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A FAMILY OF COMPACT SEMITORIC SYSTEMS WITH TWO FOCUS-FOCUS SINGULARITIES

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Abstract. About 6 years ago, semitoric systems were classified by Pelayo & Vũ Ngọc by means of five invariants. Standard examples are the coupled spin oscillator on $\mathbb{S}^2 \times \mathbb{R}^2$ and coupled angular momenta on $\mathbb{S}^2 \times \mathbb{S}^2$, both having exactly one focus-focus singularity. But so far there were no *explicit* examples of systems with more than one focus-focus singularity which are semitoric in the sense of that classification. This paper introduces a 6-parameter family of integrable systems on $\mathbb{S}^2 \times \mathbb{S}^2$ and proves that, for certain ranges of the parameters, it is a compact semitoric system with precisely two focus-focus singularities. Since the *twisting index* (one of the semitoric invariants) is related to the relationship between different focus-focus points, this paper provides systems for the future study of the twisting index.

1. Introduction

An *integrable system* is a triple (M, ω, F) where (M, ω) is a $2n$-dimensional symplectic manifold and $F\colon M \to \mathbb{R}^n$ is a smooth function, known as the *momentum map*, whose components Poisson commute and are linearly independent almost everywhere. The points at which the linear independence fails are known as *singular points*. An integrable system is *toric* if M is compact and the Hamiltonian vector fields of the components all have periodic flow of the same period; in this case the image of the momentum map $F(M)$ is a convex n-dimensional polytope (a special case of the Atiyah-Guillemin-Sternberg Theorem [2, 12]) and additionally, by the work of Delzant [7], $F(M)$ completely determines the system (M, ω, F) up to equivariant symplectomorphism.

So-called semitoric integrable systems are a special class of integrable systems on 4-manifolds for which one of the two components of its momentum map has

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a Hamiltonian vector field with periodic flow. Specifically, a *semitoric integrable system* is an integrable system $(M, \omega, F = (J, H))$ such that J is proper with periodic flow and every singular point is nondegenerate with no hyperbolic blocks (see Section 2 for a discussion of types of singular points). Semitoric integrable systems can have singular points of focus-focus type, which do not occur in toric integrable systems, and are an example of almost toric manifolds which were introduced by Symington [32].

Semitoric integrable systems were studied and classified by Pelayo & Vũ Ngọc [25, 26]. The classification is in terms of five invariants: the number of focus-focus points (which is finite according to Vũ Ngọc [35]); an infinite family of polygons known as a semitoric polygon; a Taylor series in two variables for each focus-focus point; the height of the focus-focus value in the semitoric polygon; and the twisting index, which, roughly, is an integer for each pair of focus-focus points describing the 'twist' of the singular Lagrangian fibration between them. Semitoric systems are rigid enough to admit a classification, but flexible enough to appear more frequently in physical examples and to admit more interesting dynamics. The main reason semitoric systems exhibit more interesting behavior than toric systems is the presence of the focus-focus points and the monodromy that these singularities can produce in the integral affine structure of the momentum map image $F(M)$.

While the Pelayo-Vũ Ngọc classification predicts many systems and gives certain properties of those systems, one thing that has thus far been lacking are *explicit* examples of semitoric systems giving the symplectic manifold (M, ω) and the momentum map F. Le Floch & Pelayo [18] explicitly describe the coupled angular momenta system (originally described in [31], see Example 2.12) and details of the coupled spin oscillator (see Example 2.13) are spread over several papers. These systems each have exactly one focus-focus singularity. In the present work we describe semitoric systems on $M = \mathbb{S}^2 \times \mathbb{S}^2$ which have two focus-focus singular points, generalizing the system from [18]. More precisely, the main result of this paper is the following.

Theorem 1.1. *Let $M = \mathbb{S}^2 \times \mathbb{S}^2$ be equipped with the symplectic form $\omega = -(R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$ where $\omega_{\mathbb{S}^2}$ is the standard volume form on the sphere and $0 < R_1 < R_2$ are real numbers. For $\vec{R} := (R_1, R_2)$ and $\vec{t} := (t_1, t_2, t_3, t_4) \in \mathbb{R}^4$ define $J_{\vec{R}}, H_{\vec{t}} : M \to \mathbb{R}$ by*

$$(1) \qquad \begin{cases} J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2, \\ H_{\vec{t}}\,(x_1, y_1, z_1, x_2, y_2, z_2) := t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2 \end{cases}$$

where (x_i, y_i, z_i) are Cartesian coordinates on $\mathbb{S}^2 \subset \mathbb{R}^3$ for $i = 1, 2$. Then there exist choices of $t_1, t_2, t_3, t_4, R_1, R_2$ such that $(M, \omega, (J_{\vec{R}}, H_{\vec{t}}))$ is a semitoric system with exactly two focus-focus points.

Theorem 1.1 is restated in more detail in Section 3 as Theorem 3.1. The coupled angular momenta system with coupling parameter $t \in\,]0, 1[$ is the special case of Equation (1) with $t_1 = t$, $t_3 = t_4 = 1 - t$, and $t_2 = 0$. The coupled angular momenta system describes the rotation of two vectors (with magnitudes R_1 and R_2) about the z-axis and has as a second integral a linear combination of the z-component of the first vector and the inner produce of the two vectors, while the system in Equation (1) includes additionally the z-component of the second vector and also breaks the inner product into two components, namely the projection to the z-axis and the projection to the xy-plane.

The system in Equation (1) is studied from a different point of view in mathematical physics, where it is a special case of a generalized Gaudin model. We refer the interested reader to Petrera's PhD thesis [29] and the references therein for the development since Gaudin's original work [11].

Theorem 1.1 gives explicit global formulas (defined by the same expression on the entire manifold) for a family of examples of semitoric systems with more than one focus-focus point. This family should be useful for understanding semitoric systems at a concrete, computationally amenable, context. The twisting index invariant is related to the relationship between different focus-focus singular points, so having an example with multiple focus-focus points will help in understanding this invariant (though it does actually appear in a more subtle way for systems with only one focus-focus point).

Additionally, not only the system itself, but also the method by which we produce this system is of interest. We construct it as a linear combination of four different systems of toric type (semitoric systems with no focus-focus points) and in this way one can see how it deforms into each of these four systems (see Figure 1) which correspond to four elements of the associated semitoric polygon. Let N denote the north pole of $\mathbb{S}^2$ and S denote the south pole, so that (N,N), (N,S), (S,N), and (S,S) are the four possible products of poles in $\mathbb{S}^2\times\mathbb{S}^2$. The next theorem follows from Theorem 4.4 in Section 4, in which we take $R_1 = 1$ and $R_2 = 2$ for simplicity.

Theorem 1.2. *For $s_1, s_2 \in [0,1]$ let $(J_{\vec{R}}, H_{(s_1,s_2)})$ denote the system $(J_{\vec{R}}, H_{\vec{t}})$ where*

$$t_1 = (1-s_1)(1-s_2), \qquad t_2 = s_1 s_2, \qquad t_3 = s_1 + s_2 - 2s_1 s_2, \qquad t_4 = s_1 - s_2.$$

Then $(J_{(1,2)}, H_{(s_1,s_2)})$ has the following properties:

1) *it is an integrable system for all $(s_1,s_2)\in[0,1]^2$;*
2) *it is a semitoric system when $(s_1,s_2)\in[0,1]^2\setminus\gamma$ where $\gamma\subset[0,1]^2$ is the union of four smooth curves;*
3) *the points $(N,S),(S,N)\in\mathbb{S}^2\times\mathbb{S}^2$ transition between being elliptic-elliptic, focus-focus, and degenerate depending on the value of (s_1,s_2);*
4) *it is semitoric with exactly two focus-focus points for all (s_1,s_2) in an open neighborhood of $\left(\frac{1}{2},\frac{1}{2}\right)$;*
5) *it is semitoric with no focus-focus point if $(s_1,s_2)\in\{(0,0),(0,1),(1,0),(1,1)\}$.*

The set γ represents the moment at which singular points become degenerate while they change between focus-focus and elliptic-elliptic type. Proposition 2.8 states that if the type of a singular point changes from focus-focus to elliptic-elliptic by smoothly varying the integrals (on a fixed manifold) then it must become degenerate during the transition, in fact, it is undergoing a Hamiltonian-Hopf bifurcation, see Remark 2.9. The set γ is an intersection of zero sets of discriminants of certain polynomials, see Equation (23). The image of the momentum map for the system in Theorem 1.2 is plotted in Figure 1 for various choices of (s_1,s_2) and γ is plotted in Figure 2. The coupled angular momenta system from Le Floch & Pelayo [18] is exactly the one parameter family of systems obtained from the system in Theorem 1.2 by taking $s_2 = 0$, so the momentum map image of the coupled angular momentum system is the bottom row of images in Figure 1.

Recently there has been a lot of activity relating to semitoric integrable systems, which we review briefly now. There has been work regarding quantizations of

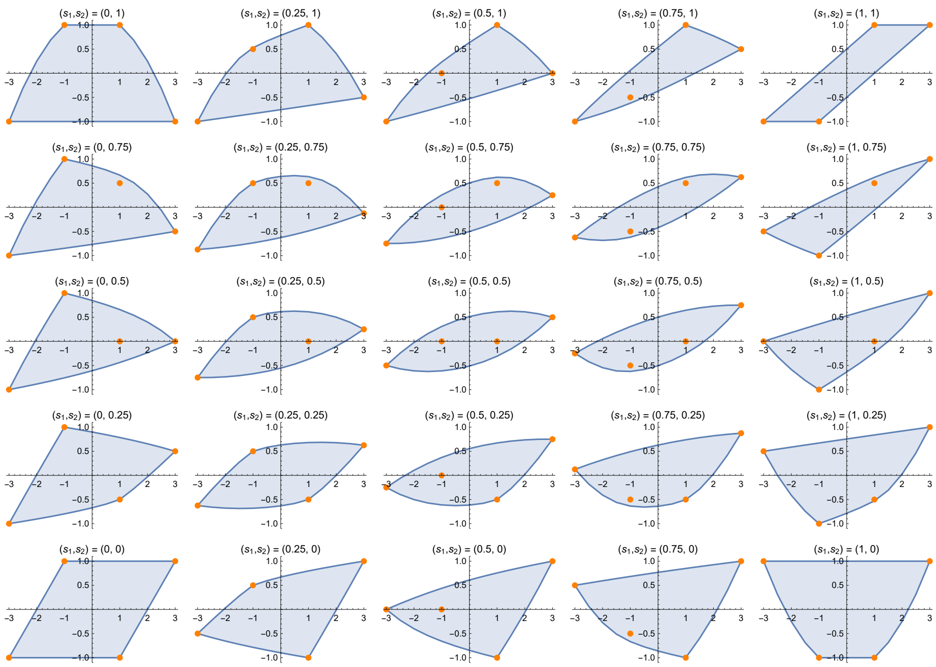

FIGURE 1. An image of the momentum map $(J_{(1,2)}, H_{(s_1,s_2)})$ with the rank 0 points marked for varying values of $s_1, s_2 \in [0,1]$. Notice that the coupled angular momenta system shown in Figure 3 is the bottom row of the system shown in this figure since the coupled angular momenta is the special case for which $s_2 = 0$.

