

DEPARTMENT OF ECONOMICS

**All but one:
How pioneers of linear economics
overlooked Perron-Frobenius mathematics**

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1. Introduction

Recently MIT Press published the final two volumes, numbers 6 and 7, of *The Collected Scientific Papers of Paul Anthony Samuelson*. Because he wrote on many and widely different topics, Samuelson has often been considered the last generalist in economics, but it is remarkable how large a proportion of the final two volumes is devoted to linear economic models of the input-output type, often called Leontief-Sraffa models. Besides Samuelson, many other Nobel laureates published on similar linear models, for example Wassily Leontief, Ragnar Frisch, Herbert Simon, Tjalling Koopmans, Kenneth Arrow, Robert Solow, Lawrence Klein, Gérard Debreu, Leonid Hurwicz, Harry Markowitz, John Hicks, Robert Mundell, James Mirrlees, Richard Stone, Joseph Stiglitz, and Amartya Sen.¹

Because the inputs in processes of production correspond to nonnegative quantities, the theory of nonnegative matrices plays a crucial role in many modern treatments of linear economics. The title of my paper refers to ‘Perron-Frobenius mathematics’ in honour of the fundamental papers by Oskar Perron (1907a, 1907b) and Georg Frobenius (1908, 1909, 1912) on positive and nonnegative matrices.²

Today these Perron-Frobenius theorems on nonnegative matrices enjoy wide applications, the most sensational perhaps being their implicit use by millions of internet surfers, who routinely activate Google’s PageRank algorithm every day (Langville & Meyer, 2006). Such a remarkable worldwide application in an internet search engine is far away from the spirit of the original theoretical articles, written more than a century ago, by Perron and Frobenius. These two German scholars preferred to concentrate on pure mathematics. Especially

¹ Some bibliographic examples: Leontief (1928, 1941, 1951), Frisch (1934a, 1934b), Hawkins & Simon (1949), Koopmans (1951), Arrow (1951, 1954), Solow (1952), Klein (1952, 1956), Debreu & Herstein (1953), Hurwicz (1955), Markowitz (1955), Dorfman, Samuelson & Solow (1958), Arrow, Hoffenberg, Markowitz & Shephard (1959), Hicks (1960), Mundell (1965), Mirrlees (1969), Stone (1970), Stiglitz (1970) and Sen (2003, 2004). If I also take into account book reviews, collaboration in Leontief’s Harvard research projects, participation in the capital controversy, and theory or applications of linear programming, then I can extend this Nobel list by including Simon Kuznets, James Tobin, Reinhard Selten, Vernon Smith, Franco Modigliani, Merton Miller, William Sharpe, Leonid Kantorovich, etc.

² I call matrices and vectors *nonnegative* if all their elements are positive or zero; *positive* if all their elements are positive. If a nonnegative matrix or vector is not zero, it is sometimes called *semipositive*. Good introductions to nonnegative matrices for economists are provided by Nikaido (1968, 1970) or Takayama (1985). Nikaido’s 1970 book contains a very long chapter, simply titled ‘the Frobenius Theorem’, which is a generic name for covering some original results by Frobenius and many related theorems on eigenvalues and eigenvectors, determinants, solutions of related equations or inequalities, etc. The relevant Chapter 4 in Takayama’s book is titled ‘Frobenius’ theorems, dominant diagonal matrices, and applications’. See also Dorfman, Samuelson & Solow (1958), Kurz & Salvadori (1995) and Bidard (2004).

Frobenius in Berlin considered applications as inferior subjects that should be relegated to technical schools.

Their preference for pure mathematics might explain why neither Frobenius, who died in 1917, nor Perron, who remained a very active scholar until half a year before his death in 1975, ever published on mathematical economics, although the Perron-Frobenius theorems on nonnegative matrices are often the crucial tools to solve many mathematical problems in linear economic models.³ In the first half of the twentieth century, numerous publications in economics (and in many other scientific disciplines) could have benefited from applying Perron-Frobenius but failed to do so, even the publications authored by mathematicians. Before the Second World War, nearly all pioneers of linear economics overlooked Perron-Frobenius: either they didn't know it, or they didn't realize its relevance. I know of only one amazing exception: the French Jesuit mathematician Maurice Potron (1872-1942) made explicit and intense use of Perron-Frobenius mathematics in the core of his economic model, as early as 1911, one year before the third and final paper by Frobenius (1912) appeared. But Potron's contemporaries didn't recognize his originality, and his economic writings fell into oblivion, until a paper by Émeric Lendjel (2000) started the process of changing Potron's status in economics from zero to hero.

The seminal publications on linear economics in the 1910s and 1920s were written by Charasoff, Potron, Bray and Remak. All four were mathematicians! All except Potron overlooked the usefulness of Perron-Frobenius. It is interesting to study how Perron-Frobenius was overlooked so long in economics, and to provide additional insight into the communication or lack of communication between the different protagonists of economics and mathematics.

My paper is organized as follows. Section 2 starts with Georg von Charasoff, because he was the first pioneer who wrote after Perron's 1907 results on positive matrices became available. For many reasons, Charasoff had a higher probability of discovering and applying Perron-Frobenius than the average mathematician or economist. Two of his mathematical interests (irreducibility of equations, and continued fractions) were close to Perron's interests, and the second part of his dissertation was close to Frobenius' classic paper on irreducibility of differential equations. But in his publications Charasoff never mentioned Perron or Frobenius,

³ A comprehensive Perron bibliography is provided by Heinhold (1980) and Frank (1982). On the historical origins of Perron-Frobenius mathematics, see Thomas Hawkins (2008, 2013).

he didn't use their matrices, and just assumed that the properties of his simple numerical examples with three commodities also hold in the general case with n commodities.

Section 3 briefly describes the now well-known case of the French Jesuit mathematician Maurice Potron, whose linear economics used Perron-Frobenius as early as 1911, in order to prove the existence of positive solutions for the relevant variables. He presented remarkable duality properties that show the connection between his quantity system and his price system, and he anticipated the famous Hawkins-Simon conditions. All his pioneering results were neglected by his contemporaries.

Section 4 contains a few remarks about Hubert Bray's 1922 paper, on Cournot's equations of exchange. Section 5 discusses the mathematically similar system presented by Robert Remak in 1929. I briefly mention his forgotten economic paper of 1918, on the repayment of the national debt, and then I focus attention on the mathematical economics of his classic 1929 paper on superposed prices. Here Remak neglected the useful results of Frobenius (his own doctoral supervisor in 1911), and moreover it turns out that most of Remak's mathematical arguments concentrate on rather bizarre economic systems, in which the most important commodities have zero prices.

Section 6 presents related topics, about the two eponyms of Leontief-Sraffa models. It is now well-known that both Wassily Leontief and Piero Sraffa started their research on such topics in the late 1920s, although both were confronted with some delays in their publication processes. Contrary to Dorfman (1973), new archival evidence suggests that Leontief was well aware of Remak's work. Leontief noticed the analogy with the mathematics of his own system, and even contacted Remak in Berlin in 1931 (maybe in vain). Ultimately Leontief presented his input-output algebra to the readers of Anglo-Saxon journals only in 1937, after an earlier paper was rejected by John Maynard Keynes for the *Economic Journal* in 1933, and by Ragnar Frisch for *Econometrica* in 1934. Even more than Leontief, Sraffa preferred to work in isolation. He didn't discuss his mathematical problems with competent economic colleagues in Cambridge, nor with the specialists of the *Econometric Society*, but preferred help from three non-economists: Frank Ramsey, Alister Watson and especially Abram Besicovitch. The latter surely knew Perron (but not his results on positive matrices), was a student of Markov and had published in 1918 a neglected paper in Russian, involving diagonally dominant matrices. The above description suggests that Besicovitch in his early

days ‘came close’ to Perron-Frobenius results, but it is well-known that he didn’t know Perron-Frobenius, and invented his own proofs for Sraffa in the 1940s.

Mathematicians like Charasoff, Bray, Remak, Besicovitch, and others didn’t use the relevant theorems on nonnegative matrices in economics. This gives additional prestige to the remarkable performance of Maurice Potron, who in 1911 provided the first application of the theorems in any discipline, twenty years before Richard von Mises (1931: 536) referred to Frobenius’ matrices in the context of probability theory.

2. Charasoff goes to the limit

Georg von Charasoff ⁴ was born in Tbilisi in 1877, in a wealthy family. After his doctoral dissertation on mathematics (Heidelberg in 1902), he was living the free life of a ‘rentier’ and an ‘independent scholar’ in Germany and Switzerland, and he published two books on Marx, *Karl Marx über die menschliche und kapitalistische Wirtschaft* (1909) and *Das System des Marxismus* (1910). Further planned work on economics was never published, due to negative changes in personal and political circumstances.⁵ His second wife poisoned herself with cyanide in 1912, and his publisher went out of business. Charasoff himself sexually approached his house-maid in Zurich, refused to pay her wage when she refused him, and then was condemned by a Swiss court on 24 March 1915. Charasoff didn’t attend the hearings, where several witnesses offered very negative descriptions of his personality. At that moment, Charasoff was already back in his birth place Tbilisi, trying to save the family wealth in turbulent political times. He never returned to Western Europe, and died in Stalin’s Soviet Union in 1931.⁶

For many decades after his death, Charasoff was classified in the lower divisions of heterodox economics. Only a small minority of Marxian studies reviewed or cited his two books, usually concentrating on minor details, without recognizing the striking originality of some analytical tools in his 1910 book *Das System des Marxismus*. In the preface, Charasoff (1910: xii) claims to have presented a ‘definitive solution’ to the main problems of the classical theory of value, thanks to his theory of *Urkapital* (‘original capital’), a composite commodity similar to Sraffa’s standard composite commodity (Sraffa 1960), but in Charasoff’s case it is computed with respect to an augmented input matrix, which includes both the material inputs of the production processes and the wage goods for the workers. In the language of modern mathematical economics, *Urkapital* corresponds to an eigenvector of the augmented input matrix.

⁴ He signs his first economic book in 1909 as Dr. Georg von Charasoff, but in his other publications (and in all his letters that I have seen), he omits the ‘von’. Some studies refer to him as G. A. Kharazov (G. A. = Georg Artemovich), Charasov, Charasow, etc.

⁵ Several extracts from his two books were later reprinted in *Die Aktion* or in *Der Gegner*; one extract even in both journals around the same time (see Charasoff 1920).

⁶ For many years, biographical details on Charasoff were scarce. Recently more information on Charasoff became available, thanks to Peter Klyukin (2008) and Christian Gehrke (2013). The well-documented biographical paper by Gehrke presents a wealth of new material on Charasoff’s activities in Heidelberg, Zurich and Lausanne, and also on his later years in Tbilisi, Baku and Moscow. Gehrke’s bibliography contains a long list of publications by and on Charasoff. Kenji Mori (2007: 132-133) discusses some letters from Charasoff in the Karl Kautsky papers, International Institute of Social History (Amsterdam), showing that Charasoff in 1907 submitted a paper on Marx to Kautsky, editor of *Die Neue Zeit*, but Charasoff’s manuscript was rejected and is not extant.

Neither the mathematicians nor the economists took notice of Charasoff's innovations. A special example of overlooking Charasoff's originality is provided in a doctoral dissertation on the economics of Bortkiewicz and related topics, written by Werner Klimpt (1900-1978), and defended at the Ruprecht-Karls-Universität Heidelberg in 1931, the same university where Charasoff had received his doctoral degree in 1902, and where Perron had been a professor from 1914 to 1922. Klimpt published his text in Berlin in 1936, and seems to be one of the few Charasoff readers who comments on Charasoff's innovating concept of *Urkapital*. Despite his mathematical background, Klimpt (1936: 119) rejects Charasoff's device as highly peculiar, complicated, confused etc.⁷ Apparently, neither Werner Klimpt, nor his doctoral referees Emil Julius Gumbel and Emil Lederer, understood Charasoff's innovations.⁸

In the 1980s the scientific reputation of Charasoff suddenly started to grow, thanks to the amazing discoveries of Egidi & Gilibert (1984). In the following years, many other authors studied Charasoff; see for example the remarkable, but unpublished text by Duffner & Huth (1987), and the publications by Kurz & Salvadori (1995, 2000), Stamatis (1999) and Mori (2007, 2011), all awarding Charasoff the status of a remarkable pioneer.

It is now well-known that Charasoff independently discovered some Sraffa-like tools. His *Grundprodukte* bear a close resemblance to Sraffa's basic commodities, his *Reproduktionsbasis* reminds us of Sraffa's subsystems, his labour values are computed via dated labour series, etc. The most fundamental new concept invented by Charasoff, i.e. his notion of *Urkapital*, was generated in a rather unexpected way. An attempt to explain Charasoff's approach requires some knowledge of his mathematical background.

