SESO UNIVERSITAIRE FACULTEITEN ST.-IGNATIUS TE ANTWERPEN

STUDIECENTRUM VOOR ECONOMISCH EN SOCIAAL ONDERZOEK

VAKGROEP ARBEIDSECONOMIE

*ON THE UNIQUENESS OF SQUARE COST-MINIMIZING TECHNIQUES**

Guido Erreygers

Report 92/274

May 1992

* *The present paper is a drastically revised version of a paper entitled "A new theorem on the uniqueness of dominant techniques", of which a preliminary first draft was written in April 1991. It is based on material first presented in my Ph.D. thesis "Terre, rente et choix de techniques. Une étude sur la théorie néo-ricardienne", Université Paris X - Nanterre, June 1990.*

Universitaire Faculteiten St. Ignatius
Prinsstraat 13 - B 2000 Antwerpen

D/1992/1169/09

ABSTRACT

The framework adopted in the paper is that of the Sraffian theory of the choice of techniques. Under the assumption that demand is price-independent, the uniqueness of cost-minimizing techniques is analyzed in terms of relations between square neighbouring techniques. It is shown that for certain types of economies, uniqueness is guaranteed if and only if all cost-minimizing techniques have the same 'colour'. The colour of a technique depends upon the rates of profits and growth, and the characteristics of the processes which constitute a technique. The result can be extended to economies using natural resources.

§ 1. Introduction

The non-substitution theorem, valid for economies with constant-returns-to-scale, single-product activities using a single non-produced resource (labour), contains a double uniqueness result with respect to the choice of techniques. First it states that, flukes apart, there never exists more than one 'stable', 'square' set of activities, whatever may the rate of profits, the rate of growth, and final demand. Second, given the rate of profits, the unique technique (if it exists) does not depend upon the rate of growth or final demand. Attempts have been made to widen the scope of the non-substitution theorem, and to extend it for instance to production by means of fixed capital (cf. Stiglitz, 1970; Salvadori, 1988). Nevertheless, it is now well-established that the non-substitution theorem does not hold for joint production economies, or for economies which use several primary factors. As a matter of fact, given the rate of profits, the stable technique may in these economies be different for different rates of growth or different vectors of final demand. In other words, the stable technique is *demand-dependent*. This is of course hardly a surprising result. A far more curious phenomenon, however, is that given the rate of profits, the rate of growth and final demand, there may be *more than one* stable square set of activities.

The purpose of the paper is to examine this last uniqueness question assuming demand to be of a 'rigid' type, i.e. composed of an investment component characterized by a uniform rate of growth for all methods of production, and a consumption component characterized by fixed proportions of final demand. The main advantage of working with these rigid demand regimes is that they allow us to concentrate on 'square' techniques. The theoretical framework used will be of Sraffian inspiration (cf. Sraffa, 1960), in particular the one that emerges from the works of Bidard (1984, 1990, 1991), Salvadori (1982, 1984), Salvadori & Steedman (1988), Schefold (1978, 1988, 1989) and Steedman (1976). In this literature, relatively few general results with regard to uniqueness have been obtained; it seems that only in the golden rule case, where the rate of profits equals the rate of growth, and in the case of "soft technologies" (cf. Bidard, 1990, pp. 850-852 for more details) uniqueness has been proved [1]. I shall demonstrate a theorem which gives, for economies possessing certain properties, necessary and sufficient conditions for uniqueness to be guaranteed.

The paper is organised as follows. First I assume there is only one non-produced resource. In §§ 2-7 I enounce the criteria which determine the choice of techniques [2] and I define cost-minimizing techniques. In §§ 8-13, the uniqueness theorem is prepared, stated and proved. In § 14 I indicate how the framework can be adapted to deal with certain types of natural resources. [3]

§ 2. Methods of production and disposal

As mentioned, I shall start to study economies which do not use natural resources. In this kind of economies, labour (supposed abundantly available and of homogeneous quality)

---

[1] The uniqueness result of Schefold (1988, pp. 102-109) seems to be based on a definition which rules out non-uniqueness; therefore, I do not take into consideration here.

[2] The criteria presented here are not the only ones which can be imagined; Hinrichsen & Krause (1978, 1981), for example, use another set of criteria, centred around the notion of labour-minimization.

[3] With regard to mathematical notation, I shall use the following convention. If $\mathbf{x} = [\, x_i \,]$ and $\mathbf{y} = [\, y_i \,]$ are vectors, $\mathbf{x} \leqq \mathbf{y}$ means $x_i \leq y_i$, $\forall\, i$; $\mathbf{x} \leq \mathbf{y}$ means $\mathbf{x} \leqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. A prime ' denotes transposition. Vectors and matrices completely composed of zeroes are represented by $\mathbf{0}$, the unit matrix by $\mathbf{I}$.

and produced commodities can be used either for the production or for the disposal of commodities. Each activity (or method, or process) *i* will be described by a non-negative [1x*k*] vector of commodity inputs $\mathbf{A}_i = [\, a_{i1}, a_{i2}, ..., a_{ik} \,]$, a non-negative scalar labour input l_i, and a [1x*k*] vector of commodity outputs $\mathbf{B}_i = [\, b_{i1}, b_{i2}, ..., b_{ik} \,]$, where *k* is the number of commodities which can be produced by the economy. Activity *i produces* commodity *j* if its output of *j* is positive ($b_{ij} > 0$); it *disposes of* commodity *j* if its output of *j* is negative ($b_{ij} < 0$). [4] A method that eliminates a commodity without using inputs will be called a free disposal method. (By assumption, free production methods cannot exist.) For each method constant returns to scale prevail. The number of available processes, *h*, is finite and at least equal to the number of commodities, *k* ($0 < k \leq h$). The inputs of all available processes together are described by a [*h*x*k*] matrix of commodity inputs $\mathbf{A} = [\, \mathbf{A}_i \,]$, a [*h*x1] vector of labour inputs $\mathbf{l} = [\, l_i \,]$, and a [*h*x*k*] matrix of commodity outputs $\mathbf{B} = [\, \mathbf{B}_i \,]$. The activity levels of the *h* processes will be given by the [*h*x1] activity vector $\mathbf{x} = [\, x_i \,]$.

§ 3. Extra-profits and extra-costs

Let the [*k*x1] vector $\mathbf{p} = [\, p_j \,]$ represent the prices of the *k* commodities, the scalar w the wage earned by labourers, and the scalar r the rate of profits. If the difference between the value of the outputs of activity *i* ($= \mathbf{B}_i\mathbf{p}$) and the costs to be paid for the inputs, i.e. the sum of replacement costs, wages and profits ($= \mathbf{A}_i\mathbf{p} + l_i w + r\mathbf{A}_i\mathbf{p}$), is positive, activity *i pays extra-profits*. If, on the contrary, this difference is negative, activity *i requires extra-*

---

[4] I assume that it is for each activity clear which amount of a commodity should be classified as an input (always a non-negative quantity) and which as output (eventually a negative quantity). Inputs are typically *means* (of production), outputs *results*.

costs. If there is no difference, the activity *breaks even*. The Sraffian theory of production prices is build upon the assumption that all processes which are in operation ($x_i > 0$) break even. For reasons of stability, it is furthermore required that all processes which are not in operation ($x_i = 0$) do not pay extra-profits. Putting $\mathbf{C}(r) \equiv \mathbf{B}-(1+r)\mathbf{A}$, we must therefore have :

$$\mathbf{C}(r)\mathbf{p} - \mathbf{l}w \leqq \mathbf{0} \tag{1}$$

$$\mathbf{x}'[\ \mathbf{C}(r)\mathbf{p} - \mathbf{l}w\ ] = 0 \tag{2}$$

§ 4. Supply and demand

In a general equilibrium framework it would be natural to assume demand to be dependent upon prices, wages, profits and activity levels. As is often done in Sraffian and von Neumann-like models, however, I shall in this paper adopt the restrictive assumption that demand is price-independent. Demand will be composed of the following elements : replacement of the means of production (= $\mathbf{x}'\mathbf{A}$), expansion at the uniform rate of growth g (= $g\mathbf{x}'\mathbf{A}$), and consumption according to a fixed final demand vector (= $\mathbf{d}'$), with g a non-negative scalar and $\mathbf{d}$ a semi-positive [kx1] vector of commodities. [5] Supply is of course the total (gross) output of the economy (= $\mathbf{x}'\mathbf{B}$). As far as quantities are concerned, equilibrium will be reached if supply equals demand, i.e. in formal terms :

$$\mathbf{x}'\mathbf{B} = \mathbf{x}'\mathbf{A} + g\mathbf{x}'\mathbf{A} + \mathbf{d}' \tag{3}$$

