



An Ascoli theorem in approach theory

R. Lowen

Department of Mathematics, University of Antwerp, Brussel, Belgium

Received 29 September 2001; received in revised form 17 April 2002

Abstract

In this paper we prove a general Ascoli theorem in the uniform setting of approach theory.
© 2003 Published by Elsevier B.V.

MSC: 54A05; 54B30; 54E15

Keywords: Ascoli; Approach space; Uniform gauge space; Contraction; Equicontractive; Compact; Precompact

1. Introduction

In this paper we will deal not only with approach spaces [5], but also and mainly with their uniform counterpart as introduced in [6]. In [5,6] various characterizations of both the local and uniform types of approach structures are given. For our purposes it is most convenient to use the characterizations which make use of ideals of pseudo-quasi-metrics, shortly pq-metrics. We note that our pq-metrics are not required to be finite and that the word ideal here has to be understood in the order-theoretic sense.

Let \mathcal{G} be an ideal of pq-metrics, then we say \mathcal{G} is locally saturated if, whenever e is a pq-metric such that

$$\forall x \in X, \forall \varepsilon > 0, \forall \omega < \infty, \exists d \in \mathcal{G}: e(x, \cdot) \wedge \omega \leq d(x, \cdot) + \varepsilon$$

it follows that $e \in \mathcal{G}$. Such a locally saturated ideal will be referred to as a *gauge*. An *approach space* then is a pair (X, \mathcal{G}) where \mathcal{G} is a gauge.

We say that an ideal \mathcal{G} consisting of p-metrics is globally (or uniformly) saturated if, whenever e is a p-metric such that

$$\forall \varepsilon > 0, \forall \omega < \infty, \exists d \in \mathcal{G}: e \wedge \omega \leq d + \varepsilon$$

E-mail address: rlow@ruca.ua.ac.be (R. Lowen).

it follows that $e \in \mathcal{G}$. Such an ideal will be called a *uniform gauge*. A pair (X, \mathcal{G}) where \mathcal{G} is a uniform gauge will be referred to as a *uniform gauge space*. Note that this terminology differs from the one used in [6].

In both cases (approach spaces and uniform gauge spaces) the associated morphisms are defined in the same way. Let (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) be approach spaces (respectively uniform gauge spaces) and let $f: X \rightarrow Y$ be a function. We say that f is a *contraction* (respectively a *uniform contraction*) if

$$\forall d \in \mathcal{G}_Y: d \circ (f \times f) \in \mathcal{G}_X.$$

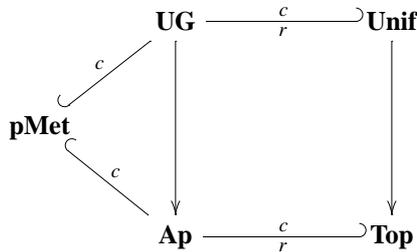
Equivalent formulations of contractivity and uniform contractivity bring the difference between local and global saturation to light. A map is a contraction if and only if

$$\forall x \in X \forall d \in \mathcal{G}_Y \forall \varepsilon > 0 \forall \omega < \infty \exists e \in \mathcal{G}_X: d(f(x), f(\cdot)) \wedge \omega \leq e(x, \cdot) + \varepsilon,$$

whereas it is a uniform contraction if and only if

$$\forall d \in \mathcal{G}_Y \forall \varepsilon > 0 \forall \omega < \infty \exists e \in \mathcal{G}_X: d \circ (f \times f) \wedge \omega \leq e + \varepsilon.$$

Approach spaces and contractions form a topological category (in the sense of [1]), and so do uniform gauge spaces and uniform contractions [5,6]. We will denote these categories, respectively, **Ap** and **UG**. The relation among these categories and well-known related categories is depicted in the following diagram:



Top (respectively **Unif**) is embedded in **Ap** (respectively **UG**) simultaneously bireflectively and coreflectively. The well-known functorial relation between **Top** and **Unif**, to a large extent, carries over to **Ap** and **UG**. The most remarkable aspect of the diagram takes place on the left side. **pMet** (with non-expansive maps) is fully and coreflectively embedded in both **Ap** and **UG**. The thus obtained subcategories are not stable under the formation of infinite products. Taking an infinite product of p-metric approach spaces in **Ap**, provides the underlying product-set with an approach structure which, in general, is neither p-metric nor topological but which has as topological coreflection precisely the product topology of the topologies underlying the p-metrics. The same situation presents itself in the uniform case. We will not dwell on this particularly important aspect of approach theory here but rather refer the interested reader to [5] and [6].