2.1. Charasoff's mathematical background

Charasoff started studies of medicine in Moscow, but then switched to mathematics in Germany, and presented his doctoral dissertation, *Arithmetische Untersuchungen über Irreduktibilität* in 1902, in the Faculty of Natural Sciences and Mathematics of the Ruprecht-Karls-Universität in Heidelberg. Charasoff cannot be found in the set of more than 300 names

⁷ In German: 'höchst eigenartig' – 'kompliziert' – 'unübersichtlich'.

⁸ On Lederer's economics, see Hagemann (2000); on Gumbel and Lederer in Heidelberg, see Brintzinger (1996). The Gumbel biography by Arthur D. Brenner (2001: 32-34) describes how the pacifist Georg Friedrich Nicolai tried to resume his university lectures in Berlin in 1920, but was shouted down by hundreds of demonstrators. When the Faculty Senate accused Nicolai of desertion and treason, only three persons signed a letter to protest against Nicolai's banning from the university: the publisher Curt Thesing, Emil Gumbel and Otto Buek. Note that Buek was Charasoff's most important helper, the only person that received thanks in the preface of Charasoff's 1909 book. Gumbel is not an important innovator in the present story, but in the 1930s he seemed to be the first scholar who came into contact with the work of four pioneers of linear economics: Charasoff, Remak, Potron and Leontief (see *infra*).

mentioned in the autobiography of his doctoral promoter Leo Königsberger (1919). The dissertation looks like a minor exercise in the research program of his promoter. Its results generated no follow up in the mathematical literature, and the dissertation fell into oblivion.

Note that the ‘irreducibility’ in Charasoff’s dissertation is different from the problem of reducible and irreducible matrices in Perron-Frobenius mathematics.⁹ In Charasoff’s dissertation, it simply refers to the irreducibility of equations.¹⁰ Although the topic itself is not connected to positive or nonnegative matrices, one of the specialists on irreducibility of equations was ...Oskar Perron (1905a, 1907c).¹¹ In the final part of his 1902 dissertation, Charasoff studied the irreducibility of differential equations. Nearly three decades earlier, a seminal paper on related kinds of differential equations had been published by a promising 23-year old German mathematician, namely...Georg Frobenius (1873).¹² Despite all this, Charasoff never referred to Perron or Frobenius, neither in his mathematical dissertation nor in his economic books.

In the late nineteenth and the early twentieth century, David Hilbert in Göttingen was one of the most important mathematicians of the world, and perhaps the most influential in the German academic networks of the time. His address at the *Second International Congress of Mathematicians* in Paris in 1900 contained 23 famous unsolved problems, many of which still fascinate mathematicians today. Numerous scholars sent letters or papers to Hilbert, who acted as one of the editors of the influential *Mathematische Annalen* from 1902 to 1939. It is unknown whether Charasoff ever sent his dissertation results of 1902 to a journal, but in 1904 he had formulated some results on another topic: continued fractions (*Kettenbrüche*). He submitted his results to Hilbert, but obtained a rejection.¹³ Charasoff very politely accepted

⁹ Consider a $n \times n$ matrix A with elements a_{ij} , where i and j belong to the index set $\{1, 2, \dots, n\}$. If this set can be partitioned in two nonempty subsets I and J such that $a_{ij} = 0$ if $i \in I, j \in J$, then A is called reducible; otherwise A is called irreducible. For example, $A = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ is reducible (take $I = \{1\}$ and $J = \{2\}$).

¹⁰ Consider for example the equation $x^2 - 5 = 0$. It is called reducible if its left hand side (a polynomial of the second degree) can be ‘reduced’ to polynomials of lower degree. If we consider all real numbers, this is possible, because $x^2 - 5 = 0$ can be written as $(x - \sqrt{5})(x + \sqrt{5}) = 0$, but if we consider only rational numbers, the equation is irreducible. Much more sophisticated cases of irreducibility exist, of course, for example in the differential equations in Charasoff’s dissertation.

¹¹ Alfred Brauer (1948: 423) writes that Perron ‘was the first who obtained criteria for the irreducibility of polynomials depending on the comparative size of the coefficients’. Studies on irreducibility of equations often mentioned Perron and Königsberger, but never referred to Charasoff.

¹² ‘Frobenius paper on irreducible differential equations was his first publication introducing a significant and entirely new concept...’ (Hawkins 2013: 23). Frobenius was born 26 October 1849; the seminal paper was dated 24 April 1873.

¹³ After Hilbert’s rejection, it remained unpublished (and unfound). The name Charasoff is not mentioned in the comprehensive bibliographies on continued fractions compiled by Wölffing (1908) and Brezinski (1991).

the verdict in his letter of 10 May 1904 to Hilbert.¹⁴ Charasoff mentioned that he had read many papers by Minkowski, but apologized for overlooking a relevant paper in which Minkowski had already worked out a similar theorem on continued fractions. No further details are given by Charasoff, who apologized for using in such an unproductive manner some of Hilbert's very precious time.¹⁵

This story shows that Charasoff was following the literature on continued fractions at that time, and that Hilbert had suggested Charasoff should get a more complete knowledge of this literature. If Charasoff was stimulated by this advice, then from 1905 on he must have noticed a series of successful publications on continued fractions by ... Oskar Perron, who would become a well-respected authority on the subject.¹⁶

Perron (1905b, 1905c) wrote two papers on continued fractions for the Academy of Sciences in Munich, and then, more importantly and more visibly, he published his Habilitation thesis on the same subject, in the long opening article in the 1907 volume of the leading journal *Mathematische Annalen* (Perron, 1907a). In the course of his study, Perron had to prove some 'auxiliary results' on matrices and their characteristic equations. His first treatment of this matrix problem was hidden in the middle of his long article, but a few months later Perron (1907b) isolated these matrix problems in a separate article 'Zur Theorie der Matrices', again in the *Mathematische Annalen*, and he gave a complete proof of the now classic Perron Theorem on the eigenvalues and eigenvectors of positive matrices.

Perron expressed some dissatisfaction with his own proofs, because he often had to rely on limit arguments. In algebra, pure mathematicians prefer purely algebraic tools, and try to avoid limits if possible. In the authoritative historical study on this story, Thomas Hawkins (2008: 679) used the expression 'Perron's Limit Lemma' for the following 'inelegant' auxiliary tool that Perron needed in his 1907 proofs:

¹⁴ See Nachlass David Hilbert, Cod. Ms. D. Hilbert 59, Abteilung Handschriften und seltene Drucke, Niedersächsische Staats- und Universitätsbibliothek, Göttingen. I thank Dr. Hunger and his staff for providing me copies of some Hilbert correspondence with Charasoff, Remak and Potron.

¹⁵ The very humble, courteous letter to Hilbert is not representative of Charasoff. The paper by Gehrke (2013) offers some remarkable stories on Charasoff's rude, unpleasant character in everyday life in Germany and Switzerland.

¹⁶ In 1905 Charasoff enrolled as an 'auditor' for Heinrich Burkhardt's lectures on elliptic functions at the University of Zurich (Gehrke 2013: 17), one of the specialties of Georg Frobenius. Note that Oskar Perron went on writing about continued fractions up until the 1970s, and his textbook *Die Lehre von den Kettenbrüchen* was often reprinted and is still recommended in some reading lists today.

Perron's Limit Lemma (Perron 1907b: 259-261): *Let the $n \times n$ matrix A have all its elements a_{ij} positive, and denote the elements of its v -th power A^v by a_{ij}^v . Then for $v \rightarrow \infty$, we can find a positive row vector $q = [q_1 \ \dots \ q_n]$, a positive column vector $p = [p_1 \ \dots \ p_n]^t$, and a positive number λ such that*

$$(i) \quad \lim_{v \rightarrow \infty} \frac{a_{1k}^v}{a_{1s}^v} = \lim_{v \rightarrow \infty} \frac{a_{2k}^v}{a_{2s}^v} = \dots = \lim_{v \rightarrow \infty} \frac{a_{nk}^v}{a_{ns}^v} = \frac{q_k}{q_s}$$

$$(ii) \quad \lim_{v \rightarrow \infty} \frac{a_{i1}^v}{a_{r1}^v} = \lim_{v \rightarrow \infty} \frac{a_{i2}^v}{a_{r2}^v} = \dots = \lim_{v \rightarrow \infty} \frac{a_{in}^v}{a_{rn}^v} = \frac{p_i}{p_r}$$

$$(iii) \quad \lim_{v \rightarrow \infty} \frac{a_{ij}^{v+1}}{a_{ij}^v} = \lambda \quad \text{for every } i \text{ and } j$$

$$(iv) \quad qA = \lambda q \text{ and } Ap = \lambda p$$

Translated in more intuitive terms: when the value of v is high, every row of the matrix A^v is nearly proportional to the row vector q , every column of the matrix A^v is nearly proportional to the column vector p , and two consecutive matrices A^{v+1} and A^v are nearly proportional to each other. For Perron, the above limit properties were just auxiliary tools, allowing him to prove that the positive matrix A has a positive eigenvalue λ that can be associated with a positive row eigenvector q , and a positive column eigenvector p ; this result belongs to Perron's Main Theorem.¹⁷ When Georg Frobenius, a pioneer in matrix algebra, read Perron's second 1907 paper on matrices, he responded to two challenges: to offer proofs of Perron's main theorem without using Perron's Limit Lemma, and to extend Perron's results on eigenvalues and eigenvectors (and minors of certain determinants) from positive matrices to nonnegative ones. Although in declining health, in the last years of his life, in three remarkable papers on the subject, Frobenius (1908, 1909, 1912) succeeded masterfully.

Many ironical aspects exist. Perron and Frobenius preferred purely algebraic proofs without limits, if possible. But for a mathematical treatment of Charasoff's economics, Perron's Limit Lemma is extremely useful. It provides the mathematical foundations of Charasoff's

¹⁷ For many details on continued fractions and the historical origins of the papers by Perron and Frobenius, I refer to Hawkins (2008, 2013). Perron's Main Theorem states the following: a square positive matrix A has a positive eigenvalue λ whose absolute value is larger than that of any other eigenvalue (hence λ is called the dominant eigenvalue), λ is a simple root of the characteristic equation, and λ can be associated with a positive row eigenvector q , and a positive column eigenvector p . These eigenvectors are unique up to a scalar multiple. Perron clearly mentioned that his results are not only valid for a positive matrix A , but also for a nonnegative matrix A if A^v is positive for some power v . Therefore Perron's results are useful for Charasoff's system.

economics. Given his mathematical background, and his interest in continued fractions, Charasoff was in a better position than the average mathematician to know Perron's work, but it is remarkable that in the hundreds of pages of his two books, he never explicitly referred to Perron or Frobenius, not even in one footnote. Charasoff limited his algebra to simple numerical examples, and assumed, without general proofs, that the properties of his examples with three commodities also hold for economic systems with n commodities.

2.2. Charasoff's system of production of commodities by means of commodities

The different chapters of Charasoff's 1910 book, *Das System des Marxismus*, started with a quotation from Marx (without further bibliographical details). For example, Charasoff placed the following quotation below the title of his brilliant chapter 10 (in German):

Coal is required for making gas, but gas lighting is used in producing coal. . . . Coal must replace the wear and tear of the steam-engine used to produce it. But the steam-engine consumes coal. Coal itself enters into the means of production of coal. Thus it replaces itself in kind. Transport by rail enters into the production costs of coal, but coal in turn enters into the production costs of the locomotive. (Charasoff, 1910: 118)

Readers of Marx's *Theories of Surplus Value* (vol. 1, p. 247) will recognize the source of this quotation. By choosing this text of Marx, Charasoff wanted to emphasize the circular nature of production. Elsewhere, in a long footnote, Charasoff (1910: 290-291) referred to a similar circular example from Pareto, also involving coal that is ultimately produced by coal.