---

[5] Cf. Schefold (1988, p. 97-99) for an attempt to justify this assumption.

or, using the shorthand $\mathbf{C}(g) \equiv \mathbf{B}\text{-}(1+g)\mathbf{A}$:

$$\mathbf{x}'\mathbf{C}(g) = \mathbf{d}' \tag{4}$$

The main advantage of the rigid demand assumption is that it reduces the number of sets of activities which could be selected by the criteria determining the choice of techniques. More precisely only sets of activities equal in number to the number of commodities, i.e. 'square' sets, can be chosen. The reason is that for a given rate of growth and a given vector of demand, normally (except by fluke) exactly k methods have to be activated in order to satisfy the system of k equations (4).

§ 5. Techniques

Furthermore, it should be clear that a demand vector chosen at random can normally only be satisfied by a square set of *independent* methods. In mathematical terms this independence means that the matrix composed of the rows $\mathbf{C}_i(g) \equiv \mathbf{B}_i\text{-}(1+g)\mathbf{A}_i$ corresponding to the methods i of a square set must have full rank. Square sets of independent methods of which at least one requires a positive labour input, will be called *techniques.* [6] Let Y be a set of k methods, and define $\mathbf{C}_Y(g)$ as the $[kxk]$ matrix formed by the rows $\mathbf{C}_i(g)$ corresponding to the k methods which belong to Y, and $\mathbf{l}_Y$ and $\mathbf{x}_Y$ the $[kx1]$ vectors formed respectively by the scalars l_i and x_i for the same set of methods. We then have the following definition :

[6] With regard to economies in which each commodity can be freely and separately disposed of, the labour input condition prevents that the set composed of the k free disposal methods would be a technique.

Definition 1 :

A square set of methods Y is a *technique* if det $\mathbf{C}_Y(g) \neq 0$ and $\mathbf{l}_Y \neq \mathbf{0}$.

Given the rate of growth, the set of all demand vectors (not necessarily semi-positive) which a technique can satisfy will be called its domain :

Definition 2 :

The *domain* of technique Y, noted $Q(Y)$, is the set of all demand vectors which the methods of Y can satisfy at the given rate of growth, i.e. :

$$Q(Y) = \{ \mathbf{d} \in \mathbb{R}^k \mid \exists\, \mathbf{x}_Y \geqq \mathbf{0} : \mathbf{x}_Y'\mathbf{C}_Y(g) = \mathbf{d}' \}$$

The dimension of the domain $Q(Y)$ is equal to the dimension of the matrix $\mathbf{C}_Y(g)$. Since it is my purpose to examine general situations, I shall assume that the rate of growth g is such that for all techniques Y matrix $\mathbf{C}_Y(g)$ has full rank. In other words, I assume that det $\mathbf{C}_Y(g) \neq 0$, if Y is a technique (this condition can always be assured by a slight move of g).

As is traditional in Sraffian models, I shall treat the rate of profits as an exogenous variable. Since by assumption all active methods must break even, each technique normally fully determines the prices and the wage. More precisely, the k equations stating that the methods of Y break even together with a numéraire equation in general suffice to determine the k prices $\mathbf{p}$ and the wage w :

$$\mathbf{C}_Y(r)\mathbf{p} - \mathbf{l}_Y w = \mathbf{0} \tag{5}$$

$$\mathbf{n'p} = 1 \tag{6}$$

where $\mathbf{n}$ is a (semi-positive) [kx1] numéraire vector. [7] Problems could occur for the values of r for which det $\mathbf{C}_Y(\mathrm{r}) = 0$. Suppose R is a root of det $\mathbf{C}_Y(\mathrm{r}) = 0$. In three cases, multiple solutions will be obtained : (i) if the rank of the matrix $\mathbf{C}_Y(\mathrm{R})$ is smaller than k-1; (ii) if the labour vector $\mathbf{l}_Y$ is not independent of the matrix $\mathbf{C}_Y(\mathrm{R})$; and (iii) if the numéraire vector $\mathbf{n}$ is not independent of the matrix $\mathbf{C}_Y(\mathrm{R})$. I shall assume that these 'pathological' cases do not occur. I therefore impose the following regularity condition [8] :

Regularity condition 1

Suppose Y is a technique. Then :

(i) all non-negative real roots of det $\mathbf{C}_Y(\mathrm{r}) = 0$ are simple;

(ii) if R is a non-negative real root of det $\mathbf{C}_Y(\mathrm{r}) = 0$ and $\mathbf{x}$ and $\mathbf{p}$ are vectors for which $\mathbf{x'C}_Y(\mathrm{R}) = \mathbf{0'}$ and $\mathbf{C}_Y(\mathrm{R})\mathbf{p} = \mathbf{0}$, we have $\mathbf{x'l}_Y \neq 0$ and $\mathbf{n'p} \neq 0$.

This condition ensures that all prices and the wage are uniquely defined even if the rate of profits happens to be a root of the equation det $\mathbf{C}_Y(\mathrm{r}) = 0$; in particular, the wage will be zero for such values of the rate of profits. Moreover, it ensures that the polynomial det $\mathbf{C}_Y(\mathrm{r})$ changes sign as r rises and passes such a root. For a given numéraire, the unique solutions of equations (5)-(6) will be denoted as $\mathbf{p}(Y,\mathrm{r})$ and $\mathrm{w}(Y,\mathrm{r})$.

---

[7] It may happen that the sign of all prices and the wage changes for different semi-positive numéraire vectors. To this problem I return in the next section.

[8] Cf. Schefold's assumption of a "regular economy" (Schefold, 1978, p. 267).

§ 6. Non-negative prices

Once the question of the determination of prices and wage has been settled, one thing remains to be clarified : whereas it is natural to require that the wage should be non-negative (who would be prepared to pay to have to work ?), should prices also be non-negative ? [9] I think a distinction has to be made between commodities which *can* be overproduced, and commodities which cannot. Given a technique Y, I shall say that commodity j can be overproduced if there exists a process $i \in Y$ that disposes of commodity j, i.e. if $b_{ij} < 0$. I propose to adopt the following rule : commodities which cannot be overproduced should have non-negative prices, but commodities which can be overproduced may have negative prices. The reason to allow a potentially overproduced commodity to have a negative price is simple : if it is costly to eliminate a good (e.g. dangerous waste-material) and people do not want to consume this good, then it is only natural that a price has to be paid to have it removed. In formal terms, let $P(Y)$ be the set of commodities which cannot be overproduced given that only the methods of Y can be activated :

$$P(Y) = \{ j \mid \forall i \in Y : b_{ij} \geq 0 \} \tag{7}$$

I require that the prices determined by technique Y are such that $\forall j \in P(Y) : p_j(Y,r) \geq 0$.

A 'freak' case which I want to exclude is the one where the wage and a commodity's price are simultaneously zero. More precisely I want this to be true with respect to commodities which cannot be overproduced. I therefore impose a second regularity condition :

---

[9] Cf. Franke (1986, pp. 303-305) for a discussion of this problem.

Regularity condition 2

If the rate of profits r is such that $w(Y,r) = 0$, then $\forall j \in P(Y) : p_j(Y,r) \neq 0$.

