

ON THE STATIONARY VIBRATIONS OF A RECTANGULAR PLATE SUBJECTED TO STRESS PRESCRIBED PARTIALLY AT THE CIRCUMFERENCE

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ABSTRACT. The stationary periodical problem of a vibrating rectangular plate, stressed at a segment while fixed elsewhere at one of its edges, is considered. Using the finite Fourier transformation, the problem is converted to a singular integral equation that in turn can be reduced to an infinite system of algebraic equations. The truncation of the algebraic system is justified.

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1. CONVERSION OF THE GOVERNING EQUATIONS INTO A SINGULAR INTEGRAL EQUATION

The boundary conditions considered in this problem are

$$\bar{\sigma}_y(x, 0; t) = -P^* e^{i\omega t}, \quad (P = \text{const.}), \quad |x| < c \quad (1.1)$$

$$\bar{v}(x, 0; t) = 0, \quad c \leq |x| < \pi \quad (1.2)$$

$$\bar{\tau}_{xy}(x, 0; t) = 0, \quad |x| < \pi \quad (1.3)$$

$$\bar{u}(x, -1; t) = 0, \quad |x| < \pi \quad (1.4)$$

$$\bar{\tau}_{xy}(x, -1; t) = 0, \quad |x| < \pi \quad (1.5)$$

Expressed in terms of the longitudinal and transversal potentials $\bar{\phi}$ and $\bar{\psi}$, respectively, the stresses and displacements satisfy (Nowacki [1]) the following equations

$$\bar{\sigma}_y = 2\mu \left(\frac{\nu}{1-2\nu} \nabla^2 \bar{\phi} + \frac{\partial^2 \bar{\phi}}{\partial y^2} - \frac{\partial^2 \bar{\psi}}{\partial x \partial y} \right), \quad (1.6)$$

$$\bar{v} = \frac{\partial \bar{\phi}}{\partial y} - \frac{\partial \bar{\psi}}{\partial x}, \quad (1.7)$$

$$\bar{\tau}_{xy} = \mu \left(2 \frac{\partial^2 \bar{\phi}}{\partial x \partial y} + \frac{\partial^2 \bar{\psi}}{\partial y^2} - \frac{\partial^2 \bar{\psi}}{\partial x^2} \right). \quad (1.8)$$

where ν is the Poisson's ration, and μ and λ are the Lamé's constants. The functions $\bar{\phi}$ and $\bar{\psi}$ satisfy the equation

$$\nabla^2 \bar{\phi} = \frac{1}{c_1^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} \tag{1.9}$$

$$\nabla^2 \bar{\psi} = \frac{1}{c_2^2} \frac{\partial^2 \bar{\psi}}{\partial t^2} \tag{1.10}$$

where c_1 and c_2 are the propagation velocities of longitudinal and transversal waves respectively.

The stationary solutions of the problem can be expressed in the form

$$\bar{\phi}(x, y; t) = e^{i\omega t} \phi(x, y), \quad \bar{\psi}(x, y; t) = e^{i\omega t} \psi(x, y) \tag{1.11}$$

$$\bar{\sigma}_y = e^{i\omega t} \sigma_y, \quad \bar{v} = e^{i\omega t} v \quad \text{and} \quad \bar{\tau}_{xy} = e^{i\omega t} \tau_{xy}$$

where

$$\nabla^2 \phi + k_1^2 \phi = 0, \quad \nabla^2 \psi + k_2^2 \psi = 0 \tag{1.12}$$

and

$$k_1^2 = \frac{\omega^2}{c_1^2}, \quad k_2^2 = \frac{\omega^2}{c_2^2}. \tag{1.13}$$

Applying the finite Fourier transform

$$F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad f(x) = \sum_{n=-\infty}^{\infty} F_n e^{inx} \tag{1.14}$$

to the equations (1.12) and solving the resulting equations, we get

$$\Phi_n = A_n e^{\sqrt{n^2 - k_1^2} y} + B_n e^{-\sqrt{n^2 - k_1^2} y} \tag{1.15}$$

$$\Psi_n = C_n e^{\sqrt{n^2 - k_2^2} y} + D_n e^{-\sqrt{n^2 - k_2^2} y}$$

The mixed boundary conditions (1.1) and (1.2) can be written in the form

$$\sigma_y(x, 0) = -P^* + G_+(x) \tag{1.16}$$

$$u(x, 0) = G_-(x) \tag{1.17}$$

where

$$G_+(x) = \begin{cases} 0 & |x| < c \\ \text{undetermined} & c < |x| < \pi \end{cases} \tag{1.18}$$

and

$$G_-(x) = \begin{cases} \text{undetermined} & |x| < c \\ 0 & c < |x| < \pi \end{cases} \tag{1.19}$$

