



Function spaces in metrically generated theories

E. Vandersmissen^{a,*}, A. Van Geenhoven^{b,2}

^a Vrije Universiteit Brussel, Vakgroep Wiskunde, Pleinlaan 2, 1050 Brussel, Belgium

^b Universiteit Antwerpen, Middelheimlaan 1, 2020 Antwerpen, Belgium

ARTICLE INFO

MSC:

18B99

54B30

54C35

54D99

Keywords:

Metrically generated theory

Function space

Completeness

Precompactness

Ascoli theorem

ABSTRACT

The theory of metrically generated constructs provides us with an excellent setting for the study of function spaces. In this paper we develop a function space theory for metrically generated constructs and, by considering different metrically generated constructs, we capture interesting examples. For instance, for uniform spaces we retrieve the uniformity of uniform convergence and its generalization to Σ -convergence and for **UG**-spaces we obtain a quantified version of these structures. Our theory also allows for many applications, in particular we are able to characterize the complete subspaces of these function spaces and we succeed in producing an appropriate Ascoli theorem.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The setting for the classical theory of function spaces can be found, for instance, in Bourbaki [2]. Given a set (or topological space) X and a uniform space Y the uniformity of uniform convergence on Y^X is the starting point for a rich theory including a generalization to Σ -convergence. Here Σ stands for a collection of subsets of X , which, in case X is a topological space is often determined by topological properties of X . This theory also includes a study of the complete subsets of the function space Y^X and a characterization of its precompact subsets, where the latter characterization is known as the classical Ascoli theorem.

By [10] we know that a uniformity on Y can be described by a gauge \mathcal{D} of pseudometrics. This gauge has to fulfill the saturation condition $\xi_U(\mathcal{D}) = \mathcal{D}$, where $\xi_U(\mathcal{D})$ is defined as the set of all pseudometrics d satisfying

$$\forall \varepsilon > 0, \exists d_1, \dots, d_n \in \mathcal{D}, \exists \delta > 0: \sup_{i=1}^n d_i(x, y) < \delta \Rightarrow d(x, y) < \varepsilon.$$

Using this description of uniform spaces, the uniformity of uniform convergence on Y^X is given by $\xi_U(\{\gamma_d \mid d \in \mathcal{D}\})$, where γ_d is the pseudometric defined as

$$\gamma_d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

In this paper we show that this process of producing a natural function space on Y^X out of the given saturated gauge on Y is in fact applicable in more general cases, where the gauge on Y can consist of other kinds of ‘metrics’, and where the saturation on the gauge can be different from ξ_U .

* Corresponding author.

E-mail addresses: Eva.Vandersmissen@vub.ac.be (E. Vandersmissen), Anneleen.VanGeenhoven@ua.ac.be (A. Van Geenhoven).

¹ The first author is a research assistant supported by the FWO-grant G.0244.05.

² The second author is a research assistant at the Research Foundation – Flanders (FWO).

An appropriate setting to deal with this topic is provided by so-called metrically generated theories [5], which will be explained in the next section. Roughly speaking, metrically generated constructs are topological constructs which are generated by their metrizable objects. Every metrically generated construct can be seen as a construct with objects structured by collections of certain generalized metrics which satisfy some saturation condition. This saturation condition moreover fully determines the construct. We will develop function space structures on Y^X where the domain X is an object in one metrically generated construct \mathbf{X} and the codomain Y is an object in another metrically generated construct \mathbf{Y} .

The main purpose of this paper is to single out the essential facts on the metrically generated constructs involved in order to produce an equally rich theory of function spaces allowing a generalization to Σ -convergence, a study of its complete subsets and a characterization of its precompact subsets. We show that a theory of function spaces with a type of uniform convergence can be developed imposing only two mild conditions. This generality allows for many applications. First of all, by considering different kinds of generalized metric spaces one obtains function space theories not only for the classical case of all (generalized) metrics but also for other classes like for instance the class of all ultrametrics. In particular the case of non-Archimedean uniformities is captured in this way.

Second, by varying the theories one also captures for example the quantified version of uniform convergence, as well as the related Σ -convergence, and a new function space theory for Lipschitz spaces [8].

Moreover since the setting of metrically generated theories has proven to be suitable for the study of completeness [6], under some further conditions we obtain a description of the complete subsets of Y^X for uniform or Σ -convergence. In particular we investigate spaces of morphisms. Finally by introducing suitable notions of “precompactness” and “equicontractivity” which naturally also depend on the theories under consideration, we prove an appropriate Ascoli theorem.

2. Metrically generated theories

The framework we will be working in is that of metrically generated constructs as introduced in [5]. In this section we gather the preliminary material to explain these constructs. Afterwards, we pay attention to instances of metrically generated constructs which will frequently appear in the sequel. We use categorical terminology as developed in [1].

A function $d : X \times X \rightarrow [0, \infty]$ is called a *quasi-pre-metric* if it is zero on the diagonal, we will drop “pre” if d satisfies the triangle inequality and we will drop “quasi” if d is symmetric. Denote by **Met** the construct of quasi-pre-metric spaces and contractions (sometimes also called non-expansive maps).

A *base category* \mathcal{C} is a full and isomorphism-closed concrete subconstruct of **Met** which is closed under initial morphisms and contains all **Met**-indiscrete spaces. In this paper we will mainly focus on base categories \mathcal{C} consisting of metric spaces, such as the base category $\mathcal{C}^{\Delta, s}$ consisting of all metric spaces and \mathcal{C}^μ , the base category consisting of all ultrametric spaces. Sometimes we will also have to deal with \mathcal{C}^Δ , the category of all quasi-metric spaces. If (X, d) is a \mathcal{C} -object, we call d a \mathcal{C} -metric and the fibre of all \mathcal{C} -metrics on X is denoted by $\mathcal{C}(X)$. For any collection \mathcal{D} of quasi-pre-metrics on a set X we put $\mathcal{D} \downarrow := \{e \in \mathbf{Met}(X) \mid \exists d \in \mathcal{D} : e \leq d\}$. A downset in **Met**(X) is a non-empty subset \mathcal{D} such that $\mathcal{D} \downarrow = \mathcal{D}$. We say that a subset \mathcal{B} of **Met**(X) is a *basis* for \mathcal{D} if $\mathcal{B} \downarrow = \mathcal{D}$.

Given a base category \mathcal{C} , a topological construct \mathbf{X} is called *\mathcal{C} -metrically generated* if there exists a concrete functor $\mathbf{K} : \mathcal{C} \rightarrow \mathbf{X}$ such that \mathbf{K} preserves initial morphisms and $\mathbf{K}(\mathcal{C})$ is initially dense in \mathbf{X} . All \mathcal{C} -metrically generated constructs have an isomorphic description with objects and morphisms expressed in terms of \mathcal{C} -metrics as we will see next.

$\mathbf{M}^{\mathcal{C}}$ is the construct with objects pairs (X, \mathcal{D}) where X is a set and \mathcal{D} is a downset in **Met**(X) with a basis consisting of \mathcal{C} -metrics. \mathcal{D} is called a *\mathcal{C} -meter (on X)* and (X, \mathcal{D}) a *\mathcal{C} -metered space*. The fibre of \mathcal{C} -meters on X is denoted by $\mathbf{M}^{\mathcal{C}}(X)$. If (X, \mathcal{D}) and (X', \mathcal{D}') are \mathcal{C} -metered spaces, then $f : (X, \mathcal{D}) \rightarrow (X', \mathcal{D}')$ is a *contraction* if d' in \mathcal{D}' implies $d' \circ f \times f \in \mathcal{D}$. Note that $\mathbf{M}^{\mathcal{C}}$ is a topological construct. The initial structure of a source $(f_j : X \rightarrow (X_j, \mathcal{D}_j))_{j \in J}$ is given by the meter $\{d \circ f_j \times f_j \mid j \in J, d \in \mathcal{D}_j\} \downarrow$.

Concretely coreflective subconstructs of $\mathbf{M}^{\mathcal{C}}$ can be described by means of expanders. We call ξ an *expander* on $\mathbf{M}^{\mathcal{C}}$ if for any set X and any \mathcal{C} -meter \mathcal{D} on X , ξ provides us with a \mathcal{C} -meter $\xi(\mathcal{D})$ on X in such a way that ξ is extensive ($\mathcal{D} \subset \xi(\mathcal{D})$), monotone ($\mathcal{D} \subset \mathcal{D}' \Rightarrow \xi(\mathcal{D}) \subset \xi(\mathcal{D}')$), idempotent ($\xi(\xi(\mathcal{D})) = \xi(\mathcal{D})$) and if $f : Y \rightarrow X$ and $\mathcal{D} \in \mathbf{M}^{\mathcal{C}}(X)$, then $\xi(\mathcal{D}) \circ f \times f \subset \xi(\mathcal{D} \circ f \times f \downarrow)$. Given an expander ξ on $\mathbf{M}^{\mathcal{C}}$, we define $\mathbf{M}_\xi^{\mathcal{C}}$ as the full concretely coreflective subconstruct of $\mathbf{M}^{\mathcal{C}}$ with objects those \mathcal{C} -metered spaces (X, \mathcal{D}) for which $\xi(\mathcal{D}) = \mathcal{D}$. All concretely coreflective subconstructs of $\mathbf{M}^{\mathcal{C}}$ are captured in this way. For an $\mathbf{M}^{\mathcal{C}}$ -object (X, \mathcal{D}) , its $\mathbf{M}_\xi^{\mathcal{C}}$ -coreflection is exactly given by $(X, \xi(\mathcal{D}))$. Given an $\mathbf{M}_\xi^{\mathcal{C}}$ -object (X, \mathcal{D}) , we will say that $\mathcal{B} \subset \mathcal{C}(X)$ is a *ξ -basis* for \mathcal{D} if $\xi(\mathcal{B} \downarrow) = \mathcal{D}$.

If we have an $\mathbf{M}_\xi^{\mathcal{C}}$ -object (X, \mathcal{D}) , and a subset A of X , the $\mathbf{M}_\xi^{\mathcal{C}}$ -subspace structure on A is given by $\xi(\mathcal{D}|_{A \times A})$, where $\mathcal{D}|_{A \times A} = \{d|_{A \times A} \mid d \in \mathcal{D}\}$.