Charasoff's lists of inputs usually replaced the workers by their wage goods, and thus, even more than in the core of Sraffa (1960), Charasoff's economic model is a system of 'production of commodities by means of commodities'. In his 1910 book, Charasoff often considered the following numerical example with three production processes. In Sraffian layout the numbers can be presented as follows:

70 units of corn + 30 units of labour time → 100 units of corn

20 units of corn + 20 units of labour time → 100 units of bread

10 units of corn + 50 units of labour time → 100 units of cakes

Workers receive only bread (wage good), one unit of bread per unit of labour time. The cakes are a luxury product eaten only by the capitalists. If we replace ‘a unit of labour time’ by its necessary input ‘a unit of bread’, and express all input magnitudes per unit of output, then we obtain the following example (compare with Charasoff, 1910: 99):

0.7 units of corn + 0.3 units of bread + 0 cakes → 1 unit of corn

0.2 units of corn + 0.2 units of bread + 0 cakes → 1 unit of bread

0.1 units of corn + 0.5 units of bread + 0 cakes → 1 unit of cakes

Charasoff provided a rather intuitive analysis, without any explicit use of tools from matrix algebra, although modern approaches prefer matrices in his context. Let A represent the ‘augmented’ input matrix (containing technically necessary inputs of corn, augmented by the quantities of bread, the wage good). Assume a production period of one year, an unchanging technology, and consider the output of 1 unit of corn. The first row of the matrix A below describes the input vector needed one year ago: Charasoff called it ‘capital of the first order’. The first row of the matrix A^2 describes the input vector needed two years ago, the ‘capital of the second order’. The first row of A^3 describes the input vector needed three years ago, the ‘capital of the third order’, etc. In a similar way, the second rows of A , A^2 , A^3 , A^4 etc. describe the dated input vectors for bread. The third rows describe the dated input vectors for cakes:

$$A = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.2 & 0 \\ 0.1 & 0.5 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0.55 & 0.27 & 0 \\ 0.18 & 0.10 & 0 \\ 0.17 & 0.13 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0.439 & 0.219 & 0 \\ 0.146 & 0.074 & 0 \\ 0.145 & 0.077 & 0 \end{bmatrix}, A^4 = \begin{bmatrix} 0.3511 & 0.1755 & 0 \\ 0.1170 & 0.0586 & 0 \\ 0.1169 & 0.0589 & 0 \end{bmatrix}$$

The numbers above can also be found in the tables constructed by Charasoff (1910: 114). If we trace back an arbitrary output vector, say the row vector X , then the inputs of one year ago are given by $X' = XA$, the inputs of two years ago by $X'' = XA^2$, etc. Charasoff calls this a *production sequence* and denotes it as $X \sim X' \sim X'' \sim \dots$ and he also refers to it as an ‘equation’ (*Gleichung*). On the one hand, this word ‘equation’ (*Gleichung*) is sometimes used by Charasoff in unexpected places. On the other hand, Charasoff fails to show us explicitly the simple system of linear equations that was used a few years earlier by Wolfgang Mühlpfordt (in 1893 and 1895) or Vladimir Karpovich Dmitriev (in 1898) to determine the labour values of n commodities. Unlike Charasoff, both Mühlpfordt and Dmitriev held no degree in mathematics. It is remarkable that the two non-mathematicians were able to write down the now classic equations for the labour values, while Charasoff’s presentation was

more ‘verbal’ (and without a reference to Mühlpfordt or Dmitriev, whose works were probably not known by Charasoff).

Note that in a productive economy, the sequence $A, A^2, A^3, A^4 \dots$ converges to a zero matrix. But if we normalize all matrices A^v in a suitable manner,¹⁸ then the limit will be a positive

multiple of the following matrix: $A^\infty = \begin{bmatrix} 6 & 3 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

In this limit situation, all rows become proportional to a basket of Charasoff’s *Urkapital*, which is a basket of 2 units of corn, 1 unit of bread, and 0 cakes (or any positive multiple of this basket).

The intuition of Charasoff (1910: 123) for the existence of *Urkapital* was as follows. ‘Capitals of the first order’ can be very different from each other. ‘Capitals of the second order’ are a sort of average of ‘capitals of the first order’, and therefore are less different from each other, etc. In the limit all capitals have the same *Urkapital* composition. Charasoff (1910: 114) suggested that in his example the composition of *Urkapital* is already obvious after four rounds. Indeed, each row of A^4 is nearly proportional to the *Urkapital* vector $q = [2 \ 1 \ 0]$. To produce q we need the input vector $qA = [1.6 \ 0.8 \ 0]$. The vectors q and qA are different quantities of the same composite commodity.

Whatever the individual prices of corn, bread and cakes, the basket q is worth 25% more than qA , and thus the rate of profit is 25%. In this way, Charasoff (1910: 105-128) showed that the rate of profit is determined by the rate of growth of *Urkapital*. Therefore some authors, for example Kurz & Salvadori (1995: 390) and Mori (2011), give him credit for discovering a simple duality between the price system and the quantity system. In mathematical terms, the equality $qA = 0.8q$ illustrates that the *Urkapital* vector q is a row eigenvector of the matrix A , corresponding to the dominant eigenvalue 0.8.

The numbers in the sequences for the different goods will also reveal the equilibrium prices. To have an exact view of this, we should go to the limit. But again, in this example, the ‘capitals of the fourth order’ already suggest the limit outcome. The ‘capital of the fourth order’ for good 1 (see matrix A^4 , row 1) is approximately three times larger than that of goods 2 and 3 (see matrix A^4 , rows 2 and 3). Intuitively speaking, infinite years ago, the ‘start’ of the

¹⁸ Divide A by its dominant eigenvalue 0.8, and obtain the matrix $1.25A$. For $v \rightarrow \infty$ the sequence $(1.25A)^v$ converges to a limit matrix that is proportional to the matrix A^∞ of my example. See also my next two footnotes.

long process leading to a unit of good 1 today, required exactly three times more Urkapital than for a unit of goods 2 or 3. Therefore good 1 is three times more expensive than goods 2 or 3, and the column vector of the prices of the three goods will be equal to or proportional to the column vector $p = [3 \ 1 \ 1]^t$. In mathematical terms, the equality $Ap = 0.8p$ illustrates that p is a column eigenvector of the matrix A , corresponding to the dominant eigenvalue 0.8.

Charasoff used his Urkapital and his production sequences for many purposes: determination of prices and the rate of profit (and the law of the rising, not falling rate of profit); distinction between *Grundprodukte* and other products, where only the production conditions of Grundprodukte (see corn and bread above) determine the rate of profit; balanced growth with profit rate equal to growth rate; convergence of the transformation of labour values or any other value vector into prices; construction of a sort of Sraffian subsystem and dated labour to determine labour values, etc. (see Mori (2011) for more details).

The latter problem is perhaps symptomatic of Charasoff's tendency to 'go to the limit', even when simpler methods are available. Since Mühlpfordt (1893, 1895), we know that labour values can be computed by a simple system of linear equations. Consider a productive economy, let M denote the non-augmented input matrix (thus without the wage goods), and ℓ the column vector of direct labour inputs. It is well-known that the labour values correspond to the column vector v that solves $v = Mv + \ell$, which can also be written as $v = (I - M)^{-1} \ell$ or as a series $v = \ell + M\ell + M^2\ell + M^3\ell + \dots$. It is strange that Charasoff never writes down an explicit system of the form $v = Mv + \ell$ (with or without matrix notation); he prefers series and limits.

In Charasoff's numerical example, the labour value system $v = Mv + \ell$ is extremely simple:

$$v_1 = 0.7 v_1 + 0.3$$

$$v_2 = 0.2 v_1 + 0.2$$

$$v_3 = 0.1 v_1 + 0.5$$

The first equation easily gives $v_1 = 1$, and then it is trivial that $v_2 = 0.4$ and $v_3 = 0.6$. Charasoff (1910: 94-95) provided only a verbal discussion of this example, without explicitly using the above mathematical notation or the general theory of simultaneous equations.

Note that Charasoff's Urkapital contains only *Grundprodukte*. Charasoff knew that these commodities played a decisive role in his system. They can be studied in a smaller system,

independently of the other commodities. In Charasoff's corn-bread-cakes example, instead of using the whole augmented matrix A , we could use the submatrix $A_{11} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$ of corn and bread only, which is a positive matrix, and then Perron's Limit Lemma can be applied without worries about zero elements.¹⁹

Nikaido (1968: 110-112) proved a Limit Lemma for semipositive, irreducible, primitive matrices, without referring to the similar findings by Perron. Nikaido's theorem is usefully applied by Egidi & Gilibert (1984), in their mathematical treatment of a Charasoff system with n commodities. Mori (2011) also makes interesting use of an analogous procedure. These authors don't mention a connection between their limit theorems and Perron's Limit Lemma.²⁰ As I already mentioned, Frobenius (1908, 1909, 1912) simplified the proof of Perron's theorem on the existence of a positive eigenvector of a positive matrix, and generalized his research to nonnegative matrices. For enthusiasts of pure algebra like Perron and Frobenius, the Limit Lemma was an 'awkward' and 'foreign' auxiliary tool. It was discarded in the simpler proofs by Frobenius and later mathematicians. In this way, Perron's Limit Lemma disappeared from the radar of many mathematicians a few years later. But if there is one person who could and should have noticed the Limit Lemma, it was Charasoff in 1907. For him, such a lemma was not an auxiliary tool. On the contrary, it provided the mathematical foundations of his economic argument. But Charasoff seemed to overlook it, and his texts never mentioned determinants, matrices, Perron or Frobenius.

Charasoff (1910: xiii) mentioned that, unlike many scholars in German speaking countries, he didn't want to intimidate his readers by means of many scholarly references and quotations.

¹⁹ I handled the whole 3×3 matrix A , which is reducible and semipositive, but not positive. In this way, I could stay close to the tables in Charasoff (1910: 114). Perron formulated his Limit Lemma for a positive matrix A , and then mentioned that it remained valid for a semipositive matrix A if some power A^p is positive. In modern terminology, the latter means that A is semipositive, irreducible and primitive ('not cyclic'). Slightly different convergence theorems can still be constructed for the reducible case. See Egidi (1992: 249-250) who does not mention Perron's Limit Lemma, but provides a related convergence theorem for some well-behaved reducible matrices. The reducible 3×3 matrix A of the Charasoff example above satisfies Egidi's requirements: it can be written in the form $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, where the submatrix A_{11} is a semipositive, irreducible, primitive matrix, whose dominant eigenvalue 0.8 is larger than that of the submatrix A_{22} (zero in this case). Hence, in Charasoff's well-behaved reducible case, the columns of the limit matrix are proportional to the positive column eigenvector p associated with the dominant eigenvalue 0.8, and the rows are proportional to the semipositive row eigenvector q associated with the same dominant eigenvalue 0.8

²⁰ The mathematical literature contains the following theorem. Consider a positive matrix A , with dominant eigenvalue λ , associated with a positive column eigenvector p and positive row eigenvector q . If $v \rightarrow \infty$, then the v th power of the matrix (A / λ) converges to the matrix $p q / q p$; see Horn & Johnson (2013: 526, 528). This is also valid if A is semipositive, irreducible and primitive (Bernstein, 2009: 299, result xxii). In the matrix $p q / q p$ every column is proportional to p and every row is proportional to q . See also Mori (2011: 79-80).

Maybe he wanted to concentrate on simple numerical examples to avoid problems for the non-mathematical readers. But why did Charasoff, who was not a modest character, never show at least one reference to Perron-Frobenius in a footnote? Mathematically competent readers would have found such a Perron-Frobenius footnote more relevant than (for example) Charasoff's footnote on the mathematics of the three-body problem in astronomy (Charasoff 1910: xi). Hence, we may assume that Charasoff was not aware of the usefulness of Perron-Frobenius. Actually, Charasoff claimed he had developed his economics on his own. The discovery of his Urkapital was first announced in a short appendix to chapter VIII of his first book (Charasoff 1909: 67-69), a book with preface dated 12 October 1908. In the preface of his second book (Charasoff 1910: xiv), dated 24 December 1909, Charasoff suggested that he had started reading Menger, Böhm-Bawerk and Walras only around 1905 or 1906, and that he had constructed his own system completely independently of them, at a much earlier date, when studying Marx. Note that memories in autobiographical stories are sometimes imperfect, as is obvious in some stories about Leontief in a later section of my paper.

3. Potron's Catholic use of Perron-Frobenius

Charasoff was not the only young mathematician who tried to impress David Hilbert. In the summer of 1902, Hilbert received five letters from a 30-year old French engineer, Maurice Potron, who expressed his plans to concentrate on pure mathematics in his first letter (13 July 1902):

As a graduate of the *École Polytechnique*, I have abandoned the career of an engineer [...] to devote myself to the teaching and the study of mathematics.²¹

When Hilbert suggested Potron to come to Göttingen, because helping by correspondence was difficult, Potron agreed in principle, but also signalled a practical problem (letter of 29 August 1902):

I am not completely free in my actions. This year and next year, I have to take courses in theology in England, in Canterbury.

Indeed, Potron was combining his activities in mathematics with the very long training program of a Jesuit (partly in Canterbury, to avoid problems with anticlericalism in France). Ultimately Potron would write a doctoral dissertation in Paris (Potron 1904), on finite group theory, with three eminent French mathematicians in his jury: Émile Picard (his supervisor), Paul Appell and Henri Poincaré. The dissertation and a few other mathematical publications by Potron are occasionally cited in the mathematical literature.²²

In the periods 1911-1914 and 1935-1942, Potron also published on economics, but only in mathematical or Roman Catholic networks, not in leading economic journals. These publications failed to make an impression during his lifetime, and were hardly noticed up until more than half a century after his death. The situation changed when Émeric Lendjel (2000) drew the attention of economists to Potron's status as a remarkable precursor of input-output analysis. This led to many other publications on Potron's economics, plus a French and an English edition of his economic texts.²³

Inspired by Roman Catholic social thought and by his knowledge of mathematics, Potron developed a highly original economic model that used the powerful Perron-Frobenius

²¹ Potron's five letters to Hilbert are conserved in the Nachlass David Hilbert, Cod. Ms. D. Hilbert 315, Göttingen. (I translated from French into English).

²² See Parys (2010) for an annotated Potron bibliography.