The application of the Fourier transform to the completed conditions (1.16) and (1.17) whose left-hand sides are given explicitly by (1.6) and (1.7), together with (1.15) yields

$$-2\mu[-n^2(A_n + B_n) + in\sqrt{n^2 - k_2^2}(C_n - D_n)] - (\lambda + 2\mu)k_1^2(A_n + B_n) = -P_{n-}^* + G_{n+} \tag{1.20}$$

$$\sqrt{n^2 - k_1^2}(A_n - B_n) - in(C_n + D_n) = G_{n-}. \tag{1.21}$$

The Fourier components P_{n-} belong to the extension $P^*(x)$ of the constant P^* by defining it to be zero outside $|\theta| < c$. Carrying out procedures similar to those used in obtaining equations (1.20) and (1.21),

the uniform conditions (1.3)-(1.5) lead to an additional three equations of the same type. Solving the system which consists of these additional three equations together with eq. (1.21) for the coefficients A_n, B_n, C_n and D_n , and then substituting their values in (1.20), we get

$$Q^*(|n|, k_1, k_2)G_{n-} = -P_{n-}^* + G_{n+} \tag{1.22}$$

where

$$Q^*(|n|, k_1, k_2) = \frac{\{2\mu n^2 - (\lambda + 2\mu)k_1^2\} (k_2^2 - 2n^2)}{k_2^2 \sqrt{n^2 - k_1^2}} \coth \sqrt{n^2 - k_1^2} + \frac{4\mu n^2 \sqrt{n^2 - k_2^2}}{k_2^2} \coth \sqrt{n^2 - k_2^2} - 2\mu \frac{k_2^2 - k_1^2}{k_2^2} |n| + O(|n|^{-1}) \tag{1.23}$$

Finally, the discrete problem (1.22) can be written in the standard form

$$|n| G_{n-} + Q_{|n|} G_{n-} = -P_{n-}^* + G_{n+}^* \tag{1.24}$$

where

$$Q_{|n|} = \frac{k_2^2}{2\mu(k_2^2 - k_1^2)} Q^*(|n|, k_1, k_2) - |n|, \\ P_{n-}^* = \frac{k_2^2}{2\mu(k_2^2 - k_1^2)} P_{n-}^* \quad \text{and} \quad G_{n+}^* = \frac{k_2^2}{2\mu(k_2^2 - k_1^2)} G_{n+}^* \tag{1.25}$$

2. REDUCTION OF THE DISCRETE PROBLEM TO AN ALGEBRAIC SYSTEM OF EQUATIONS

Applying the inverse Fourier transform to eq. (1.24), we get

$$\frac{1}{\pi i} \frac{d}{dx} \int_{-\pi}^{\pi} \frac{G_-(x) dt}{1 - e^{i(x-t)}} + \sum_{n=-\infty}^{\infty} Q_{|n|} G_{n-} e^{inx} = -P + P_+^*(x) \tag{2.1}$$

where

$$P = \frac{k_2^2}{2\mu(k_2^2 - k_1^2)} P^*, \quad G_+^*(x) = \frac{k_2^2}{2\mu(k_2^2 - k_1^2)} G_+(x) \tag{2.2}$$

Recalling that $G_+^*(x) = 0$ for $-c < x < c$ and integrating eq. (2.1) with respect to θ , we obtain over the interval $-c < x < c$

$$\frac{1}{\pi i} \int_{-c}^c \frac{G_-(x) dt}{1 - e^{i(x-t)}} + Q_0 G_{0-} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{ni} e^{inx} + \sum_{n=-\infty}^{\infty} Q_{|n|} G_{n-} \frac{e^{inx}}{in} = -P \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{ni} e^{inx} + \alpha \tag{2.3}$$

where α is constant and the primes over the summation symbols mean that the value $n = 0$ is not included. Here we have used the expansion

$$x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{ni} e^{inx} \tag{2.4}$$

The singular integral equation with the Cauchy kernel (2.3) possesses a bounded solution $G_-(x)$ which can be written in the form (Eckhardt and El-Sheikh [2])

$$G_-(x) = -R(x) \left[Q_0 G_{0-} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} I_n(x) + \sum_{n=-\infty}^{\infty} Q_{|n|} \frac{G_{n-}}{n} I_n(x) + P \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} I_n(x) - i\alpha I_0(x) \right] \tag{2.5}$$

where

$$I_n(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i(n+1)\xi} d\xi}{R(\xi)(e^{i\xi} - e^{ix})}, \tag{2.6}$$

$$R(x) = \lim_{\substack{z \rightarrow e^{ix} \\ |z| < 1}} \sqrt{(z - e^{ic})(z - e^{-ic})}. \tag{2.7}$$