semitoric integrable systems, specifically related to the problem of recovering the classical system from the quantum one (see for instance Le Floch & Pelayo & Vũ Ngọc [19]). Hohloch & Sabatini & Sepe [13] answer the question of how the classification of semitoric systems is linked to Karshon's classification [17] of Hamiltonian $\mathbb{S}^1$-spaces. The question of lifting a Hamiltonian $\mathbb{S}^1$-action to a semitoric system is an ongoing project by Hohloch & Sabatini & Sepe & Symington and has been the topic of several conference talks. There has been work to determine the convexity of the momentum map image with respect to its intrinsic integral affine structure by Ratiu & Wacheux & Zung [30]. Alonso & Dullin & Hohloch [1] are computing higher order terms of the Taylor series invariant of the focus-focus point in the coupled spin-oscillator (Example 2.13 of the present paper). Deformations of semitoric systems have been studied by endowing the moduli space with a topology, see Palmer [22]. Kane & Palmer & Pelayo [15, 16] used combinatorial methods to study blowups/downs and minimal models of semitoric systems. Generalizations of semitoric systems are achieved in Pelayo & Ratiu & Vũ Ngọc [24] and Hohloch & Sabatini & Sepe & Symington [14]. Additionally, work has begun to extend the theory of semitoric systems to higher dimensional manifolds in Wacheux [36]. Surgery techniques for semitoric systems are an ongoing project by Hohloch & Sabatini & Sepe & Symington. Presently, hyperbolic singularities are excluded from semitoric

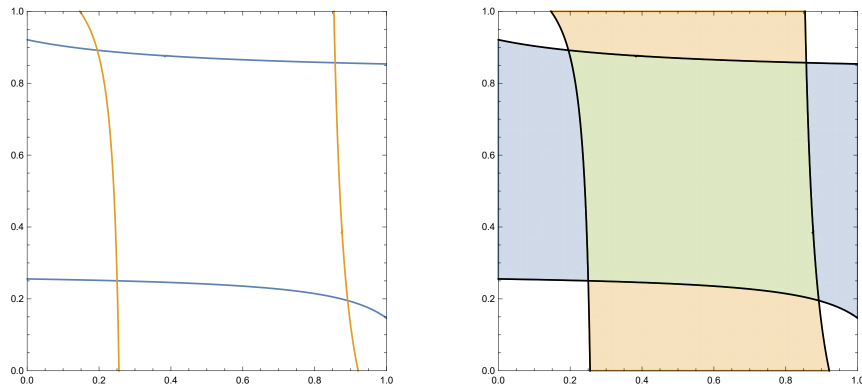

FIGURE 2. Left: a plot of the set γ, which is the union of $\gamma_{(S,N)}$ (blue) and $\gamma_{(N,S)}$ (orange), see Equation 23. Right: Values of (s_1, s_2) for which the system $(J_{(1,2)}, H_{(s_1,s_2)})$ has focus-focus values at: only the point (S,N) (blue), only the point (N,S) (orange), or at both points (green). The system is degenerate on the black curves. Compare with Figure 1.

integrable systems, but Dullin & Pelayo [8] have produced a smooth family of systems with transition from being semitoric to having a family of hyperbolic singular points. A reader new to integrable systems can consult the surveys Pelayo & Vũ Ngọc [27] and Pelayo [23], or the books Marsden & Ratiu [20] and Cushman & Bates [5].

Structure of the article: In Section 2 we review the required background, including integrable systems, singular points, and semitoric integrable systems. In Section 3 we introduce the new system and prove Theorem 1.1. In Section 4 we discuss the choice of parameters for which the system can be seen as a linear combination of four systems of toric type, and prove Theorem 1.2.

Figures: All figures and associated numerical computations in this article were made with the computer program *Mathematica*.

2. Fundamental definitions

In Section 2.1 we introduce standard notions related to integrable systems and non-degenerate points. A reader familiar with these topics can skip directly to Section 2.2.

2.1. Integrable systems and non-degenerate singular points.

2.1.1. *Integrable systems.* Given a symplectic manifold (M, ω) recall that associated to any function $f \in C^\infty(M)$ there is a vector field denoted by X^f, called the *Hamiltonian vector field associated to* f, and defined by

$$\omega(X^f, \cdot) = -df(\cdot).$$

Moreover, recall the Poisson bracket $\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by $\{f, g\} = \omega(X^f, X^g)$. An *integrable system* is a symplectic $2n$-manifold (M, ω) along with a collection of functions $f_1, \ldots, f_n$ which Poisson commute (i.e. $\{f_i, f_j\} = 0$

for all i, j) and for which the associated Hamiltonian vector fields $X^{f_1}, \dots, X^{f_n}$ are linearly independent almost everywhere. The function $F = (f_1, \dots, f_n)\colon M \to \mathbb{R}^n$ is known as the *momentum map* of this system.

In this article, we will focus on the case $n = 2$, so an integrable system will be a 4-dimensional symplectic manifold (M, ω) with a function $F\colon (M, \omega) \to \mathbb{R}^2$ whose components $F = (J, H)$ are such that $\{J, H\} = 0$ and $X^J(p)$ and $X^H(p)$ are linearly independent for almost all $p \in M$.

The points at which linear independence of the components of the momentum map fails are known as the *singular points* of the system and the *rank* of a singular point is the rank of the differential of the momentum map dF at that point. There is a natural notion of non-degeneracy for such singular points which we review now. Rank 0 singular points are known as *fixed points* since they are fixed under the flow of the Hamiltonian vector fields of the components of the momentum map; we will start with the classification of those.

2.1.2. *Rank 0 singular points, i.e., fixed points.* Let $p \in M$ be a fixed point and let $\mathcal{Q}(T_pM)$ denote the vector space of quadratic forms on T_pM. The symplectic form on M gives $\mathcal{Q}(T_pM)$ the structure of a Lie algebra which is isomorphic to $\mathfrak{sp}(4, \mathbb{R})$, see Bolsinov & Fomenko [3]. Recall that a Cartan subalgebra is a nilpotent and self-normalizing subalgebra.

Definition 2.1. A fixed point $p \in M$ is *non-degenerate* if the Hessians $d^2J(p)$ and $d^2H(p)$ span a Cartan subalgebra of the Lie algebra of quadratic forms on T_pM.

In practice, this condition can be checked by use of the following lemma.

Lemma 2.2 (Bolsinov & Fomenko [3])**.** *Let $p \in M$ be a fixed point. Denote by ω_p the matrix of the symplectic form with respect to a basis of T_pM and let d^2J and d^2H denote the matrices of the Hessians of J and H with respect to the same basis. Then p is non-degenerate if and only if d^2J and d^2H are linearly independent and there exists a linear combination of $\omega_p^{-1}d^2J$ and $\omega_p^{-1}d^2H$ which has four distinct eigenvalues.*

Sketch of proof. The result follows from the fact that an abelian subalgebra of $\mathfrak{sp}(4, \mathbb{R})$ is a Cartan subalgebra if and only if it is two dimensional and contains a regular element, in which case it is the centralizer for this regular element. The span of $\omega_p^{-1}d^2J$ and $\omega_p^{-1}d^2H$ is an abelian subspace of $\mathfrak{sp}(T_pM) \cong \mathfrak{sp}(4, \mathbb{R}) \cong \mathcal{Q}(T_pM)$ because J and H Poisson commute (since they form an integrable system) and a regular element is any matrix with four distinct eigenvalues. We conclude that if $\omega_p^{-1}d^2J$ and $\omega_p^{-1}d^2H$ are linearly independent and their span includes an element with four eigenvalues then the span is a two-dimensional abelian subalgebra which contains a regular element, and is thus Cartan. $\square$

2.1.3. *Rank 1 singular points.* To define rank 1 non-degenerate singular points we will again follow Bolsinov & Fomenko [3, Section 1.8.2]. Suppose that p is a singular point of rank 1 in a 4-dimensional integrable system $(M, \omega, F = (J, H))$. Then there exists some $\mu, \lambda \in \mathbb{R}$ such that $\mu dH + \lambda dJ = 0$ at p and the $\mathbb{R}^2$-action defined by flowing along the vector fields of J and H has a one-dimensional orbit through p. Let $L \subset T_pM$ be the tangent line of this orbit at p and let L' be the symplectic orthogonal complement to L. Notice that $L \subset L'$ and since J and H Poisson commute they are invariant under the $\mathbb{R}^2$-action and thus the operator $\mu d^2H + \lambda d^2J$ descends to the quotient L'/L.

Definition 2.3 (Bolsinov & Fomenko [3])**.** The rank 1 critical point p is *non-degenerate* if $\mu d^2H + \lambda d^2J$ is invertible on L'/L.

Now suppose that the flow of X^J is periodic. Recall that the symplectic quotient of M by J at the level c, which we denote $M /\!\!/ \mathbb{S}^1$, is the symplectic manifold $J^{-1}(c)/\mathbb{S}^1$ where the $\mathbb{S}^1$-action on $J^{-1}(c)$ is the one which comes from the flow of the Hamiltonian vector field of J.

Lemma 2.4. *If $p \in M$ is a rank 1 singular point such that $dJ \neq 0$ then p is non-degenerate if and only if d^2H is invertible at the image of p in the symplectic quotient of M by J at the level $J(p)$.*

Proof. Let L and L' be as above and let $c = J(p)$. Notice that $dJ \neq 0$ and $\dim(L) = 1$ implies that L is spanned by X^J. Thus $v \in L'$ if and only if $\omega_p(v, X^J) = 0$. By the definition of the Hamiltonian vector field this is equivalent to $v(J) = 0$, so $v \in T_p(J^{-1}(c))$. Thus $L' = T_p(J^{-1}(c))$. Furthermore, L is the tangent space to the orbit of the $\mathbb{S}^1$-action through p so $L'/L = T_p(J^{-1}(c)/\mathbb{S}^1)$ and the result follows. □

Lemma 2.4 implies the following.

Corollary 2.5. *If $dJ \neq 0$ at all points of nonzero rank then all rank 1 points of $(M, \omega, F = (J, H))$ are non-degenerate if and only if H descends to a Morse function on all possible symplectic quotients by J.*

See Bolsinov & Fomenko [3] for a description of non-degenerate points for dimensions greater than four and a description of rank 1 non-degenerate points in terms of Cartan subalgebras.

2.1.4. *Classification of non-degenerate points.* Williamson [37] classified Cartan subalgebras of $\mathfrak{sp}(n, \mathbb{R})$, which in turn implies a classification of the possible subalgebras $\mathfrak{c}$ generated by the Hessians in $T_pM \cong \mathfrak{sp}(n, \mathbb{R})$ at a non-degenerate singular points. Eliasson [10] and Miranda & Zung [21] extended Williamson's pointwise classification to a local classification, which classifies the possible forms of the momentum map in local symplectic coordinates around a fixed point p, often known as the *Eliasson-Miranda-Zung normal form.*

Theorem 2.6 (Eliasson [10], Miranda & Zung [21])**.** *If $p \in M$ is a non-degenerate singular point of an n-dimensional integrable system (M, ω, F) then there exist local symplectic coordinates $(x, y) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p such that there exist $q_1, \ldots, q_n \colon M \to \mathbb{R}$ where each q_i is given by one of:*

1) *elliptic:* $q_i(x, y) = \frac{1}{2}(x_i^2 + y_i^2)$,
2) *hyperbolic:* $q_i(x, y) = x_i y_i$,
3) *focus-focus:* $\begin{cases} q_i(x, y) = x_i y_{i+1} - x_{i+1} y_i, \\ q_{i+1}(x, y) = x_i y_i + x_{i+1} y_{i+1}, \end{cases}$
4) *non-singular:* $q_i(x, y) = y_i$,

such that $\{f_i, q_j\} = 0$ for all i, j.