The main result of [5] states that a topological construct is \mathcal{C} -metrically generated if and only if it is concretely isomorphic to $\mathbf{M}_\xi^{\mathcal{C}}$ for some expander ξ on $\mathbf{M}^{\mathcal{C}}$. In this paper we will only consider expanders implying saturation for finite suprema. We will now discuss the theories which will appear in the sequel.

2.1. The expander ξ_U on $\mathbf{M}^{\mathcal{C}}$

Uniform theories are essentially determined by the expander ξ_U , defined as follows. Let $(X, \mathcal{D}) \in \mathbf{M}^{\mathcal{C}}$.

$$d \in \xi_U(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \forall \varepsilon > 0, \exists d_1, \dots, d_n \in \mathcal{D}, \exists \delta > 0, \forall x, y \in X: \\ \sup_{i=1}^n d_i(x, y) < \delta \Rightarrow e(x, y) < \varepsilon.$$

If we apply ξ_U to $\mathbf{M}^{C^{\Delta, s}}$, we find a category which is concretely isomorphic to the construct of uniform spaces **Unif**. Applying the expander ξ_U to the base categories $\mathcal{C} = \mathcal{C}^\Delta$ and \mathcal{C}^μ leads to isomorphic descriptions of the construct of quasi-uniform spaces **qUnif** [7] and the construct of non-Archimedean uniform spaces **naUnif** [14].

The isomorphism between **Unif** and $\mathbf{M}_{\xi_U}^{C^{\Delta, s}}$ gives occasion to a characterization of uniform spaces by means of ξ_U -saturated $C^{\Delta, s}$ -meters. Given a uniform space (X, \mathcal{U}) , the corresponding $\mathbf{M}_{\xi_U}^{C^{\Delta, s}}$ -structure on X is given by

$$\{d \in C^{\Delta, s}(X) \mid \mathcal{U}_d \subset \mathcal{U}\} \downarrow.$$

Conversely, with an $\mathbf{M}_{\xi_U}^{C^{\Delta, s}}$ -object (X, \mathcal{D}) we associate the uniform space (X, \mathcal{U}) where the uniformity \mathcal{U} is generated by taking

$$V_\varepsilon^d = \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}$$

with $d \in \mathcal{D}$ and $\varepsilon > 0$ as subbasic sets. By replacing $C^{\Delta, s}$ and **Unif** by C^Δ and **qUnif**, we obtain the transitions which describe the isomorphism between **qUnif** and $\mathbf{M}_{\xi_U}^{C^\Delta}$.

In the sequel we will make no distinction between **Unif** and $\mathbf{M}_{\xi_U}^{C^{\Delta, s}}$ (resp. **qUnif** and $\mathbf{M}_{\xi_U}^{C^\Delta}$) and we will frequently describe uniform spaces by means of ξ_U -saturated $C^{\Delta, s}$ -meters (resp. quasi-uniform spaces by means of ξ_U -saturated C^Δ -meters).

2.2. The expander ξ_T on \mathbf{M}^C

Topological theories are essentially determined by the expander ξ_T , defined as follows. Let $(X, \mathcal{D}) \in \mathbf{M}^C$.

$$d \in \xi_T(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \forall x \in X, \forall \varepsilon > 0, \exists d_1, \dots, d_n \in \mathcal{D}, \exists \delta > 0, \forall y \in X: \\ \sup_{i=1}^n d_i(x, y) < \delta \Rightarrow e(x, y) < \varepsilon.$$

When applied to \mathbf{M}^C respectively with $\mathcal{C} = C^\Delta$, $C^{\Delta, s}$ and C^μ , the expander ξ_T gives rise to constructs \mathbf{M}_ξ^C that are isomorphic to the construct of all topological spaces **Top**, the construct of completely regular topological spaces **CReg** and the construct of zero-dimensional topological spaces **ZDim**. For a topological space (X, \mathcal{T}) , its isomorphic copy in $\mathbf{M}_{\xi_T}^{C^\Delta}$ is given by the C^Δ -metered space $(X, \{d \in C^\Delta(X) \mid \mathcal{T}_d \subset \mathcal{T}\} \downarrow)$. Conversely, the topological space associated with an $\mathbf{M}_{\xi_T}^{C^\Delta}$ -object (X, \mathcal{D}) is given by (X, \mathcal{T}) where $\mathcal{T} = \bigvee_{p \in \mathcal{P}} \mathcal{T}_p$ when \mathcal{P} is a C^Δ -basis of \mathcal{D} . By replacing C^Δ and **Top** by $C^{\Delta, s}$ and **CReg**, we obtain the transitions which describe the isomorphism between **CReg** and $\mathbf{M}_{\xi_T}^{C^{\Delta, s}}$.

In the sequel we will frequently use the equivalent characterization of topological spaces (resp. completely regular topological spaces) by means of ξ_T -saturated C^Δ -meters (resp. $C^{\Delta, s}$ -meters) to denote the objects of **Top** (resp. **CReg**).

2.3. The expanders ξ_A and ξ_{UG} on \mathbf{M}^C

The category of all approach spaces **Ap** and the construct of uniform approach spaces **UAp**, both with contractions, were introduced in [11] as quantified counterparts of the constructs **Top** and **CReg**. The constructs **qUnif** and **Unif** have as quantified counterparts the construct of all quasi-uniform gauge spaces **qUG** and the construct of all uniform gauge spaces **UG** in the sense of [13,16]. These constructs are all metrically generated, and are given by the expanders ξ_A and ξ_{UG} . Let $(X, \mathcal{D}) \in \mathbf{M}^C$.

$$d \in \xi_A(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \forall x \in X, \forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \dots, d_n \in \mathcal{D}, \forall y \in X: \\ e(x, y) \wedge \omega \leq \sup_{i=1}^n d_i(x, y) + \varepsilon, \\ d \in \xi_{UG}(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \forall \varepsilon > 0, \forall \omega < \infty, \exists d_1, \dots, d_n \in \mathcal{D}, \forall x, y \in X: \\ e(x, y) \wedge \omega \leq \sup_{i=1}^n d_i(x, y) + \varepsilon.$$

When applied to \mathbf{M}^{C^Δ} and $\mathbf{M}^{C^{\Delta, s}}$, ξ_A gives rise to **Ap** and **UAp**. If we apply ξ_{UG} to \mathbf{M}^{C^Δ} and $\mathbf{M}^{C^{\Delta, s}}$ we retrieve isomorphic descriptions of **qUG** and **UG**.

2.4. The expander ξ_D on \mathbf{M}^C

The theories of generalized metric spaces are given by the following expander. Let $(X, \mathcal{D}) \in \mathbf{M}^C$.

$$d \in \xi_D(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } e \leq \bigvee \mathcal{D},$$

where $\bigvee \mathcal{D} = \sup_{d' \in \mathcal{D}} d'$. By applying ξ_D on \mathbf{M}^C with $\mathcal{C} = \mathcal{C}^\Delta$, $\mathcal{C}^{\Delta, s}$ and \mathcal{C}^μ , we get isomorphic descriptions of the constructs \mathcal{C}^Δ , $\mathcal{C}^{\Delta, s}$ and \mathcal{C}^μ themselves.

2.5. The expander ξ_L on \mathbf{M}^C

The category of Lipschitz spaces is built on the definition of Lipschitz structures as given by Fraser in [8]. Fraser introduces Lipschitz structures in two different equivalent ways. A first description is given by collections of $\mathcal{C}^{\Delta, s}$ -metrics, a second one is given by means of sequences of entourages of the diagonal. His description by means of collections of $\mathcal{C}^{\Delta, s}$ -metrics is already very similar to our representation of objects of metrically generated constructs.

Definition. A Lipschitz structure on a set X is a non-empty collection \mathcal{L} of $\mathcal{C}^{\Delta, s}$ -metrics on X for which

- $d_1, d_2 \in \mathcal{L} \Rightarrow d_1 + d_2 \in \mathcal{L}$,
- $d \in \mathcal{L}, e$ is a $\mathcal{C}^{\Delta, s}$ -metric for which $e \leq d \Rightarrow e \in \mathcal{L}$,
- $\min\{d, 1\} \in \mathcal{L} \Rightarrow d \in \mathcal{L}$.

We call (X, \mathcal{L}) a Lipschitz space.

We define a relation on the set $\mathcal{C}^{\Delta, s}(X)$ as follows:

$$e \ll d \Leftrightarrow \exists \delta, K > 0, \forall x, y \in X: d(x, y) < \delta \Rightarrow e(x, y) \leq Kd(x, y).$$

A Lipschitz function between Lipschitz spaces is then defined as a function $f : (X, \mathcal{L}) \rightarrow (X', \mathcal{L}')$ for which $\forall e \in \mathcal{L}', \exists d \in \mathcal{L}: e \circ f \times f \ll d$. Since it is straightforward to check that $d \in \mathcal{L}$ and $e \ll d$, with e a $\mathcal{C}^{\Delta, s}$ -metric, implies $e \in \mathcal{L}$, we can see that f is a Lipschitz function if and only if $\forall e \in \mathcal{L}': e \circ f \times f \in \mathcal{L}$.

It turns out that **Lip** is a $\mathcal{C}^{\Delta, s}$ -metrically generated theory with the following expander. Let $(X, \mathcal{D}) \in \mathbf{M}^C$.

$$d \in \xi_L(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \exists d_1, \dots, d_n \in \mathcal{D}, \exists \delta, K > 0, \forall x, y \in X: \\ \sup_{i=1}^n d_i(x, y) < \delta \Rightarrow e(x, y) < K \sup_{i=1}^n d_i(x, y).$$

The corresponding concrete functor **K** is given by

$$\mathbf{K} : \mathcal{C}^{\Delta, s} \rightarrow \mathbf{Lip} : (X, d) \mapsto (X, M(d)),$$

where $M(d) := \{e \in \mathcal{C}^{\Delta, s}(X) \mid e \ll d\}$.