§ 7. Cost-minimizing techniques

We have now introduced all the criteria upon which the choice of techniques is based : all active processes break even; no known process pays extra-profits; demand is satisfied; the wage is non-negative; and commodities which cannot be overproduced do not have negative prices. Techniques which satisfy these criteria will be called cost-minimizing (or dominant). Two different notions will be distinguished, according to whether the final demand vector is known or not :

Definition 3

Given the rates of profits and growth, technique Y is *cost-minimizing with respect to demand vector* **d** if there exists a semi-positive numéraire vector **n**, an activity vector $\mathbf{x}_Y$, a price vector $\mathbf{p}(Y,r)$ and a wage $w(Y,r)$ such that :

$$\mathbf{x}_Y'\mathbf{C}_Y(g) = \mathbf{d}'$$

$$\mathbf{C}_Y(r)\mathbf{p}(Y,r) - \mathbf{l}_Y w(Y,r) = \mathbf{0}$$

$$\mathbf{C}(r)\mathbf{p}(Y,r) - \mathbf{l}w(Y,r) \leqq \mathbf{0}$$

$$\mathbf{n}'\mathbf{p}(Y,r) = 1$$

$$\forall j \in P(Y) : p_j(Y,r) \geq 0$$

$$w(Y,r) \geq 0.$$

$$\mathbf{x}_Y \geqq \mathbf{0}.$$

In what follows, final demand will usually not be specified. Even then, however, it makes sense to speak of cost-minimizing techniques. I shall use the following definition :

Definition 4

Given the rates of profits and growth, technique Y is a *cost-minimizing technique* if it is cost-minimizing with respect to some semi-positive demand vector **d.**

§ 8. Neighbours and rivals

The central idea of the paper is to tackle the problem of uniqueness in terms of relationships between neighbouring techniques. What are neighbouring techniques ? In the present circumstances it would suffice to say they are techniques which have k-1 methods in common. In view of the inclusion of natural resources, however, I prefer to give a definition which refers to the notion of domain of a technique. I recall that the domain of a technique is the set of all net output vectors which a technique can satisfy, given the rate of growth. Mathematically, the domain of a technique is a convex polyhedral cone, the intersection of k half-spaces bounded by the hyperplanes $H(Y,i)$, $i \in Y$:

$$H(Y,i) = \{ \mathbf{d} \in \mathbb{R}^k \mid \exists\, x_Y : x_Y' C_Y(g) = \mathbf{d}',\ x_i = 0 \} \tag{8}$$

The *faces* of the domain of a technique are the parts of these hyperplanes which effectively belong to the domain. Formally, the face $F(Y,i)$ of the domain of technique Y associated to method $i \in Y$ is equal to :

$$F(Y,i) = \{ \mathbf{d} \in \mathbb{R}^k \mid \exists\, x_Y \geq 0 : x_Y' C_Y(g) = \mathbf{d}',\ x_i = 0 \} \tag{9}$$

This enables us to give the following definition of neighbouring techniques :

Definition 5

Techniques are *neighbours* if their domains have one face in common.

Next we may wonder whether two neighbouring techniques have more than just a face in common. For this purpose I define the *interior domain* of a technique, noted $Q(Y)^*$, as the domain less its faces, i.e. :

$$Q(Y)^* = \{ \mathbf{d} \in \mathbb{R}^k \mid \exists\, x_Y > 0 : x_Y' C_Y(g) = \mathbf{d}' \} \tag{10}$$

If two neighbouring techniques Y and Z have parts of their interior domains in common, which means $Q(Y)^* \cap Q(Z)^* \neq \emptyset$, then I propose to call them rivalling techniques :

Definition 6

Two neighbouring techniques are *rivals* if their interior domains overlap.

Overlapping of interior domains can only occur if both domains are on the same 'side' of the face they have in common (fig. 1).

[INSERT FIGURE 1 HERE]

In mathematical terms this property can be expressed as follows :

Lemma 1

Two neighbouring techniques Y and Z, with $Y = \{i\} \cup [Y \cap Z]$ and $Z = \{j\} \cup [Y \cap Z]$, are rivals if and only if $\mathbf{C}_i(g)\mathbf{v}(Y,Z)/\mathbf{C}_j(g)\mathbf{v}(Y,Z) > 0$, where the $[k\mathrm{x}1]$ vector $\mathbf{v}(Y,Z)$ is such that $\mathbf{C}_{Y\cap Z}(g)\mathbf{v}(Y,Z) = \mathbf{0}$.

Proof.

Since $\mathbf{C}_Y(g)$ and $\mathbf{C}_Z(g)$ have by assumption full rank, the homogeneous system of k-1 equations $\mathbf{C}_{Y\cap Z}(g)\mathbf{v} = \mathbf{0}$ has, up to a factor of indifference, only one solution, $\mathbf{v}(Y,Z)$ say. The face in common between the two neighbours, i.e. $F(Y,i) = F(Z,j)$, can also be defined as :

$$F(Y,i) = F(Z,j) = \{ \mathbf{d} \in \mathbb{R}^k \mid \exists\, \mathbf{x}_{Y\cap Z} \geqq \mathbf{0} : \mathbf{x}_{Y\cap Z}'\mathbf{C}_{Y\cap Z}(g) = \mathbf{d}' \} \tag{11}$$

Given that $\mathbf{C}_{Y\cap Z}(g)\mathbf{v}(Y,Z) = \mathbf{0}$, it follows quite easily that this face is part of the hyperplane $H(Y,Z)$:

$$H(Y,Z) = \{ \mathbf{d} \in \mathbb{R}^k \mid \mathbf{d}'\mathbf{v}(Y,Z) = 0 \} \tag{12}$$

Let $\bar{\mathbf{d}}$ be a net output vector which is part of the interior domain of Y. This means there exists a vector $\bar{\mathbf{x}}_{Y\cap Z} > \mathbf{0}$ and a scalar $\bar{x}_i > 0$ such that :

$$\bar{\mathbf{x}}_{Y\cap Z}'\mathbf{C}_{Y\cap Z}(g) + \bar{x}_i\mathbf{C}_i(g) = \bar{\mathbf{d}}' \tag{13}$$

Post-multiplication by $\mathbf{v}(Y,Z)$ leads to :

$$\bar{x}_i\mathbf{C}_i(g)\mathbf{v}(Y,Z) = \bar{\mathbf{d}}'\mathbf{v}(Y,Z) \tag{14}$$

Since $\bar{\mathbf{d}}'\mathbf{v}(Y,Z) \neq 0$, we must have either $\mathbf{C}_i(g)\mathbf{v}(Y,Z) > 0$ or $\mathbf{C}_i(g)\mathbf{v}(Y,Z) < 0$. In the first case, we will say that the domain of technique Y lies 'above' the hyperplane $H(Y,Z)$, and in the second case, that it lies 'underneath' it. By a similar reasoning we find that the domain of technique Z lies 'above' the hyperplane $H(Y,Z)$ if $\mathbf{C}_j(g)\mathbf{v}(Y,Z) > 0$, and that it lies 'underneath' it if $\mathbf{C}_j(0)\mathbf{v}(Y,Z) < 0$. The domains of two neighbouring techniques overlap if and only if they are on the same side of the face or hyperplane $H(Y,Z)$ in common. This leads to the conclusion that the neighbours Y and Z are rivals if and only if $\mathbf{C}_i(g)\mathbf{v}(Y,Z)/\mathbf{C}_j(g)\mathbf{v}(Y,Z) > 0$. ■

§ 9. Well-conditioned and connected economies

The next step in the argument consists of linking the notion of uniqueness to that of rivalry. A few assumptions are necessary to do this. After a formal enunciation of these assumptions, I interpret them in the light of the problem we are dealing with.

Let D be a non-empty set of techniques. I define the following two properties :

Definition 7

Given the rate of growth, set D will be called *well-conditioned* if the following two conditions are satisfied :

(i) if there exist techniques $Y \in D$ and $Z \in D$ which are rivals, then there exist techniques $T_1 \in D$ and $T_2 \in D$ and a semi-positive vector $\mathbf{d}$ such that $\mathbf{d} \in Q(T_1)^*$ and $\mathbf{d} \in Q(T_2)^*$;

(ii) if there exist techniques $Y \in D$ and $Z \in D$ such that $Q(Y)^{*} \cap Q(Z)^{*} \neq \emptyset$, then there exist techniques $T_1 \in D$ and $T_2 \in D$ which are rivals.