The classification of a non-degenerate singular point can be detected by computing the eigenvalues of any associated regular element.

Proposition 2.7 (Vũ Ngọc [34, Chapter 3])**.** *If A is a regular element in the Cartan subalgebra generated by the Hessians of the components of the momentum map (i.e., A has $2n$ distinct eigenvalues) at a fixed point then the eigenvalues of A come in three distinct types of groups:*

1) a pair of imaginary roots $\pm \mathrm{i}\beta$, called an elliptic block,
2) a pair of real roots $\pm\alpha$, called a hyperbolic block,
3) a quadruple of complex roots $\pm\alpha \pm \mathrm{i}\beta$, called a focus-focus block,
where $\alpha, \beta \in \mathbb{R}$.

The types of the groups of eigenvalues of A agree with the classification of the Cartan subalgebra in Theorem 2.6. Thus they do not depend on the choice of the regular element A, they only depend on the Cartan subalgebra.

2.1.5. *Degenerate points.* Changing the integrable system on a fixed symplectic manifold cannot cause a rank 0 point to transition from being focus-focus type to being elliptic-elliptic type without passing through a degeneracy.

Proposition 2.8. *Fix a 4-dimensional symplectic manifold (M, ω). Let $t_0 \in \mathbb{R}$ and let $J_t, H_t \colon M \to \mathbb{R}$ be smooth functions which depend smoothly on $t \in \mathbb{R}$. Suppose (J_t, H_t) is an integrable system for all $t \in \mathbb{R}$ in an open interval around t_0 and $p \in M$ is a rank 0 fixed point of (J_t, H_t) for all $t \in \mathbb{R}$, which is of type elliptic-elliptic for $t > t_0$ and type focus-focus for $t < t_0$. Then (J_{t_0}, H_{t_0}) has a degenerate fixed point at p.*

Proof. Suppose that p is a non-degenerate fixed point of (J_{t_0}, H_{t_0}). Then there exists some $\gamma, \delta \in \mathbb{R}$ such that $\omega^{-1}(\gamma d^2 H_{t_0} + \delta d^2 J_{t_0})$ has four distinct eigenvalues at p. Fix such γ and δ. Since $\gamma d^2 H_t + \delta d^2 J_t$ is symmetric we see that the characteristic polynomial of $\omega^{-1}(\gamma d^2 H_{t_0} + \delta d^2 J)$ is a constant multiple of a polynomial of the form

$$g_t(X) = X^4 + b_t X^2 + c_t$$

where $b_t, c_t \in \mathbb{R}$ depend continuously on t. The zeros of g_t are given by $\pm\sqrt{\kappa_\pm}$ where

$$\kappa_\pm = \frac{-b_t \pm \sqrt{b_t^2 - 4c_t}}{2}$$

and since there are four distinct eigenvalues when $t = t_0$ we see that $b_{t_0}^2 - 4c_{t_0} \neq 0$. Thus we see that g_t has four distinct eigenvalues for all t in a neighborhood of t_0. Since the Williamson type of a fixed point does not depend on the choice of linear combination as long as one with four distinct eigenvalues is chosen we see that g_t has zeros of the form $\pm\mathrm{i}\alpha$, $\pm\mathrm{i}\beta$ for $t > t_0$ which means that $b_t^2 - 4c_t > 0$. Similarly, we see that g_t has zeros of the form $\alpha \pm \mathrm{i}\beta$ for $t < t_0$ which means that $b_t^2 - 4c_t < 0$. Thus, since $b_t - 4c_t$ varies continuously with t, we see that $b_{t_0} - 4c_{t_0} = 0$ contradicting our original claim. $\square$

Similar arguments to the one in the proof of Proposition 2.8 are used in Dullin-Pelayo [8] and in particular in Figure 4 in that paper.

Remark 2.9. When a point changes between being of elliptic-elliptic and focus-focus type it is undergoing what is known as the *Hamiltonian-Hopf bifurcation*, see [4].

We are grateful to Heinz Hanßmann and James Montaldi for bringing to our attention the Hamiltonian-Hopf bifurcation and informing us that our system is undergoing this transformation.

2.2. **Semitoric systems.** Pelayo & Vũ Ngọc [25, 26] extend the Delzant classification of toric integrable systems by introducing and classifying what are now known as semitoric systems.

Definition 2.10. A *semitoric system* is a integrable system of dimension four $(M, \omega, (J, H))$ such that:

1) J is proper,
2) the Hamiltonian flow of J (i.e. the flow of X^J) is periodic,
3) all singular points of (J, H) are non-degenerate and have no hyperbolic blocks.

A semitoric system is *simple* if there is at most one critical point in $J^{-1}(x)$ for all $x \in \mathbb{R}$.

Every semitoric system we consider in this article is a simple semitoric system.

Note that J is automatically proper in the case that M is compact. Concerning item (2)), we may assume that 2π is the minimal period. Note that this means the flow of X^J generates a faithful action of $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$.

If $(M, \omega, (J, H))$ is a semitoric integrable system and $p \in M$ is a rank zero singular point then there are exactly two possibilities for p: either p is elliptic--elliptic or focus-focus. Thus, if A is a regular element in the associated Cartan subalgebra then the eigenvalues of A must either come in two pairs $\pm \mathrm{i}\alpha$, $\pm \mathrm{i}\beta$ in which case p is elliptic-elliptic or come in one quadruple $\pm\alpha \pm \mathrm{i}\beta$ in which case p is focus-focus, where $\alpha, \beta \in \mathbb{R}$ in each case. If p is non-degenerate of rank 1 then it must be of elliptic type.

The Pelayo-Vũ Ngọc classification of simple semitoric integrable systems is in terms of five invariants, which we briefly describe now:

1) the *number of focus-focus points invariant*: $m_f \in \mathbb{Z}_{\geq 0}$ denotes the number of focus-focus singular points (which is finite by Vũ Ngọc [35]),
2) the *semitoric polygon*: a family of polygons (analogous to the Delzant polygon of a toric integrable system) which encode information about the integral-affine structure of the system. Each element is the image of a toric momentum map defined on all of M except certain subsets (which are the union of submanifolds of dimension at most three) related to the focus-focus points,
3) the *Taylor series invariant*: a Taylor series in two variables for each focus-focus point, which encodes the dynamics of the flow of the Hamiltonian vector fields as they approach the focus-focus fiber (originally introduced and described in Vũ Ngọc [33]),
4) the *volume* or *height invariant*: a real number for each focus-focus point which encodes the height of the focus-focus value in semitoric polygon,
5) the *twisting index*: an integer assigned to each focus-focus point for each element of the semitoric polygon, which encodes the relationship between the toric momentum map used to produce the element of the semitoric polygon and a preferred local momentum map around the focus-focus point.

An abstract list of such datas is known as a *list of semitoric ingredients*. Given semitoric systems $(M_i, \omega_i, (J_i, H_i))$ for $i = 1, 2$ an *isomorphism of semitoric systems* is a symplectomorphism $\Phi\colon M_1 \to M_2$ such that $\Phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and $\partial_y f > 0$ everywhere.

Theorem 2.11 (Classification by Pelayo & Vũ Ngọc [25, 26])**.** *The following hold:*

1) *Two simple semitoric systems are isomorphic if and only if they have the same five semitoric invariants,*
2) *Given a list of semitoric ingredients there exists a simple semitoric system which has those as its five invariants.*

For standard examples of semitoric systems, we refer to Section 2.4.

2.3. **The symplectic structure on $\mathbb{S}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.** In order to avoid, on the one hand, confusion concerning the various conventions in the literature and, on the other hand, to provide a precise and complete reference, the following calculations are provided in full.

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $(x_1, y_1, z_1, x_2, y_2, z_2)$ be Cartesian coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$. We consider the 4-dimensional manifold $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with symplectic form

$$\omega := \omega^{R_1 R_2} := -(R_1 \omega_{\mathbb{S}^2} \oplus R_2 \omega_{\mathbb{S}^2})$$

where $R_1, R_2 \in \mathbb{R}^{>0}$ and $\omega_{\mathbb{S}^2}$ is the standard symplectic form on $\mathbb{S}^2$. Geometrically, the symplectic form $\omega_{\mathbb{S}^2}$ on $\mathbb{S}^2$ is given in $p \in \mathbb{S}^2$ by

$$(\omega_{\mathbb{S}^2})_p(u, v) = \langle p, u \times v \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$, $p = (p_1, p_2, p_3) \in \mathbb{S}^2$ the basepoint and $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in T_p\mathbb{S}^2$ tangent vectors, i.e., $\langle p, u \rangle = 0 = \langle p, v \rangle$. To express $\omega_{\mathbb{S}^2}$ in Cartesian coordinates, we calculate

$$\begin{aligned}\langle p, u \times v \rangle &= p_1 \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} + p_2 \det \begin{pmatrix} u_3 & v_3 \\ u_1 & v_1 \end{pmatrix} + p_3 \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \\ &= p_1 (dy \wedge dz)(u, v) + p_2 (dz \wedge dx)(u, v) + p_3 (dx \wedge dy)(u, v)\end{aligned}$$

and thus

$$\omega_{\mathbb{S}^2} = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

This implies

$$\omega = -\sum_{i=1}^{2} R_i x_i dy_i \wedge dz_i + R_i y_i dz_i \wedge dx_i + R_i z_i dx_i \wedge dy_i$$

in Cartesian coordinates on $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$. We want to use charts on $\mathbb{S}^2$ that parametrise the upper and lower hemisphere as graphs over the 2-dimensional unit disk $\mathbb{D}^2$. To keep track of signs, we use $e \in \{+1, -1\}$ in the charts and have $\varphi_e : \mathbb{D}^2 \to \mathbb{S}^2$ with

$$\varphi_e(x, y) := (x, y, z_e(x, y)) := \left(x, y, e\sqrt{1 - x^2 - y^2}\right)$$

such that φ_{+1} covers the northern hemisphere and φ_{-1} the southern one. Denoting the north and south pole of $\mathbb{S}^2$ by N and S, we get charts for the 'double hemispheres' around $(N, N), (N, S), (S, N), (S, S) \in \mathbb{S}^2 \times \mathbb{S}^2$ via choosing $e_1, e_2 \in \{+1, -1\}$ accordingly and setting

$$\varphi_{e_1, e_2} : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{S}^2 \times \mathbb{S}^2, \tag{2}$$

$$\varphi_{e_1 e_2}(x_1, y_1, x_2, y_2) := \left(x_1, y_1, e_1\sqrt{1 - x_1^2 - y_1^2}, x_2, y_2, e_2\sqrt{1 - x_2^2 - y_2^2}\ \right).$$

For better readability, let us drop the subscripts e, e_1, and e_2 whenever the context allows, and introduce a function $z_i(x_i, y_i)$, i.e., we write

$$\varphi = \varphi_e, \quad z(x,y) = e\sqrt{1-x^2-y^2}, \quad \varphi = \varphi_{e_1e_2}, \quad z_i(x_i,y_i) := e_i\sqrt{1-x_i^2-y_i^2}$$

whenever possible. Now we express $\omega_{\mathbb{S}^2}$ in the new coordinates φ_e. We compute

$$\partial_x z(x,y) = \frac{-ex}{\sqrt{1-x^2-y^2}} = \frac{-x}{z(x,y)} \tag{3}$$

$$\partial_y z(x,y) = \frac{-ey}{\sqrt{1-x^2-y^2}} = \frac{-y}{z(x,y)} \tag{4}$$

yielding

$$d(z(x,y)) = \frac{-x}{z(x,y)}dx + \frac{-y}{z(x,y)}dy \tag{5}$$

leading to

$$\varphi^*\omega_{\mathbb{S}^2} = \left(\frac{x^2}{z(x,y)} + \frac{y^2}{z(x,y)} + z(x,y)\right) dx \wedge dy = \frac{1}{z(x,y)} dx \wedge dy.$$