2.6. The expander β on \mathbf{M}^C

In the course of our investigations we will, for technical reasons, need to consider the expander β on \mathbf{M}^C which is a further adaptation of ξ_{UG} . Let $(X, \mathcal{D}) \in \mathbf{M}^C$.

$$d \in \beta(\mathcal{D}) \Leftrightarrow \exists e \in \mathcal{C}(X): d \leq e \text{ and } \forall \varepsilon > 0, \exists d_1, \dots, d_n \in \mathcal{D}, \forall x, y \in X: \\ e(x, y) \leq \sup_{i=1}^n d_i(x, y) + \varepsilon.$$

3. Uniform convergence on Y^X

In this section we will develop a technique to construct function space structures of “uniform convergence” in metrically generated constructs \mathbf{M}_ξ^C with $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$. We will study the relationship between this function space structure and the uniformity of uniform convergence. Therefore we consider the expanders ξ_U and ξ_T on $\mathbf{M}^{\mathcal{C}^{\Delta, s}}$ and we make use of the natural transition which interprets an \mathbf{M}_ξ^C -object (X, \mathcal{D}) as an $\mathbf{M}^{\mathcal{C}^{\Delta, s}}$ -object and sends it to the uniform space $(X, \xi_U \mathcal{D})$. This transition is given by the functor **G_U**, which is defined as follows:

$$\begin{array}{ccc} \mathbf{Gu} : & \mathbf{M}_\xi^C & \xrightarrow{\quad} \mathbf{Unif}, \\ & (Y, \mathcal{D}) \mapsto & (Y, \xi_U \circ e(\mathcal{D})), \\ & (Y, \mathcal{D}) \xrightarrow{f} (Y', \mathcal{D}') \mapsto & (Y, \xi_U \circ e(\mathcal{D})) \xrightarrow{f} (Y', \xi_U \circ e(\mathcal{D}')), \end{array}$$

where $e(\mathcal{D})$ equals the meter \mathcal{D} itself, but interpreted as a $\mathcal{C}^{\Delta, s}$ -meter, so that we can apply the $\mathbf{M}^{\mathcal{C}^{\Delta, s}}$ -expander ξ_U on it to obtain a uniformity.

Note that $\xi_T \circ e(\mathcal{D})$ represents the underlying topology of the uniformity $\xi_U \circ e(\mathcal{D})$. Whenever it is clear from the context, we write $\xi_U \mathcal{D}$ and $\xi_T \mathcal{D}$ instead of $\xi_U \circ e(\mathcal{D})$ and $\xi_T \circ e(\mathcal{D})$.

Given sets X and Y and a function $d \in [0, \infty]^{Y \times Y}$, we define

$$\gamma_d^X : Y^X \times Y^X \rightarrow [0, \infty]: (f, g) \mapsto \sup_{x \in X} d(f(x), g(x)).$$

Note that if (Y, d) belongs to **Met**, then the source

$$(ev_x : (Y^X, \gamma_d^X) \rightarrow (Y, d): f \mapsto f(x))_{x \in X}$$

is a product in **Met**. When no confusion can occur we simply denote the function γ_d^X by γ_d .

In order to rely on this construction to build a function space structure in a metrically generated category \mathbf{M}_ξ^C , we restrict to those base categories $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$ for which the following assumption holds:

$$[\mathbf{A1}] \quad \text{for sets } X, Y \text{ and for } d \in [0, \infty]^{Y \times Y}: d \in \mathcal{C}(Y) \Rightarrow \gamma_d \in \mathcal{C}(Y^X).$$

The base categories $\mathcal{C}^{\Delta, s}$ and \mathcal{C}^μ satisfy this condition.

Definition 3.1. Let a base category $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$ which satisfies **[A1]** and an expander ξ on \mathbf{M}^C be given. If X is a set and $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$, then we define $(Y^X, \mathcal{D}_u^{X, \xi})$ to be the \mathbf{M}_ξ^C -object, with $\mathcal{D}_u^{X, \xi}$ the ξ -saturation of the meter $\{\gamma_d^X \mid d \in \mathcal{D}\} \downarrow$.

Whenever it is clear from the context, we omit the superscripts ξ and X and write \mathcal{D}_u^ξ or \mathcal{D}_u instead of $\mathcal{D}_u^{X, \xi}$.

In general we cannot restrict to a ξ -basis of \mathcal{D} when constructing the function space structure $\mathcal{D}_u^{X, \xi}$. In order to remedy this we put an assumption on the expander ξ :

$$[\mathbf{A2}] \quad \text{for sets } X, Y, \text{ for } \mathcal{B} \subset \mathcal{C}(Y) \text{ and } e \in \mathcal{C}(Y): e \in \xi(\mathcal{B} \downarrow) \Rightarrow \gamma_e \in \xi(\{\gamma_d \mid d \in \mathcal{B}\} \downarrow).$$

The following result shows that the space $(Y^X, \mathcal{D}_u^{X, \xi})$ is unambiguously determined by the choice of a ξ -basis for \mathcal{D} if the expander ξ satisfies **[A2]**.

Proposition 3.2. Let $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$ be a base category which satisfies **[A1]** and let ξ be an expander on \mathbf{M}^C which satisfies **[A2]**. If X is a set and $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ has ξ -basis \mathcal{B} , then

$$\mathcal{D}_u^{X, \xi} = \xi(\{\gamma_d \mid d \in \mathcal{B}\} \downarrow).$$

Proof. By **[A2]** we have that $\{\gamma_d \mid d \in \mathcal{D}\} \subset \xi(\{\gamma_d \mid d \in \mathcal{B}\} \downarrow)$. Using the fact that ξ is idempotent and monotone the result follows. \square

Examples 3.3.

- (1) $\xi_D, \xi_U, \xi_{UG}, \xi_L$ and β on $\mathbf{M}^{\mathcal{C}^{\Delta, s}}$ all satisfy condition **[A2]**, but ξ_T and ξ_A do not satisfy **[A2]**.
- (2) If $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$ is a base category which satisfies **[A1]** and ξ is an expander on $\mathbf{M}^{\mathcal{C}^{\Delta, s}}$ which satisfies **[A2]**, then the modification of ξ to \mathbf{M}^C also satisfies **[A2]**. Hence the expanders $\xi_D, \xi_U, \xi_{UG}, \xi_L$ and β on \mathbf{M}^{C^μ} , with C^μ the base category consisting of ultrametrics, also satisfy **[A2]**.

As we will see now, for most theories \mathbf{M}_ξ^C there exists a strong relationship between the structure $\mathcal{D}_u^{X, \xi}$ on Y^X and the uniformity of uniform convergence $(\xi_U \mathcal{D})_u^{X, \xi_U}$ on Y^X derived from the uniformity $\xi_U \mathcal{D}$ on Y . For this purpose, we first recall the following proposition concerning uniform spaces.

Proposition 3.4. For any set X and $(Y, \mathcal{D}) \in \mathbf{Unif}$ we have that the uniformity \mathcal{D}_u^{X, ξ_U} on Y^X is precisely the uniformity of uniform convergence on Y^X derived from the uniformity \mathcal{D} on Y .

Proof. Let \mathcal{U} be the collection of entourages corresponding to the uniform space (Y, \mathcal{D}) . Recall from [2] that the uniformity of uniform convergence \mathcal{U}_u on Y^X derived from the uniformity \mathcal{U} on Y is generated by the subbase $\{W(V) \mid V \in \mathcal{W}\}$, where \mathcal{W} is a subbase of \mathcal{U} and

$$W(V) = \{(f, g) \in Y^X \times Y^X \mid \forall x \in X: (f(x), g(x)) \in V\}.$$

Since $\{V_\varepsilon^d \mid d \in \mathcal{D}, \varepsilon > 0\}$ is a subbase of \mathcal{U} and $W(V_\varepsilon^d) \subset V_\varepsilon^{\gamma_d} \subset W(V_\varepsilon^d)$ for all $d \in \mathcal{D}$ and $\varepsilon > 0$, it follows that the meter \mathcal{D}_u^{X, ξ_U} and the collection of entourages \mathcal{U}_u define the same uniformity. \square

Now consider for every theory \mathbf{M}_ξ^C and every set X the following functor:

$$\begin{aligned} \mathbf{F}_\xi^X : \quad & \mathbf{M}_\xi^C \longrightarrow \mathbf{M}_\xi^C, \\ & (Y, \mathcal{D}) \longmapsto (Y^X, \mathcal{D}_u^{X, \xi}), \\ & (Y, \mathcal{D}) \xrightarrow{f} (Y', \mathcal{D}') \longmapsto (Y^X, \mathcal{D}_u^{X, \xi}) \xrightarrow{\mathbf{F}_\xi^X f} (Y'^X, \mathcal{D}'^{X, \xi}), \end{aligned}$$

where $\mathbf{F}_\xi^X f(g) := f \circ g$, for any $g \in Y^X$. The following proposition states that for every theory \mathbf{M}_ξ^C and every set X the diagram

$$\begin{array}{ccc} \mathbf{M}_\xi^C & \xrightarrow{\mathbf{F}_\xi^X} & \mathbf{M}_\xi^C \\ \mathbf{G}_U \downarrow & & \downarrow \mathbf{G}_U \\ \mathbf{Unif} & \xrightarrow{\mathbf{F}_{\xi_U}^X} & \mathbf{Unif} \end{array}$$

commutes. Recall that ξ_U is considered as an expander on $\mathbf{M}^{C^{\Delta, s}}$.

Proposition 3.5. *If $\mathcal{C} \subset C^{\Delta, s}$ is a base category which satisfies [A1] and if ξ is an expander on \mathbf{M}^C for which $\xi_U \circ e \circ \xi = \xi_U \circ e$ then for any set X and $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ we have that*

- (1) $\xi_U(\mathcal{D}_u^{X, \xi})$ coincides with the uniformity of uniform convergence on Y^X derived from the uniformity $\xi_U \mathcal{D}$ on Y ;
- (2) $\xi_T(\mathcal{D}_u^{X, \xi})$ coincides with the topology of uniform convergence on Y^X derived from the uniformity $\xi_U \mathcal{D}$ on Y .

Proof. It is sufficient to prove the first statement. From the condition on ξ it follows that $\xi_U(\mathcal{D}_u^{X, \xi}) = (\xi_U(\mathcal{D}))_u^{X, \xi_U}$ and the result follows. \square

For $\xi = \xi_{UG}$, ξ_L or β on \mathbf{M}^C the condition that $\xi_U \circ e \circ \xi = \xi_U \circ e$ is satisfied. If $\xi = \xi_{UG}$ we retrieve function space structures which were studied before in [12,16]. For general ξ on \mathbf{M}^C we call $\mathcal{D}_u^{X, \xi}$ the \mathbf{M}_ξ^C -structure of uniform convergence on Y^X derived from (Y, \mathcal{D}) .