(Notice that in this definition we may have $Y = T_1$ and/or $Z = T_2$.)

Definition 8

Given the rate of growth, set D will be called *connected* if for any pair of techniques $Y \in D$ and $Z \in D$ ($Y \neq Z$), either Y is a neighbour of Z, or there exists a chain of techniques $T_1 \in D, T_2 \in D, ..., T_s \in D$ such that Y is a neighbour of T_1, T_1 is a neighbour of T_2, ..., and T_s is a neighbour of Z.

Let us now define the set $D(g,r)$ as the set of all cost-minimizing techniques, given the rate of growth g and the rate of profit r. For the theorems that follow, I shall assume either that $D(g,r)$ is well-conditioned, or that $D(g,r)$ is well-conditioned *and* connected. (I shall also use the equivalent expressions that the economy is well-conditioned, or well-conditioned and connected.)

It may be useful to explain these notions in a less formal way. A well-conditioned economy possesses the following two essential properties :

1) If there is rivalry among cost-minimizing techniques (i.e. if there exists at least one couple of rivalling cost-minimizing techniques), at least one semi-positive final demand vector belongs to the interior domain of more than one cost-minimizing technique. This assumption excludes that rivalling cost-minimizing techniques have interior domains which overlap only outside the non-negative orthant. A numerical example may clarify this problem. Suppose we have 4 processes to produce 3 commodities, and suppose the rate of growth is such that $C_1(g) = [\ -1, -2, 2\ ]$, $C_2(g) = [\ -2, -1, 2\ ]$, $C_3(g) = [\ 5, 1, 1\ ]$,

$\mathbf{C}_4(g) = [\ 1, 5, 1\ ]$. If $Y = \{1,2,3\}$ and $Z = \{1,2,4\}$ were the only cost-minimizing techniques, the condition would not be verified. It is clear that Y and Z are neighbours; they are also rivals, because for the vector $\mathbf{v}(Y,Z)' = [\ 2, 2, 3\ ]$ we find that $\mathbf{C}_3(g)\mathbf{v}(Y,Z)/\mathbf{C}_4(g)\mathbf{v}(Y,Z) = 1 > 0$. (E.g. the vector $\mathbf{d}' = [\ -\frac{1}{2}, -\frac{1}{2}, 1\ ]$ belongs to both interior domains.) But there exists no semi-positive $\mathbf{d}$ which can be produced by both Y and Z.
2) If the interior domains of two cost-minimizing techniques overlap, there is rivalry among cost-minimizing techniques. The purpose of this condition is to exclude cases where overlapping of interior domains of cost-minimizing techniques occurs without rivalry. Again a numerical example may be useful. [10] Suppose there exist 4 processes to produce 2 commodities, with $\mathbf{A}_1 = [\ 1, 2\ ]$, $\mathbf{A}_2 = [\ 2, 1\ ]$, $\mathbf{A}_3 = [\ 0, 0\ ]$, $\mathbf{A}_4 = [\ 0, 0\ ]$, $\mathbf{B}_1 = [\ 5, 7\ ]$, $\mathbf{B}_2 = [\ 7, 9/2\ ]$, $\mathbf{B}_3 = [\ -1, 0\ ]$, $\mathbf{B}_4 = [\ 0, -1\ ]$, $l_1 = l_2 = 1$, $l_3 = l_4 = 0$. For $r = 3$ and $g = 0$, we find that the only cost-minimizing techniques are $Y = \{1,4\}$ and $Z = \{2,3\}$. Although their interior domains overlap (e.g. $\mathbf{d}' = [\ 1, 1\ ]$ can be produced by both), they are not rival, since they are not even neighbours.

The notion of a connected economy, on the other hand, implies the fundamental property that *in the set of cost-minimizing techniques*, each cost-minimizing technique is directly or indirectly neighbour of every other cost-minimizing technique. The numerical example which we have just considered is an example of an economy which is not connected for the rate of growth $g = 0$ and the rate of profits $r = 3$. It must of course be kept in mind that the properties of well-conditionedness and connectedness are 'local' properties : an economy may be well-conditioned and/or connected for some sets of values of (r, g) but not for others.

---

[10] This is a modified version of an example given by Franke (1986, pp. 306-307).

§ 10. Uniqueness and rivalry

My purpose is to find necessary and sufficient conditions for uniqueness of cost-minimizing techniques to be guaranteed. I shall say uniqueness of cost-minimizing techniques is guaranteed if, exception made for the final demand vectors located on the boundaries between domains of neighbouring cost-minimizing techniques, no final demand vector can be satisfied by more than one cost-minimizing technique. In other words : if there do not exist cost-minimizing techniques Y and Z and a semi-positive vector $\mathbf{d}$ such that $\mathbf{d}$ belongs to the interior domains of both Y and Z. [11] It is now easy to proof the following theorem :

THEOREM 1

In a well-conditioned economy the uniqueness of cost-minimizing techniques is guaranteed if and only if there is absence of rivalry among the cost-minimizing techniques.

Proof.

If the uniqueness of cost-minimizing techniques is guaranteed, there cannot be found two cost-minimizing techniques which are rivals, otherwise the set *D(g,r)* could not possess property (i) of Definition 7. Alternatively, if there is no rivalry between the cost-minimizing techniques, there cannot be found two cost-minimizing techniques which have

[11] As 'points of substitution' between neighbouring techniques, the final demand vectors situated on the boundaries (i.e. the faces) of the domains represent exceptional cases. There is little harm in ignoring them; the set of all final demand vectors located on the boundaries is negligible in comparison to the set of all final demand vectors which can be satisfied by the cost-minimizing techniques. For a numerical example of such 'points of substitution', cf. Salvadori (1988, pp. 12-13, Example 3).

overlapping interior domains, otherwise *D(g,r)* could not possess property (ii) of Definition 7. Hence uniqueness is guaranteed. ■

§ 11. <u>Colour of a technique</u>

A new concept will allow us to derive a sharper result. This new concept is what I call the 'colour' of a technique :

Definition 9

Given the rates of profits and growth, technique *Y* is called *white* if one of the following conditions is satisfied :

(i) $\det C_Y(r) / \det C_Y(g) > 0$;

(ii) $\det C_Y(r) = 0$ and there exists an arbitrarily small scalar ϵ such that the sign of $\det C_Y(r+\epsilon) / \det C_Y(g)$ is equal to the sign $w(Y,r+\epsilon) / p_j(Y,r+\epsilon)$ for all $j \in P(Y)$.

If none of these conditions is verified, technique *Y* is called *black*.

A few words may be in place to explain condition (ii). Its sole purpose is to determine the colour of a technique in case the wage is zero. We shall need it in the following situation. Suppose $R > 0$ is a root of the equation $\det C_Y(r) = 0$. Suppose that for the rate of growth $g = G$ and a rate of profits $r \in]R, R'[$, where $R < R'$, technique *Y* is cost-minimizing *and* $\det C_Y(r)$ does not change sign. Then condition (ii) ensures that, given the rate of growth $g = G$, technique *Y* has the same colour for all rates of profits $r \in [R, R'[$; in addition, it will be cost-minimizing for $g = G$ and $r = R$. (A similar reasoning would be valid if R' < R instead of R < R'.)

Manifestly, the colour of a technique depends upon the rates of profits and growth. In the case of several non-negative roots of the equation det $C_Y(r) = 0$, the map of the values of r and g for which a technique is black or white resembles a deformed chequerboard (fig. 2).

[INSERT FIGURE 2 HERE]

§ 12. The main theorem

We are now in a position to state and demonstrate the main result of the paper.

THEOREM 2

In a well-conditioned and connected economy the uniqueness of cost-minimizing techniques is guaranteed (flukes apart) if and only if all cost-minimizing techniques have the same colour.

Proof.