Subsequently we get for ω in coordinates $\varphi = \varphi_{e_1e_2}$ the expression

$$\begin{aligned}\varphi^*\omega &= -\ \varphi^*(R_1\omega_{\mathbb{S}^2} \oplus R_2\omega_{\mathbb{S}^2}) \\ &= -\left(\frac{R_1}{z_1(x_1,y_1)}\ dx_1 \wedge dy_1 + \frac{R_2}{z_2(x_2,y_2)}\ dx_2 \wedge dy_2\right)\end{aligned}$$

and thus in matrix form we have

$$\omega = \begin{pmatrix} 0 & \frac{-R_1}{z_1} & 0 & 0 \\ \frac{R_1}{z_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-R_2}{z_2} \\ 0 & 0 & \frac{R_2}{z_2} & 0 \end{pmatrix} \quad \text{and} \quad \omega^{-1} = \begin{pmatrix} 0 & \frac{z_1}{R_1} & 0 & 0 \\ \frac{-z_1}{R_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{z_2}{R_2} \\ 0 & 0 & \frac{-z_2}{R_2} & 0 \end{pmatrix}. \tag{6}$$

Suppose $f : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$. Using the charts $\varphi_{e_1e_2}$, we compute for $h := f \circ \varphi_{e_1e_2} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ the differential

$$dh = \sum_{i=1}^{2} \partial_{x_i} h\ dx_i + \partial_{y_i} h\ dy_i$$

and can solve $\omega(X^h, \cdot) = -dh$ for X^h via

$$(X^h)^T = -\left(\partial_{x_1} h,\ \partial_{y_1} h, \partial_{x_2} h,\ \partial_{y_2} h\right)\ \omega^{-1}$$

so

$$X^h(x_1,y_1,x_2,y_2) = \begin{pmatrix} \frac{\partial_{y_1} h(x_1,y_1,x_2,y_2)\ z_1(x_1,x_2)}{R_1} \\ -\frac{\partial_{x_1} h(x_1,y_1,x_2,y_2)\ z_1(x_1,x_2)}{R_1} \\ \frac{\partial_{y_2} h(x_1,y_1,x_2,y_2)\ z_2(x_2,y_2)}{R_2} \\ -\frac{\partial_{x_2} h(x_1,y_1,x_2,y_2)\ z_2(x_2,y_2)}{R_2} \end{pmatrix}. \tag{7}$$

2.4. **Explicit examples of semitoric systems.** Consider the manifold $\mathbb{S}^2 \times \mathbb{S}^2$ with symplectic form $\omega := -(R_1\omega_{\mathbb{S}^2} \oplus R_2\omega_{\mathbb{S}^2})$ where $\omega_{\mathbb{S}^2}$ is the standard volume form on $\mathbb{S}^2$ and $0 < R_1 < R_2$ are real numbers.

Example 2.12 (**Coupled angular momenta).** The coupled angular momenta system is given by $J_{\vec{R}}, H_t \colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ with

$$\begin{cases} J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2, \\ H_t(x_1, y_1, z_1, x_2, y_2, z_2) := (1-t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2) \end{cases} \tag{8}$$

where (x_i, y_i, z_i) are Cartesian coordinates on $\mathbb{S}^2 \subset \mathbb{R}^3$ for $i = 1, 2$, $t \in [0,1]$ is the *coupling parameter*, and $\vec{R} = (R_1, R_2) \in \mathbb{R}^2$ with $0 < R_1 < R_2$.

This system was originally introduced in Sadovskií & Zĥilinskií [31] and studied in detail in Le Floch & Pelayo [18], where it is shown that there exist two fixed values $t_-, t_+ \in (0,1)$ with $t_- < t_+$ which depend on R_1, R_2 such that

1) if $t_- < t < t_+$ then $(J_{\vec{R}}, H_t)$ is a semitoric system with exactly one focus-focus point,
2) if $t > t_+$ or $t < t_-$ the $(J_{\vec{R}}, H_t)$ is a semitoric system with exactly zero focus-focus points (these are known as systems of *toric type*, see Section 2 of Vũ Ngọc [35]),
3) if $t = t_-$ or $t = t_+$ then $(J_{\vec{R}}, H_t)$ has a degenerate singular point, and thus is not a semitoric system.

The image of the momentum map for Example 2.12 with varying values of t is shown in Figure 3.

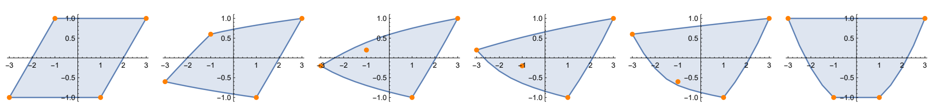

FIGURE 3. The momentum map image for the coupled angular momenta system with the rank zero points marked. As the coupling parameter t changes one of the rank zero points transitions from being elliptic-elliptic to being focus-focus and then back to elliptic-elliptic.

Another standard example of a semitoric system is

Example 2.13 (**Coupled spin oscillator).** The coupled spin oscillator system is given by $J, H : \mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{R}$ where

$$J(x,y,z,u,v) := \frac{1}{2}(u^2+v^2) + z \quad \text{and} \quad H(x,y,z,u,v) := \frac{1}{2}(ux+vy)$$

with Cartesian coordinates (x,y,z) on $\mathbb{S}^2$ and (u,v) on $\mathbb{R}^2$.

See Pelayo & Vũ Ngọc [28] for a detailed investigation of Example 2.13.

Remark 2.14. The spherical pendulum consists of $J, H : \mathrm{T}^*\mathbb{S}^2 \to \mathbb{R}$ with

$$J(q_1,q_2,q_3,p_1,p_2,p_3) := q_1p_2 - q_2p_1,$$
$$H(q_1,q_2,q_3,p_1,p_2,p_3) := \frac{1}{2}(p_1^2+p_2^2+p_3^2) + q_3$$

and satisfies nearly all of the requirements to be semitoric, but the J is not proper since the momentum map image contains unbounded vertical lines. However, the spherical pendulum is a so-called *generalized semitoric system*, as discussed in Pelayo & Ratiu & Vũ Ngọc [24]. For this same reason, the quadratic spherical pendulum (see for example Cushman & Vũ Ngọc [6] and Efstathiou & Martynchuk [9]) is not a semitoric integrable system.

3. A family of systems with two focus-focus points

In this section we introduce the system which is the subject of this paper and prove Theorem 1.1, our main result. This system is *minimal* in the sense of Kane & Palmer & Pelayo [16], i.e., it is not possible to perform a *blowdown of toric type* on the system (see Kane & Palmer & Pelayo [16, Section 4.1] for a description of this operation). Minimal semitoric integrable systems are classified in Kane & Palmer & Pelayo [16] and the system discussed in the present paper is minimal of type (2), using the terminology of that paper.

3.1. **The system.** Consider $R_1, R_2 \in \mathbb{R}^{>0}$ as scaling of radii with $R_1 < R_2$ and endow $\mathbb{S}^2 \times \mathbb{S}^2$ with the symplectic form $\omega = \omega^{R_1 R_2}$. Let $\vec{t} := (t_1, t_2, t_3, t_4) \in \mathbb{R}^4$ be parameters, let $\vec{R} = (R_1, R_2)$, and define $\Phi := (J_{\vec{R}}, H_{\vec{t}}) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^2$ in Cartesian coordinates by

$$\begin{cases} J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2, \\ H_{\vec{t}}(x_1, y_1, z_1, x_2, y_2, z_2) := t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2 \end{cases}$$

as in Equation (1).

Unless we explicitly need the parameters we often write $J := J_{\vec{R}}$ and $H := H_{\vec{t}}$ for brevity. The main result of this section is

Theorem 3.1. *The following hold:*

1) *The system (1) is a compact integrable system for all choices of parameters with $t_3 \neq 0$,*
2) *The system (1) is semitoric and has two focus-focus points for parameters in a neighborhood of*

$$R_1 = 1,\ R_2 = 2,\ t_1 = \frac{1}{4},\ t_2 = \frac{1}{4},\ t_3 = \frac{1}{2},\ t_4 = 0.$$

Theorem 3.1 is a combination of Propositions 3.9 and 3.13 and Corollary 3.15 which we prove in the remainder of this section.

Remark 3.2. At the parameters for which the system in Equation (1) is has two focus-focus points it enjoys a certain sense of uniqueness. As shown in [16, Theorem 2.5], up to scaling the lengths of the sides, there is only one semitoric polygon for which the corresponding system is compact with two focus-focus points such that J has isolated fixed points. Thus, this semitoric polygon is the one associated to the system in Equation (1). By evaluating J on the rank zero points (see Lemma 3.4) we can easily find the semitoric polygon for the system (1), as shown in Figure 4.

Remark 3.3. At first, we considered the system

$$J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2,$$
$$H_{(\ell_1,\ell_2)}(x_1, y_1, z_1, x_2, y_2, z_2) := (1 - \ell_1) z_1 + (1 - \ell_2) z_2 + \ell_1 \ell_2 (x_1 x_2 + y_1 y_2 + z_1 z_2)$$

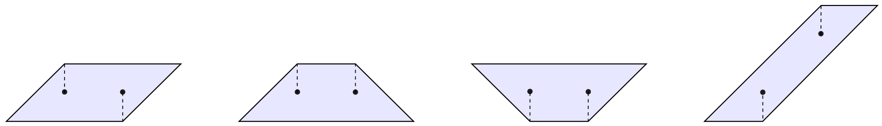

FIGURE 4. Four semitoric polygons associated to the system (1). The slanted edges all have slope ± 1. For each polygon the x-coordinates of the vertices, from left to right, are $-R_1 - R_2$, $R_1 - R_2$, $-R_1 + R_2$, and $R_1 + R_2$ (since we assume $R_1 < R_2$).

for $\ell_1, \ell_2 \in [0,1]$ hoping to generalize the construction of the coupled angular momentum, but numerical evidence strongly suggests that while there are two different points that become focus-focus for certain values there is no choice of ℓ_1 and ℓ_2 for which both points are focus-focus simultaneously.

3.2. **Rank 0 points and their nondegeneracy.** In the chart φ, the integrals J and H are given by

$$J(x_1, y_1, x_2, y_2) = R_1 z_1 + R_2 z_2, \tag{9}$$

$$H(x_1, y_1, x_2, y_2) = t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2. \tag{10}$$

where each $z_i = z_i(x_i, y_i)$ is a function of x_i and y_i for $i = 1, 2$. Using equations (3), (4), and (7), the Hamiltonian vector fields are given by

$$X^J(x_1, y_1, x_2, y_2) = \begin{pmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \end{pmatrix}, X^H(x_1, y_1, x_2, y_2) = \begin{pmatrix} \frac{-t_1 y_1 + t_3 y_2 z_1 - t_4 z_2 y_1}{R_1} \\ \frac{t_1 x_1 - t_3 x_2 z_1 + t_4 z_2 x_1}{R_1} \\ \frac{-t_2 y_2 + t_3 y_1 z_2 - t_4 z_1 y_2}{R_2} \\ \frac{t_2 x_2 - t_3 x_1 z_2 + t_4 z_1 x_2}{R_2} \end{pmatrix}. \tag{11}$$

Recall that N denotes the north pole of $\mathbb{S}^2$ and S the south pole.