²³ See for example Abraham-Frois & Lendjel (2004, 2006), Bidard & Erreygers (2007, 2010), Mori (2008), Bidard, Erreygers & Parys (2006, 2009), etc.

mathematics of nonnegative matrices as early as 1911.²⁴ Potron's 1904 dissertation had used Frobenius' work on group theory, and this might explain why Potron also paid attention to Frobenius' publications on positive and nonnegative matrices. In the opening article of the April issue of *Le Mouvement Social* ('Revue Catholique Internationale'), Potron (1912a) stressed that the problems of just prices and just wages are interdependent, and therefore must be studied together in systems of equalities or inequalities. But are all the equalities or inequalities compatible? That was Potron's fundamental problem, and Perron-Frobenius mathematics offered the solution:

By a curious coincidence, to obtain its solution, I had to do nothing more than using and complementing the results, of a purely theoretical and abstract order, obtained recently (in 1908 and 1909) by two mathematicians, one Swedish, Mr Perron, the other German, Mr Frobenius. (Potron, 1912a: 291)²⁵

Note that, for many decades, this citation, and other important references to Perron-Frobenius in other Potron articles, failed to attract the attention they deserved.²⁶

In recent years, Potron's economic model has finally become well-known. I will give a short description of its core, with a minimum of variables.²⁷ To shorten my exposition, I assume

²⁴ The fundamental mathematical economic text is Potron (1913), but its results were already announced earlier (Potron 1911a, 1911b, 1911c, 1912a, 1912b).

²⁵ Translation by Bidard & Erreygers (2010: 82). The citation suggests that Potron had read Frobenius (1908, 1909), but not Perron (1907a, 1907b). I can't explain why Potron wrote that Perron was Swedish; Perron was a German, without any special private or professional connection to Sweden. Gårding (1998) never mentions Perron's name in his history of mathematics in Sweden, and Sweden doesn't figure in the Perron biographies by Heinhold (1980) and Frank (1982). An amateurish psychological explanation of Potron's lapse: maybe Potron in 1912 was sloppy and just assumed that Perron's given name, Oskar, connected him with Sweden, a country that had been ruled by Oscar I from 1844 till 1859, and by Oscar II from 1872 till 1907. In reality, there was no doubt about Perron being 'a real German', and therefore Perron was able to keep his chair in Munich and stay in the German academic networks under Nazism, although his intellectual integrity led to some difficult working conditions. When the German publishing company de Gruyter wanted to produce a third edition of Perron's successful book *Irrationalzahlen* during the war, it tried to delete all bibliographical references to 'Jewish' authors (Cantor, Kronecker, Minkowski, etc.), but Perron bravely kept on objecting to this practice, and the uncensored third edition appeared only after the war (Remmert & Schneider 2010: 239-240).

²⁶ In the social sciences literature I discovered only one reader who in 1912 immediately drew attention to the above Potron reference to Perron-Frobenius: an anonymous reviewer in the *Rivista Internazionale di Scienze Sociali e Discipline Ausiliarie*. This reviewer provided an interesting summary of Potron (1912a), emphasizing the interdependence of just wages and just prices, the problem of their compatibility, and the relevance of the mathematical results of 'due scrittori, l'uno svedese, il Terron [sic], l'altro tedesco, il Frobenine'. (Anonymous 1912: 69). I guess that this anonymous reviewer was either Filippo Ermini (a philologist who usually took care of the section that contained summaries of international journal articles) or Giuseppe Toniolo (an economist from the University of Pisa; he was the driving force behind the *Rivista Internazionale*, and, just like Potron, a supporter of the corporatist system that was proposed in the encyclical *Rerum Novarum* by Pope Leo XIII).

²⁷ My notations for vector inequalities are: $x \geq y$ means that the vector $x - y$ is nonnegative, $x > y$ means that $x - y$ is positive. The expression $x \geq y$ is equivalent to $x = y + b$, $b \geq 0$, where b is a nonnegative vector of slack variables. Potron uses many slack variables, but in my simplified representation I will omit them. I often replace the awkward expression 'nonnegative and nonzero' by *semipositive*.

that all matrices in the Potron model are semipositive and irreducible (Potron's analysis is more sophisticated and lengthier),²⁸ and that the dominant eigenvalue of A is smaller than one.²⁹

3.1. The economico-social state

Potron characterizes an *economico-social state* by the following constants (three matrices and one special number):

- the $n \times k$ matrix L of labour input coefficients ℓ_{ih}
- the $n \times n$ matrix A of commodity input coefficients a_{ij}
- the $k \times n$ matrix B of annual consumption coefficients b_{hj}
- the crucial number N (each labourer works at most N hours per year)

Potron's system of production assumes constant returns to scale, a production period of one year, circulating capital and heterogeneous labour. Production process i ($i = 1, 2, \dots, n$) produces one unit of commodity i by means of a_{ij} units of commodity j ($j = 1, 2, \dots, n$) and ℓ_{ih} ($h = 1, 2, \dots, k$) hours of labour of type h . The annual consumption baskets of the labourers depend on the social group they belong to: the annual basket for a worker of type h consists of b_{hj} units of commodity j ($j = 1, 2, \dots, n$).

3.2. Satisfactory regimes of production and labour

Let the number of workers in social group h be represented by x_h and the gross output of process i by y_i . These variables³⁰ are collected in the $1 \times k$ vector x and in the $1 \times n$ vector y . A so-called *satisfactory regime of production and labour* must satisfy two fundamental principles:

- *the principle of sufficient production*: $y \geq yA + xB$
- *the right to rest*: $Nx \geq yL$

²⁸ I borrow most of the notation and the mathematical content from Bidard, Erreygers & Parys (2009).

²⁹ Same assumption in Potron. Then $(I - A)^{-1}$ is positive, and the economy can produce a positive net output, which then must be divided between workers and 'non-workers'. Note that Potron's matrix A contains the technical inputs only. Potron collects the consumption coefficients of the workers in another matrix B .

³⁰ On the one hand, Potron treats L , A , B and N as constants, but on the other hand, he makes the strong assumption that x is a variable, which means that workers of any group can quickly be converted to any other group (and adopt the consumption basket of their new group).

On the one hand the gross output y must be sufficient for industrial and domestic consumption, but on the other hand we must respect the maximum number of N working hours per year for each type of labour. Are these two fundamental principles compatible? Can both inequalities be satisfied with positive y and x ?

Suppose we put $x = \frac{1}{N}yL$, then by substitution the principle of sufficient production means $y \geq y\left(A + \frac{1}{N}LB\right)$. The latter is an expression of the form $yQ \leq y$ where $Q = A + \frac{1}{N}LB$ is a semipositive irreducible matrix. Then Perron-Frobenius mathematics guarantees that such an inequality has a positive solution y if the dominant eigenvalue of the matrix Q is smaller than or equal to one.³¹ In the present context this is expressed by the following fundamental condition:

$$dom\left(A + \frac{1}{N}LB\right) \leq 1 \quad (1)$$

Potron also considers the following alternative method. The principle of sufficient production, $y \geq yA + xB$, leads to $y \geq xB(I - A)^{-1}$ and then substitution in $Nx \geq yL$ (right to rest) gives $Nx \geq xB(I - A)^{-1}L$. The latter is an expression of the form $xP \leq Nx$ where the matrix $P = B(I - A)^{-1}L$ is a semipositive irreducible matrix. Then Perron-Frobenius mathematics guarantees that such an inequality has a positive solution x if the dominant eigenvalue v of the matrix P is smaller than or equal to N :

$$v = dom[B(I - A)^{-1}L] \leq N \quad (2)$$

In Potron's terminology, the dominant eigenvalue v is called *the characteristic number of the economico-social state*.

In this way, Potron presents two equivalent 'dominant eigenvalue' conditions (1) and (2) for the existence of satisfactory regimes of production and labour. If the economico-social state, i.e. the set of the values of L , A , B and N , is 'too bad' for the existence of a satisfactory solution, then we must try to decrease some elements of the matrices L , A , or B (more efficient production or less consumption) or increase the maximum number of N working hours per year.

³¹ More generally, for a given scalar α and semipositive irreducible matrix M , there exists a positive row vector z that solves the vector inequality $zM \leq \alpha z$ if and only if $(dom)M \leq \alpha$. This result is applied to the matrices $Q = A + \frac{1}{N}LB$ and $P = B(I - A)^{-1}L$ in the present section. By means of a vector b of slack variables the inequality system $zM \leq \alpha z$, $z \geq 0$ can also be written $z(\alpha I - M) = b$, $z \geq 0$, $b \geq 0$. The latter formulation is closer to Potron, but requires more variables.

Intuitively, the characteristic number v describes the ‘average’ number of hours per year that a worker of an ‘average’ social group has to contribute to produce the annual consumption of the workers. If every worker performs N hours per year, and v is much smaller than N , then the surplus labour time can be used for surplus production that can support many ‘non-workers’, also called ‘simple consumers’ or ‘rentiers’ by Potron. Given Potron’s dislike of the class struggle, he would have been astonished if he had known that a century after his original analysis, some economists suddenly started to use parts of his ‘Roman Catholic’ model for Marxian theorems on surplus labour and exploitation in economies with heterogeneous labour.³²

3.3. Simply satisfactory regimes of prices and wages

Let p_i be the price of a unit of commodity i and w_h the hourly wage of a worker of type h . These variables are collected in the $n \times 1$ vector p and the $k \times 1$ vector w . A so-called *simply satisfactory regime of prices and wages* must satisfy two fundamental principles:

- *the principle of justice in exchange:* $p \geq Ap + Lw$

- *the right to life:* $Nw \geq Bp$

On the one hand, justice in exchange requires that the prices of the commodities must be high enough to cover their material input costs plus the wage costs. On the other hand, the right to life means that wages must be high enough to cover the price of the consumption goods. Are these two fundamental principles compatible? Can both vector inequalities be satisfied by positive p and w ? Potron applied Perron-Frobenius mathematics again, and established a remarkable case of duality.

3.4. Duality

Potron arrived at the same dominant eigenvalue conditions (1) or (2) as before. In other words, the existence conditions for satisfactory regimes of production and labour are exactly the same as the existence conditions for simply satisfactory regimes of prices and wages. In this way Potron (1911c, 1913) is the first who proved duality results between quantity systems and price systems for the general case of n commodities.³³

³² See for example Mori (2008).

³³ I described only the case of ‘simply satisfactory regimes of prices and wages’, which refers to the simple case where every worker performs exactly N hours per year. Potron is more general.

3.5. Anticipating the Hawkins-Simon conditions

Potron also thought about practical applications. He suggested that a *Bureau de Calculs* should collect the relevant coefficients of the matrices L , A and B . Was it possible to check whether the dominant eigenvalue of the matrix $Q = A + \frac{1}{N}LB$ is smaller than one, without computing the eigenvalue itself via an immense characteristic equation? Inspired by Frobenius' results on matrices and principal minors, Potron (1913: 62-63) developed the following criterion: it is necessary and sufficient that all principal minors of the matrix $I - Q$ are positive. Later, Potron (1937b, 1937c, 1939) formulated this fundamental criterion in an equivalent and more efficient way: it is necessary and sufficient that the *leading* principal minors of $I - Q$ are positive.³⁴

The Potron criterion anticipated the famous Hawkins-Simon conditions by many years (Hawkins & Simon 1949). Moreover, when David Hawkins and Herbert Simon published their often quoted *Econometrica* article in 1949, they were not familiar with Perron-Frobenius or Potron. After their article, jointly written by correspondence, without seeing each other, Hawkins and Simon concentrated on widely different topics. Hawkins was a philosopher of science, who had contributed to many different disciplines, including economics (Hawkins 1948, 1949). In a letter to Richard Trahair, dated 8 June 1993, Hawkins was very modest about the status of the 1949 Hawkins-Simon conditions: 'The theorem [...] was, it turned out later, first published long before by the mathematician Frobenius.'³⁵ Hawkins described himself as follows: 'I was an amateur in the field and have published very little else on economics.' It is obvious that Hawkins didn't know Potron.

3.6. Overlooking Potron

In his articles from 1911-1914 Potron immediately introduced a full-fledged open input-output system and the associated quantity and price equations for the general n-commodity case. Leontief introduced his open price and quantity system in the *Quarterly Journal of Economics* as late as the 1940s (Leontief 1944, 1946a, 1946b). Potron was still unknown. His

³⁴ Georgescu-Roegen (1950: 220; 1966: 336) presented this criterion as the economical version of Hawkins-Simon, but overlooked Potron. Note that Potron (1939) is listed in the remarkable bibliography of Woodbury (1954), without any further explanation.

³⁵ Trahair was working on an encyclopedia of eponymous terms in the social sciences, and had written to Hawkins about the Hawkins-Simon conditions. A copy of Hawkins' letter is in the David Hawkins Papers, Box 17, Folder 14 (University of Colorado at Boulder Libraries, Archives Dept.). I thank David M. Hays (Archivist) and his assistants for their efficient and friendly help in Boulder.

style of exposition was difficult, and his articles had appeared in non-economic journals that were overlooked by economists.