The integrals $I_n(x)$ are to be understood in the sense of the principal value. For $n > 0$, [3, p. 215], we have

$$I_n(x) = -e^{i(n-1)x} \sum_{j=0}^{n-1} \frac{e^{-ixj}}{2^j} \sum_{m=0}^j \frac{(2m-1)!![2(j-m)-1]!!}{m!(j-m)!e^{imc}e^{-i(j-m)c}}, \quad n > 0 \tag{2.8}$$

and

$$I_0(x) = 0. \tag{2.9}$$

Further, it has already been shown (Muskhelishvili [4]) that

$$I_{-n}(x) = -e^{-ix} I_n(-x). \tag{2.10}$$

In relation (2.8), the convention $0!! = (-1)!! = 1$ is adopted. By virtue of (2.9) and the fact that $\frac{G_{n-}}{n}$ is odd,

the solution (2.5) can be written in the form

$$G_{-}(x) = -R(x) \left[Q_0 G_{0-} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \{I_n(x) - I_{-n}(x)\} + \sum_{n=1}^{\infty} Q_n \frac{G_{n-}}{n} \{I_n(x) - I_{-n}(x)\} + P \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \{I_n(x) - I_{-n}(x)\} \right]. \tag{2.11}$$

Substituting in the formula

$$G_{l-} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{-}(x) e^{itx} dx \tag{2.12}$$

according to (2.11), we get the algebraic system

$$G_{l-} + Q_0 G_{0-} N_{0l} + \sum_{n=1}^{\infty} Q_n \frac{G_{n-}}{n} [N_{nl} - N_{-nl}] = P N_{0l}, \quad l = 0, 1, 2, \dots \tag{2.13}$$

where for $n \neq 0$ we have used the notation

$$\begin{aligned} N_{-nl} &= \frac{1}{2i} \int_{-\infty}^{\infty} R(x) I_{-n}(x) e^{-itx} dx \\ &= \sum_{j=0}^{n-1} A_{j-n-l-1} \sum_{m=0}^j \frac{(2m-1)!![2(j-m)-1]!!}{2m!!(2j-2m)!!e^{i(j-2m)c}}; \\ N_{nl} &= -\sum_{j=0}^{n-1} A_{n-j-l-2} \sum_{m=0}^j \frac{(2m-1)!![2(j-m)-1]!!}{2m!!(2j-2m)!!e^{-i(j-2m)c}}. \end{aligned} \tag{2.14}$$

where

$$A_k = \frac{1}{2\pi i} \int_{AB} \sqrt{(t-A)(t-B)} t^k dt, \quad A = e^{-ic}, \quad B = e^{ic} \tag{2.15}$$

and the arc AB is directed along the unit circle. The values of these integrals are given by the formulas [3, p. 216]

$$\begin{aligned}
 A_{-1} &= \frac{e^{-c} + e^{ic}}{4} - \frac{1}{2} = A_{-2}, \\
 A_{-k} &= \frac{1}{2^k} \left[\frac{(2k-5)!! [e^{-i(k-1)c} + e^{i(k-1)c}]}{(k-1)!} \right. \\
 &\quad \left. - \sum_{m=1}^{k-2} \frac{(2m-3)!!(2k-2m-5)!!}{m!(k-1-m)!} e^{i(k-1-2m)c} \right], \quad k > 2.
 \end{aligned}
 \tag{2.16}$$

and for $k > 0$ we have [4]

$$A_k = A_{-(k+3)}. \tag{2.17}$$

Finally, N_{0l} is given by

$$N_{0l} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \{N_{nl} - N_{-nl}\} \tag{2.18}$$

It has been shown [2] that $N_{-ln}(N_{ln})$ tends to zero as $l \rightarrow \infty$ more rapidly than $\frac{1}{l}$ and indeed, it tends to zero as $n \rightarrow \infty$ since

$$\frac{1}{l} N_{-ln}(N_{ln}) = \frac{1}{n} N_{-nl}(N_{nl}). \tag{2.19}$$

This establishes not only that the coefficients of G_{n-} , $n = 0, 1, 2, \dots$ in system (2.13) will tend to zero as $n, l \rightarrow \infty$, but also that the expression of N_{0l} will converge so quickly that it can be calculated through truncation at a suitable order.