All functors in the diagram are identities on morphisms and hence are completely determined by their actions on objects. If (X, \mathcal{G}) is an approach space, then the topological coreflection of (X, \mathcal{G}) is determined by the closure operator: $x \in \overline{A} \iff \forall \varepsilon > 0, \forall d \in \mathcal{G}, \exists y \in A: d(x, y) < \varepsilon$. If (X, \mathcal{G}) is a uniform gauge space then the uniform coreflection has as uniformity simply the one generated by \mathcal{G} . If (X, d) is a p(q)-metric space then both

the gauge (in the case of a pq-metric) and the uniform gauge (in the case of a p-metric) are given by the principal ideal generated by $\{d\}$. If (X, \mathcal{T}) is a topological space then it is embedded into **Ap** by associating with it the gauge consisting of all pq-metrics which generate topologies coarser than \mathcal{T} . A uniform space is embedded into **UG** simply by associating with it the set of all uniformly continuous p-metrics, which, as can easily be verified, actually is a uniform gauge in our sense.

It is clear that a gauge generated by p-metrics is also a uniform gauge. Hence the subcategory of **Ap** consisting of all subspaces of products of p-metric spaces is actually embedded into **UG**. This embedding is even full and it extends the well-known embedding of **Creg** into **Unif** via the fine uniformity. Given a uniform gauge the underlying gauge is the smallest gauge which is, in general, strictly finer. It is obtained by saturating the uniform gauge for the local saturation property.

In order to prove an Ascoli theorem in the setting of approach theory we now require three main items. First of all we need function spaces, second we need a notion of precompactness and third we need a notion of equicontractivity.

2. Function spaces, precompactness and equicontractivity

Suppose we are given two uniform gauge spaces (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) and let Σ be a cover of X which is closed under finite unions. Let \mathcal{H} be any collection of functions from X to Y . Then we define a uniform gauge on \mathcal{H} as follows. For any $A \in \Sigma$ and any $d \in \mathcal{G}_Y$ we define

$$D_{d,A} : \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty] : (f, g) \mapsto \sup_{x \in A} d(f(x), g(x)).$$

Clearly these functions are p-metrics and hence they determine a uniform gauge. Actually they form a basis for an ideal and the uniform gauge generated by this basis is obtained by saturating the set $\{D_{d,A} \mid d \in \mathcal{G}_Y, A \in \Sigma\}$ for the uniform saturation property. We will denote this uniform gauge by $\langle \Sigma, \mathcal{G}_Y \rangle$. The following proposition tells us what is the relation between this structure on \mathcal{H} and well-known function space uniformities and topologies. We leave the easy verification to the reader.

Proposition 2.1. *If (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) are uniform gauge spaces, $\mathcal{H} \subset Y^X$ and we consider the uniform gauge $\langle \Sigma, \mathcal{G}_Y \rangle$ on \mathcal{H} then the following hold:*

- (1) *The topological (respectively uniform) coreflection is structured with the topology (respectively uniformity) of uniform convergence on Σ -sets.*
- (2) *The p-metric coreflection (both in **Ap** and in **UG**) is structured with the supremum p-metric.*

For any set X we denote by $2^{(X)}$ the set of all finite subsets of X . In [5] a measure of compactness $\mu_c(X)$ for an approach space (X, \mathcal{G}) was defined by

$$\mu_c(X) := \sup_{\varphi \in \mathcal{G}^X} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} \varphi(x)(x, z).$$

Actually this definition extends the well-known measure introduced by K. Kuratowski in [3] (there called measure of non-compactness). This measure has most properties which one might want it to have. In particular, a Tychonoff theorem holds for it, for topological spaces $\mu_c(X) = 0$ if and only if X is compact, for metric spaces $\mu_c(X) = 0$ if and only if X is precompact and $\mu_c(X) < \infty$ if and only if X is bounded. For a proof of these facts we refer the interested reader to [4] and [5].

If X is the approach reflection of a uniform gauge space (X, \mathcal{G}) , then the foregoing formula for $\mu_c(X)$ need not be changed. In other words, instead of the gauge generated by \mathcal{G} , a generating subset, in this case \mathcal{G} itself can be taken.