4. Σ -convergence on Y^X

From now on \mathcal{C} always denotes an arbitrary base category contained in $C^{\Delta, s}$, satisfying [A1], and ξ stands for an arbitrary expander on \mathbf{M}^C , satisfying [A2].

Given sets X and Y and a subset A of X , consider the restriction map $r_A : Y^X \rightarrow Y^A$ defined by $r_A(f) = f|_A$. For a given function $d \in [0, \infty]^{Y \times Y}$, let $\gamma_{d,A} : Y^X \times Y^X \rightarrow [0, \infty]$ be defined as

$$\gamma_{d,A} = \gamma_d^A \circ r_A \times r_A.$$

Lemma 4.1.

- (1) For sets X and Y , for $A \subset X$ and for $d \in [0, \infty]^{Y \times Y}$ we have that

$$d \in \mathcal{C}(Y) \quad \Rightarrow \quad \gamma_{d,A} \in \mathcal{C}(Y^X).$$

- (2) For sets X and Y , for $A \subset X$ and for $\mathcal{B} \subset \mathcal{C}(Y)$ and $e \in \mathcal{C}(Y)$ we have that

$$e \in \xi(\mathcal{B} \downarrow) \quad \Rightarrow \quad \gamma_{e,A} \in \xi(\{\gamma_{d,A} \mid d \in \mathcal{B}\} \downarrow).$$

Proof. In order to prove (1) note that $r_A : (Y^X, \gamma_{d,A}) \rightarrow (Y^A, \gamma_d^A)$ is initial in **Met**. By condition [A1] the space (Y^A, γ_d^A) belongs to \mathcal{C} . Since \mathcal{C} is closed under **Met**-subobjects also $(Y^X, \gamma_{d,A})$ belongs to \mathcal{C} .

In order to prove (2) let $\mathcal{B} \subset \mathcal{C}(Y)$, $e \in \mathcal{C}(Y)$ and $e \in \xi(\mathcal{B} \downarrow)$. Again consider the function $r_A : Y^X \rightarrow Y^A$. Further let $\mathcal{D} = \{\gamma_d^A \mid d \in \mathcal{B}\} \downarrow$. Since ξ is an expander we have $\xi(\mathcal{D}) \circ r_A \times r_A \subset \xi(\mathcal{D} \circ r_A \times r_A \downarrow)$. By condition [A2] we have $\gamma_e^A \in \xi(\mathcal{D})$, hence in view of $\gamma_{e,A} = \gamma_e^A \circ r_A \times r_A$ we can conclude that $\gamma_{e,A} \in \xi(\{\gamma_d^A \circ r_A \times r_A \mid d \in \mathcal{B}\} \downarrow) = \xi(\{\gamma_{d,A} \mid d \in \mathcal{B}\} \downarrow)$. \square

Definition 4.2. If X is a set, $\Sigma \subset 2^X$ is a cover of X , and if (Y, \mathcal{D}) is an \mathbf{M}_ξ^C -object then we define $(Y^X, \mathcal{D}_\Sigma^\xi)$ to be the \mathbf{M}_ξ^C -object with \mathcal{D}_Σ^ξ the initial lift of the source $(r_A : Y^X \rightarrow (Y^A, \mathcal{D}_u^{A,\xi}))_{A \in \Sigma}$ in \mathbf{M}_ξ^C .

Whenever it is clear from the context we write \mathcal{D}_Σ instead of \mathcal{D}_Σ^ξ .

The meter \mathcal{D}_Σ^ξ can be described in terms of an arbitrary ξ -basis of \mathcal{D} .

Proposition 4.3. Given a set X , a cover $\Sigma \subset 2^X$ of X and an \mathbf{M}_ξ^C -object (Y, \mathcal{D}) with ξ -basis \mathcal{B} , we have that

$$\mathcal{D}_\Sigma^\xi = \xi(\{\gamma_{d,A} \mid d \in \mathcal{B}, A \in \Sigma\} \downarrow).$$

Proof. Since the meters $(\mathcal{D}_u^{A,\xi})_{A \in \Sigma}$ do not depend on the choice of a particular ξ -basis for \mathcal{D} , without loss of generality we can use the same ξ -basis \mathcal{B} for the construction of each of them. The initial structure \mathcal{D}_Σ^ξ is the ξ -saturated meter

$$\begin{aligned} \mathcal{D}_\Sigma^\xi &= \xi(\{e \circ r_A \times r_A \mid A \in \Sigma, e \in \mathcal{D}_u^{A,\xi}\} \downarrow) \\ &= \xi(\{\gamma_d^A \circ r_A \times r_A \mid A \in \Sigma, d \in \mathcal{B}\} \downarrow) \\ &= \xi(\{\gamma_{d,A} \mid d \in \mathcal{B}, A \in \Sigma\} \downarrow). \quad \square \end{aligned}$$

We write $\langle \Sigma \rangle$ for the ideal in $(2^X, \subset)$ generated by Σ , i.e. the smallest subset of 2^X containing Σ and closed under the operations of taking finite unions and subsets. Since the expander ξ saturates for finite suprema it follows that $\mathcal{D}_{\langle \Sigma \rangle} = \mathcal{D}_\Sigma$. This means that, without loss of generality, we can suppose that Σ is an ideal in $(2^X, \subset)$. From now on we will also require that Σ is a cover of X .

Remarks 4.4.

- (1) If we choose Σ to be $\{X\}$ (or equivalently $\Sigma = 2^X$), then $\mathcal{D}_\Sigma^\xi = \mathcal{D}_u^\xi$.
- (2) If $\Sigma \subset \Sigma' \subset 2^X$, then $\text{id}_{Y^X} : (Y^X, \mathcal{D}_{\Sigma'}) \rightarrow (Y^X, \mathcal{D}_\Sigma)$ is a morphism in \mathbf{M}_ξ^C since clearly $\mathcal{D}_\Sigma \subset \mathcal{D}_{\Sigma'}$. In particular, we have that $\mathcal{D}_\Sigma \subset \mathcal{D}_u$ for all Σ , hence $\text{id}_{Y^X} : (Y^X, \mathcal{D}_u) \rightarrow (Y^X, \mathcal{D}_\Sigma)$ is a morphism.
- (3) If we take for ξ the expander ξ_U on $\mathbf{M}^{C^{\Delta,s}}$, then the structure \mathcal{D}_Σ corresponds to the uniformity of Σ -convergence.

In Proposition 3.5 we showed that the underlying uniformity (resp. topology) of the \mathbf{M}_ξ^C -structure $\mathcal{D}_u^{X,\xi}$ is the uniformity (resp. topology) of uniform convergence derived from the uniformity $\xi_U \mathcal{D}$ on Y . We are now able to prove a similar proposition which states that \mathcal{D}_Σ^ξ corresponds to Σ -convergence.

Proposition 4.5. Let ξ be an expander on \mathbf{M}^C such that $\xi_U \circ e \circ \xi = \xi_U \circ e$, with ξ_U the usual expander on $\mathbf{M}^{C^{\Delta,s}}$. For any set X , for any cover $\Sigma \subset 2^X$ and for any $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ we have that

- (1) $\xi_U(\mathcal{D}_\Sigma^\xi)$ coincides with the uniformity of Σ -convergence derived from the uniformity $\xi_U(\mathcal{D})$ on Y .
- (2) $\xi_T(\mathcal{D}_\Sigma^\xi)$ coincides with the topology of Σ -convergence derived from the uniformity $\xi_U(\mathcal{D})$ on Y .

Proof. Let \mathcal{B} be a ξ -basis of \mathcal{D} . Since \mathcal{C} is a subcategory of $\mathcal{C}^{\Delta,s}$, it is obvious that the source

$$(r_A : (Y^X, \{\gamma_{d,C} \mid d \in \mathcal{B}, C \in \Sigma\} \downarrow) \rightarrow (Y^A, \{\gamma_d^A \mid d \in \mathcal{B}\} \downarrow))_{A \in \Sigma}$$

is initial in $\mathbf{M}^{C^{\Delta,s}}$. The category **Unif** is concretely coreflectively embedded in $\mathbf{M}^{C^{\Delta,s}}$. Hence

$$(r_A : (Y^X, \xi_U \{\gamma_{d,C} \mid d \in \mathcal{B}, C \in \Sigma\} \downarrow) \rightarrow (Y^A, \xi_U \{\gamma_d^A \mid d \in \mathcal{B}\} \downarrow))_{A \in \Sigma}$$

is initial in **Unif**. By the assumption on ξ this is exactly the source

$$(r_A : (Y^X, \xi_U(\mathcal{D}_\Sigma^\xi)) \rightarrow (Y^A, \xi_U(\mathcal{D}_u^{A,\xi})))_{A \in \Sigma}.$$

So by Proposition 3.5 it follows that $\xi_U(\mathcal{D}_\Sigma^\xi)$ is the initial structure on

$$(r_A : Y^X \rightarrow (Y^A, (\xi_U \mathcal{D})_u^{A, \xi_U}))_{A \in \Sigma},$$

where $(\xi_U \mathcal{D})_u^{A, \xi_U}$ is the uniformity of uniform convergence on Y^A derived from the uniform space $(Y, \xi_U \mathcal{D})$. \square

Again we can formulate this in terms of a commutative diagram of functors. For a particular set X and $\Sigma \subset 2^X$, consider the functor

$$\begin{array}{ccc} \mathbf{F}_\xi^{X, \Sigma} : & \mathbf{M}_\xi^C & \longrightarrow \mathbf{M}_\xi^C, \\ & (Y, \mathcal{D}) \longmapsto & (Y^X, \mathcal{D}_\Sigma^\xi), \\ & (Y, \mathcal{D}) \xrightarrow{f} (Y', \mathcal{D}') \longmapsto & (Y^X, \mathcal{D}_\Sigma^\xi) \xrightarrow{\mathbf{F}_\xi^{X, \Sigma} f} (Y'^X, \mathcal{D}'_\Sigma^\xi), \end{array}$$

where $\mathbf{F}_\xi^{X, \Sigma} f(g) := f \circ g$, for any $g \in Y^X$.