The 1914-1918 war stopped Potron's activities on economics. After 1918 he concentrated on his mathematical teachings and research. His scientific network consisted essentially of Jesuits and alumni of the *École Polytechnique*. In the 1930s, the latter group organized regular meetings at the *Centre Polytechnicien d'Études Économiques*. In this context, in 1935 Potron heard of recent work in *Econometrica* by Ragnar Frisch (1934a, 1934b), who used nonnegative matrices in his economic model, without Perron-Frobenius: 'Frisch drew on mathematical results which he proved as he went along, without checking up on the literature' (Bjerkholt & Knell 2006: 408n12).

Ironically, Frisch's former assistant Olav Reiersøl, trying to extend statistical results by Koopmans (1937), was the first to show the power of Frobenius' results to the econometricians, in his 1941 *Econometrica* paper on statistical confluence analysis. Reiersøl (1941: 8) commented on his discovery of Frobenius matrices: 'I vainly attempted to generalize the proof of Koopmans already mentioned until I happened to read a paper by Frobenius...' It is symptomatic that Reiersøl, a graduate in mathematics, hadn't picked up the relevant matrix algebra in his university courses, but only later: 'I hadn't learned anything about matrices in my studies of mathematics, so I had to read up this subject during my summer holidays in 1935' (Reiersøl 2000: 115). Note that Reiersøl's statistical confluence analysis is an econometric topic, unrelated to Potron's system or to Frisch's model of circulation planning of 1934.

When Potron heard about the Frisch matrix in 1935, he obviously recognized the link between Frisch and his own use of nonnegative matrices in 1911-1914. In this way, Potron again became interested in (highly individual) research on economics, and tried to spread his economic ideas in some new writings (Potron 1935, 1936, 1937a, 1937b, 1937c, 1939, 1942).

Although he was hampered by a speech defect, Potron organized a series of six lectures in Paris in 1937, to propagate his economic model. Two of these lectures were attended by François Divisia and René Roy, both alumni of the *École Polytechnique*, and important figures in the international community of mathematical economists (for example, in the *Econometric Society*). However, neither Divisia nor Roy ever disseminated Potron's message. In his book on the economic contributions of the French engineers, Divisia (1951) never mentions the name Potron.

One year before the 1937 lectures in Paris, Potron had made his only ‘international live appearance’ ever, at the *International Congress of Mathematicians* (Oslo, 13-18 July 1936). About five hundred mathematicians (plus two hundred members of family) attended this congress. Most of the time was devoted to plenary sessions or speeches, but more specialized subjects were treated in different special sections. According to Dunnington (1936: 91) about forty sectional sessions took place, papers were limited to 15 minutes (plus discussion), with the result that 227 papers were presented. Georgiadou (2004: 317) also mentions the 15 minutes rule, and adds that ‘summaries were to be published in the congress files and the manuscripts themselves had to be submitted before the end of the congress’.

In Oslo, Potron presented one paper on mathematical analysis, and one on his economic model. The economic paper (Potron 1936, 1937a) was located in Special Section IV, called *Probability Theory, Mathematical Statistics, Insurance Mathematics, and Econometrics*, organized by Ragnar Frisch and Birger Meidell. Only three papers treated economics: Potron on his linear system, Ragnar Frisch on price index comparisons, and Franz Alt on utility measurement. Arthur Bowley presented two papers on statistics, so did Emil Gumbel and Herman Wold. Émile Borel delivered one paper on games, William Feller on stochastic processes, Harald Cramér on the central limit theorem, Maurice Fréchet on probability, etc. All in all this was an impressive list of first rate scholars. Here Potron had his 15 minutes of potential fame. Both his unpublished Oslo manuscript (Potron 1936) and the summary published in the proceedings (Potron 1937a) are very difficult to read, and (as usual) Potron provided only a quick reference to Frobenius, and not a single reference to the mainstream economic literature. Neither his texts nor his oral presentation did convey the originality of his contribution, and he failed to make an impression.

Probably, not only Potron’s speech defect and his difficult style of writing were to blame. In addition, his situation was intrinsically difficult. Compare with George Dantzig in 1947, for example. Dantzig (partly inspired by Leontief’s input-output model) made the first public announcement of his discoveries on linear programming in the December 1947 meeting of the *American Statistical Association & Institute of Mathematical Statistics*. This happened in a session with three other speakers on other topics. Each speaker had a 20-minute time constraint, and in this situation Dantzig’s paper ‘was so sketchy that a listener who was not already familiar with the background would not be likely to perceive the importance and the difficulty of the problem whose solution was announced’ (Dorfman 1984: 292).

Franz Alt (1910-2011), participant in Karl Menger's famous Mathematical Colloquium in Vienna in the 1930s, was the longest survivor of 'Potron's 1936 Oslo session', and wrote in 2007:

I attended the Oslo International Congress of 1936 (as well as several other Congresses of the International Mathematical Union) and I do remember speaking there (and many times subsequently) with Ragnar Frisch and some other participants, but the name of Potron does not come to mind.³⁶

In the twentieth century, Potron remained unknown in the Anglo-Saxon networks. Paul Samuelson (1915-2009) was very clear on this:

I have an excellent memory and I know that I and my associates in Cambridge, Massachusetts, and at RAND Corporation in California, did not know anything about the name Potron.

We should not be surprised that different scholars, unknown to each other, should converge toward some common mathematical truths.³⁷

³⁶ Mail from Franz Alt to the author, 24 May 2007.

³⁷ Letter from Paul Samuelson to the author, 5 August 2009.

4. Bray on Cournot's equations of exchange

Probably the earliest use of simultaneous equations in economics appeared in the work of Achilles Nicolas Isnard (1781), hidden in the middle of a nondescript lay-out, nearly unnoticeable for readers who quickly browse through the nearly 700 pages of the *Traité des Richesses*. Both the notation and the mathematical content look primitive compared to the more professional and more general approach of modern writers. More sophisticated simultaneous equations appeared in Cournot (1838), in his Chapter Three, on computing the rates of exchange between the currencies of different countries. Here Cournot, probably for the first time in the history of mathematical economics, introduced the modern *double-index notation*, in his nonnegative coefficients m_{ik} (i and k running from 1 to r).³⁸ Cournot constructed a general system of r linear equations, but failed to use the general theory of determinants to solve it, and limited his solution to a system with three equations.³⁹

Nearly a century later, the general solution to Cournot's system was formulated by Hubert Evelyn Bray (1889-1978) in his article *Rates of Exchange* in the *American Mathematical Monthly* of November-December 1922. A few years earlier, the Rice Institute (today Rice University) in Houston had awarded its first Ph.D. in mathematics to Bray in 1918, on a dissertation *A Green's Theorem in Terms of Lebesgue Integrals*. Bray stayed at Rice throughout his whole career, mostly concentrating on pure mathematics. His 1922 paper on Cournot's problem was inspired by his mathematical colleague Griffith Conrad Evans, who had discussed these equations in a course at Rice from 1920 to 1921.⁴⁰

Bray was able to determine the unknown rates of exchange via a homogeneous system of n linear equations. Equation i ($i = 1, \dots, n$) described that for currency i total purchases must equal total sales. From a mathematical point of view, Bray's system of n currencies is similar to Remak's system of n commodities that I will consider in more detail in the next section. Although all his coefficients m_{ik} were nonnegative, Bray did not use Perron-Frobenius, and developed his own analysis of the special matrix of his system.⁴¹

³⁸ Cournot (1838) separated the two indices i and k by a comma. Cauchy (1815: 53-54) had introduced such a notation in his mathematical study of determinants. On Cauchy's determinants, see also Hawkins (2013: 86-93).

³⁹ It is very probable that Léon Walras studied the simultaneous equations of Isnard and Cournot at an early stage of his career (see Walras 1905, Jaffé 1969).

⁴⁰ See the first footnote of Bray's article (1922) and the preface of Evans' book (1930). For more details on Evans 'among the Econ', see Weintraub (1998).

⁴¹ Recently Maroscia (2008) studied Sraffa's Chapter 1 by means of Bray's method (thus without Perron-Frobenius).

Bray was first and foremost a mathematician, and received some international recognition, for example, by being elected a foreign member of the *Société Mathématique de France* in 1930. After his 1922 article on Cournot, he showed occasional interest in economics in 1937, when he participated (without a paper) in the *Third Annual Research Conference on Economics and Statistics*. This was a yearly meeting in Colorado Springs, organized by the *Cowles Commission*, less formal than the meetings of the *Econometric Society*. The list of speakers in 1937 included Bray's former colleague Griffith Conrad Evans (now at Berkeley), Karl Menger, Ragnar Frisch, Jacob Marschak, Harold Davis, Joseph Schumpeter, and others.⁴²

Bray's conference timing was not perfect. Perhaps the 1938 or the 1940 conferences would have been more interesting for him. In 1940 Leontief presented results of his input-output research, announcing his forthcoming 1941 book.⁴³ So Bray was 'too early' to meet Leontief's closed input-output system, whose algebra was mathematically similar to the Cournot-Bray equations of currency exchange. The same bad timing occurred with respect to the 1938 conference, which celebrated the centenary of Cournot's classic, without Bray's participation. In one of the Cournot sessions, René Roy drew special attention to Cournot's mathematical discussion of the equations of currency exchange:

This discussion is the first example of a synthesis which permits clear analysis of the nature and number of variables considered in the economic system, as well as of the relations which exist between them. In other words, Cournot here pointed the way to the founders of the theory of economic equilibrium, who try to show exactly the degrees of freedom of the economic system by comparing the number of variables with the number of relations which connect them. (Roy 1938: 70)

Roy didn't refer to Bray's work or the existence of positive solutions. In a postwar *Econometrica* paper on rates of exchanges and other problems, Roy (1946) mentioned only the three equations system of Evans' book, and tackled the case of n equations without referring to Bray's generalization of 1922.

⁴² All reports of these Cowles conferences are available at <http://cowles.econ.yale.edu/archive/reprints/index.htm>

⁴³ In the conference report, Leontief announced a forthcoming book titled *Structure of American Industries, 1919-1929*, which of course became his 1941 classic *The Structure of American Economy 1919-1929*.

In point of fact, in the first decades after 1922, Evans seemed to be one of the few who referred to Bray's paper.⁴⁴ Evans was first and foremost a mathematician (Ph. D. from Harvard), but he paid more intense and long standing attention to economics than Bray, and became one of the founding members of the *Econometric Society*. In his well-known textbook *Mathematical Introduction to Economics*, Evans (1930) included a reference to Bray's general treatment of Cournot's rates of exchange, in an exercise at the end of his chapter VII (On Rates of Exchange). But this chapter didn't attract much attention, and Henry Schultz (1931) in his long book review concentrated on other topics. However, note that Leontief was interested in Evans' book; in his letter of 5 January 1932 Leontief thanked Schultz for sending a copy of his review.⁴⁵ Even Piero Sraffa took notice of Evans' book. Sraffa's *Cambridge Pocket Diary 1930-1931* has many empty pages, and contains few bibliographical references (less than 10 for the whole year). But on Thursday 15 January 1931, Sraffa's diary makes a reference to 'Evans, Math. Introd. to Econ. McGraw-Hill' (see my section 6 for more details on Sraffa and Leontief).

⁴⁴ See also the short references to Bray in Debreu & Herstein (1953), Woodbury (1954), Morgenstern (1959), Gale (1960) and Dorfman (1973). A long discussion on equations of currency exchange is given by Lorenzen (1969). For many interesting historical remarks on Cournot-Bray and Walras on exchange, see Gilibert (1981).

⁴⁵ The letter is in the Wassily Leontief Papers, Harvard University Archives, Accession # 12255, Box 10, Ancient Correspondence 1930-32. Another folder (HUG 4517.45, Box 2, Folder Spring 1948 - Econ 4A) shows that Evans was one of the six books on the reading list of Leontief's course 'Introduction to the Mathematical Treatment of Economic Theory'. I thank the staff of the Harvard University Archives for their kind help in the Pusey Library.

5. Remak fails to normalize

Just like Potron in Paris, Remak in Berlin wrote a doctoral dissertation on finite group theory (Remak, 1911).⁴⁶ The choice of this abstract dissertation topic was clearly influenced by his supervisor, who was none other than Frobenius himself. In 1929 Remak used his mathematical abilities to study the existence of positive solutions in his linear system of ‘superposed prices’, without ever referring to the tailor made theorems of his former master Frobenius (who had died in 1917).