Lemma 3.4. *The set of rank 0 points of (J, H), i.e., the set of fixed points, is given by $\{(N,N), (N,S), (S,N), (S,S)\}$.*

Proof. Geometrically, J is the sum of the height function on each factor of the product $\mathbb{S}^2 \times \mathbb{S}^2$ scaled by R_1 and R_2 respectively. Thus J gives rise to horizontal rotations on each of the two spheres and its Hamiltonian flow has fixed points exactly at $\{(N,N), (N,S), (S,N), (S,S)\}$. The function J reaches its global maximum, $R_1 + R_2$, at (N,N) and its global minimum, $-(R_1 + R_2)$, at (S,S). The corresponding fibers $J^{-1}(R_1 + R_2)$ and $J^{-1}(-(R_1 + R_2))$ consist exactly of the singletons $\{(N,N)\}$ and $\{(S,S)\}$.

Fixed points of $(J,H) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ require $\operatorname{rk} D(J,H) = 0$. Therefore they must have $DJ = 0$ which is equivalent to $X^J = 0$, i.e., when we look for fixed points of (J,H), the only candidates are the points (N,N), (N,S), (S,N), and (S,S) for which we have to check if additionally $DH = 0$ or equivalently $X^H = 0$ holds.

Since all possible fixed points lie in the range of the charts $\varphi_{e_1 e_2}$ we can check the values of X^H by using formula (3), (4), and (11). The corresponding point in the domain is in all cases $(x_1, y_1, x_2, y_2) = (0,0,0,0)$ and we compute that $X^H(0,0,0,0)$ vanishes and thus $\{(N,N), (N,S), (S,N), (S,S)\}$ is indeed the fixed point set of (J,H). $\square$

Keep in mind from the above proof that the rank 0 points correspond to the origin in the charts in (2).

Lemma 3.5. *At the origin $p = (0,0,0,0)$ in the charts in (2), we find*

$$\omega_p^{-1} d^2 J(p) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{12}$$

and

$$\omega_p^{-1} d^2 H(p) = \begin{pmatrix} 0 & -\frac{t_1+e_2t_4}{R_1} & 0 & \frac{e_1t_3}{R_1} \\ \frac{t_1+e_2t_4}{R_1} & 0 & -\frac{e_1t_3}{R_1} & 0 \\ 0 & \frac{e_2t_3}{R_2} & 0 & -\frac{t_2+e_1t_4}{R_2} \\ -\frac{e_2t_3}{R_2} & 0 & \frac{t_2+e_1t_4}{R_2} & 0 \end{pmatrix}. \tag{13}$$

Proof. We compute the Hessians of J and H using (9) and (10). Since derivatives are additive we can first calculate the Hessians of their components seperately. Recall from (3) and (4) that $\partial_{x_i} z_i = \frac{-x_i}{z_i}$ and $\partial_{y_i} z_i = \frac{-y_i}{z_i}$, yielding

$$\partial^2_{x_ix_i} z_i = \frac{-z_i + x_i\partial_{x_i} z_i}{z_i^2}, \quad \partial^2_{x_iy_i} z_i = \frac{x_i\partial_{y_i} z_i}{z_i^2}, \quad \text{and} \quad \partial^2_{y_iy_i} z_i = \frac{-z_i + y_i\partial_{y_i} z_i}{z_i^2}.$$

Since z_1 does not depend on x_2, y_2 and z_2 does not depend on x_1 and y_1 we obtain for the Hessian of z_i w.r.t. the variables x_1, y_1, x_2, y_2 in p

$$d^2 z_1(p) = \begin{pmatrix} -e_1 & 0 & 0 & 0 \\ 0 & -e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d^2 z_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -e_2 & 0 \\ 0 & 0 & 0 & -e_2 \end{pmatrix}.$$

Next we consider the term $x_1x_2 + y_1y_2$ and get

$$d^2(x_1x_2 + y_1y_2)(p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the term z_1z_2, we get

$$d^2(z_1z_2)(p) = \begin{pmatrix} -e_1e_2 & 0 & 0 & 0 \\ 0 & -e_1e_2 & 0 & 0 \\ 0 & 0 & -e_1e_2 & 0 \\ 0 & 0 & 0 & -e_1e_2 \end{pmatrix}.$$

The equations (9) and (10) together with the above calculations yield

$$d^2 J(p) = \begin{pmatrix} -e_1R_1 & 0 & 0 & 0 \\ 0 & -e_1R_1 & 0 & 0 \\ 0 & 0 & -e_2R_2 & 0 \\ 0 & 0 & 0 & -e_2R_2 \end{pmatrix}$$

and

$$d^2 H(p) = \begin{pmatrix} -t_1e_1 - t_4e_1e_2 & 0 & t_3 & 0 \\ 0 & -t_1e_1 - t_4e_1e_2 & 0 & t_3 \\ t_3 & 0 & -t_2e_2 - t_4e_1e_2 & 0 \\ 0 & t_3 & 0 & -t_2e_2 - t_4e_1e_2 \end{pmatrix}.$$

Evaluating ω^{-1} from (6) at p yields

$$\omega_p^{-1} = \begin{pmatrix} 0 & \frac{e_1}{R_1} & 0 & 0 \\ -\frac{e_1}{R_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e_2}{R_2} \\ 0 & 0 & -\frac{e_2}{R_2} & 0 \end{pmatrix}$$

and therefore, using $e_1^2 = 1 = e_2^2$, we get the desired results for $\omega_p^{-1}J(p)$ and $\omega_p^{-1}H(p)$. $\square$

Given a polynomial of the form $ay^2 + by + c$, the expression $b^2 - 4ac$ is called the *discriminant* of the polynomial. Thus, a straightforward calculation yields

Corollary 3.6. *Denote by I the 4×4 identity matrix. Then the characteristic polynomial of $\omega_p^{-1} d^2 H(p)$ is given by*

$$\begin{aligned} \chi(X) &:= \det\big(\omega_p^{-1} d^2 H(p) - XI\big) \\ &= X^4 + \frac{1}{R_1^2 R_2^2}\big(R_1^2(t_2 + e_1 t_4)^2 + 2e_1 e_2 R_1 R_2 t_3^2 + R_2^2(t_1 + e_2 t_4)^2\big) X^2 \\ &\quad + \frac{1}{R_1^2 R_2^2}\big((t_2 + e_1 t_4)^2 (t_1 + e_2 t_4)^2 - 2e_1 e_2 (t_2 + e_1 t_4)(t_1 + e_2 t_4) t_3^2 + t_3^4\big) \end{aligned}$$

which is a polynomial of second degree in $Y := X^2$ with discriminant

(14)

$$\begin{aligned} \triangle := &\triangle_{\vec{R},\vec{t},e_1,e_2} \\ = &\left(\frac{1}{R_1^2 R_2^2}\big(R_1^2(t_2 + e_1 t_4)^2 + 2e_1 e_2 R_1 R_2 t_3^2 + R_2^2(t_1 + e_2 t_4)^2\big)\right)^2 \\ &- \frac{4}{R_1^2 R_2^2}\big((t_2 + e_1 t_4)^2 (t_1 + e_2 t_4)^2 - 2e_1 e_2 (t_2 + e_1 t_4)(t_1 + e_2 t_4) t_3^2 + t_3^4\big). \end{aligned}$$

Now we want to determine the type of the rank 0 points located at (N, N), (N, S), (S, N), (S, S), i.e., if they are nondegenerate or not and, in case they are nondegenerate, if they are focus-focus or elliptic-elliptic or something else. We will see that the type of the rank 0 points highly depends on the choice of parameters $\vec{R}$ and $\vec{t}$.

Proposition 3.7 (**Rank 0 Criterion**)**.** *Suppose $p \in \mathbb{S}^2 \times \mathbb{S}^2$ has z-coordinates $(e_1, e_2) \in \{-1, 1\}^2$. Then p is a rank 0 singular point of $(J_{\vec{R}}, H_{\vec{t}})$. If $\triangle_{\vec{R},\vec{t},e_1,e_2} < 0$ then p is non-degenerate of focus-focus type, and if $\triangle_{\vec{R},\vec{t},e_1,e_2} > 0$ then p is non-degenerate and is of type elliptic-elliptic, elliptic-hyperbolic, or hyperbolic-hyperbolic.*

Proof. We already know that the set of rank 0 point are exactly those with z-coordinates ± 1 by Lemma 3.4. Note that the characteristic polynomial of $\omega_p^{-1} d^2 H(p)$ has zeros

$$X = \pm\sqrt{\frac{-1}{2R_1^2 R_2^2}\left(R_1^2(t_2 + e_1 t_4)^2 + 2e_1 e_2 R_1 R_2 t_3^2 + R_2^2(t_1 + e_2 t_4)^2\right) \pm \frac{\sqrt{\triangle}}{2}}$$

where $\triangle := \triangle_{\vec{R},\vec{t},e_1,e_2}$ is as in Equation (14).

If $\triangle < 0$ then there are four eigenvalues which take the form $\alpha \pm \mathrm{i}\beta$ for $\alpha, \beta \in \mathbb{R}$, and thus p is focus-focus by Proposition 2.7. If, $\triangle > 0$ then p is a non-degenerate fixed point which is not focus-focus, so it is either elliptic-elliptic, hyperbolic-hyperbolic, or hyperbolic-elliptic. ☐

Note that in the case $\triangle = 0$ the point can still be non-degenerate, but Proposition 3.7 does not give us any information in this case. The following statement implies that there exist parameter values for which the system has four nondegenerate rank 0 points, two of them elliptic-elliptic and two focus-focus, and is proved by plugging the values into the criterion in Proposition 3.7.

Corollary 3.8. *For the parameter values*

$$R_1 = 1, \quad R_2 = 2, \quad t_1 = \frac{1}{4}, \quad t_2 = \frac{1}{4}, \quad t_3 = \frac{1}{2}, \quad t_4 = 0, \tag{15}$$

the matrix $\omega_p^{-1} d^2 H(p)$ has four distinct eigenvalues at each of the points $p \in \{(N,N),(S,S),(N,S),(S,N)\}$ given by

$$Eig(N,N) = Eig(S,S) = \left\{ \pm \frac{\mathrm{i}}{8}\sqrt{\frac{21+3\sqrt{33}}{2}},\ \pm\frac{\mathrm{i}}{8}\sqrt{\frac{21-3\sqrt{33}}{2}} \right\},$$

$$Eig(N,S) = Eig(S,N) = \left\{ \left(\sqrt{\frac{5}{32}}\right)\left(\pm\cos\left(\frac{1}{2}\arctan\left(\frac{3\sqrt{31}}{11}\right)\right) \pm \mathrm{i}\sin\left(\frac{1}{2}\arctan\left(\frac{3\sqrt{31}}{11}\right)\right)\right)\right\},$$

and thus p is a nondegenerate fixed point according to Lemma 2.2. In particular, (N,N) and (S,S) are elliptic-elliptic and (N,S) and (S,N) are focus-focus.

Since nonvanishing and noncoinciding are open conditions, there exist in fact intervals around the parameters (15) where the systems continues to have two focus-focus and two elliptic-elliptic points.