Then the diagram

$$\begin{array}{ccc} \mathbf{M}_\xi^C & \xrightarrow{\mathbf{F}_\xi^{X, \Sigma}} & \mathbf{M}_\xi^C \\ \mathbf{G}_U \downarrow & & \downarrow \mathbf{G}_U \\ \mathbf{Unif} & \xrightarrow{\mathbf{F}_{\xi_U}^{X, \Sigma}} & \mathbf{Unif} \end{array}$$

commutes. Because of this proposition we will call \mathcal{D}_Σ^ξ the \mathbf{M}_ξ^C -structure of Σ -convergence on Y^X derived from the space (Y, \mathcal{D}) .

We can now prove that (Y, \mathcal{D}) is embedded in the function space $(Y^X, \mathcal{D}_\Sigma)$.

Proposition 4.6. *Let X be a set, let $\Sigma \subset 2^X$ be a cover of X and let (Y, \mathcal{D}) be an \mathbf{M}_ξ^C -object. If $c : Y \rightarrow Y^X : y \mapsto \mathbf{y}$, where \mathbf{y} is the constant map $X \rightarrow Y : x \mapsto y$, then the subspace $c(Y)$ of $(Y^X, \mathcal{D}_\Sigma)$ is isomorphic with (Y, \mathcal{D}) .*

Proof. Clearly c is an injective map. Since $\gamma_{d,A} \circ c \times c = d$ for any $d \in \mathcal{D}$ and $A \in \Sigma$, we find that $\mathcal{D}_\Sigma \circ c \times c = \mathcal{D}$, from which it follows that c is an initial morphism in \mathbf{M}_ξ^C . \square

Proposition 4.7. *If X is a set, if $\Sigma \subset 2^X$ covers X and if (Y, \mathcal{D}) is an \mathbf{M}_ξ^C -object, then the evaluation map $ev_x : (Y^X, \mathcal{D}_\Sigma) \rightarrow (Y, \mathcal{D}) : f \mapsto f(x)$ is a morphism in \mathbf{M}_ξ^C , for any $x \in X$.*

Proof. This is immediate, since for any $d \in \mathcal{D}$ and any $f, g \in Y^X$ we have $d(ev_x(f), ev_x(g)) = d(f(x), g(x)) \leq \gamma_{d,A}(f, g)$ if $x \in A$, which proves that $d \circ (ev_x \times ev_x) \in \mathcal{D}_\Sigma$. \square

5. Complete subsets of $(Y^X, \mathcal{D}_\Sigma)$

In this section we will characterize the uniformly and the metrically complete subsets of \mathbf{M}_ξ^C -spaces of Σ -convergence. Again let ξ_U and ξ_T be the usual expanders on $\mathbf{M}^{C, \Delta, s}$ and suppose that e sends a \mathcal{C} -meter \mathcal{D} on the meter \mathcal{D} itself interpreted as a $\mathcal{C}^{\Delta, s}$ -meter. Recall from [6] the definitions of uniform and metric completeness.

Definition 5.1. An object (X, \mathcal{D}) in \mathbf{M}_ξ^C is uniformly complete if the associated uniform space $(X, \xi_U \circ e(\mathcal{D}))$ is complete in the usual sense.

Definition 5.2. An object (X, \mathcal{D}) in \mathbf{M}_ξ^C is metrically complete if the uniform space $(X, \xi_U(\{\bigvee \mathcal{D}\} \downarrow))$ is complete in the usual sense.

In order to avoid repetition of the arguments, we will use a common notation for the two constructions which lie at the basis of these completeness notions. We denote by $h\mathcal{D}$ the transformation of \mathcal{D} , by $\xi_U h\mathcal{D}$ the associated uniformity and by $\xi_T h\mathcal{D}$ its associated topology. Hence for uniform completeness $h = e$ and for metric completeness h stands for the transformation which sends a ξ -saturated \mathcal{C} -meter \mathcal{D} to the $\mathcal{C}^{\Delta, s}$ -meter $\{\bigvee \mathcal{D}\} \downarrow$. The terminology “ h -complete” will be used to describe either uniform completeness or metric completeness.

In [6] and [15] it turned out that for those expanders ξ which satisfy

$$[\mathbf{A3}] \quad \xi_T \circ h = \xi_T \circ h \circ \xi,$$

there exists a completion theory which is firm in the sense of [3,4]. This extra assumption on ξ will also play an important role in the study of h -complete subsets of function space structures of Σ -convergence. Among other things it enables us to generalize the classical result which states that a closed subset of a complete uniform space is again complete.

Proposition 5.3. *Let ξ be an expander which satisfies $[\mathbf{A3}]$. If (Y, \mathcal{D}) is an h -complete \mathbf{M}_ξ^C -object, then each subset Z of Y which is closed in the topology $\xi_T h \mathcal{D}$ is also h -complete.*

Proof. Denote by \mathcal{D}' the \mathbf{M}_ξ^C -subspace structure on Z induced by (Y, \mathcal{D}) . By the assumption on the expander ξ , the topology $\xi_T h \mathcal{D}'$ coincides with the subspace topology of $\xi_T h \mathcal{D}$ on Z . Let \mathcal{F} be a filter on Z which is $\xi_U h \mathcal{D}'$ -Cauchy. Then $\text{stack}_Y \mathcal{F}$ is $\xi_U h \mathcal{D}$ -Cauchy and hence converges to a point x of Y in $\xi_T h \mathcal{D}$. Moreover $x \in Z$, since Z is closed in $\xi_T h \mathcal{D}$ and hence \mathcal{F} converges to x in the topology $\xi_T h \mathcal{D}'$. \square

Examples 5.4.

- (1) In the case that $h = e$, the condition $[\mathbf{A3}]$ is fulfilled by the expanders ξ_U , ξ_{UG} , ξ_L and β on \mathbf{M}^C , but ξ_D on \mathbf{M}^C does not satisfy $[\mathbf{A3}]$.
- (2) If h sends a meter \mathcal{D} on a set X to the meter $\{\bigvee \mathcal{D}\} \downarrow$, then $[\mathbf{A3}]$ is satisfied by the expanders β , ξ_{UG} and ξ_D on \mathbf{M}^C , but not by ξ_U and ξ_L on \mathbf{M}^C .

Once we require ξ to satisfy $[\mathbf{A3}]$, we are also able to characterize the convergent filters of the function space Y^X with respect to the topology $\xi_T h \mathcal{D}_\Sigma$. If Φ is a filter on Y^X , we denote by $\Phi(x)$ the filter base on Y formed by the sets $\text{ev}_x(\mathcal{H})$ as \mathcal{H} runs through Φ .

Proposition 5.5. *Let a set X , $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ and a cover $\Sigma \subset 2^X$ be given. If Φ is a filter on Y^X and $f \in Y^X$, then the following conditions are equivalent:*

- (1) Φ converges to f in $(Y^X, \xi_T h \mathcal{D}_\Sigma)$;
- (2) Φ is $\xi_U h \mathcal{D}_\Sigma$ -Cauchy and for every $x \in X$: $\text{stack } \Phi(x)$ converges to $f(x)$ for $\xi_T h \mathcal{D}$.

Proof. The proof of (1) \Rightarrow (2) is straightforward. To see (2) \Rightarrow (1) we first consider the case $h = e$. Let Φ be a $\xi_U \mathcal{D}_\Sigma$ -Cauchy filter on Y^X such that, for every x , $\text{stack } \Phi(x)$ converges to $f(x)$ with respect to $\xi_T \mathcal{D}$. Due to the assumption on ξ is the collection $\{B_{\gamma_{d,A}}(g, \varepsilon) \mid g \in Y^X, \varepsilon > 0, d \in \mathcal{D}, A \in \Sigma\}$ a basis for the collection of open sets of the topology $\xi_T \mathcal{D}_\Sigma$ on Y^X . Let d be a \mathcal{C} -metric in \mathcal{D} , $A \in \Sigma$ and $\varepsilon > 0$. From the $\xi_U \mathcal{D}_\Sigma$ -Cauchyness of Φ it follows that there exists a $g \in Y^X$ such that $B_{\gamma_{d,A}}(g, \varepsilon/4) \in \Phi$. Hence $B_d(g(x), \varepsilon/4) \in \text{stack } \Phi(x)$, for all $x \in A$, what induces that $\gamma_{d,A}(g, f) < \varepsilon/2$. So $B_{\gamma_{d,A}}(g, \varepsilon/4) \subset B_{\gamma_{d,A}}(f, \varepsilon)$ from which we obtain that $B_{\gamma_{d,A}}(f, \varepsilon) \in \Phi$. We can conclude that Φ converges to f for $\xi_T \mathcal{D}_\Sigma$.

In the case that h sends a \mathcal{C} -meter \mathcal{D} to the $\mathcal{C}^{\Delta, s}$ -meter $\{\bigvee \mathcal{D}\} \downarrow$, we have that $\xi_T \{\bigvee \mathcal{D}_\Sigma\} \downarrow$ is the underlying topology of the metric $\gamma_{\bigvee \mathcal{D}, X}$ and $\xi_T \{\bigvee \mathcal{D}\} \downarrow$ is the underlying topology of the metric $\bigvee \mathcal{D}$. Since the uniformity $\xi_U \{\bigvee \mathcal{D}_\Sigma\} \downarrow$ is finer than the underlying uniformity of the metric $\gamma_{\bigvee \mathcal{D}, X}$, we find the result by analogous reasoning as in the case that $h = e$. \square

This result leads to a characterization of the h -complete subsets of $(Y^X, \mathcal{D}_\Sigma)$.

Theorem 5.6. *Let a set X , $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ and a cover $\Sigma \subset 2^X$ be given. A subspace \mathcal{H} of $(Y^X, \mathcal{D}_\Sigma)$ is h -complete if and only if for every $\xi_U h \mathcal{D}_\Sigma$ -Cauchy filter Φ on \mathcal{H} there exists an $f \in \mathcal{H}$ such that $\text{stack } \Phi(x)$ converges to $f(x)$ in $\xi_T h \mathcal{D}$ for every $x \in X$.*

This proposition enables us to formulate a condition under which h -completeness of a subset \mathcal{H} of Y^X for \mathcal{D}_Σ implies h -completeness with respect to $\mathcal{D}_{\Sigma'}$, with Σ and Σ' subsets of 2^X .