I) The 'flukes' of the theorem are the finite number of cases where the rate of profits is such that there is 'switching' (or indifference) between cost-minimizing techniques. Rates of profits for which there is switching are those for which two neighbouring cost-minimizing techniques pay the same wage (measured in a common numéraire). Without loss of generality, I therefore assume that any two neighbouring cost-minimizing techniques pay a different wage.

II) According to theorem 1, we have to proof that there is absence of rivalry among cost-minimizing techniques. To do this, I will concentrate on an arbitrary couple of neighbouring cost-minimizing techniques Y and Z, and determine under what conditions they are rivals. I take $Y \cap Z = M$, $Y = \{i\} \cup M$, and $Z = \{j\} \cup M$. By assumption $w(Y,r) \neq w(Z,r)$; I assume that $0 \leq w(Y,r) < w(Z,r)$. [12] Since Y is cost-minimizing, we know that $s_j(Y,r) = C_j(r)\mathbf{p}(Y,r) - l_j w(Y,r) < 0$; likewise, we have $s_i(Z,r) = C_i(r)\mathbf{p}(Z,r) - l_i w(Z,r) < 0$. Define $e(Y,Z,r)$ as follows :

$$e(Y,Z,r) = s_i(Z,r)w(Y,r)/s_j(Y,r)w(Z,r) \tag{15}$$

It is clear that $w(Y,r) > 0 \leftrightarrow e(Y,Z,r) > 0$ and $w(Y,r) = 0 \leftrightarrow e(Y,Z,r) = 0$.

III) By assumption Y and Z are neighbours; we have seen earlier that they would be rivals if and only if :

$$C_i(g)\mathbf{v}(Y,Z)/C_j(g)\mathbf{v}(Y,Z) > 0 \tag{16}$$

Let us for a moment assume that the rate of profits is equal to the rate of growth ($r = g$). Prices and wage associated to technique Y and Z would then be determined by the following equations (jointly with the numéraire equation) :

$$\mathbf{C}_Y(g)\mathbf{p}(Y,g) - \mathbf{l}_Y w(Y,g) = \mathbf{0} \tag{17}$$

$$\mathbf{C}_Z(g)\mathbf{p}(Z,g) - \mathbf{l}_Z w(Z,g) = \mathbf{0} \tag{18}$$

[12] Throughout this proof I assume that only one numériare $\mathbf{n}$ is used, and that it is such that $p_f(Y,r) \geq 0$, $\forall f \in P(Y)$ and $p_f(Z,r) \geq 0$, $\forall f \in P(Z)$.

Let us concentrate on the *k-1* methods which *Y* and *Z* have in common, and which are grouped in the set *M*. It is not difficult to check that :

$$\mathbf{C}_M(g)\ [\ w(Z,g)\mathbf{p}(Y,g) - w(Y,g)\mathbf{p}(Z,g)\ ] = \mathbf{0} \tag{19}$$

The vector $[\ w(Z,g)\mathbf{p}(Y,g) - w(Y,g)\mathbf{p}(Z,g)\ ]$ is therefore proportional to the vector $\mathbf{v}(Y,Z)$ which I have used to determine whether *Y* and *Z* are rivals. It is also easy to deduce that :

$$\mathbf{C}_i(g)\ [\ w(Z,g)\mathbf{p}(Y,g) - w(Y,g)\mathbf{p}(Z,g)\ ] = -\ w(Y,g)s_i(Z,g) \tag{20}$$

$$\mathbf{C}_j(g)\ [\ w(Z,g)\mathbf{p}(Y,g) - w(Y,g)\mathbf{p}(Z,g)\ ] = w(Z,g)s_j(Y,g) \tag{21}$$

Knowing that $e(Y,Z,g) = s_i(Z,g)w(Y,g)/s_j(Y,g)w(Z,g)$, we conclude that there is rivalry between *Y* and *Z* if and only if $e(Y,Z,g) < 0$.

IV) We should therefore know how the sign of $e(Y,Z,r)$ behaves as r changes. Let us try to express $e(Y,Z,r)$ in a more transparant way.

The prices $\mathbf{p}(Y,r)$ and the wage $w(Y,r)$ associated to technique *Y* are determined by the following *k+1* equations :

$$\mathbf{C}_Y(r)\mathbf{p}(Y,r) - \mathbf{l}_Y w(Y,r) = \mathbf{0} \tag{22}$$

$$\mathbf{n}'\mathbf{p}(Y,r) = 1 \tag{23}$$

Let us again concentrate on the *k-1* methods which *Y* and *Z* have in common, and let us define :

$$\mathbf{F}(r) \equiv \begin{bmatrix} \mathbf{n}' \\ \mathbf{C}_M(r) \end{bmatrix} \qquad \mathbf{g} \equiv \begin{bmatrix} 0 \\ \mathbf{l}_M \end{bmatrix} \tag{24}$$

On the basis of (22) and (23) we can deduce that :

$$\mathbf{F}(r)\mathbf{p}(Y,r) - \mathbf{g}w(Y,r) = \mathbf{I}^1 \tag{25}$$

where $(\mathbf{I}^1)' = [\ 1, 0, 0, ..., 0\ ]$. It easily follows that we have :

$$\mathbf{p}(Y,r) = [\mathbf{F}(r)]^{-1}\mathbf{g}w(Y,r) + [\mathbf{F}(r)]^{-1}\mathbf{I}^1 \tag{26}$$

Since $s_i(Y,r) = \mathbf{C}_i(r)\mathbf{p}(Y,r) - l_i w(Y,r) = 0$, we finally arrive at :

$$w(Y,r) = \frac{\mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1}{l_i - \mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{g}} \tag{27}$$

By analogy we obtain :

$$\mathbf{p}(Z,r) = [\mathbf{F}(r)]^{-1}\mathbf{g}w(Z,r) + [\mathbf{F}(r)]^{-1}\mathbf{I}^1 \tag{28}$$

$$w(Z,r) = \frac{\mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1}{l_j - \mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{g}} \tag{29}$$

Given that $s_i(Z,r) = \mathbf{C}_i(r)\mathbf{p}(Z,r) - l_i w(Z,r)$ and $s_j(Y,r) = \mathbf{C}_j(r)\mathbf{p}(Y,r) - l_j w(Y,r)$, we then have :

$$s_i(Z,r) = \{ l_i - \mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{g} \} [w(Y,r) - w(Z,r)] \tag{30}$$

$$s_j(Y,r) = - \{ l_j - \mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{g} \} [w(Y,r) - w(Z,r)] \tag{31}$$

As $w(Y,r) \neq w(Z,r)$, this means that :

$$\frac{s_i(Z,r)}{s_j(Y,r)} = - \frac{l_i - \mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{g}}{l_j - \mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{g}} \tag{32}$$

and so we find that :

$$e(Y,Z,r) = - \frac{\mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1}{\mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1} \tag{33}$$

This allows us to conclude that :

$$\frac{e(Y,Z,r)}{e(Y,Z,g)} = \frac{\mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1/\mathbf{C}_i(g)[\mathbf{F}(g)]^{-1}\mathbf{I}^1}{\mathbf{C}_j(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1/\mathbf{C}_j(g)[\mathbf{F}(g)]^{-1}\mathbf{I}^1} \tag{34}$$

V) We have seen that $e(Y,Z,r)$ is either positive or zero. Let us begin by assuming that $e(Y,Z,r)$ is positive.

The sign of $e(Y,Z,g)$ - either positive or negative - will be equal to the sign of the right-hand side of expression (34). From (27) it follows that :

$$w(Y,r) = 0 \leftrightarrow \mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1 = 0 \tag{35}$$

Since $w(Y,r) \leftrightarrow \det \mathbf{C}_Y(r) = 0$, this means that the polynomial $\mathbf{C}_i(r)[\mathbf{F}(r)]^{-1}\mathbf{I}^1$ changes sign whenever det $C_Y(r)$ changes sign, and therefore that the sign of the numerator of the right-hand side of (34) is equal to the sign of [det $C_Y(r)$ / det $C_Y(g)$]. By analogy we find that the sign of the denominator is equal to the sign of [det $C_Z(r)$ / det $C_Z(g)$]. We can therefore conclude that the sign of $e(Y,Z,g)$ is *positive* if and only if Y and Z have the same colour (either both black or both white). The sign of $e(Y,Z,g)$ is *negative* if and only if they have a different colour (either Y black and Z white, or Y white and Z black).