Proposition 3.9. *There exists an open set $U \subset \mathbb{R}^6$ which contains the point $(1, 2, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ such that for all $(R_1, R_2, t_1, t_2, t_3, t_4) \in U$ the system given in Equation (1) has elliptic-elliptic points at (N,N) and (S,S) and focus-focus points at (N,S) and (S,N).*

3.3. **Rank 1 points.** We want to study rank 1 points by means of cylindrical coordinates. To avoid the problems with cylindrical coordinates near poles we state the following observation.

Lemma 3.10. *Let t_1, t_2, t_3, $t_4 \in \mathbb{R}$, with $t_3 \neq 0$ and let R_1, $R_2 \in \mathbb{R}^{>0}$. If $(x_1, y_1, z_1, x_1, y_2, z_2) \in \mathbb{S}^2 \times \mathbb{S}^2$ is a critical point of rank 1 of (1) then $z_1, z_2 \neq \{\pm 1\}$.*

Proof. Critical points of (J, H) from (1) are those $p \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $dH(p)$ and $dJ(p)$ are linearly dependent, which is equivalent to the existence of a nonzero $\lambda \in \mathbb{R}$ such that $d(H - \lambda J)(p) = 0$ since $dJ = 0$ only occurs at the rank 0 points. Defining $f_1, f_2 : \mathbb{R}^6 \to \mathbb{R}$ by $f_i(x_1, y_1, z_1, x_2, y_2, z_2) := x_i^2 + y_i^2 + z_i^2$ for $i = 1, 2$, this is equivalent to looking for critical points of $H - \lambda J : \mathbb{R}^6 \to \mathbb{R}$ on the set $f_1^{-1}(1) \cap f_2^{-1}(1)$, i.e., critical points can be computed by means of Lagrangian multipliers, i.e., a critical point $p := (x_1, y_1, z_1, x_1, y_2, z_2)$ satisfies the equations

$$\begin{cases} \nabla H(p) = \lambda \nabla J(p) + \mu_1 \nabla f_1(p) + \mu_2 \nabla f_2(p), \\ x_1^2 + y_1^2 + z_1^2 = 1, \\ x_2^2 + y_2^2 + z_2^2 = 1 \end{cases}$$

for some $\lambda, \mu_1, \mu_2 \in \mathbb{R}$. Using the gradient with respect to the Euclidean metric, we obtain

$$\begin{pmatrix} t_3 x_2 \\ t_3 y_2 \\ t_1 + t_4 z_2 \\ t_3 x_1 \\ t_3 y_1 \\ t_2 + t_4 z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda R_1 \\ 0 \\ 0 \\ \lambda R_2 \end{pmatrix} + \begin{pmatrix} 2\mu_1 x_1 \\ 2\mu_1 y_1 \\ 2\mu_1 z_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2\mu_2 x_2 \\ 2\mu_2 y_2 \\ 2\mu_2 z_2 \end{pmatrix}. \tag{16}$$

Recall that the rank 0 points are precisely those with $z_1 \in \{\pm 1\}$ and $z_2 \in \{\pm 1\}$ simultaneously. Suppose that $z_1 \in \{\pm 1\}$ which implies $x_1 = y_1 = 0$ since $(x_1, y_2, z_1) \in \mathbb{S}^2$. Then, recalling that $t_3 \neq 0$, we see that Equation (16) implies $x_2 = y_2 = 0$ which in turn implies $z_2 \in \{\pm 1\}$ so the only solution is in fact a rank 0 point. The same argument works if we assume $z_2 \in \{\pm 1\}$. $\square$

We now introduce cylindrical coordinates on $\mathbb{S}^2 \times \mathbb{S}^2$ via

$$(x_i, y_i, z_i) \mapsto \left(\sqrt{1 - z_i^2} \cos(\theta_i), \quad \sqrt{1 - z_i^2} \sin(\theta_i), \quad z_i \right)$$

where $i \in \{1, 2\}$ and θ_i is the counterclockwise angle between the x_i-axis and (x_i, y_i) in $\mathbb{R}^2$. In these coordinates, the system (1) becomes

$$J(\theta_1, z_1, \theta_2, z_2) = R_1 z_1 + R_2 z_2,$$
$$H(\theta_1, z_1, \theta_2, z_2) = t_1 z_1 + t_2 z_2 + t_3 \sqrt{(1 - z_1^2)(1 - z_2^2)} \cos(\theta_1 - \theta_2) + t_4 z_1 z_2$$

and the symplectic form is

$$\omega = R_1 dz_1 \wedge d\theta_1 + R_2 dz_2 \wedge d\theta_2. \tag{17}$$

According to Lemma 3.10 these coordinates are valid where rank 1 points may occur (if the rank 1 point occurs at the discontinuity of θ_i, then shift the domain of definition of θ_i). We compute the derivative

$$dJ(\theta_1, z_1, \theta_2, z_2) = \begin{pmatrix} 0, & R_1, & 0, & R_2 \end{pmatrix} \tag{18}$$

which never vanishes. Therefore we have Corollary 2.5 at our disposal.

Let us compute the symplectic quotient $(\mathbb{S}^2 \times \mathbb{S}^2) /\!\!/ \mathbb{S}^1$ where the $\mathbb{S}^1$-action is induced by J. Given $c \in \left] -(R_1 + R_2), (R_1 + R_2) \right[$, which is the set of regular values of J, we can solve for z_1 on the level set $J^{-1}(c)$ to find

$$z_1 = \frac{c - R_2 z_2}{R_1}.$$

By Equations (18) and (17) we see that $X^J = \partial_{z_1} + \partial_{z_2}$ so the flow of J rotates θ_1 and θ_2 by a common angle. Thus, the $\mathbb{S}^1$-action produced by the flow of X^J preserves the angle difference $\theta_1 - \theta_2$. Now consider the chart on the quotient $J^{-1}(c)/\mathbb{S}^1$ with coordinates (ζ, ϑ) given by

$$\zeta := z_1 \quad \text{and} \quad \vartheta := \theta_1 - \theta_2$$

where

$$-1 < \zeta < 1 \qquad \text{and} \qquad \frac{c-R_2}{R_1} < \zeta < \frac{c+R_2}{R_1}$$

since $-1 < z_1, z_2 < 1$. All rank 1 critical points occur in this chart since by Lemma 3.10 rank 1 points do not occur when $z_1 = \pm 1$ or $z_2 = \pm 1$. We now let H descend to the symplectic quotient $(\mathbb{S}^2 \times \mathbb{S}^2) /\!\!/ \mathbb{S}^1$ where it reads

$$\begin{aligned} &H(\zeta,\vartheta)\\ &= t_1\zeta + t_2\frac{c-R_1\zeta}{R_2} + t_3\cos(\vartheta)\sqrt{(1-\zeta^2)\left(1-\left(\frac{c-R_1\zeta}{R_2}\right)^2\right)} + t_4\frac{c-R_1\zeta}{R_2}\zeta\\ &= \frac{t_2 c}{R_2} + \frac{t_1R_2 - t_2R_1 + t_4 c}{R_2}\zeta - \frac{t_4R_1}{R_2}\zeta^2 + t_3\cos(\vartheta)\sqrt{(1-\zeta^2)\left(1-\left(\frac{c-R_1\zeta}{R_2}\right)^2\right)}. \end{aligned}$$

We abbreviate the term under the last root by

$$A(\zeta) := A(\zeta, c, R_1, R_2) := (1-\zeta^2)\left(1-\left(\frac{c-R_1\zeta}{R_2}\right)^2\right)$$

and its derivatives with respect to ζ by

$$A' := \partial_\zeta A \qquad \text{and} \qquad A'' := \partial^2_{\zeta\zeta} A.$$

We note $A(\zeta) \geq 0$ with $A(\zeta) = 0$ if and only if $\zeta = \pm 1$ or $\zeta = \frac{c\pm R_2}{R_1}$. Because of the bounds on ζ we always have $A(\zeta) > 0$. In order to find the critical points of H on the symplectic quotient we calculate the partial derivatives

$$\begin{aligned} \partial_\vartheta H(\zeta,\vartheta) &= -t_3\sin(\vartheta)\sqrt{A(\zeta)},\\ \partial_\zeta H(\zeta,\vartheta) &= \frac{t_1R_2 - t_2R_1 + t_4c}{R_2} - \frac{2t_4R_1}{R_2}\zeta + t_3\cos(\vartheta)\frac{A'(\zeta)}{2\sqrt{A(\zeta)}}. \end{aligned}$$

Lemma 3.11. *(ζ,ϑ) is a critical point of H on the symplectic quotient if and only if*

$$\vartheta \in \pi\mathbb{Z} \quad \textit{and} \quad 0 = t_1R_2 - t_2R_1 + t_4c - 2t_4R_1\zeta + t_3R_2\cos(\vartheta)\frac{A'(\zeta)}{2\sqrt{A(\zeta)}}.$$

Proof. The point (ζ,ϑ) is critical if and only if $\partial_\vartheta H(\zeta,\vartheta) = 0 = \partial_\zeta H(\zeta,\vartheta)$. Since $A(\zeta)$ and t_3 are nonzero $\partial_\zeta H(\zeta,\vartheta) = 0$ is equivalent to $\sin(\vartheta) = 0$ meaning $\vartheta \in \pi\mathbb{Z}$ and $\cos(\vartheta) = \pm 1$. Together with $\partial_\vartheta H(\zeta,\vartheta) = 0$ we get the desired result. □

3.4. **Integrability.** We consider the system (1) in the chart $\varphi = \varphi_{e_1e_2}$ defined in (2). By means of (7), we obtain as Hamiltonian vector fields in these coordinates

$$X^{x_i} = -\frac{z_i}{R_i}\partial_{y_i}, \quad X^{y_i} = \frac{z_i}{R_i}\partial_{x_i}, \quad X^{z_i} = -\frac{y_i}{R_i}\partial_{x_i} + \frac{x_i}{R_i}\partial_{y_i}$$

for $i \in \{1,2\}$. Moreover, we deduce from (5)

$$dz_i = \frac{-x_i}{z_i}dx_i + \frac{-y_i}{z_i}dy_i.$$

This yields

$$
\left\{
\begin{aligned}
&\{z_i, x_i\} = -dz_i(X^{x_i}) = -\frac{y_i}{R_i},\\
&\{z_i, y_i\} = -dz_i(X^{y_i}) = \frac{x_i}{R_i},\\
&\{z_i, x_j\} = \{z_i, y_j\} = 0 \quad \text{for } i \neq j,\\
&\{z_i, z_i\} = -dz_i(X^{z_i}) = -\frac{x_i y_i}{R_i z_i} + \frac{x_i y_i}{R_i z_i} = 0,\\
&\{z_1, z_2\} = -dz_1(X^{z_2}) = 0.
\end{aligned}
\right.
\tag{19}
$$

Now we are ready to show

Lemma 3.12. *$\{J, H\} = 0$ for all $R_1, R_2 \in \mathbb{R}^{>0}$ and all $t = (t_1, t_2, t_3, t_4) \in \mathbb{R}^4$.*

Proof. We recall that the Poisson bracket is linear and that we have the identities in (19). Then we compute in the coordinates given in (2)

$$
\begin{aligned}
\{J, H\} &= \{R_1 z_1 + R_2 z_2,\ t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2\}\\
&\overset{(19)}{=} R_1(t_3\{z_1, x_1 x_2\} + t_3\{z_1, y_1 y_2\} + t_4\{z_1, z_1 z_2\})\\
&\quad + R_2(t_3\{z_2, x_1 x_2\} + t_3\{z_2, y_1 y_2\} + t_4\{z_2, z_1 z_2\}).
\end{aligned}
$$