Corollary 5.7. *Let a set X and $(Y, \mathcal{D}) \in \mathbf{M}_\xi^C$ be given. If $\Sigma \subset \Sigma' \subset 2^X$ are covers of X , then every $\mathcal{H} \subset Y^X$ which is h -complete in $(Y^X, \mathcal{D}_\Sigma)$ is also h -complete in $(Y^X, \mathcal{D}_{\Sigma'})$.*

Proof. This follows from Theorem 5.6 since every $\xi_U h \mathcal{D}_{\Sigma'}$ -Cauchy filter is also $\xi_U h \mathcal{D}_\Sigma$ -Cauchy. \square

Corollary 5.8. *If $\mathcal{H} \subset Y^X$ is such that for every $x \in X$ the $\xi_T h \mathcal{D}$ -closure of $\text{ev}_x(\mathcal{H})$ is h -complete in (Y, \mathcal{D}) , then the $\xi_T h \mathcal{D}_\Sigma$ -closure of \mathcal{H} is h -complete in $(Y^X, \mathcal{D}_\Sigma)$.*

Corollary 5.9. *Let X be a set with a cover Σ , and let (Y, \mathcal{D}) be an \mathbf{M}_ξ^C -object. If (Y, \mathcal{D}) is h -complete, then also $(Y^X, \mathcal{D}_\Sigma)$ is h -complete.*

Proof. This follows by combining Proposition 5.3 and the preceding corollary. \square

Examples 5.10.

- (1) If we take for $\xi = \xi_U$ and $C = C^{\Delta, s}$ (or C^μ), we obtain that whenever (Y, \mathcal{U}) is a complete (non-Archimedean) uniform space, also the uniform space of Σ -convergence $(Y^X, \mathcal{U}_\Sigma)$ is a complete (non-Archimedean) uniform space.
- (2) If we take for $\xi = \xi_{UG}$ and $C = C^{\Delta, s}$, then when (Y, \mathcal{G}) is a uniformly (resp. metrically) complete uniform gauge space, also the uniform gauge space of Σ -convergence $(Y^X, \mathcal{G}_\Sigma)$ is uniformly (resp. metrically) complete.
- (3) If we take for $\xi = \xi_L$ and $C = C^{\Delta, s}$, then when (Y, \mathcal{L}) is a uniformly complete Lipschitz space, also the Lipschitz space of Σ -convergence is uniformly complete.

6. Spaces of contractions

If (X, T) is a topological space and (Y, \mathcal{U}) a complete uniform space with underlying topology T' , then it is well known (see for example [2]) that the collection of continuous maps between (X, T) and (Y, T') endowed with the uniformity of uniform convergence, is complete as well. In this section we will see that this is merely a special case of a far more general result for metrically generated theories. We consider an h -complete \mathbf{M}_ξ^C -object instead of a complete uniform space and an \mathbf{M}_η^C -object instead of a topological space, with η an arbitrary expander on \mathbf{M}^C . Recall that C is supposed to be a base category contained in $C^{\Delta, s}$ which satisfies [A1] and \mathbf{M}_ξ^C is a theory which satisfies [A2] and [A3]. To state that for an \mathbf{M}_η^C -object (X, \mathcal{D}) and an h -complete \mathbf{M}_ξ^C -object (Y, \mathcal{G}) , the collection

$$\{f \in Y^X \mid f : (X, \mathcal{D}) \rightarrow (Y, \eta\mathcal{G}) \text{ is a morphism in } \mathbf{M}_\eta^C\}$$

endowed with the \mathbf{M}_ξ^C -subspace structure of \mathcal{G}_u^ξ is h -complete, we will need to put an extra condition on η . It turns out to be sufficient to require that every \mathbf{M}_η^C -object (X, \mathcal{D}) is closed under taking C -metrics on X which are “almost” contained in \mathcal{D} , in the sense that a C -metric d on X is contained in \mathcal{D} if for all $\varepsilon > 0$: $(d - \varepsilon) \vee 0 \in \mathcal{D}$, where $(d - \varepsilon) \vee 0 : X \times X \rightarrow [0, \infty]$: $(d(x, y) - \varepsilon) \vee 0$. This condition on the expander η is equivalent with the expression $\beta \leq \eta$ on \mathbf{M}^C and is satisfied by the expanders $\xi_{UG}, \xi_U, \xi_D, \xi_T$ and ξ_A on \mathbf{M}^C . Nevertheless an $\mathbf{M}_{\xi_L}^C$ -object does not have to be β -saturated.

Proposition 6.1. *Given a theory \mathbf{M}_η^C such that $\beta \leq \eta$ on \mathbf{M}^C , an \mathbf{M}_η^C -object (X, \mathcal{D}) and an \mathbf{M}_ξ^C -object (Y, \mathcal{G}) ,*

$$\mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G}))) = \{f \in Y^X \mid f : (X, \mathcal{D}) \rightarrow (Y, \eta(\mathcal{G})) \text{ is a morphism in } \mathbf{M}_\eta^C\}$$

is a closed subset of $(Y^X, \xi_T h \mathcal{G}_u^\xi)$.

Proof. Denote by $\overline{\mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G})))}$, the closure of $\mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G})))$ in $\xi_T h \mathcal{G}_u^\xi$. Let $f \in \overline{\mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G})))}$ and let d be a C -metric contained in \mathcal{G} . For every $\varepsilon > 0$ we have that $B_{\gamma_d}(f, \varepsilon)$ is open in the topology $\xi_T h \mathcal{G}_u^\xi$ on Y^X and hence there exists a $g \in \mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G})))$ such that $\gamma_d(f, g) < \varepsilon$. By applying the triangle inequality and the symmetry of d we find that $\forall x, y \in X : d(f(x), f(y)) \leq d(g(x), g(y)) + 2\varepsilon$. Since $d \circ g \times g \in \mathcal{D}$, it follows that $(d \circ f \times f) - 2\varepsilon \vee 0 \in \mathcal{D}$. By arbitrariness of ε , it follows that $d \circ f \times f \in \beta\mathcal{D} = \mathcal{D}$. So $\mathcal{G} \circ f \times f \subset \mathcal{D}$. Since η is an expander, we can conclude that $\eta(\mathcal{G}) \circ f \times f \subset \mathcal{D}$, hence $f : (X, \mathcal{D}) \rightarrow (Y, \eta(\mathcal{G}))$ is a morphism in \mathbf{M}_η^C . \square

Together with Proposition 5.3 and Corollary 5.9 we can conclude:

Corollary 6.2. *Let \mathbf{M}_η^C be a theory for which $\beta \leq \eta$ on \mathbf{M}^C , let $(X, \mathcal{D}) \in \mathbf{M}_\eta^C$ and let (Y, \mathcal{G}) be an h -complete \mathbf{M}_ξ^C -object.*

Then $\mathbf{M}_\eta^C((X, \mathcal{D}), (Y, \eta(\mathcal{G})))$ equipped with the subspace structure of (Y^X, \mathcal{G}_u^ξ) is h -complete.

Examples 6.3.

- (1) Let X be a set, (Y, \mathcal{U}) a uniform space which is complete in the classical sense (hence uniformly complete) with underlying topology (Y, T) and let \mathcal{U}_u be the uniformity of uniform convergence on Y^X derived from (Y, \mathcal{U}) .
 - (a) If (X, T') is a topological space, then the collection of continuous functions $\mathbf{Top}((X, T'), (Y, T))$ equipped with the subspace uniformity of (Y^X, \mathcal{U}_u) is complete.
 - (b) If (X, \mathcal{U}') is a uniform space, then the collection of uniformly continuous functions $\mathbf{Unif}((X, \mathcal{U}'), (Y, \mathcal{U}))$ equipped with the subspace uniformity of (Y^X, \mathcal{U}_u) is complete.
- (2) Let X be a set, (Y, \mathcal{G}) a uniformly complete (resp. metrically complete) **UG**-space with underlying approach space (Y, δ) and let \mathcal{G}_u be the **UG**-structure of uniform convergence on Y^X derived from (Y, \mathcal{G}) .

- (a) If (X, δ') is an approach space, then the collection of contractions $\mathbf{Ap}((X, \delta'), (Y, \delta))$ equipped with the subspace \mathbf{UG} -structure of (Y^X, \mathcal{G}_u) is uniformly complete (resp. metrically complete).
- (b) If (X, \mathcal{D}) is a \mathbf{UG} -space, then the collection of uniform contractions $\mathbf{UG}((X, \mathcal{D}), (Y, \mathcal{G}))$ equipped with the subspace \mathbf{UG} -structure of (Y^X, \mathcal{G}_u) is uniformly complete (resp. metrically complete).

Proposition 6.1 can also be extended to \mathbf{M}_ξ^C -spaces of Σ -convergence.

Proposition 6.4. *Let η be an expander on \mathbf{M}^C such that $\beta \leq \eta$ on \mathbf{M}^C , $(X, \mathcal{D}) \in \mathbf{M}_\eta^C$, $\Sigma \subset 2^X$ a cover of X and $(Y, \mathcal{G}) \in \mathbf{M}_\xi^C$. Denote the collection of all $f \in Y^X$ for which*

$$\forall A \in \Sigma: f|_A : (A, \eta(\mathcal{D}|_{A \times A})) \rightarrow (Y, \eta(\mathcal{G})) \text{ is an } \mathbf{M}_\eta^C\text{-morphism}$$

as $\text{Mor}_\eta^\Sigma(X, Y)$. Then $\text{Mor}_\eta^\Sigma(X, Y)$ is a closed subset of $(Y^X, \xi_T h \mathcal{G}_\Sigma^\xi)$.