In [6] a measure of precompactness for uniform gauge spaces was defined in terms of towers of semi-uniformities (which provide another way of characterizing uniform gauge spaces). We rephrase this definition in terms of uniform gauges. For a uniform gauge space (X, \mathcal{G}) its measure of precompactness is defined by

$$\mu_{pc}(X) = \sup_{d \in \mathcal{G}} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} d(x, z).$$

It is interesting to compare the formulas for μ_c (for the underlying approach space) and μ_{pc} (for the uniform gauge space itself). What we see is that they are entirely the same except for the first supremum, which in the case of compactness ranges over the set \mathcal{G}^X and in the case of precompactness ranges over the set \mathcal{G} . In the approach case, which just as topology is a local theory, the p-metrics must be allowed to vary from point to point, and in the uniform gauge case, which just as uniformity is a global theory, the same p-metric has to be chosen in every point. It was already shown in [6], but is easily deduced from the above formula that for the measure of precompactness too a number of good consistency results hold. Thus, e.g., $\mu_{pc} \leq \mu_c$, for uniform spaces $\mu_{pc}(X) = 0$ if and only if X is precompact, for metric spaces $\mu_{pc}(X) = 0$ if and only if X is precompact and $\mu_{pc}(X) < \infty$ if and only if X is bounded. The following example shows that even in a metric space the measure of precompactness, just as the measure of compactness, can attain any value. Let $[0, a]$ be equipped with the Euclidean metric and consider the supremum-metric on $X := [0, a]^{\mathbb{N}}$. Then it can easily be verified that $\mu_{pc}(X) = a$. Also, the inequality $\mu_{pc} \leq \mu_c$ is, in general, strict. To see this, it suffices to take a precompact non-compact uniform space and embed it in **UG**. Then one measure is zero whereas the other one is infinite. One small supplementary result which we require is the following. The easy proof is again left to the reader.

Proposition 2.2. *If X is a uniform gauge space and A_1, \dots, A_n are subsets of X then $\mu_{pc}(\bigcup_{i=1}^n A_i) \leq \sup_{i=1}^n \mu_{pc}(A_i)$.*

Next we need a concept of equicontractivity. If (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) are uniform gauge spaces, then $\mathcal{H} \subset Y^X$ is called *uniformly equicontractive* if

$$\forall d \in \mathcal{G}_Y, \exists e \in \mathcal{G}_X, \forall f \in \mathcal{H}: d \circ (f \times f) \leq e,$$

and it is clear that it is sufficient that the condition be satisfied for all d in a basis for \mathcal{G}_Y .

Clearly, if \mathcal{H} is uniformly equicontractive then each $f \in \mathcal{H}$ is a uniform contraction, and a subset of a uniformly equicontractive set is again uniformly equicontractive. The

following result is easily verified and we leave this to the reader. Note the remarkable characterization in the metric case.

Proposition 2.3. *In the case of uniform spaces X and Y a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive if and only if it is uniformly equicontinuous, and in the case of p -metric spaces X and Y a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive simply when it consists of uniform contractions, i.e., non-expansive maps.*

Since in what follows we will explicitly be working with the measure of precompactness, the nicest formulation of an Ascoli theorem is obtained if we also work with natural measures of uniform contractivity and equicontractivity. Taking into account the second characterization of uniform contractions given in the first section, the following measures seem natural. For any $f \in Y^X$ we define the measure of uniform contractivity of f as

$$\mu_{uc}(f) := \inf\{\delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X: d \circ (f \times f) \leq e + \delta\},$$

and obviously then, for any $\mathcal{H} \subset Y^X$ we define the measure of uniform equicontractivity of \mathcal{H} as

$$\mu_{uec}(\mathcal{H}) := \inf\{\delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X \forall f \in \mathcal{H}: d \circ (f \times f) \leq e + \delta\}.$$

It is beyond the scope of this paper to show this, but there are several consistency results proving that these are indeed natural concepts in approach theory. In accordance with Proposition 2.3 it is easily seen that in the case of p -metric spaces $\mu_{uec}(\mathcal{H}) = \sup_{f \in \mathcal{H}} \mu_{uc}(f)$. The following example however shows that, in general, this is not the case, and moreover that the measure of uniform equicontractivity can attain all possible values, even for a set where all the individual functions are uniform contractions. Consider again $X := [0, a]^{\mathbb{N}}$ but now equipped with the product uniform gauge. Then all projections $\text{pr}_n : [0, a]^{\mathbb{N}} \rightarrow [0, a]$ are uniform contractions but it can be verified that with $\mathcal{H} := \{\text{pr}_n \mid n \in \mathbb{N}\}$ we have $\mu_{uec}(\mathcal{H}) = a$.