5.1. Eccentric and independent thinker

Unlike Frobenius, Remak enjoyed participating in discussions on economics. In Göttingen he even attended some undergraduate lectures in economics. His nonstandard dress and behaviour earned him the nickname ‘the communist’, and the young students didn’t like Remak’s attendance in the economics lectures. This caused some incidents in the lecture room, and Remak was ejected from the university in Göttingen. Not only the students, but also some professors (Hilbert for example) preferred seminars without the rather extrovert and critical interventions by Remak.⁴⁷

Remak’s academic career proceeded slower than normal for a mathematician of his capacities, just like the slow career of his father and his grand-father.⁴⁸ Because of Remak’s eccentric personality, his first two attempts to obtain his Habilitation failed, despite a sufficient number of mathematical publications in German top journals. A typical example of Remak’s despair is shown in a letter of 2 May 1925, from Remak to Hilbert, begging to end the problems caused by conflicts in Göttingen in 1918-20. Faculty rumours in Berlin gave Remak the impression that Hilbert’s influence had contributed to the first two rejections in Berlin. Remak couldn’t believe that the Göttingen troubles could have such grave consequences in Berlin: ‘You said to me in 1920 that you would put nothing in the way of my career at other

⁴⁶ The short book by de Séguier & Potron (1938) contains twenty-seven bibliographical references on group theory, including four by Remak (1911, 1913, 1930a, 1930b), but it is highly probable that Potron and Remak never knew anything about each other’s economics.

⁴⁷ Numerous stories on Remak’s unusual behaviour exist. See for example the memories by Pinl (1969), Behnke (1978) and Fenchel (1980). More bibliographical references can be found in Merzbach (1992), Hagemann & Punzo (2007) and Siegmund-Schultze (2009). Note that Behnke and Fenchel attended the Oslo conference in 1936, just like Potron.

⁴⁸ The Jewish Remak family in Berlin generated four well-known intellectuals. The brother of the mathematician Robert Remak (1888-1942) was Georg Remak (1890-1979), a lawyer, and senior federal prosecutor at the end of his career. Their father Ernst Julius Remak (1849-1911) and especially their grandfather Robert Remak (1815-1865) were famous neurologists, and are still mentioned today in medical journals. See for example the article on the two Remak neurologists, published in *The Lancet* by J.M.S. Pearce (1996).

universities.’ Remak also wrote he didn’t believe Hilbert approved of the ‘year-long persecutions’ (‘Verfolgungen’) that he suffered in the academic world, and he asked Hilbert a letter attesting the qualities of his scientific work.⁴⁹ Only on 11 January 1929 Remak finally obtained his Habilitation in Berlin (Biermann, 1988: 210).

Four years later the Nazi government withdrew his right to give lectures, and even publishing papers became more difficult. For example, in 1938 Remak submitted two papers to the *Journal für die reine und angewandte Mathematik*. The editor Helmut Hasse suffered from a shortage of strong contributions, and should normally have accepted the very valuable papers from Remak. But in October 1938 he didn’t dare to publish papers by ‘Jewish’ authors (Remmert & Schneider 2010: 238-239)

Remak was able to move to Amsterdam in 1939. His lack of emotional intelligence worried some mathematical colleagues in Holland. Hans Freudenthal, another migrant from Germany to Holland, received a letter from L.E.J. Brouwer,⁵⁰ dated 2 March 1940, warning him that Remak was in danger to be expelled by the Dutch government, on the grounds of bad behaviour, a few months *before* Holland was occupied by the German troops. Some even claimed that the only way to save Remak’s life was to put him in a psychiatric hospital (van Dalen 2013: 656-657). Such a difficult person had a less than average chance of surviving after the German occupation of Holland in May 1940. Remak died on transport to Auschwitz on 13 November 1942 (Vogt 1998: 342).

According to Merzbach (1992: 500), Remak had visited Louis Joel Mordell in Manchester in 1939, and was to go to Cambridge in October 1940 for a little while. However, Remak was trapped in Holland when the German troops occupied that country in May 1940. Normally, a mathematician with his set of publications could have obtained an excellent position in an Anglo-Saxon top university, if he had possessed average social skills. Suppose Remak had been able to settle in Cambridge, and to enter into contact there with some other Jewish scholars who had left their dictatorial home countries: Besicovitch, Wittgenstein, Sraffa ...

⁴⁹ David Hilbert Nachlass - Cod. Ms. D. Hilbert 326, Göttingen. (I translated from German into English).

⁵⁰ The Dutch mathematician Luitzen Egbertus Jan (‘Bertus’) Brouwer (1881-1966), the founder of intuitionism, is often cited in economics with reference to his fixed-point theorem. Hans Freudenthal (1905-1990) had already left Germany in 1930, to become Brouwer’s assistant in Amsterdam. Unlike many other Jewish inhabitants of Amsterdam, Freudenthal was able to survive the German occupation.

5.2. Remak's 1918 paper on the national debt

On 23 May 1918 Remak published what is probably his first economic paper, *Vorschläge zur Tilgung der Reichsschuld* (Proposals for Repayment of the National Debt), in the *Europäische Staats- und Wirtschaftszeitung*. This journal was founded in Berlin in 1916, and its weekly issues contained many contributions on political and economic problems, written by among others Edgar Jaffé (the editor), Carl Ballod, Eduard Bernstein, Lujo Brentano, Karl Diehl, Robert René Kuczynski, Adolf Löwe, Otto Neurath, Franz Oppenheimer, Max Weber, etc.

Remak provided only one contribution, and its only bibliographical reference was a comprehensive book on wealth taxation, written by the historian and economist Ignaz Jastrow (1917). Jastrow's proposals for a wealth tax also drew the attention of his American colleague Frank W. Taussig. Both Jastrow and Taussig had shown a common interest in the building of business schools at the academic level, like the *Handelshochschule* in Berlin (Jastrow) and the *Harvard Business School* (Taussig).⁵¹ After the war, Jastrow (1920, 1923) wrote two articles in the *Quarterly Journal of Economics* (Taussig was its editor), in which he compared the proposals of his 1917 book with the actual capital levy tax in Germany.

Remak's 1918 article had no influence on economists in Berlin or elsewhere. He probably published the article because of his wide-ranging political and social interests, including public and private finance. Born in a wealthy family, originally Remak's private capital was sufficient to live the life of a 'private scholar', just like Charasoff and Potron, but due to the German hyperinflation in 1923 he was obliged to take a part time job at the *Deutsche Bank* (Vogt, 1998: 337). Perhaps, this led to a further improvement of Remak's practical and theoretical knowledge of finance, and explains the unexpected success of his introductory lectures on insurance mathematics, in the winter semester of 1931-32. These lectures earned Remak more than a thousand Reichsmark of tuition fees, more than ten times more than his earlier lectures on group theory (Vogt 2008: 414-415).

I draw attention to the 1918 article, because I have never seen it in other Remak studies, but its topic has few connections to Remak's mathematical economic model of 1929. With some goodwill, the first paragraphs of the 1918 paper can be used to illustrate a dated input series or the working of the multiplier. Further in the text, Remak (1918: 389) asserted that 'the state and the empire have the obligation to shorten and to cheapen the way from producers to consumers'. Such a precept can also be considered one of the principles of Remak's well-

⁵¹ See Redlich (1957).

known system of 1929, which contained many critical remarks against the high mark ups by merchants, because they caused non ideal prices and depressed aggregate consumption.

5.3. The Remak price system

In his short 1933 note, Remak emphasized the following:

A price does not emerge from supply and demand, it is rather a number which has to satisfy certain conditions. The price of a commodity must cover the prices of the expenses contained in it including the cost of living, which may be taken to be known, of the people participating in its production. This leads to the superposed price systems.⁵²

This unorthodox statement showed a striking resemblance to the following by Maurice Potron two years later:

...the prices, the wages, [...] are not imposed by any *physical* necessity. They result from *conventions* which, at least theoretically, are *free*.⁵³

As is well explained by Hagemann & Punzo (2007), one source of inspiration for Remak's 1929 article was provided by the debates on capitalism and socialism, and on the calculation of rational prices in socialism. Remak seems to sustain the optimistic idea that his system of superposed prices provides an 'exact' criterion to judge the *extremal* character of an economic system. In modern parlance, Remak's notion of an extremal economy refers to a system with Pareto efficient output, where it is impossible to increase the output of any good, except by decreasing the output of at least one other.

The link between Remak's superposed prices and the extremal output quantities was not clearly visible; there was no mathematics of the quantity system. Remak presented verbal remarks on crises, regional differences, Taylor's system of scientific management, birth control and other demographic measures, heterogeneous labour, differences in the cost of living between city and countryside, profit sharing, high profits, etc. But he didn't offer a formal treatment of these subjects. The mathematical part of his 1929 paper was limited to the price system of an economy with no surplus (no profits), analogous to the first pages of Sraffa's 1960 book.

⁵² Remak (1933: 840), as translated by Kurz & Salvadori (1995: 398n19).

⁵³ Potron (1935: 63), as translated by Bidard & Erreygers (2010: 169).

In Remak's system there are n sectors,⁵⁴ each producing a single output. Sector μ produces commodity μ . To this end sector μ buys $t_{\mu\lambda}$ units of commodity λ from sector λ ($\mu, \lambda = 1, 2, \dots, n$). Remak's 'superposed' prices x_1, x_2, \dots, x_n require that for each sector μ the value of its inputs equals the value of its output (total purchases = total sales):

$$t_{\mu 1}x_1 + t_{\mu 2}x_2 + \dots + t_{\mu n}x_n = x_\mu(t_{1\mu} + t_{2\mu} + \dots + t_{n\mu}) \quad \mu = 1, 2, \dots, n \quad (3)$$

In this way, Remak obtains a system of n homogeneous equations with n unknowns x_1, x_2, \dots, x_n . In an obvious matrix notation, I can write it as $Tx = Dx$, or $(D - T)x = 0$, where D is a diagonal matrix with the μ th main diagonal element equal to $t_{1\mu} + t_{2\mu} + \dots + t_{n\mu}$. In general, $(D - T)x = 0$ is not an eigenvalue system. At this point, Remak fails to normalize the units of measurement.⁵⁵ By a clever choice of units, he could have put the gross output of each sector equal to one, and then he could have obtained the well-known simple form $Ap = p$, by putting $p = Dx$ and $A = TD^{-1}$. Observe that p is an eigenvector of A , and that A is stochastic (every column sums to 1).⁵⁶

Because the determinant of $D - T$ is zero, Remak (1929: 726) knows that his homogeneous system has at least one nonzero solution. But only nonnegative solutions are acceptable: 'on this subject the familiar theorem on determinants is silent' (Remak 1929: 726). He should have added: 'But in the final years of his life, my late Professor Frobenius provided us with the matrix tools to obtain a quick answer to this problem'.

Perhaps Remak overlooked Perron-Frobenius, because he failed to normalize the units of measurement for the different commodities, and thus he considered no eigenvalue system. And it is unclear to what extent Remak knew the details of Frobenius 1912 paper; Remak finished his dissertation in 1911, Frobenius died in 1917, and in his otherwise positive Habilitation report on Remak, Issai Schur wrote: 'Despite the wide range of his interests, Herr Remak has only a relatively small knowledge of the literature' (Vogt 1998: 338).

If, on the contrary, he was aware of the relevance of the Frobenius eigenvalue systems, then Remak paid insufficient attention to the normal scholarly standards: without ever referring to Frobenius, Remak presented a time consuming proof of his own, using the above system (3)

⁵⁴ At the start of his argument, Remak assumes a system with n different individuals, each producing a different commodity, but we can also interpret his equations as a system with n different sectors.

⁵⁵ Such a normalization is a routine operation in modern economics; see for example the Remak system in Gale (1960: 261) or Kurz & Salvadori (1995: 398).

⁵⁶ Another eigenvalue system is $Fx = x$, with $F = D^{-1}T$, but in general F is not stochastic.

of n homogeneous equations. Remak first noticed that it had a zero determinant, thus a nonzero solution vector x . Then he assumed for a moment that not all elements of x have the same sign, and he partitioned the economic system into three subsets: goods with a positive price, goods with a negative price and goods with a zero price. Just as for an individual industry, the purchases of any subset of industries must equal its sales. By applying this principle patiently, Remak then shows that it is always possible to construct at least one semipositive price vector x , and then he investigates how many such solutions exist, by looking at the pattern of interdependence between the industries.

The best way to visualize this interindustrial interdependence is to use Frobenius' theory of reducible matrices. A typical strategy for analysing a reducible input matrix A is to partition it in a *lower triangular block form*, already used by Frobenius (1912: §11), and therefore often called a *Frobenius normal form*:

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

where all the submatrices A_{ii} on the main diagonal are square and irreducible.⁵⁷

The logic of Remak's problem led him to a somewhat similar procedure, but without employing matrices. In his lengthy argument he used the following terminology and results. When sector μ supplies commodities (directly or indirectly) to sector λ , and vice versa, then Remak said that μ and λ are equivalent indices, and he partitioned the set of n sectors into m different subsets, which he called 'groups of equivalent indices'. Such a partition corresponds to the m irreducible submatrices of the Frobenius normal form above. In Remak's terminology, a group K is higher than a group Λ if there is a (direct or indirect) delivery of commodities from Λ to K . Group K is a *highest group* if there exists no other group that is higher than K . There is at least one highest group. Several groups can be highest, but only if there is no connection between them. By a patient study of all possible outcomes, Remak arrived at the following results (my notation):

⁵⁷ Some of these submatrices on the main diagonal can be 1 x 1 matrices (zero or nonzero). I follow the same tradition as Hawkins (2008: 683-686) and Horn & Johnson (2013: 402), and I call every 1 x 1 matrix irreducible. Such a 1 x 1 matrix on the main diagonal corresponds to a group of one industry. If such a 'singleton' does not deliver its output to other groups, it must be included in the list of 'highest' groups, in order to retain Remak's results on the number of linearly independent semipositive price vectors. Remak never explicitly mentioned such groups of one industry. Paragraph 5.5 below, on 'extreme beans systems', presents examples where beans form a 'singleton' that is highest.