VI) Next let us examine the special case $e(Y,Z,r) = 0$. The appropriate way to do this is to observe the system for a value of the rate of profits $r+\epsilon$ close to r, keeping the rate of growth fixed at g. Let ϵ be such that $w(Y,r+\epsilon) > 0$; since Y is by assumption cost-minimizing for the rate of profits r, this choice of ϵ ensures that $w(Y,r+\epsilon)/p_f(Y,r+\epsilon) > 0$, $\forall f \in P(Y)$. [13] In addition, let ϵ (positive or negative) be small enough to have $w(Y,r+\epsilon) < w(Z,r+\epsilon)$ and [det $C_Z(r+\epsilon)$ / det $C_Z(r)$] > 0. So doing, we know on the basis of (15) that the sign of $e(Y,Z,r+\epsilon)$ is equal to the sign of $w(Y,r+\epsilon)$, that is : positive. Two cases must now be distinguished. First suppose that for the rate of profits r, technique Y is white. According to definition 9, this means that [det $C_Y(r+\epsilon)$ / det $C_Y(g)$] and $w(Y,r+\epsilon)/p_f(Y,r+\epsilon)$, $\forall f \in P(Y)$, have the same sign. Given our choice of ϵ, this implies that [det $C_Y(r+\epsilon)$ / det $C_Y(g)$] is positive. Hence, the sign of $e(Y,Z,r+\epsilon)/e(Y,Z,g)$ is equal to the sign of [det $C_Z(r+\epsilon)$ / det $C_Z(g)$]. It follows that the sign of $e(Y,Z,g)$ is positive if Z is white, and negative if Z is black. In the second place, suppose that for the rate of profits r, technique Y is black. Then [det $C_Y(r+\epsilon)$ / det $C_Y(g)$] and $w(Y,r+\epsilon)/p_f(Y,r+\epsilon)$, $\forall f \in P(Y)$, have opposite sign, and therefore the sign of $e(Y,Z,r+\epsilon)/e(Y,Z,g)$ is opposite to the sign of [det $C_Z(r+\epsilon)$ / det $C_Z(g)$]. Now the conclusion is that the sign of $e(Y,Z,g)$

[13] Regularity condition 2 guarantees that $p_f(Y,r) > 0$, $\forall f \in P(Y)$; by continuity, this property remains valid for rates of profits close to r.

is positive if Z is black, and negative if Z is white.

VII) In each case, it turns out that there is rivalry between neighbouring techniques (i.e. $e(Y,Z,g) < 0$) if and only if the techniques in question are not of the same colour. Since we assume that the economy is connected, this means that there is absence of rivalry among cost-minimizing techniques if and only if they are all of the same colour. ■

§ 13. The golden rule regime

In the literature on choice of techniques, special attention has often been given to the case where the rate of growth equals the rate of profits ($g = r$), which reflects the situation where labourers do not save and capitalists do not consume (cf. Abraham-Frois & Berrebi, 1987, pp. 268-278; Bidard, 1986; 1990, pp. 844-845; Salvadori, 1982, p. 286; Schefold, 1978; Steedman, 1976). One of the conclusions of this research is that the golden rule regime has a lot in common with single-product systems; in particular, uniqueness always seems to be guaranteed. Theorem 2 allows us to draw a similar conclusion. Observing that for $g = r$ all cost-minimizing techniques are white, we immediately obtain the following result :

THEOREM 3

Under golden rule conditions, if the economy is well-conditioned and connected, the uniqueness of cost-minimizing techniques is (flukes apart) guaranteed.

§ 14. Natural resources

Finally and as promised, I shall indicate how the previous results can be extended to economies which use natural resources. To keep things simple, I shall confine myself to resources which are available in fixed supply and 'indestructible', i.e. do not undergo qualitative change irrespective of whether they are used for production or disposal, or left unused [14]. I assume there are *m* ($m > 0$) such natural resources. Let the positive scalar t_j represent the fixed supply of resource *j* ($j=1,2,...,m$), and let the [mx1] vector $\mathbf{t} = [\, t_j \,]$ be the vector representing the supplies of all *m* natural resources. The description of the methods of production and disposal will now include a component which specifies how much each of them uses of the *m* natural resources. Let $\mathbf{T}_i = [\, t_{ij} \,]$ be the [1xm] vector of natural resource inputs of method *i*; method *i* can then be schematically described as $(\mathbf{A}_i, \mathbf{T}_i, l_i) \rightarrow \mathbf{B}_i$.

The usual assumption with regard to natural resources (and this distinguishes them from the other primary factor of the model, labour) is that a positive rent can be paid for their use only if they are scarce. In other words, if part of the available supply of resource *j* is lying 'fallow', i.e. is not being used for the production or disposal of commodities, the rent z_j to be paid for their use will be zero. In addition, all rents should be non-negative.

These two requirements allow us to express the fact that a portion of the available supply of a resource is idle as if a fictituous process were used at a positive level. The process in question has only one positive input (one unit of a resource) and no outputs. It simply describes what happens if a unit of a resource is *not* used for the production

[14] Natural resources of this type prevent the economy from growing steadily at a *positive* balanced growth rate. It therefore seems necessary to assume a zero growth rate ($g = 0$).

or the disposal of commodities. There will be m such fictituous processes; let the process corresponding to the idleness of resource j be represented as the $(h+j)$-th process. With respect to resource 1, for instance, we then have :

$$\mathbf{A}_{h+1} = \mathbf{0}', \quad \mathbf{T}_{h+1} = [\ 1, 0, ..., 0\ ], \quad l_{h+1} = 0, \quad \mathbf{B}_{h+1} = \mathbf{0}' \tag{36}$$

The economy is now composed of $h+m$ processes; accordingly, the information relevant to the m fictituous processes will be included in the matrices **A**, **B** and **T**, and the vectors **l** and **x**.

Let us briefly consider how the natural resources and the fictituous processes enter into the price and quantity systems. The price equations (1)-(2) must now be written as follows :

$$\mathbf{C}(r)\mathbf{p} - \mathbf{Tz} - \mathbf{l}w \leq \mathbf{0} \tag{37}$$

$$\mathbf{x}'[\ \mathbf{C}(r)\mathbf{p} - \mathbf{Tz} - \mathbf{l}w\ ] = 0 \tag{38}$$

where $\mathbf{z} = [\ z_j\ ]$ is the $[m\text{x}1]$ vector of rents. Notice that the m last equations of (37) express that rents must be non-negative. With regard to the quantity equations, we now have to look for activity vectors **x** which satisfy demand *and* use up *exactly* the available supplies of the natural resources ('exactly' because of the inclusion of the m idleness processes). We must therefore replace (3) by :

$$\mathbf{x}'\mathbf{C}(g) = \mathbf{d}' \tag{39}$$

$$\mathbf{x}'\mathbf{T} = \mathbf{t}' \qquad (40)$$

It is easy to see that techniques must now be composed of $(k+m)$ processes instead of k processes; if not, equations (39)-(40) could only be verified by fluke. A 'square' technique is in the present circumstances a technique composed of processes equal in number to the sum of the number of commodities *and* the number of natural resources (cf. Sraffa, 1960, p. 78).