3. An Ascoli theorem

Let (X, \mathcal{G}_X) and (Y, \mathcal{G}_Y) be uniform gauge spaces, let Σ be a cover of X and let $\mathcal{H} \subset Y^X$ be an arbitrary collection of maps. Further let \mathcal{H} be equipped with the uniform gauge structure induced by $\langle \Sigma, \mathcal{G}_Y \rangle$. For any $x \in X$ the “point- x -evaluation map” is denoted as follows:

$$\text{ev}_x : \mathcal{H} \rightarrow Y: f \mapsto f(x).$$

Further, if $A \subset X$ then we put $\mathcal{H}|_A := \{f|_A \mid f \in \mathcal{H}\}$. Ascoli’s theorem basically describes (pre)compact subsets of function spaces, but a few peripheral results are also well known and used in this context, see e.g [2]. A first one states that: “if \mathcal{H} is precompact then so, for any $x \in X$, is $\text{ev}_x(\mathcal{H})$ ”. This is a consequence of the following general result.

Proposition 3.1. *The following inequality holds:*

$$\sup_{x \in X} \mu_{pc}(\text{ev}_x(\mathcal{H})) \leq \mu_{pc}(\mathcal{H}).$$

Proof. For any $x \in X$ choose $A \in \Sigma$ such that $x \in A$. Then it follows that $d \circ (f \times f) \leq D_{d,A}$. Hence, for any $x \in X$, the map $\text{ev}_x: \mathcal{H} \rightarrow Y$ is a uniform contraction. Hence, the result is an immediate consequence of [6] where it was shown that, under uniform contractions, the measure of precompactness decreases. \square

Another result in this context states that: “for any $A \in \Sigma$, if all the functions in \mathcal{H} have uniformly continuous restrictions to A and if \mathcal{H} itself is precompact, then \mathcal{H} is uniformly equicontinuous”. Again, this is a consequence of the following general result.

Proposition 3.2. For any $A \in \Sigma$ the following inequality holds:

$$\mu_{uc}(\mathcal{H}|_A) \leq 2\mu_{pc}(\mathcal{H}) + \sup_{f \in \mathcal{H}} \mu_{uc}(f|_A).$$

Proof. We suppose that all values on the right-hand side are finite, otherwise there is nothing to prove, and we choose α such that

$$\mu_{pc}(\mathcal{H}) = \sup_{d \in \mathcal{G}_Y, A \in \Sigma} \inf_{\mathcal{K} \in 2^{(\mathcal{H})}} \sup_{f \in \mathcal{H}} \inf_{g \in \mathcal{K}} \sup_{a \in A} d(f(a), g(a)) < \alpha. \quad (1)$$

Next we also choose β such that for all $h \in \mathcal{H}$

$$\mu_{uc}(h|_A) = \inf \{ \delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X: d \circ (h|_A \times h|_A) \leq e + \delta \} < \beta. \quad (2)$$

Now fix $d \in \mathcal{G}_Y$ and $A \in \Sigma$. From (1) it follows that there exists a finite subset $\mathcal{K} \subset \mathcal{H}$ such that $\sup_{f \in \mathcal{H}} \inf_{g \in \mathcal{K}} \sup_{a \in A} d(f(a), g(a)) < \alpha$. From (2), for any $g \in \mathcal{K}$ there exists $e_g \in \mathcal{G}_X$ such that $d \circ (g|_A \times g|_A) \leq e_g + \beta$. Put $e := \sup_{g \in \mathcal{K}} e_g \in \mathcal{G}_X$. Then, if $f \in \mathcal{H}$ we can find $g \in \mathcal{K}$ such that for all $a \in A$, $d(f(a), g(a)) < \alpha$. Hence it follows that for all $x, y \in A$:

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), g(x)) + d(g(x), g(y)) + d(g(y), f(y)) \\ &\leq \alpha + e_g(x, y) + \beta + \alpha \leq e(x, y) + (2\alpha + \beta). \quad \square \end{aligned}$$

The classical theorem of Ascoli, [2] states that: “if each set in Σ is precompact, if for each set $A \in \Sigma$ the collection $\mathcal{H}|_A$ is uniformly equicontinuous and if for each $x \in X$, $\text{ev}_x(\mathcal{H})$ is precompact, then \mathcal{H} is precompact”. This, finally, is a consequence of the following Ascoli theorem, which, analogously to the foregoing two propositions, again has no conditions, since they are “encapsulated in the inequality”.