Fundamental solutions: Each highest group h generates a semipositive $n \times 1$ price vector p^h , with the following properties:

$$p_j^h > 0 \text{ if commodity } j \text{ belongs to group } h$$

$$p_j^h = 0 \text{ otherwise}$$

Remak called this a *fundamental solution* ('Grundlösung') of his price system. He also proved that a fundamental solution generated by group h is unique up to a scalar multiple.

All solutions: All semipositive price vectors are linear combinations of fundamental solutions. If there are k highest groups, there are k linearly independent semipositive price vectors. Positive price vectors exist if and only if all m groups are highest.

Note that there always is at least one highest group, thus at least one semipositive solution.

In modern parlance, Remak spent nearly all his mathematical efforts on sophisticated reducible matrices, but then he offered a numerical example involving only a very simple positive (and thus irreducible) matrix. This example (Remak 1929: 731-733) studied a system of three individuals, each specializing in the production of one product. Producer 1 produces 27 tons of good 1, with price per ton equal to x_1 . For his production process or for his own consumption, he needs 3 units of his own product 1, plus 5 units of product 2, plus 7 units of product 3. He sells 3 units of his output to himself, plus 11 units to producer 2, plus 13 units to producer 3. The first equation below states that the value of the inputs of producer 1 must be equal to the value of his output. Similar explanations hold for producers 2 and 3:

$$3x_1 + 5x_2 + 7x_3 = x_1(3 + 11 + 13)$$

$$11x_1 + 2x_2 + x_3 = x_2(5 + 2 + 7)$$

$$13x_1 + 7x_2 + 6x_3 = x_3(7 + 1 + 6)$$

All the input coefficients are positive, which means that every sector directly delivers to every sector. In this extreme case, the only 'group of equivalent indices' is $\{1, 2, 3\}$, the set of all indices itself, a trivial case with one highest group, thus one fundamental solution. After slow calculations and controls, Remak found the superposed price vector $x = [89 \quad 101 \quad 233]^t$, which is unique up to a scalar multiple.

For theoretical purposes, it is better to normalize the units of measurement. The gross output of good 1 is 27; let us call it 27 tons of iron. Then introduce a new unit of measurement for iron: 1 unit of iron = 27 tons, with price $p_1 = 27x_1$. In a similar way, I normalize the units for goods 2 and 3, with $p_2 = 14x_2$ and $p_3 = 14x_3$. Such a normalization of gross outputs is a routine one in economics, but Remak didn't think of it in his mathematical proofs. The above normalization would have transformed Remak's example into the following

$$\frac{3}{27}p_1 + \frac{5}{14}p_2 + \frac{7}{14}p_3 = p_1$$

$$\frac{11}{27}p_1 + \frac{2}{14}p_2 + \frac{1}{14}p_3 = p_2$$

$$\frac{13}{27}p_1 + \frac{7}{14}p_2 + \frac{6}{14}p_3 = p_3$$

This is a simple Perron-Frobenius system of the form $Ap = p$, with a stochastic matrix A (every column sums to 1). We find a positive solution $p = [2403 \quad 1414 \quad 3262]^t$, which is unique up to a scalar multiple. Observe that Remak's numerical example was a poor illustration of his theory, because it used a positive matrix A , whereas Remak's mathematical proofs allowed for some zero input coefficients and some zero prices.

5.4. The David Gale example

The 7 by 7 matrix example by Gale (1960: 267-271) can be used to give more insight into the general structure of Remak's theorems. After renumbering the seven commodities, I obtain the following input matrix in Frobenius normal form:

$$\left[\begin{array}{cc|ccc|cc} 0.2 & 0.5 & | & 0 & 0 & 0 & | & 0 & 0 \\ 0.5 & 0.3 & | & 0 & 0 & 0 & | & 0 & 0 \\ \hline 0.1 & 0 & | & 0.2 & 0.3 & 0 & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 0.4 & | & 0 & 0 \\ 0 & 0 & | & 0.8 & 0.7 & 0.6 & | & 0 & 0 \\ \hline 0.2 & 0 & | & 0 & 0 & 0 & | & 0.5 & 0.9 \\ 0 & 0.2 & | & 0 & 0 & 0 & | & 0.5 & 0.1 \end{array} \right]$$

The system is partitioned into three groups, $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5\}$, and $G_3 = \{6, 7\}$. The groups G_2 and G_3 are the two highest groups, generating the two fundamental solutions $p^2 = [0 \quad 0 \quad 15 \quad 40 \quad 100 \quad 0 \quad 0]^t$ and $p^3 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 18 \quad 10]^t$. All semipositive price vectors are linear combinations of the vectors p^2 and p^3 . No positive price vector exists, because not all groups are highest.

Gale presented this example in a slightly different setting, in a linear model of international trade between seven countries. Neither Gale nor Remak raised an alarm about the following point. In all superposed price vectors the first two components are zero. The first two goods never played a role in Remak's fundamental solutions. In Gale's approach G_2 and G_3 were called 'irreducible subsets', while G_1 seemed to be treated as less important; it was relegated to a less important part of Gale's matrix. Gale himself didn't use the above Frobenius normal form, his terminology and lay out were different from Frobenius or from the current Sraffian literature, and his book never referred to Frobenius.

5.5. Extreme Beans systems

Readers of Sraffa (1960) will object to the bizarre hierarchy of the seven commodities in the Remak-Gale approach.⁵⁸ In Sraffa's terminology, goods 1 and 2 above are not negligible, quite the contrary, they are the two basic commodities of the economic system, because they are directly or indirectly necessary for the production of every commodity. The five other goods are Sraffa's nonbasic commodities. Gale's example has the same status as my following *Extreme Beans* example, which is a sort of extreme version of Sraffa's Appendix B on self-reproducing beans:

0.5 units of corn + 0 units of beans → 1 unit of corn

0.5 units of corn + 1 unit of beans → 1 unit of beans

Note that in the Remak story we consider an augmented input matrix, and thus the above example implies that workers eat only corn. The Remak and Gale solutions present a zero price for corn (basic) and a positive price for beans (nonbasic). But will we ever observe such a bizarre system? Not in Sraffa's world context of uniform rates of profit. The corn producers, or more generally, the basic sectors, form the substance of the economic system, producing the important commodities. A country with such a bad technology for beans will concentrate on corn production only. The corn producers will earn a nice profit rate of 100%, and none of them would invest in beans production. If beans are produced, we live in a bizarre country where corn and labour are worthless (price of corn and thus also the wages are zero).

⁵⁸ See also Newman (1962), Zaghini (1967), Bharadwaj (1970) and Opocher (2006).

In the Remak-Gale approach, even nonbasics have zero prices if they are not ‘highest’. Consider the following system:

$$0.5 \text{ corn} + 0 \text{ horses} + 0 \text{ beans} \rightarrow 1 \text{ corn}$$

$$0.2 \text{ corn} + 0.5 \text{ horses} + 0 \text{ beans} \rightarrow 1 \text{ horse}$$

$$0.3 \text{ corn} + 0.5 \text{ horses} + 1 \text{ bean} \rightarrow 1 \text{ bean}$$

Here the Remak-Gale solution gives zero prices for corn (basic) and horses (nonbasic, but not ‘highest’), and a positive price for beans only.

Remak spent most of his mathematical efforts on such systems, and the zero prices for various goods.⁵⁹ His explanation of the relevance of his results was poor. In his two articles Remak (1929: 734; 1933: 839) seemed to draw parallels between the zero prices in his system and the very low prices that countries with very weak currencies receive, when they exchange their products with strong countries that own large stocks of gold and silver.

5.6. Remak’s visibility in economics

Unlike Charasoff and Potron in earlier years, Remak presented his system in a journal that had wide visibility among economists, the prestigious *Jahrbücher für Nationalökonomie und Statistik*. Moreover, four years later he could add a few pages in the same journal, on the practical computability of his price system.

Even though it appeared in an economic journal, Remak’s 1929 article was also reviewed in the *Jahrbuch über die Fortschritte der Mathematik*. The review was a neutral summary, written by Erika Pannwitz (1929), who didn’t mention the missing link with Perron-Frobenius. At that time, Pannwitz wrote her doctoral dissertation in Berlin (Ph.D. in 1931). Many years later, Waldemar Wittmann consulted her on the origin of Remak’s paper. Pannwitz suggested that economists from the network around Ladislaus von Bortkiewicz had contacted Remak, about the existence of positive solutions in linear price systems. Remak became interested, and then wrote the 1929 paper (Wittmann 1967: 401). It is possible that Remak’s article provided a challenge for John von Neumann to start the construction of his own more general model of growth (Kurz & Salvadori 1995: 412-414). Remak’s results were

⁵⁹ Potron (1913: §15) devoted one long footnote to ‘partially reducible matrices’ with dominant eigenvalue equal to one, but spent the rest of his mathematical efforts on more relevant systems.

also discussed by other colleagues of the *Mathematische Institut* in Berlin, and some ridiculed his economic proposals (Wittmann 1967: 407).⁶⁰

Hans Freudenthal recommended Remak's article to Tjalling Koopmans (1951: 33), David Gale (1960) included Remak's model in his well-known textbook, Peter Newman (1962) referred to it in his widely cited review of Sraffa's book, and Baumol & Goldfeld (1968) provided a partial English translation of Remak (1929). Unlike Charasoff and Potron, Remak was not in need of a 'rediscovery'. Some connoisseurs knew his work from the beginning, for example Emil Gumbel, who showed wide interests in mathematical statistics, politics, economics and many other topics, and Wassily Leontief (see next section).

An important letter from Remak to Gumbel was recently published by Annette Vogt (2008). The letter is dated 20 April 1931. About two months earlier, on 19 February 1931, Gumbel had acted as the doctoral promoter in the oral defence of Klimpt's dissertation on Bortkiewicz (including comments on Charasoff: see my section 2). Gumbel must have liked this sort of topics, because he had told Remak about his plans to encourage his doctoral students in Heidelberg to work on the Remak system. Remak's letter of 20 April 1931 thanked Gumbel for his interest in the 1929 paper, and then provided some additional comments on it.

Just like his article, Remak's letter emphasized the problem of distortions in the actual prices due to exaggerated mark ups and profits.⁶¹ It would take many statisticians and mathematicians to construct and solve the necessary equations and compute Remak's superposed prices. Remak (1929: 735) hoped that improved electrical circuits would help to solve the equations somewhere in the future. In his letter to Gumbel, he thought about the day when this can be done for the first time: 'it will be a big event for us, somewhat like the breakthrough in the middle of the Simplon tunnel' (Remak 1931). Two years later, Remak still seemed optimistic; earth scientists can perform even more extensive geodetic computations (Remak 1933: 841). In economics a lot of data collection is necessary, but

⁶⁰ My paper doesn't discuss Bortkiewicz. His important publications on Marx were already finished before the Perron-Frobenius story started. Von Neumann (1937) also falls outside the scope of my paper. Note that the existence theorems in von Neumann's very general dynamic model, with joint production and choice of technique, needed stronger tools than Frobenius' theorems of 1912.

⁶¹ Remak's expressions like 'die Kaufmännische Zuschläge' literally mean 'the merchant's mark ups', but he also thought of the high rates of profit. Markups or profits depress the purchasing power of the workers, and lead to a sort of underconsumption and depression. I doubt whether Remak saw the following problem: suppose equal value composition of capital and uniform rates of profit exist in all sectors, then *relative* prices would not be distorted by high or low rates of profit (of course, the income distribution would be different).

Remak hoped that this problem could be treated by other scholars who were closer than he to practical statistics (Remak 1933: 842).⁶²

Both Gumbel and Remak shared similar political outlooks; both were expelled from the German academic networks in the 1930s. Gumbel first fled to France, and later to the U.S., where he became a well-respected statistician, especially after the publication of his classic *Statistics of Extremes* (Gumbel 1958). Remak did not survive the transport to Auschwitz in 1942.

⁶² Remak and Potron both wanted statistical help for the practical computations of the input-output data. Potron (1914: 168) was optimistically comparing this task with the computation of ten volumes of logarithmic tables. In the 1930s Potron was more pessimistic, because of the massive amount of numbers required (Bidard, Erreygers & Parys 2009: 143-148).