Since the Poisson bracket satisfies the product rule $\{a, bc\} = \{a, b\}c + \{a, c\}b$ it follows

$$
\overset{(19)}{=} R_1\left(t_3\left(\frac{-y_1}{R_1} x_2 + \frac{x_1}{R_1} y_2\right)\right) + R_2\left(t_3\left(\frac{-y_2}{R_2} x_1 + \frac{x_2}{R_2} y_1\right)\right) = 0.
$$

The charts in (2) are not defined for $z_i = 0$. To show $\{J, H\} = 0$ there, consider Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ and choose charts given by

$$
(x_1, z_1, x_2, z_2) \mapsto \left(x_1, e_1\sqrt{1 - x_1^2 - z_1^2}, z_1, x_2, e_2\sqrt{1 - x_2^2 - z_2^2}, z_2\ \right)
$$

etc. The calculations are completely analogous. □

Proposition 3.13. *The system $(\mathbb{S}^2 \times \mathbb{S}^2, \omega, (J_{\vec{R}}, H_{\vec{t}}))$ given in Equation (1) is integrable for all parameter values with $0 < R_1 < R_2$ and $t_1, t_2, t_3, t_4 \in \mathbb{R}$ with $t_3 \neq 0$.*

Proof. Fix parameter values and let $J = J_{(R_1,R_2)}$ and $H = H_{(t_1,t_2,t_3,t_4)}$ with $t_3 \neq 0$. In Lemma 3.12 we showed that $\{J, H\} = 0$. It follows from Lemma 3.11 and the fact that $t_3 \neq 0$ that the rank 1 critical points occupy a set of measure zero since there are only finitely many on each symplectic quotient. By Lemma 3.4 there are only finitely many rank 0 points and thus J and H are linearly independent almost everywhere. □

3.5. **Nondegeneracy of rank 1 points.** Now we want to study nondegeneracy of the rank 1 critical points. Therefore we have to compute the Hessian of H on the symplectic quotient. We get

$$\begin{aligned}
\partial^2_{\vartheta\vartheta} H(\zeta,\vartheta) &= -t_3\cos(\vartheta)\sqrt{A(\zeta)},\\
\partial^2_{\vartheta\zeta} H(\zeta,\vartheta) &= -t_3\sin(\vartheta)\frac{A'(\zeta)}{2\sqrt{A(\zeta)}},\\
\partial^2_{\zeta\zeta} H(\zeta,\vartheta) &= -\frac{2t_4R_1}{R_2} + \frac{t_3\cos(\vartheta)}{2}\,\frac{A''(\zeta)\sqrt{A(\zeta)} - A'(\zeta)\frac{A'(\zeta)}{2\sqrt{A(\zeta)}}}{A(\zeta)}\\
&= \frac{-2t_4R_1}{R_2} + t_3\cos(\vartheta)\,\frac{2A''(\zeta)A(\zeta)-(A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}}
\end{aligned}$$

Now we come to a criterion for nondegeneracy. Let $pr_c\colon J^{-1}(c)\to J^{-1}(c)/\mathbb{S}^1$ denote the quotient map for each $c\in J(\mathbb{S}^2\times\mathbb{S}^2)$.

Proposition 3.14 (Rank 1 Criterion). *Suppose $p\in\mathbb{S}^2\times\mathbb{S}^2$ is a rank 1 critical point and denote $c=J(p)$ and $pr_c(p)=(\zeta,\vartheta)$. Then p is non-degenerate if and only if $\partial^2_{\zeta\zeta}H(\zeta,\vartheta)\neq 0$. In particular, p is non-degenerate and of elliptic-regular type if*

$$\frac{2t_4R_1}{t_3R_2}\cos(\vartheta) > \frac{2A''(\zeta)A(\zeta)-(A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}},$$

non-degenerate and of hyperbolic-regular type if

$$\frac{2t_4R_1}{t_3R_2}\cos(\vartheta) < \frac{2A''(\zeta)A(\zeta)-(A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}},$$

and degenerate otherwise.

Proof. We start by computing the symplectic form on the symplectic quotient. Let $j\colon J^{-1}(c)\to\mathbb{S}^2\times\mathbb{S}^2$ be the inclusion map. Recall $\omega=\sum_{i=1}^2 R_i dz_i\wedge d\theta_i$ so we have

$$j^*\omega = R_1dz_1\wedge d\theta_1 + R_2 d\left(\frac{c-R_1z_1}{R_2}\right)\wedge d\theta_2 = R_1dz_1\wedge d(\theta_1-\theta_2)$$

and thus on the reduced space $\mathbb{S}^2\times\mathbb{S}^2 /\!\!/ \mathbb{S}^1$ in the coordinates (ζ,ϑ) we have the symplectic form

$$\omega_{\mathrm{red}} = R_1 d\zeta\wedge d\vartheta \qquad \text{with matrix} \qquad \omega_{\mathrm{red}} = R_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Since (ζ,ϑ) is a critical point Lemma 3.11 implies that $\sin(\vartheta)=0$, so $\partial_{\vartheta\zeta}H(\zeta,\vartheta)=0$ and thus

$$\begin{aligned}
\omega_{\mathrm{red}}^{-1}d^2H(\zeta,\vartheta) &= \frac{1}{R_1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial^2_{\zeta\zeta}H(\zeta,\vartheta) & 0\\ 0 & \partial^2_{\vartheta\vartheta}H(\zeta,\vartheta)\end{pmatrix}\\
&= \frac{1}{R_1}\begin{pmatrix}0 & -\partial^2_{\theta\theta}H(\zeta,\vartheta)\\ \partial^2_{\zeta\zeta}H(\zeta,\vartheta) & 0\end{pmatrix}
\end{aligned}$$

which has eigenvalues

$$\lambda_\pm = \pm\frac{1}{R_1}\sqrt{-\partial^2_{\zeta\zeta}H(\zeta,\vartheta)\partial^2_{\vartheta\vartheta}H(\zeta,\vartheta)}.$$

Since $\cos(\vartheta)=\pm1$ we see that $\partial^2_{\vartheta\vartheta}H(\zeta,\vartheta)\neq 0$ and so the eigenvalues are distinct if and only if $\partial^2_{\zeta\zeta}H(\zeta,\vartheta)\neq 0$, establishing the first part of the claim. To complete the proof we notice that $\lambda_\pm$ are purely imaginary if $\partial^2_{\zeta\zeta}H(\zeta,\vartheta)\partial^2_{\vartheta\vartheta}H(\zeta,\vartheta)>0$, which implies that p is elliptic-regular, and purely real otherwise, implying that p is hyperbolic-regular. We compute

$$\begin{aligned}
&\partial^2_{\zeta\zeta} H(\zeta,\vartheta)\partial^2_{\vartheta\vartheta} H(\zeta,\vartheta)\\
&= \left(\frac{-2t_4R_1}{R_2} + t_3\cos(\vartheta)\,\frac{2A''(\zeta)A(\zeta)-(A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}}\right)\left(-t_3\cos(\vartheta)\sqrt{A(\zeta)}\right)\\
&= \sqrt{A(\zeta)}\left(\frac{2t_3t_4R_1}{R_2}\cos(\vartheta) - t_3^2\,\frac{2A''(\zeta)A(\zeta)-(A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}}\right),
\end{aligned}$$

and the result follows because $\sqrt{A(\zeta)} > 0$ for the bounds on ζ. □

The following is established by plugging the specific values into the inequality from Proposition 3.14 (for more details see the proof of Lemma 4.2).

Corollary 3.15. *For the parameter values*

$$R_1 = 1, \quad R_2 = 2, \quad t_1 = \frac{1}{4}, \quad t_2 = \frac{1}{4}, \quad t_3 = \frac{1}{2}, \quad t_4 = 0, \tag{20}$$

all rank 1 points are non-degenerate and of elliptic-regular type.

4. A linear combination of systems of toric type

In this section we apply the results of the previous section to a special choice of parameters of the system. Let $\vec{s} := (s_1, s_2) \in [0,1]^2$ and consider the system $(J_{\vec{R}}, H_{\vec{s}})$ on $\mathbb{S}^2 \times \mathbb{S}^2$ using the same $J_{\vec{R}}$ as before but using $H_{\vec{s}} := H_{\vec{t}}$ where

$$t_1 = (1-s_1)(1-s_2), \qquad t_2 = s_1s_2, \qquad t_3 = s_1+s_2-2s_1s_2, \qquad t_4 = s_1-s_2,$$

i.e., we consider

$$\begin{cases} J_{\vec{R}}(x_1,y_1,z_1,x_2,y_2,z_2) := R_1z_1 + R_2z_2,\\ H_{\vec{s}}(x_1,y_1,z_1,x_2,y_2,z_2) := (1-s_1)(1-s_2)z_1 + s_1s_2z_2\\ \qquad\qquad + s_1(1-s_2)(x_1x_2+y_1y_2+z_1z_2)\\ \qquad\qquad + s_2(1-s_1)(x_1x_2+y_1y_2-z_1z_2). \end{cases} \tag{21}$$

Thinking of R_1 and R_2 as fixed, this produces a two parameter family of systems $\{(J_{\vec{R}}, H_{(s_1,s_2)}) \mid s_1, s_2 \in [0,1]\}$. This family is of interest because it shows the system $(J_{\vec{R}}, H_{(\frac{1}{2},\frac{1}{2})})$, which is a semitoric integrable system with exactly two focus-focus points by Theorem 3.1, as a linear combination of systems of toric type. The systems $(J_{\vec{R}}, H_{(0,0)})$, $(J_{\vec{R}}, H_{(0,1)})$, $(J_{\vec{R}}, H_{(1,0)})$, and $(J_{\vec{R}}, H_{(1,1)})$ are systems of toric type whose associated polygons agree (as subsets of $\mathbb{R}^2$) with four elements of the semitoric polygon of the semitoric system $(J_{\vec{R}}, H_{(\frac{1}{2},\frac{1}{2})})$. The images of the momentum maps for these systems are shown in Figure 1 and a plot describing the number of focus-focus points for different values of $s_1, s_2 \in [0,1]$ is show in Figure 2.

In the following series of lemmas we apply the various general results developed in Section 3 to the special case of the system (21).