Theorem 3.3. The following inequality holds:

$$\frac{1}{2}\mu_{pc}(\mathcal{H}) \leq \sup_{x \in X} \mu_{pc}(\text{ev}_x(\mathcal{H})) + \sup_{A \in \Sigma} \mu_{pc}(A) + \sup_{A \in \Sigma} \mu_{uc}(\mathcal{H}|_A).$$

Proof. Again, we suppose that all values on the right-hand side are finite, and we let α and β be such that, for all $x \in X$ and $A \in \Sigma$:

$$\mu_{pc}(\text{ev}_x(\mathcal{H})) = \sup_{d \in \mathcal{G}_Y} \sup_{\mathcal{F} \in 2^{(\mathcal{H})}} \inf_{f \in \mathcal{H}} \sup_{g \in \mathcal{F}} d(f(x), g(x)) < \alpha, \quad (3)$$

$$\mu_{pc}(A) = \sup_{e \in \mathcal{G}_X} \sup_{B \in 2^{(A)}} \inf_{a \in A} \sup_{b \in B} e(a, b) < \beta. \quad (4)$$

Next we also choose γ such that for all $A \in \Sigma$, $\mu_{uec}(\mathcal{H}|_A) < \gamma$, which implies that

$$\forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X \forall k \in \mathcal{H}|_A: d \circ (k \times k) \leq e + \gamma. \tag{5}$$

Now fix $d \in \mathcal{G}_Y$ and $A \in \Sigma$. From (5) it follows that there exists $e \in \mathcal{G}_X$ such that for all $f \in \mathcal{H}$:

$$d \circ (f|_A \times f|_A) \leq e + \gamma.$$

For this e , from (4) it then follows that there exists a finite subset $B \subset A$ and a function $A \rightarrow B: a \mapsto b_a$ such that $e(a, b_a) < \beta$.

Let $Z := \bigcup_{b \in B} \text{ev}_b(\mathcal{H}) \subset Y$. Then it follows from (3) and Proposition 2.2 that

$$\mu_{pc}(Z) = \mu_{pc}\left(\bigcup_{b \in B} \text{ev}_b(\mathcal{H})\right) \leq \sup_{b \in B} \mu_{pc}(\text{ev}_b(\mathcal{H})) < \alpha.$$

Hence, there exists a finite subset $C \subset Z$ and a function $Z \rightarrow C: z \mapsto c_z$ such that $d(z, c_z) < \alpha$. For any $h \in C^B$ let $\mathcal{B}(h) := \{f \in \mathcal{H} \mid \forall b \in B: d(f(b), h(b)) < \alpha\}$.

Now, fix $f \in \mathcal{H}$, and consider the function $h_f : B \rightarrow C: b \mapsto c_{f(b)}$. It then follows that $f \in \mathcal{B}(h_f)$. Let $\mathcal{K} := \{h \in C^B \mid \mathcal{B}(h) \neq \emptyset\}$. Then the foregoing shows that the collection $\{\mathcal{B}(h) \mid h \in \mathcal{K}\}$ is a finite cover of \mathcal{H} . Now for each $h \in \mathcal{K}$ we choose an arbitrary function $g_h \in \mathcal{B}(h)$ and we let $\mathcal{F} := \{g_h \mid h \in \mathcal{K}\}$. Then \mathcal{F} is a finite subset of \mathcal{H} and, by the foregoing, we obtain that for any $a \in A$

$$\begin{aligned} d(f(a), g_{h_f}(a)) &\leq d(f(a), f(b_a)) + d(f(b_a), h_f(b_a)) \\ &\quad + d(h_f(b_a), g_{h_f}(b_a)) + d(g_{h_f}(b_a), g_{h_f}(a)) \\ &= d(f(a), f(b_a)) + d(f(b_a), c_{f(b_a)}) \\ &\quad + d(h_f(b_a), g_{h_f}(b_a)) + d(g_{h_f}(b_a), g_{h_f}(a)) \\ &\leq (e(a, b_a) + \gamma) + \alpha + \alpha + (e(a, b_a) + \gamma) \leq 2(\alpha + \beta + \gamma), \end{aligned}$$

which by the arbitrariness of respectively $a \in A$, $d \in \mathcal{G}_Y$ and $A \in \Sigma$ shows that $\mu_{pc}(\mathcal{H}|_A) \leq 2(\alpha + \beta + \gamma)$. \square

References

- [1] J. Adamek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, Wiley–Interscience, New York, 1990.
- [2] N. Bourbaki, Topologie Générale, Chapitre 10. Espaces fonctionnels, Hermann, Paris, 1961.
- [3] K. Kuratowski, Sur les espaces complets, Fund. Math. 15 (1930) 301–309.
- [4] R. Lowen, Kuratowski’s measure of non-compactness revisited, Quarterly J. Math. Oxford 39 (1988) 235–254.
- [5] R. Lowen, Approach Spaces. The Missing Link in the Topology-Uniformity-Metric Triad, Clarendon Press, Oxford, 1997.
- [6] R. Lowen, B. Windels, A-Unif: a common supercategory of UNIF and MET, Internat. J. Math. Math. Sci. 21 (1998) 1–18.