For economies using indestructible, fixed supply natural resources the definition of a cost-minimizing technique then becomes :

Definition 10

Given the available amounts of the natural resources $\mathbf{t}$ as well as the rates of profits and growth, technique Y is *cost-minimizing with respect to demand vector* $\mathbf{d}$ if there exists a semi-positive numéraire vector $\mathbf{n}$, an activity vector $\mathbf{x}_Y$, a price vector $\mathbf{p}(Y,r)$, a rent vector $\mathbf{z}(Y,r)$ and a wage $w(Y,r)$ such that :

$$\mathbf{x}_Y'\mathbf{C}_Y(g) = \mathbf{d}'$$

$$\mathbf{x}_Y'\mathbf{T}_Y = \mathbf{t}'$$

$$\mathbf{C}_Y(r)\mathbf{p}(Y,r) - \mathbf{T}_Y\mathbf{z}(Y,r) - \mathbf{l}_Y w(Y,r) = \mathbf{0}$$

$$\mathbf{C}(r)\mathbf{p}(Y,r) - \mathbf{T}\mathbf{z}(Y,r) - \mathbf{l}w(Y,r) \leqq \mathbf{0}$$

$$\mathbf{n}'\mathbf{p}(Y,r) = 1$$

$$\forall j \in P(Y) : p_j(Y,r) \geq 0$$

$$w(Y,r) \geq 0$$

$$\mathbf{x}_Y \geqq \mathbf{0}.$$

The question is : do these manipulations permit the extension of the analysis of §§ 2-13 to economies with natural resources ? The answer is yes, provided we replace in the appropriate places the matrix **C**(.) and the vectors **d**, **p** and **n** respectively by $\tilde{\mathbf{C}}$(.), $\tilde{\mathbf{d}}$, $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{n}}$, defined as follows : $\tilde{\mathbf{C}}(.) \equiv [\ \mathbf{C}(.) \quad -\mathbf{T}\ ]$, $\tilde{\mathbf{d}}' \equiv [\ \mathbf{d}' \quad -\mathbf{t}'\ ]$, $\tilde{\mathbf{p}}' \equiv [\ \mathbf{p}' \quad \mathbf{z}'\ ]$ and $\tilde{\mathbf{n}}' \equiv [\ \mathbf{n}' \quad \mathbf{0}'\ ]$. [15] For instance, the domain of a technique *Q(Y)* must now be defined as :

$$Q(Y) = \{\ \mathbf{d} \in \mathbb{R}^k \mid \exists\ \mathbf{x}_Y \geq \mathbf{0} : \mathbf{x}_Y'\tilde{\mathbf{C}}_Y(g) = \tilde{\mathbf{d}}'\ \} \tag{41}$$

In essence, this is all that is required to incorporate natural resources.

The existence and uniqueness of cost-minimizing techniques in economies with natural resources have been studied by Bidard (1987), D'Agata (1983, 1984), Salvadori (1986, 1987), Saucier (1981, 1984), and others. It has been pointed out that in these economies the uniqueness of cost-minimizing techniques is not even guaranteed in the special case where all available methods of production are single-product processes (cf. D'Agata, 1983, 1984). For uniqueness to be guaranteed in that case three additional assumptions seem to be required : (i) natural resources enter into the production equations of only one of the *k* commodities (typically corn); (ii) each process which produces that commodity uses only one natural resource; and (iii) each natural resource is used by only one production process. It should by now be clear - and I think the integration of natural resources suggested above confirms this - that even a very 'moderate' presence of natural resources makes economies behave as if they were joint

[15] Likewise, vector $\mathbf{v}(Y,Z)$ of Lemma 1 must be replaced by the $[(k+m)\mathrm{x}1]$ vector $\tilde{\mathbf{v}}(Y,Z)$ which satisfies $\tilde{\mathbf{C}}_{Y\cap Z}(g)\tilde{\mathbf{v}}(Y,Z) = \mathbf{0}$.

production economies.

§ 15. Concluding remarks

The uniqueness result obtained in this paper, dealing with the theory of choice of techniques from a Sraffian perspective and assuming demand to be price-independent, can be summarized as follows : in a well-conditioned and connected economy, the uniqueness of cost-minimizing techniques is guaranteed if and only if all cost-minimizing techniques are of the same colour. This result remains valid for economies using indestructible, fixed-supply natural resources. Both the colour of a technique and the properties of well-conditionedness and connectedness depend upon the rates of profits and growth. The colour of a technique can be determined quite easily : all that is required is a description of the processes which make up the technique. It is far more difficult to ascertain whether an economy is well-conditioned or connected, since information is needed about *all* cost-minimizing techniques. To enhance the usefulness of the uniqueness theorem, further research should therefore try to clarify under which conditions an economy possesses these properties.

REFERENCES

ABRAHAM-FROIS, G. & E. BERREBI (1987), *Prix, Profits, et Rythmes d'Accumulation*, Paris, Economica.

BIDARD, Ch. (1984), "Choix techniques en production jointe", in : Ch. BIDARD (Ed.), *La Production Jointe. Nouveaux Débats*, Paris, Economica, pp. 186-207.

BIDARD, Ch. (1986), "Is Von Neumann square ?", *Zeitschrift für Nationalökonomie*, vol. 46(4), pp. 407-419.

BIDARD, Ch. (1987), "Il n'y a plus de rente", in : Ch. BIDARD (Ed.), *La Rente. Actualité de l'Approche Classique*, Paris, Economica, pp. 187-207.

BIDARD, Ch. (1990), "An algorithmic theory of the choice of techniques", *Econometrica*, vol. 58(4), pp. 839-859.

BIDARD, Ch. (1991), *Prix, Reproduction, Rareté*, Paris, Dunod.

D'AGATA, A. (1983), "The existence and unicity of cost-minimizing systems in intensive rent theory", *Metroeconomica*, vol. 35(1), pp. 147-158.

D'AGATA, A. (1984), "Molteplicità di merci agricole e rendita differenziale estensiva", *Richerche Economiche*, vol. 38(1), pp. 78-94.

FRANKE, R. (1986), "Some problems concerning the notion of cost-minimizing systems in the framework of joint production", *The Manchester School*, vol. 54(3), pp. 298-307.

HINRICHSEN, D. & U. KRAUSE (1978), "Choice of techniques in joint production models", *Operations Research Verfahren*, vol. 34, pp. 155-171.

HINRICHSEN, D. & U. KRAUSE (1981), "A substitution theorem for joint production models with disposal processes", *Operations Research Verfahren*, vol. 41, pp. 287-291.

SALVADORI, N. (1982), "Existence of cost-minimizing systems within the Sraffa framework", *Zeitschrift für Nationalökonomie*, vol. 42(3), pp. 281-298.

SALVADORI, N. (1984), "Le choix de techniques chez Sraffa : le cas de la production jointe", in : Ch. BIDARD (Ed.), *La Production Jointe. Nouveaux Débats*, Paris, Economica, pp. 175-185.

SALVADORI, N. (1986), "Land and choice of techniques within the Sraffa framework", *Australian Economic Papers*, vol. 25(46), pp. 94-105.

SALVADORI, N. (1987), "Les ressources naturelles rares dans la théorie de Sraffa", in : Ch. BIDARD (Ed.), *La Rente. Actualité de l'Approche Classique*, Paris, Economica, pp. 161-176.

SALVADORI, N. (1988), "Fixed capital within the Sraffa framework", *Zeitschrift für Nationalökonomie*, vol. 48(1), pp. 1-17.

SALVADORI, N. & I. STEEDMAN (1988), "Joint production analysis in a Sraffian framework", *Bulletin of Economic Research*, vol. 40(3), pp. 165-195.

SAUCIER, Ph. (1981), *Le Choix des Techniques en Situation de Limitations de Ressources*, Thèse pour le doctorat d'Etat, Université de Paris II.

SAUCIER, Ph. (1984), "L'évolution des rentes dans une économie en croissance", in : G. ABRAHAM-FROIS (Ed.), *L'Economie Classique. Nouvelles Perspectives*, Paris, Economica, pp. 165-179.

SCHEFOLD, B. (1978), "On counting equations", *Zeitschrift für Nationalökonomie*, vol. 38(3-4), pp. 253-285.