3. JUSTIFICATION OF THE TRUNCATION

The infinite system (2.13) can be solved by truncating it at a suitable order m . This is equivalent to reducing (2.3) to the more simple equation

$$\tilde{K} \tilde{G}_- = \tilde{f} \tag{3.1}$$

where \tilde{K} and \tilde{G} are "approximations" to the operator K and the function $G_-(x)$, respectively, in the left-hand side of the original equation (2.3)

$$\tilde{K} \tilde{G}_- = \frac{1}{\pi i} \int_{-c}^c \frac{\tilde{G}_-(x) dt}{1 - e^{i(x-t)}} + Q_0 \tilde{G}_0 - \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{ni} e^{inx} + \sum_{n=-m}^m Q_{|n|} \tilde{G}_n - \frac{e^{inx}}{in} \tag{3.2}$$

and

$$\tilde{f} = -P \sum_{n=-m}^m \frac{(-1)^n}{ni} e^{inx} + \tilde{\alpha}. \tag{3.3}$$

Equation (2.13) itself can be denoted similarly in the form

$$K G_- = f. \tag{3.4}$$

We shall now outline some features about equations (3.1) and (3.4). First, eq. (3.1) has a unique bounded solution defined by eq. (2.5) truncated at the m -th order and in which $\{G_{n-}, n = 0, 1, \dots, m\}$ is the solution of system (2.13) truncated at the m -th order. Second, the Banach space $L_2[-c, c]$ is the domain and range of both the operators K and \tilde{K} . Finally, the operator $\tilde{K}^{-1}(K - \tilde{K})$ is bounded in $L_2[-c, c]$ with the norm

$$\|\tilde{K}^{-1}(K - \tilde{K})\| < 1, \tag{3.5}$$

provided m is sufficiently large. This will be established if we show that $\|K - \tilde{K}\|$ can be made arbitrarily small. We have

$$(K - \bar{K})G(x) = \frac{1}{\pi} \sum_{n=m}^{\infty} \frac{Q_n}{n} \int_{\bar{c}}^c \sin n(x-t)G(t)dt$$

and consequently,

$$\begin{aligned} \|(K - \bar{K})G(x)\| &= \int_{\bar{c}}^c \left| \frac{1}{\pi} \sum_{n=m}^{\infty} \frac{Q_n}{n} \int_{\bar{c}}^c \sin n(x-t)G(t)dt \right|^2 dx \\ &\leq \frac{1}{\pi^2} \int_{\bar{c}}^c \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \int_{\bar{c}}^c |G(t)| \cdot |\sin n(x-t)| dt \right)^2 dx \\ &\leq \frac{1}{\pi^2} \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \int_{\bar{c}}^c |G(t)| dt \right)^2 \int_{\bar{c}}^c dx \\ &= \frac{2c}{\pi^2} \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \int_{\bar{c}}^c |G(t)| dt \right)^2 \\ &= \frac{2c}{\pi^2} \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \right)^2 \left(\int_{\bar{c}}^c |G(t)| dt \right)^2 \\ &\leq \frac{2c}{\pi^2} \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \right) \int_{\bar{c}}^c dt \int_{\bar{c}}^c |G(t)|^2 dt \\ &\leq 4 \left(\sum_{n=m}^{\infty} \frac{Q_n}{n} \right)^2 \|G\|^2 \end{aligned}$$

from which it follows that

$$\|K - \bar{K}\| \leq 2 \sum_{n=m}^{\infty} \frac{Q_n}{n}. \quad (3.6)$$

The right-hand side in the last expression tends to zero since $Q_n = 0\left(\frac{1}{n}\right)$ according to the definitions (1.25) and (1.23), and the relation (3.5) will eventually be fulfilled.

In view of the above considerations (Cherski [5]), it follows that equation (2.13) has the unique solution

$$G_- = \bar{G}_- + [I + \bar{K}^{-1}(K - \bar{K})]^{-1} \bar{K}^{-1}(f - K\bar{G}_-), \quad (3.7)$$

where I is the unit operator. Further, the resulting error due to the truncation can be estimated according to the formula

$$\|G_- - \bar{G}_-\| \leq \frac{\|\bar{K}^{-1}(f - K\bar{G}_-)\|}{1 - \|\bar{K}^{-1}(K - \bar{K})\|}. \quad (3.8)$$

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