Lemma 4.1. *For any choice of parameters $s_1, s_2 \in [0,1]$ the system in Equation (21) is integrable.*

Proof. Recall that (21) is a special case of (1) with $t_3 = s_1 + s_2 - 2s_1s_2$, so by Proposition 3.13 we know the result holds for all $s_1, s_2 \in [0,1]$ such that $s_1 + s_2 -$

$2s_1s_2 \neq 0$. This only leaves the cases of $s_1 = s_2 = 0$ and $s_1 = s_2 = 1$. The case $s_1 = s_2 = 0$ leads to the system

$$J_{(1,2)}(x_1, y_1, z_1, x_2, y_2, z_2) = z_1 + 2z_2, \qquad H_{(0,0)}(x_1, y_1, z_1, x_2, y_2, z_2) = z_1$$

and the case $s_1 = s_2 = 1$ to the system

$$J_{(1,2)}(x_1, y_1, z_1, x_2, y_2, z_2) = z_1 + 2z_2, \qquad H_{(1,1)}(x_1, y_1, z_1, x_2, y_2, z_2) = z_2,$$

which are each known to be toric integrable systems. $\square$

Lemma 4.2. *For any choice of parameters $s_1, s_2 \in [0,1]$, all rank 1 critical points of $(J_{(1,2)}, H_{(s_1,s_2)})$ are nondegenerate and of elliptic-regular type.*

Proof. The cases of $s_1 = s_2 = 0$ and $s_1 = s_2 = 1$ produce toric systems as described in the proof of Lemma 4.1, so all rank 1 points in these systems are non-degenerate and of elliptic-regular type. Now consider $(s_1, s_2) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$ which implies $s_1 + s_2 - 2s_1s_2 > 0$. Substituting $R_1 = 1$, $R_2 = 2$, $t_3 = s_1 + s_2 - 2s_1s_2$ and $t_4 = s_1 - s_2$ into the criterion in Proposition 3.14 we see that it is sufficient to show

$$\frac{s_1 - s_2}{s_1 + s_2 - 2s_1s_2}\cos(\vartheta) > \frac{2A''(\zeta)A(\zeta) - (A'(\zeta))^2}{4(A(\zeta))^{\frac{3}{2}}}. \tag{22}$$

Standard calculus shows that the value of the left-hand-side of Equation (22) is in the interval $[-1,1]$ for all s_1, s_2, ϑ and the value of the right-hand-side of Equation (22) can be seen to be in the interval $]-\infty,-1[$ for all $(\zeta, c) \in\ ]-1,1[\ \times\ ]-3,3[$ by plotting it in Mathematica (see Figure 5), so the inequality is verified. $\square$

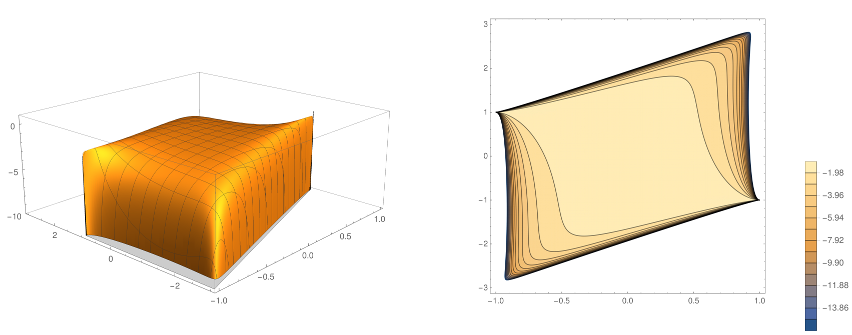

FIGURE 5. This figure analyses the right hand side of Equation (22): The plot on the left shows the graph of the right hand side of Equation (22) which is always below -1.06066. The contour plot on the right displays the associated level sets.

For $R_1 = 1$, $R_2 = 2$, $\vec{s} = (s_1, s_2) \in [0,1]^2$ and $\vec{e} = (e_1, e_2) \in \{0,1\}^2$ consider the discriminant from (14) given by

$$\begin{aligned}\triangle_{(\vec{s},\vec{e})} := \Bigg(\frac{1}{4}\Big(&(s_1 s_2 + e_1(s_1 - s_2))^2 + 4e_1e_2(s_1 + s_2 - 2s_1s_2)^2 \\ &+ 4((1-s_1)(1-s_2) + e_2(s_1 - s_2))^2\Big)\Bigg)^2 \\ &- \Bigg((s_1s_2 + e_1(s_1 - s_2))^2((1-s_1)(1-s_2) + e_2(s_1 - s_2))^2 \\ &- 2e_1e_2(s_1s_2 + e_1(s_1 - s_2))((1-s_1)(1-s_2) + e_2s_1 - s_2)(s_1 + s_2 - 2s_1s_2)^2 \\ &+ (s_1 + s_2 - 2s_1s_2)^4\Bigg)\end{aligned}$$

and set

$$\begin{aligned}\gamma_{(N,S)} &:= \{(s_1, s_2) \in [0,1]^2 \mid \triangle_{(s_1,s_2,1,-1)} = 0\}, \\ \gamma_{(S,N)} &:= \{(s_1, s_2) \in [0,1]^2 \mid \triangle_{(s_1,s_2,-1,1)} = 0\}, \\ \gamma &:= \gamma_{(N,S)} \cup \gamma_{(S,N)}.\end{aligned} \tag{23}$$

The sets are plotted in Figure 2.

Lemma 4.3. *The system $(J_{(1,2)}, H_{(s_1,s_2)})$, $s_1, s_2 \in [0,1]$, has exactly four critical points of rank 0, namely $\{(N,N),(N,S),(S,N),(S,S)\}$. The points (N,N) and (S,S) are non-degenerate and of elliptic-elliptic type for all $s_1, s_2 \in [0,1]^2$. The point (N,S) is non-degenerate except when $(s_1,s_2) \in \gamma_{(N,S)}$ and the point (S,N) is non-degenerate except when $(s_1,s_2) \in \gamma_{(S,N)}$. In particular, for $s_1, s_2 \in \{0,1\}$, all four points are elliptic-elliptic and for $s_1 = s_2 = \frac{1}{2}$ the points (N,S) and (S,N) are both focus-focus.*

Proof. Using Corollary 3.6, we study the behaviour of the discriminant $\triangle_{(\vec{s},\vec{e})}$ for the parameter values in question. If $(e_1,e_2) \in \{(1,1),(-1,-1)\}$, we are in the chart around (N,N) or (S,S) and $\triangle_{(\vec{s},\vec{e})}$ is positive. Figure 6 shows a plot of the case $(e_1,e_2) = (1,1)$. If $(e_1,e_2) \in \{(1,-1),(-1,1)\}$, we are in the chart around (N,S) or (S,N) and $\triangle_{(\vec{s},\vec{e})}$ vanishes along two curves. Figure 7 shows a plot of the case $(e_1,e_2) = (1,-1)$. □

Lemmas 4.1, 4.2, and 4.3 combine to form the following, which implies Theorem 1.2.

Theorem 4.4. *The system $(J_{(1,2)}, H_{(s_1,s_2)})$ has the following properties:*

1) *for all $s_1, s_2 \in [0,1]^2$ it is an integrable system such that, with the possible exception of (N,S) and (S,N) (depending on s_1 and s_2), all of the singular points are non-degenerate of type elliptic-elliptic or elliptic-regular;*
2) *the points (N,S) and (S,N) are rank 0 singular points which transition between being of focus-focus, elliptic-elliptic, and degenerate as (s_1,s_2) varies, and they are only degenerate on a set $\gamma \subset [0,1]^2$ which is the union of four smooth curves.*

Thus, $(J_{(1,2)}, H_{(s_1,s_2)})$ is a semitoric system for all $(s_1,s_2) \in [0,1]^2 \setminus \gamma$. In particular, if $(s_1,s_2) \in \{(0,0),(0,1),(1,0),(1,1)\}$ then $(J_{(1,2)}, H_{(s_1,s_2)})$ is a semitoric system with no focus-focus points and the system $(J_{(1,2)}, H_{(\frac{1}{2},\frac{1}{2})})$ is a semitoric system with exactly two focus-focus points.

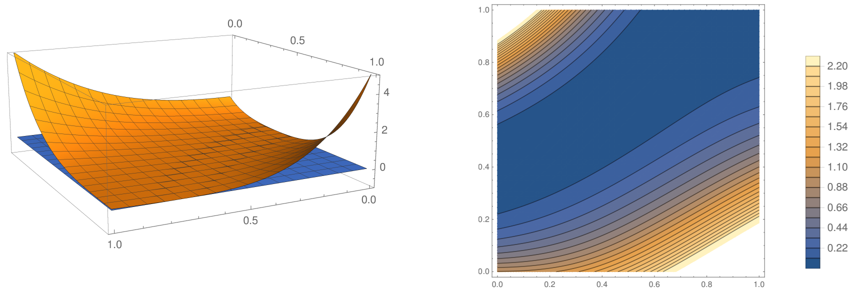

FIGURE 6. Case $(e_1, e_2) = (1, 1)$: on the left, the graph of $(s_1, s_2) \mapsto \triangle_{((s_1,s_2),(1,1))}$ (orange) and a plane through zero (blue) are displayed. On the right, the associated level sets of $(s_1, s_2) \mapsto \triangle_{((s_1,s_2),(1,1))}$ are shown.

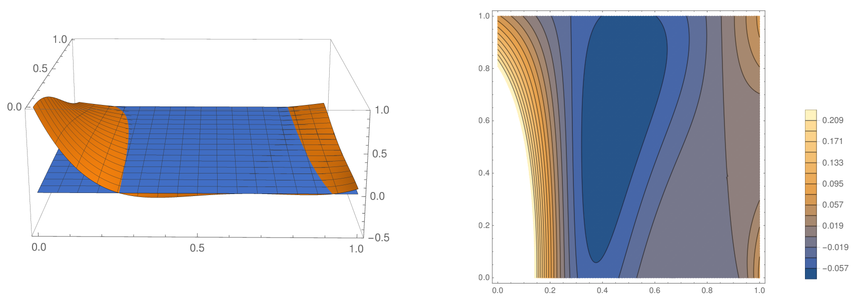

FIGURE 7. Case $(e_1, e_2) = (1, -1)$: on the left, the graph of $(s_1, s_2) \mapsto \triangle_{((s_1,s_2),(1,-1))}$ (orange) and a plane through zero (blue) are displayed. On the right, the associated level sets of $(s_1, s_2) \mapsto \triangle_{((s_1,s_2),(1,-1))}$ are shown.

Note that the set γ is given in Equation (23) and is plotted in Figure 2.

4.1. **A degenerate point.** By Proposition 2.8 we know that for each $\vec{R}$ there exist some values of $s_1, s_2 \in [0, 1]$ such that $(J_{\vec{R}}, H_{(s_1,s_2)})$ is a degenerate system because the points (N, S) and (S, N) transition between being focus-focus and being elliptic-elliptic.

Example 4.5. Assume that $s_1 = s_2$. Since $(J_{(1,2)}, H_{(0,0)})$ and $(J_{(1,2)}, H_{(1,1)})$ have no focus-focus points and $(J_{(1,2)}, H_{(\frac{1}{2},\frac{1}{2})})$ has focus-focus points at (N, S) and (S, N) there must exist at least two values of $s \in \,]0, 1[$ such that $(J_{(1,2)}, H_{(s,s)})$ has a degenerate rank 0 point by Proposition 2.8. Plugging $t_1 = (1-s)^2$, $t_2 = s^2$, $t_3 = 2s(1-s)$, $t_4 = 0$, $e_1 = -1$, and $e_2 = 1$ into $\omega_p^{-1} d^2 H$ in Equation 13 and taking the discriminant of the characteristic polynomial equal to zero gives exactly two solutions in the range $]0, 1[$. These solutions are s_+ and s_- where

$$s_{\pm} = \frac{1}{31}\left(\pm 8\sqrt{5} + 14 \mp \sqrt{82 \mp 24\sqrt{5}}\right)$$

and $s_+ \approx 0.856953$, $s_- \approx 0.250291$. Since there must be at least two degenerate points and these are the only points for which $\omega_p^{-1} d^2 H$ has less than four distinct eigenvalues we conclude that $(J_{(1,2)}, H_{(s_+,s_+)})$ and $(J_{(1,2)}, H_{(s_-,s_-)})$ have a degenerate point at (S, N).

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