $a, b \in B$ are homogeneous and $d = \deg(b)$. Thus B_τ is the k-algebra generated by x_τ, y_τ, t_τ and z_τ subject to the relations

$$x_\tau t_\tau = t_\tau x_\tau,\ y_\tau t_\tau = t_\tau y_\tau,\ x_\tau y_\tau - p^2 q^{-2} y_\tau x_\tau = q^{-1}(f t_\tau^2 + d t_\tau z_\tau + e z_\tau^2),$$
$$z_\tau x_\tau = q x_\tau z_\tau,\ z_\tau y_\tau = q^{-1} y_\tau z_\tau,\ z_\tau t_\tau = t_\tau z_\tau.$$

The element t_τ is central and the *dehomogenization* C, say, of B_τ is obtained by factoring out the ideal generated by $t_\tau - 1$. Thus it is the k-algebra generated by x_τ, y_τ and z_τ subject to the relations

$$z_\tau x_\tau = q x_\tau z_\tau,\ z_\tau y_\tau = q^{-1} y_\tau z_\tau,\ x_\tau y_\tau - p^2 q^{-2} y_\tau x_\tau = q^{-1}(f + d z_\tau + e z_\tau^2).$$

Comparing these with (36), the defining parameters q and p are replaced by q^{-1} and $p^2 q^{-2}$ respectively. If $e \neq 0$, the algebra C is of the same general form as A with v quadratic, but if $e = 0$, it is of the original form specified in §6.1 with the degree of v dropping from 2 to 1. This is the situation with Witten's two deformations. His first deformation is of the form A, with v quadratic with zero constant term, both before and after the change of variables which makes t normal, and his second is the corresponding algebra formed by homogenization, twisting and dehomogenization.

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The original parameters, labelled q and p in the general discussion above, are q^{-1} and 1 and so become q and q^2 in C.

We conclude this section by showing that, under certain conditions on the parameters, the only situations where there are isomorphisms between algebras of the forms $V_{p,q}$ and $W_{p,q}$ are those covered by Remark 6.6 and Lemma 6.5.

6.10. Theorem. *Let $p, q, p_1, q_1 \in k^*\backslash\{1\}$ be such that the subgroups $\langle p, q\rangle$ and $\langle p_1, q_1\rangle$ of k^* have positive rank, $p \neq q$ and $p_1 \neq q_1$.*

(i) $W_{p,q} \not\simeq V_{p_1,q_1}$.

(ii) *If $\{p_1, q_1\} \neq \{p, q\}$ and $\{p_1, q_1\} \neq \{p^{-1}, q^{-1}\}$, then $W_{p,q} \not\simeq W_{p_1,q_1}$ and $V_{p,q} \not\simeq V_{p_1,q_1}$.*

Proof. (i) This follows from the determination of the finite-dimensional simple modules for $V_{p,q}$ and $W_{p,q}$ given by [22, 3.1]. In the notation of [22], $v_d = (1 + pq^{-1} + \ldots + (pq^{-1})^{d-1})t$ for $V_{p,q}$ and $v_d = (1 + pq^{-1} + \ldots + (pq^{-1})^{d-1})t + 1 + p + \ldots + p^{d-1}$ for $W_{p,q}$. The only positive integers s for which there exist s-dimensional simple $V_{p,q}$-modules are 1, for which there are infinitely many, and, if it exists, the least positive integer d such that $q^d = p^d$, for which there are again infinitely many. On the other hand, $W_{p,q}$ also has infinitely many 1-dimensional simple $W_{p,q}$-modules but, for $d > 1$, there is at most one d-dimensional simple module. For $d = 2$, this module exists precisely when $((1 + pq^{-1})t + 1 + p)k[t] \neq (t + 1)k[t]$. As $q \neq 1$, there is a unique two-dimensional simple $W_{p,q}$-module.

(ii) We begin with the determination of the non-zero normal elements which generate prime ideals of $W_{p,q}$. Let $W = W_{p,q}$ and let $S = R(k[t^{\pm 1}], \sigma, t, p)$ be the localization of W at the powers of the normal element t. As $p \neq 1$ and $p \neq q$, the element z of Lemma 2.5 exists and the results of [17] apply. Both zW and tW are prime ideals of W. Let $\mathcal{N}(W) = \{w \in W : wW = Ww$ is a non-zero prime ideal of $W\}$ and define $\mathcal{N}(S)$ in a similar way.

First consider the case where the subgroup $\langle p, q\rangle$ of k^* has rank two. By [17, 2.21], $\mathcal{N}(S)$ consists of associates of z and hence, by standard localization theory, for example [26, 2.1.16], $\mathcal{N}(W) = k^*z \cup k^*t$.

Now consider the case where $\langle p, q\rangle$ has rank one. Without loss of generality, we may assume that q is not a root of unity so that [17, 2.21] is applicable to S. Let n be the minimal positive integer such that $p^n \in \langle q\rangle$ and let i be the integer such that $p^n = q^i$. If $i < 0$, then $t^{-i}z^n$ is central in W and, for all $\eta \in k^*$, $t^{-i}z^n - \eta \in \mathcal{N}(W)$. If $i \geq 0$, then $t^i - \eta z^n \in \mathcal{N}(W)$ for all $\eta \in k^*$. It follows from [17, 2.21] that $\mathcal{N}(S)$ consists of the associates of z and those of elements of the form $t^{-i}z^n - \eta$ or $t^i - \eta z^n$, $\eta \in k^*$, depending on whether $i > 0$. It then follows that $\mathcal{N}(W) = k^*z \cup k^*t \cup (\bigcup_{\eta\in k^*} k^*(t^{-i}z^n - \eta))$ if $i < 0$ and $\mathcal{N}(W) = k^*z \cup k^*t \cup (\bigcup_{\eta\in k^*} k^*(t^i - \eta z^n))$ if $i > 0$.

In either case, for $m \geq 0$, let $W_m = k[t, z]x^m$ and $W_{-m} = y^m k[t, z]$. Then $W = \sum_{m\in\mathbb{Z}} W_m$ and each W_m is simultaneously an eigenspace for all the automorphisms of W of the form $w \mapsto a^{-1}wa$, $a \in \mathcal{N}(W)$. For each $j \in \mathbb{Z}$, the eigenvalues for W_j are q^j for $a = t$, p^j for $a = z$, 1 when a has the form $t^{-i}z^n - \eta$ and $p^{jn} = q^{ji}$ when a has the form $t^i - \eta z^n$. It follows that the set $\{\{p, q\}, \{p^{-1}, q^{-1}\}\}$ is determined by W. This proves the result for $W_{p,q}$ and the proof for $V_{p,q}$ is similar. □

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