Proof. This goes along the same lines as the proof of Proposition 6.1. \square

Corollary 6.5. *Let \mathbf{M}_η^C be a theory such that $\beta \leq \eta$ on \mathbf{M}^C , (X, \mathcal{D}) an \mathbf{M}_η^C -object, Σ a cover, and (Y, \mathcal{G}) an h -complete \mathbf{M}_ξ^C -object. Then $\text{Mor}_\eta^\Sigma(X, Y)$ equipped with the subspace structure of $(Y^X, \mathcal{G}_\Sigma^\xi)$ is h -complete.*

7. An Ascoli theorem in metrically generated constructs

Since we have developed a satisfying notion of function space structures of Σ -convergence for metrically generated constructs \mathbf{M}_ξ^C (which satisfy [A1] and [A2]), it is important now to investigate whether we can formulate an Ascoli theorem in this setting and to investigate which conditions, if any, we hereto need to impose on a metrically generated construct.

Theorem 7.1. ([2, Bourbaki version of Ascoli's theorem]) *Let (X, \mathcal{T}) be a topological space (resp. uniform space), let Σ be a cover of X , let (Y, \mathcal{U}) be a uniform space, and let \mathcal{H} be a set of functions of X into Y such that for each function $u \in \mathcal{H}$ and each $A \in \Sigma$, the restriction of u to A is continuous (resp. uniformly continuous). If the sets $A \in \Sigma$ are compact (resp. precompact), then \mathcal{H} is precompact with respect to the uniform structure of Σ -convergence if and only if the following conditions are satisfied:*

- (1) *For each $A \in \Sigma$, the set $\mathcal{H}|_A$ of restrictions to A of functions of \mathcal{H} is equicontinuous (resp. uniformly equicontinuous);*
- (2) *For each $x \in X$, the set $\text{ev}_x(\mathcal{H})$ is precompact.*

Before a study of Ascoli's theorem in the setting of metrically generated theories is possible, it is necessary to develop suitable counterparts of the concepts of (uniform) equicontinuity and (pre-)compactness for arbitrary metrically generated constructs. An extension of the concept of uniform equicontinuity to the metrically generated category \mathbf{UG} is already known: in [12] the concept of uniform equicontractivity was introduced for uniform gauge spaces. Given two uniform gauge spaces (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) , a subset \mathcal{H} of Y^X is called uniformly equicontractive if $\forall d \in \mathcal{G}_Y, \exists e \in \mathcal{G}_X, \forall f \in \mathcal{H}: d \circ (f \times f) \leq e$. For uniform spaces this concept coincides with uniform equicontinuity.

The possibility to represent the objects of metrically generated constructs by means of metered spaces allows for a unifying treatment of the concepts of equicontinuity for topological spaces, uniform equicontinuity for uniform spaces and uniform equicontractivity for uniform gauge spaces by means of the concept ζ -equicontractivity in a metrically generated category $\mathbf{M}_\xi^\mathcal{E}$.

Definition 7.2. Let \mathcal{E} be a base category and let ζ be an expander on $\mathbf{M}^\mathcal{E}$. Further let (X, \mathcal{D}) be an $\mathbf{M}_\xi^\mathcal{E}$ -object and let (Y, \mathcal{D}') be an $\mathbf{M}^\mathcal{E}$ -object. A subset \mathcal{H} of Y^X is called ζ -equicontractive if for all $d \in \mathcal{D}': \sup_{f \in \mathcal{H}} d \circ f \times f \in \mathcal{D}$.

We note that when a subset \mathcal{H} of Y^X is ζ -equicontractive, then for any $f \in \mathcal{H}$ the map $f : (X, \mathcal{D}) \rightarrow (Y, \zeta(\mathcal{D}'))$ is a contraction in $\mathbf{M}_\xi^\mathcal{E}$.

If we apply the notion of ζ -equicontractivity to the case that $\zeta = \xi_T$ on \mathbf{M}^{C^Δ} , we retrieve the classical notion of equicontinuity for topological spaces. If $\zeta = \xi_U$ (resp. ξ_{UG}) on $\mathbf{M}^{C^\Delta, s}$, we recover the notion of uniform equicontinuity (resp. uniform equicontractivity).

A concept of precompactness for metrically generated constructs is also needed. Recall that a quasi-uniform space (X, \mathcal{U}) is called precompact if $\forall U \in \mathcal{U}, \exists A \subset X$ finite: $\bigcup_{x \in A} U(x) = X$ [7]. A C^Δ -metric d on X is called precompact if the quasi-uniformity induced by the metric d is precompact i.e. if $\forall \varepsilon > 0, \exists A \subset X$ finite: $\bigcup_{x \in A} B_d(x, \varepsilon) = X$. If \mathcal{D} is the corresponding meter of the quasi-uniformity \mathcal{U} , then precompactness of \mathcal{U} is equivalent with the claim that \mathcal{D} has a basis of precompact C^Δ -metrics. This formulation leads us to an adequate notion of precompactness for objects of metrically generated constructs $\mathbf{M}_\xi^\mathcal{E}$ with $\mathcal{E} \subset C^\Delta$, which we call ζ -precompactness.

Definition 7.3. Given a base category $\mathcal{E} \subset \mathcal{C}^\Delta$ and a theory $\mathbf{M}_\xi^\mathcal{E}$, an object (X, \mathcal{D}) of $\mathbf{M}_\xi^\mathcal{E}$ is called ζ -precompact if \mathcal{D} has a basis of precompact \mathcal{C}^Δ -metrics. A subset A of X is called ζ -precompact if the $\mathbf{M}_\xi^\mathcal{E}$ -subspace object $(A, \zeta(\mathcal{D}|_{A \times A}))$ is ζ -precompact.

It is clear that when an object (X, \mathcal{D}) of $\mathbf{M}_\xi^\mathcal{E}$ is ζ -precompact, we have that for all $d \in \mathcal{D} : \forall \varepsilon > 0, \exists A \subset X$ finite: $\bigcup_{x \in A} B_d(x, \varepsilon) = X$. If ξ_U is the usual expander on $\mathbf{M}^{\mathcal{C}^\Delta}$ and e' associates with a \mathcal{C} -meter \mathcal{D} the meter itself interpreted as a \mathcal{C}^Δ -meter, then it is obvious that an $\mathbf{M}_\xi^\mathcal{E}$ -object (X, \mathcal{D}) is ζ -precompact if and only if the associated quasi-uniform space $(X, \xi_U \circ e'(\mathcal{D}))$ is precompact in the classical sense. Hence a topological space (X, \mathcal{T}) is ξ_T -precompact if and only if its fine quasi-uniformity is precompact, which exactly means that the topological space (X, \mathcal{T}) is compact [7].

These definitions of equicontractivity and precompactness for metrically generated constructs allow us to formulate an appropriate Ascoli theorem. Therefore we consider:

- a base category $\mathcal{C} \subset \mathcal{C}^{\Delta, s}$ which satisfies [A1];
- a base category \mathcal{E} such that $\mathcal{C} \subset \mathcal{E} \subset \mathcal{C}^\Delta$;
- an expander ξ on $\mathbf{M}^\mathcal{C}$ satisfying [A2] and for which $\xi_U \circ e \circ \xi = \xi_U \circ e$;
- a theory $\mathbf{M}_\eta^\mathcal{E}$ for which $\beta \leq \eta$.

Note that we do not impose any relation between the two expanders ξ and η . The assumption $\xi_U \circ e \circ \xi = \xi_U \circ e$ on ξ is strictly stronger than condition [A3] and guarantees that a subset A of an $\mathbf{M}_\xi^\mathcal{E}$ -object (X, \mathcal{D}) is ξ -precompact if and only if $\mathcal{D}|_{A \times A}$ has a basis of precompact \mathcal{C} -metrics.

Under these assumptions on the theories $\mathbf{M}_\xi^\mathcal{E}$ and $\mathbf{M}_\eta^\mathcal{E}$ we are able to characterize the precompact subsets of $\mathbf{M}_\xi^\mathcal{E}$ -structures of Σ -convergence.

Theorem 7.4 (Ascoli theorem for metrically generated constructs). Suppose we have an $\mathbf{M}_\eta^\mathcal{E}$ -object (X, \mathcal{G}_X) , an $\mathbf{M}_\xi^\mathcal{C}$ -object (Y, \mathcal{G}_Y) , a cover Σ of X and a collection of functions $\mathcal{H} \subset Y^X$, such that for each function $f \in \mathcal{H}$ and each $A \in \Sigma : f|_A : (A, \mathcal{G}_X|_A) \rightarrow (Y, \eta(\mathcal{G}_Y))$ is an $\mathbf{M}_\eta^\mathcal{E}$ -morphism.

If every element A of Σ is η -precompact, then \mathcal{H} is ξ -precompact as a subspace of $(Y^X, (\mathcal{G}_Y)_\Sigma)$ if and only if

- (1) for every $A \in \Sigma ; \mathcal{H}|_A \subset Y^A$ is η -equicontractive;
- (2) for every $x \in X : ev_x(\mathcal{H})$ is ξ -precompact.

This result can be deduced from the following three propositions.

Proposition 7.5. Let an $\mathbf{M}_\eta^\mathcal{E}$ -object (X, \mathcal{G}_X) , an $\mathbf{M}_\xi^\mathcal{C}$ -object (Y, \mathcal{G}_Y) , a cover Σ of X and a collection of functions $\mathcal{H} \subset Y^X$ be given. If \mathcal{H} is ξ -precompact then so for any $x \in X$ is $ev_x(\mathcal{H})$.

Proof. Let $x \in X, d \in \mathcal{G}_Y, \varepsilon > 0$. Choose $A \in \Sigma$ such that $x \in A$. Since \mathcal{H} is ξ -precompact, there exists a finite subset $\mathcal{F} \subset \mathcal{H}$ such that $\mathcal{H} \subset \bigcup_{f \in \mathcal{F}} B_{\gamma_{d,A}}(f, \varepsilon)$. Now it follows that $ev_x(\mathcal{H}) \subset \bigcup_{f \in \mathcal{F}} B_d(f(x), \varepsilon)$. \square

Proposition 7.6. Given an $\mathbf{M}_\eta^\mathcal{E}$ -object (X, \mathcal{G}_X) , an $\mathbf{M}_\xi^\mathcal{C}$ -object (Y, \mathcal{G}_Y) , a cover Σ of X and a collection of functions $\mathcal{H} \subset Y^X$, then for each $A \in \Sigma, \mathcal{H}|_A$ is η -equicontractive, if the following conditions are fulfilled:

- (1) for every $f \in \mathcal{H}, A \in \Sigma : f|_A \in \mathbf{M}_\eta^\mathcal{C}((A, \eta(\mathcal{G}_X|_{A \times A})), (Y, \eta(\mathcal{G}_Y)))$;
- (2) \mathcal{H} is ξ -precompact.