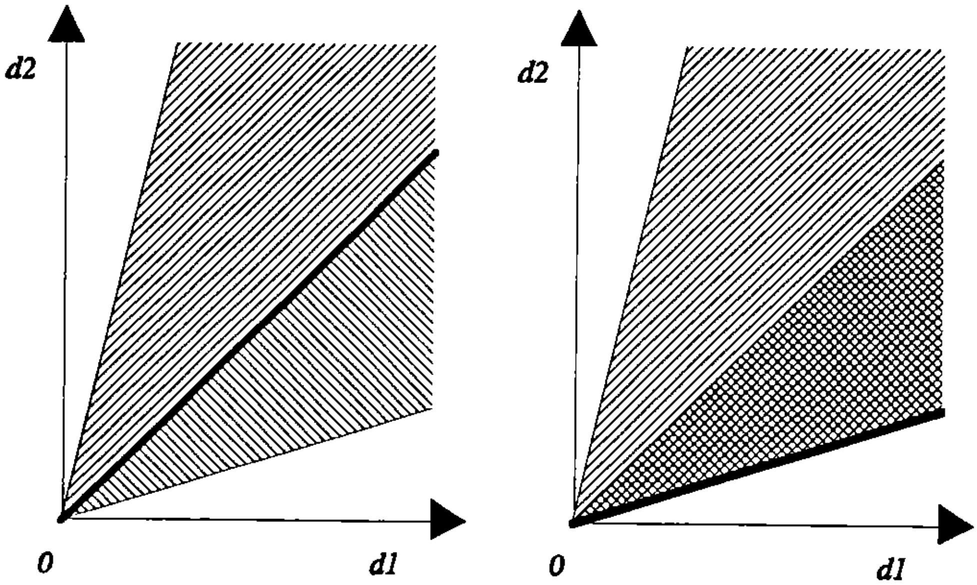
SCHEFOLD, B. (1988), "The dominant technique in joint production systems", *Cambridge Journal of Economics*, vol. 12(1), pp. 97-123.

SCHEFOLD, B. (1989), *Mr Sraffa on Joint Production and Other Essays*, London, Unwin Hyman.

SRAFFA, P. (1960), *Production of Commodities by Means of Commodities*, Cambridge, Cambridge University Press.

STEEDMAN, I. (1976), "Positive profits with negative surplus value : a reply to Wolfstetter", *Economic Journal*, vol. 86(344), pp. 873-876.

STIGLITZ, J.E. (1970), "Non-substitution theorems with durable capital goods", *Review of Economic Studies*, vol. 37(4), no. 112, pp. 543-553.



(a) no overlapping *(b) overlapping*

Legend :

- *domain of technique Y*
- *domain of technique Z*
- *face in common between Y and Z*

Figure 1

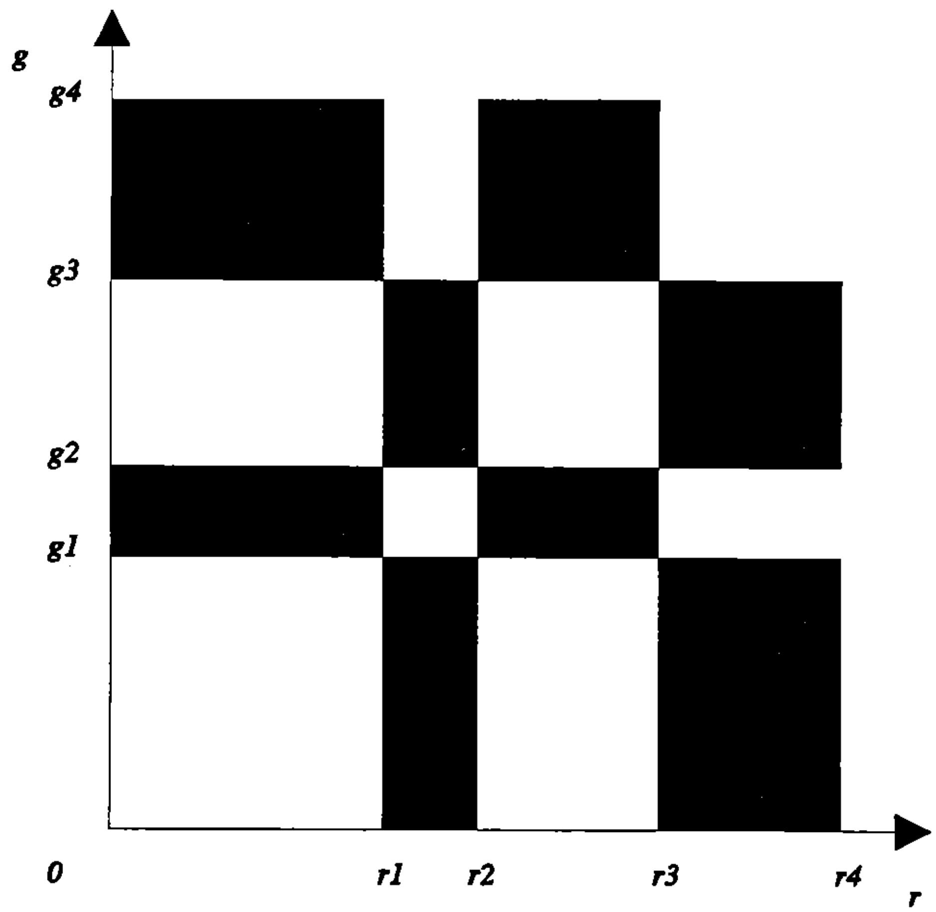


Figure 2

LIJST VAN RECENTE SESO-RAPPORTEN

KONINGS J., Experimentele economie als onderzoeksmethodologie : basisprincipes en resultaten van een duopolie-experiment, juni 1991, 28 blz. **(91/259)**

VAN TRIER W., "State Bonus" or basic income in the age of reconstruction, September 1991, 145 blz. **(91/260)**

PAUWELS W., Some properties of the Hicks and Morishima elasticities of substitution, September 1991, 22 blz. **(91/261)**

MEULEPAS M. en J. PLASMANS, ESTIMATING AND FORECASTING EXCHANGE RATES BY MEANS OF PPP AND UIP, An application on the Belgian Franc and the Dutch Guilder vis à vis the DM and US Dollar, September 1991, 49 blz. **(91/262)**

PEERSMAN G. en A. VAN POECK, De Belgische competitiviteitswet : een betwistbaar instrument, november 1991, 23 blz. **(91/263)**

PLASMANS J. en H. PAUWELS, Modelling and forecasting Belgian stock market prices, December 1991, 59 blz. **(91/264)**

ITO Y., PLASMANS J., DE ZEEUW A., CERVIO PINHO A. & A. MARKINK, Hierarchical optimal control for policy evaluation using econometric models for the European Community, November 1991, 48 blz. **(91/265)**

JENNES G. en G. DIERICKX, Belastingstelsels en belastingbeleid (1955-1988). Een vergelijkende analyse, januari 1992, 51 blz. **(92/266)**

JENNES G. en G. DIERICKX, Het belastingbeleid in 15 OESO-landen (1955-1988) : alternatieve verklaringen, januari 1992, 103 blz. **(92/267)**

DE BORGER B., KERSTENS K., MOESEN W. en J. VANNESTE, Explaining differences in productive efficiency : an application to Belgian unicipalities, February 1992, 27 blz. **(92/268)**

DE GRAEVE D., HUYBRECHTS K., DESSERS L. & J. HEYRMAN, Laboratory testing in the general practitioner's office : economic consequences, February 1992, 29 blz. **(92/269)**

BRUGGEMAN A. & F. HEYLEN, Indicators of the stance of monetary and fiscal policy - A pragmatic approach for the OECD countries in the 1980s, March 1992, 37 blz. **(92/270)**

HUYBRECHTS K., Economische evaluatie van antibioticaprofylaxis bij de implantatie van een heupprothese, maart 1992, 26 blz. **(92/271)**

MEERSMAN H. & I. VANNESTE, Statusverschillen van mannelijke en vrouwelijke UFSIA-TEW-ALUMNI, maart 1992, 60 blz. **(92/272)**

NONNEMAN W., Rendabiliteit van onderwijs in België, april 1992, 32 blz. **(92/273)**