Proof. Let $A \in \Sigma, d$ \mathcal{C} -metric in \mathcal{G}_Y and $\varepsilon > 0$. Since \mathcal{H} is precompact, there exists a finite subset $\mathcal{F} \subset \mathcal{H}$ such that $\mathcal{H} \subset \bigcup_{f \in \mathcal{F}} B_{\gamma_{d,A}}(f, \varepsilon/2)$. So for an arbitrary $g \in \mathcal{H}$, there exists an $f_g \in \mathcal{F}$ such that $\gamma_{d,A}(f_g, g) < \varepsilon/2$. By applying the symmetry and the triangle inequality we find for every $x, y \in A$ that

$$\begin{aligned} d(g(x), g(y)) &\leq d(g(x), f_g(x)) + d(f_g(x), f_g(y)) + d(f_g(y), g(y)) \\ &\leq d(f_g(x), f_g(y)) + \varepsilon. \end{aligned}$$

Hence we deduce that $\sup_{g \in \mathcal{H}} d \circ g|_A \times g|_A \leq \sup_{f \in \mathcal{F}} d \circ f|_A \times f|_A + \varepsilon$, so $\sup_{g \in \mathcal{H}} d \circ g|_A \times g|_A - \varepsilon \vee 0 \in \eta(\mathcal{G}_X|_{A \times A})$. Since an η -saturated meter is supposed to be β -saturated, we can conclude that $\sup_{g \in \mathcal{H}} d \circ g|_A \times g|_A \in \eta(\mathcal{G}_X|_{A \times A})$. \square

All we still have to do is to prove the other implication of Theorem 7.4.

Proposition 7.7. Let an $\mathbf{M}_\eta^\mathcal{E}$ -object (X, \mathcal{G}_X) , an $\mathbf{M}_\xi^\mathcal{C}$ -object (Y, \mathcal{G}_Y) , a cover Σ of X and a collection of functions $\mathcal{H} \subset Y^X$ be given. If each set $A \in \Sigma$ is η -precompact, if for each set $A \in \Sigma$ the collection $\mathcal{H}|_A$ is η -equicontractive and if for each $x \in X$, $ev_x(\mathcal{H})$ is ξ -precompact, then \mathcal{H} is ξ -precompact too.

Proof. It is sufficient to verify that for any d \mathcal{C} -metric in \mathcal{G}_Y , for any $A \in \Sigma$: $\gamma_{d,A}|\mathcal{H} \times \mathcal{H}$ is a precompact \mathcal{C} -metric in order to conclude ξ -precompactness of \mathcal{H} . Let d be a \mathcal{C} -metric in \mathcal{G}_Y , $A \in \Sigma$, $\varepsilon > 0$. $\mathcal{H}|_A$ is η -equicontractive, hence $e := \sup_{f \in \mathcal{H}} d \circ f|_A \times f|_A \in \eta(\mathcal{G}_X|_{A \times A})$. From the symmetry of d it follows that e is also symmetric. Since $(A, \eta(\mathcal{G}_X|_{A \times A}))$ is η -precompact, there exists a finite subset B of A such that $A = \bigcup_{x \in B} B_e(x, \varepsilon/5)$. Hence $Z := \bigcup_{x \in B} ev_x(\mathcal{H})$ is a finite union of ξ -precompact subsets of (Y, \mathcal{G}_Y) , so Z is ξ -precompact too. It follows that there exists a finite subset C of Z such that $Z \subset \bigcup_{y \in C} B_d(y, \varepsilon/5)$. Consider for every function $h \in \mathcal{C}^B$, the set $\mathcal{B}(h) := \{f \in \mathcal{H} \mid \forall b \in B: d(f(b), h(b)) < \varepsilon/5\}$. For $f \in \mathcal{H}$, choose a function $h_f : B \rightarrow C$ such that $d(f(b), h_f(b)) < \varepsilon/5$, for any $b \in B$. Clearly $f \in \mathcal{B}(h_f)$ and hence $\bigcup_{h \in \mathcal{C}^B} \mathcal{B}(h) = \mathcal{H}$. For every function $h : B \rightarrow C$ for which $\mathcal{B}(h)$ is not empty we can choose an element $g_h \in \mathcal{B}(h)$. Then $\{g_{h_f} \mid f \in \mathcal{H}\}$ is finite, since \mathcal{C}^B is finite. Take $f \in \mathcal{H}$ and $a \in A$, then there exists an element x of B such that $e(x, a) < \varepsilon/5$. Then

$$\begin{aligned} d(f(a), g_{h_f}(a)) &\leq d(f(a), f(x)) + d(f(x), h_f(x)) + d(h_f(x), g_{h_f}(x)) + d(g_{h_f}(x), g_{h_f}(a)) \\ &\leq e(a, x) + d(f(x), h_f(x)) + d(h_f(x), g_{h_f}(x)) + e(x, a) \\ &< 4\varepsilon/5. \end{aligned}$$

So $\gamma_{d,A}(f, g_{h_f}) < \varepsilon$ and we can conclude that $\{B_{\gamma_{d,A}|\mathcal{H} \times \mathcal{H}}(g_{h_f}, \varepsilon) \mid f \in \mathcal{H}\}$ is a finite cover of \mathcal{H} . \square

Example 7.8. If we apply Theorem 7.4 to $\mathcal{C} = \mathcal{C}^{\Delta, s}$, $\xi = \xi_U$, $\mathcal{E} = \mathcal{C}^\Delta$ (resp. $\mathcal{C}^{\Delta, s}$) and $\eta = \xi_T$ (resp. ξ_U) we retrieve the Bourbaki version of Ascoli's theorem.

Theorem 7.4 can also be used to characterize the ξ_{UG} -precompact subsets of **UG**-structures of Σ -convergence. So far it is not known whether 0-compactness of an approach space (X, δ) (see [11]) coincides with ξ_A -precompactness, i.e. precompactness of (X, \mathcal{UD}) with \mathcal{D} the fine **qUG**-space of (X, δ) , but in [9] it is proved that 0-compactness implies ξ_A -precompactness.

Example 7.9. Let a set X , a **UG**-space (Y, \mathcal{G}) , a cover Σ of X and a subset \mathcal{H} of Y^X be given, and let \mathcal{G}_Σ be the **UG**-structure of Σ -convergence derived from (Y, \mathcal{G}) .

- (1) If (X, δ) is an approach space, if for any $A \in \Sigma$, $f \in \mathcal{H}$: $f|_A$ is a contraction and if for any $A \in \Sigma$: $(A, \delta|_A)$ is 0-compact (or even ξ_A -precompact), then \mathcal{H} is ξ_{UG} -precompact for \mathcal{G}_Σ if and only if for any $A \in \Sigma$: $\mathcal{H}|_A \in Y^A$ is ξ_A -equicontractive and for any $x \in X$, $ev_x(\mathcal{H})$ is ξ_{UG} -precompact.
- (2) If \mathcal{D} is a **qUG**-structure on X , if for any $A \in \Sigma$, $f \in \mathcal{H}$: $f|_A$ is a **qUG**-morphism and if for any $A \in \Sigma$ the space $(A, \xi_{UG}(\mathcal{D}|_{A \times A}))$ is ξ_{UG} -precompact, then \mathcal{H} is ξ_{UG} -precompact for \mathcal{G}_Σ if and only if for any $A \in \Sigma$: $\mathcal{H}|_A \in Y^A$ is ξ_{UG} -equicontractive and for any $x \in X$, $ev_x(\mathcal{H})$ is ξ_{UG} -precompact.

References

- [1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, Pure Appl. Math. (N.Y.), John Wiley & Sons, New York, 1990, The Joy of Cats, A Wiley-Interscience Publication.
- [2] N. Bourbaki, Éléments de mathématique. Fasc. X. Première partie. Livre III: Topologie générale. Chapitre 10: Espaces fonctionnels, Deuxième édition, entièrement refondue, Actualités Sci. Indust., vol. 1084, Hermann, Paris, 1961.
- [3] G.C.L. Brümmer, E. Giuli, A categorical concept of completion of objects, Comment. Math. Univ. Carolin. 33 (1) (1992) 131–147.
- [4] G.C.L. Brümmer, E. Giuli, H. Herrlich, Epireflections which are completions, Cah. Topol. Geom. Differ. Categ. 33 (1) (1992) 71–73.
- [5] E. Colebunders, R. Lowen, Metrically generated theories, Proc. Amer. Math. Soc. 133 (5) (2005) 1547–1556 (electronic).
- [6] E. Colebunders, R. Lowen, E. Vandersmissen, Uniqueness of completion for metrically generated constructs, Topology Appl. 155 (2007) 39–55.
- [7] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Lect. Notes Pure Appl. Math., vol. 77, Marcel Dekker, New York, 1982.
- [8] R.B. Fraser Jr., Axiom systems for Lipschitz structures, Fund. Math. 66 (1969) 15–24.
- [9] A. Gerlo, Separation, completeness and compactness in metrically generated theories, PhD thesis, Vrije Universiteit Brussel, 2007.
- [10] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer-Verlag, 1976.
- [11] R. Lowen, Approach spaces, in: The Missing Link in the Topology-Uniformity-Metric Triad, in: Oxford Math. Monogr., The Clarendon Press/Oxford University Press/Oxford Science Publications, New York, 1997.
- [12] R. Lowen, An Ascoli theorem in approach theory, in: IV Iberoamerican Conference on Topology and its Applications, Topology Appl. 137 (1–3) (2004) 207–213.
- [13] R. Lowen, B. Windels, AUnif: A common supercategory of pMet and Unif, Int. J. Math. Math. Sci. 21 (1998) 1–18.
- [14] A.F. Monna, Remarques sur les métriques non-archimédiennes. II, Nederl. Akad. Wetensch. 53 (1950) 625–637; Indag. Math. (N.S.) 179–191 (1950) 1950.
- [15] E. Vandersmissen, Completion via nearness for metrically generated constructs, Appl. Categ. Structures 15 (5–6) (2007) 633–645.
- [16] B. Windels, Uniform approach theory, PhD thesis, Universiteit Antwerpen, 1997.