6. The two eponyms of this age of Leontief and Sraffa

A few additional comments are in order about the two eponyms of ‘Leontief-Sraffa models’ or ‘this age of Leontief and Sraffa’.⁶³ Both Leontief and Sraffa started their pioneering work on linear economics in the late 1920s, and both preferred to do so in isolation, but sometimes they tried to benefit from discussions with competent mathematicians.⁶⁴

6.1. Leontief and Remak

In his article on Leontief’s Nobel Prize, Robert Dorfman (1973: 431) wrote:

I still remember Leontief’s gleeful excitement when he came across the work of Remak, who proposed a theoretical input-output formulation of an economy seven years before Leontief’s earliest paper on the subject. A mathematician, H. E. Bray, had written in similar vein seven years before that.

Dorfman apparently had no access to Leontief’s 1928 dissertation, published one year before Remak’s paper. More corrections to Dorfman’s story are necessary in the light of archival evidence. Leontief’s letters to Adolf Löwe (3 June 1931) and to Joseph Schumpeter (7 June 1931) provided some news about the economists Leontief had visited while being back in Germany. Leontief also had met Emil Gumbel, and mentioned that discussions with Gumbel had helped to clarify some mathematical questions. Moreover, Gumbel had referred Leontief to Remak’s work, and a few days later (11 June 1931) Leontief wrote to Remak himself:

In a recent conversation Prof. Gumbel signaled me that you turned your mind, from the mathematical point of view, to the same problems of mathematical political economy that I have arrived at from the economic point of view: determinant analysis of the economic systems of equations, etc. I would enjoy it very much if I had the, alas so rare, opportunity to discuss all these questions and to see you occasionally. I stay in Berlin till the end of August and would be very grateful for your answer.⁶⁵

It is unclear what happened next. Anyway, the Dorfman story of Leontief’s ‘gleeful excitement’ about Remak reminds us of the stories of Leontief’s ‘amusement’ about his exact

⁶³ This expression was coined by Samuelson (1971: 400).

⁶⁴ After visiting Harvard in 1933, Richard Kahn wrote to Joan Robinson that Leontief was really brilliant and doomed to isolation (Rosselli 2005: 265). In an interview of 5 April 1997 by DeBresson (2004: 146), Leontief said: ‘I think very separately from my friends and colleagues’. His wife added: ‘When Wassily’s mind is at work, he puts a wall around himself’.

⁶⁵ Wassily Leontief Papers, Harvard University Archives, Accession 12255, Box 10, Folder: Ancient Correspondence 1930-1932 (I translated from German into English).

year and place of birth. The latter story appeared in many recent papers. For example, Baumol & ten Raa (2009: 511) wrote: ‘His date of birth was presumed to be 5 August 1906, but after the collapse of the Soviet Union Leontief apparently first found out, and reported with much amusement, that it was precisely one year earlier, in 1905.’

In his very early days, Leontief indeed thought that he was born in St. Petersburg in 1906, but the Harvard archives contain ample evidence that he was in Munich in 1905, and that Leontief knew the exact details many decades before the fall of the Soviet Union. In a letter to the Local Board No. 68 in Berkeley, dated 14 June 1943, Leontief wrote that his parents, who were students in Munich in 1905, had just informed him about his exact year and place of birth. Leontief enclosed a correct birth certificate in the letter (the Harvard file contains a certificate dated October 1932).⁶⁶

6.2. Keynes rejects Leontief

Schumpeter at Harvard thought very highly of Leontief. In a letter of 3 December 1932 to Keynes, editor of the *Economic Journal*, Schumpeter announced that Leontief ‘will furnish a very interesting piece of work during the Christmas recess. It would make an article of about fifteen pages. Would you care to have it for the Journal?’ After four months of waiting, Keynes must have sent Leontief a card to ask him about the article, because Leontief (in a letter of 12 April 1933) thanked Keynes for sending such a card, and for remembering his article, and of course Leontief apologized for the delay. So did Schumpeter who gave more details in a letter to Keynes (19 April 1933): ‘I want to apologize for Leontief. He has indeed produced what I consider a most original and interesting piece of work but then instead of finishing the article he got married in spite of my disapproval of this step. This explains the delay.’

Wesley Mitchell (letter to Leontief, 27 April 1933) had heard from Simon Kuznets that Leontief expected ‘to have an article on the Theory of International Trade in the next number of the *Economic Journal*’. In his reply to Mitchell (28 April 1933), Leontief explained that the relevant paper was not ‘his small note on international trade’, which he called ‘a rather unimportant by-product of my lectures’.⁶⁷ Leontief wanted to impress the readers of the *Economic Journal* (then the top journal for economics) with a much more fundamental item.

⁶⁶ Wassily Leontief Papers, Harvard University Archives, HUG 4517.6, Box 1, Folder: Misc., Subfolder: Various Important Documents.

⁶⁷ The small note would appear in the next issue of the *Quarterly Journal of Economics* (Leontief 1933).

He wrote Mitchell that the relevant item was ‘a condensed article for the *Economic Journal* dealing with the theoretical scheme on which the statistical part of the analysis of my work for the National Bureau is based’

Given Leontief’s research reports in that period, given the nature of his 1928 doctoral dissertation in Berlin, given his letter to Remak of 11 June 1931 (see Section 6.1 above), this article must have been Leontief’s first attempt to present a sort of theoretical input-output model to the Anglo-Saxon readers. The paper was finally submitted to Keynes, accompanied by a letter of 2 June 1933: ‘I am sending you under separate cover the manuscript of my article *Economic Changes and General Equilibrium*’.

Instead of appearing in the next issue of the *Economic Journal*, Leontief’s paper was rejected within a few weeks. In his letter to Keynes, dated 25 July 1933, Leontief was very polite: ‘I have received my manuscript and I thank you for the letter, which accompanied it. I am sorry the paper did not meet your approval, still I feel that you owe me no apology for this’.⁶⁸

It is possible that part of the content of his rejected and lost manuscript of 1933 was used in his 1937 article in the *Review of Economic Statistics*, which is the first paper that presented the algebraic equations of Leontief’s closed input-output system.⁶⁹ Contrary to Remak in 1929, Leontief (1937) didn’t discuss the positivity of the solutions of his equations, but he presented many comparative static exercises, showing the effect of changes in some input coefficients. Maybe Keynes rejected the rather abstract paper because it was too far outside the Marshallian mainstream, or because it contained too many equations, presented in a rather long-winded manner, or too many partial derivatives and determinants?

Recent research by Bjerkholt (2013), using plenty of unpublished material from Frisch’s editorial files, revealed that Leontief had submitted a paper with the same title in November 1933 to Frisch for *Econometrica*. The Leontief paper itself is still lost, but Bjerkholt found two referee reports by Frisch himself, i.e. one long letter on Leontief’s November 1933 version and one letter on Leontief’s revised January 1934 version. From these comments by Frisch, it seems that Leontief initially tried to include a lot of nonlinearities in his circular

⁶⁸ The two quoted Schumpeter letters (and many others) were published in chronological order in Schumpeter (2000). The letters from Leontief to Keynes are in the Harvard University Archives, HUG 4517.5, Box 1, Folder: General Correspondence 1932-1941. At Harvard, I couldn’t find Leontief’s manuscript and Keynes’s letter of rejection.

⁶⁹ The famous paper of one year earlier (Leontief 1936) contained no equations, only empirical results. Note that Leontief’s mathematics didn’t concentrate on the existence of positive solutions. Frobenius’ theorem appeared in Leontief’s publications for the first time in 1961, in a study on the stability of dynamic systems (Leontief 1961: 660).

model, and his use of differential calculus seemed rather messy, which Frisch tried to improve upon, by making many suggestions for changes. Leontief felt unable to publish ‘Frisch’s version of his article’ and that finished it (Bjerkholt, 2013: 45). We still have not seen a copy of the Leontief paper itself, but Bjerkholt’s findings suggest that the lost paper was even more mathematical than Leontief’s 1937 publication in the *Review of Economic Statistics*. The 1937 publication assumed linearity throughout, the 1933 version seemed to contain many nonlinear complications that Leontief apparently hadn’t presented in a polished way, and therefore Frisch patiently tried to generate a more elegant mathematical treatment of Leontief’s ideas. Keynes surely spent less time on deciphering Leontief’s unpolished mathematics of 1933, and quickly rejected the paper. Moreover, it is well-known that Keynes was not applying positive discrimination in favour of submissions containing more mathematics than average and/or emanating from America (Moggridge 1990).

6.3. Sraffa and the mathematicians in Cambridge

In 1928, a few years before rejecting Leontief’s paper, Keynes was able to read a draft of the opening propositions of *Production of Commodities by Means of Commodities* by Piero Sraffa (1960). Although this book required more than three decades of preparation,⁷⁰ the preface expressed thanks to only three scholars, namely Frank Ramsey, Alister Watson and especially Abram Besicovitch, for their mathematical help. Note that all three are ‘non-economists’. Thanks to extensive archival research by Kurz and Salvadori, we now have a better picture of Sraffa’s contacts with these three ‘mathematical friends’ in Cambridge (U.K.). It is now clear that Ramsey, Watson and Besicovitch overlooked the powerful Perron-Frobenius theorem:

Sraffa’s papers would seem to imply that none of his mathematical friends referred him to this theorem (Kurz & Salvadori 2001: 264).

As early as 1928, Sraffa presented to Frank Ramsey a system of simultaneous equations involving the prices of three commodities and a uniform rate of profit (rate of interest).

⁷⁰ From early 1930 on Sraffa spent a massive amount of time on his edition of Ricardo (Pollitt 1998; Gehrke & Kurz 2002; Naldi 2005). It is probably not widely known that in 1929 Dobb and Sraffa made plans to edit an English translation of Marx’s *Theorien über den Mehrwert*, and asked Kautsky’s permission to use his German edition as a starting point (letter from Dobb & Sraffa to Kautsky, 2 December 1929, and letter from Dobb to Kautsky, 15 February 1930, Karl Kautsky Papers D IX 316-317, International Institute of Social History, Amsterdam). I suppose Sraffa quickly abandoned the idea of the Marx translation project, in order to concentrate on the academically more prestigious Ricardo edition. By a remarkable coincidence, Sidney Webb’s letter recommending Dobb as Marx-editor to Kautsky was written on the same day that the *Royal Economic Society* entrusted the Ricardo edition to Sraffa: 13 February 1930. The rest of the story is well-known: Dobb never translated Marx’s *Theorien über den Mehrwert*, but greatly helped Sraffa to edit Ricardo.

Ramsey then wrote down the characteristic equation of the underlying 3 by 3 input matrix, but didn't use matrix algebraic terminology, and ignored Perron-Frobenius mathematics (Kurz & Salvadori 2001: 262-264).

Kurz and Salvadori also showed how, on various occasions in the 1940s and 1950s, Alister Watson was of great help to Sraffa to tackle the problems of basics and nonbasics in the model of single product industries.⁷¹ In modern parlance, the problem was the study of a reducible input matrix, its economically meaningful eigenvalues, its nonnegative eigenvectors, the connections between the group of basic commodities and the different groups of interconnected nonbasics, in other words, the connections between the different parts of the Frobenius normal form of the input matrix. However, Watson never referred Sraffa to the relevant mathematical literature, and discussed it in more intuitive terms. Finally, in a letter dated 9 June 1961, one year after Sraffa's 1960 book was published, Watson wrote to Sraffa:

I have recently come across a paper giving a brief statement and bibliography of the theorems of the type you prove and use that have been dealt with mathematically (cited by Kurz & Salvadori, 2001: 278).

According to Kurz & Salvadori (2001: 284), Sraffa wrote 'yes, send bibliography' on the top of Watson's letter, but no further traces were found in the Sraffa papers. The Watson story reveals how Sraffa's research was done in isolation from some of his economic colleagues in Cambridge.⁷² Sraffa could easily have obtained the Frobenius' reference from his colleague Richard Goodwin, who was appointed in Cambridge (U.K.) in 1950, after Harvard denied him tenure. Goodwin (1950) used Frobenius' results in his paper on the oscillation of the matrix multiplier. According to Goodwin (1953: 83)⁷³ it was Göran Ohlin, a Harvard student of his, who had drawn his attention to Frobenius. In this context Velupillai (1982: 78-79) provided the English translation of part of a Swedish letter from Göran Ohlin to Björn Thalberg, dated 5 October 1976: the letter showed that Goodwin had challenged his Harvard students to solve a certain matrix problem, and promised an A score to anyone who solved it. Ohlin succeeded thanks to his knowledge of the Frobenius' results. Later, Ohlin became a successful professor of economics and international civil servant, specializing in problems of economic

⁷¹ Kurz & Salvadori (2001) discuss Sraffa's meetings with Watson from 1945 onward. When looking at Sraffa's diaries in Cambridge, I noticed that there are also a number of meetings with Watson in 1938 and 1939, but I have no further information on their relevance for Sraffa's book.

⁷² On the Cambridge of that time, see Marcuzzo, Naldi, Rosselli & Sanfilippo (2008).

⁷³ Goodwin presented this paper at the first input-output conference in Driebergen (Holland) in September 1950.

development and demography. He didn't concentrate on linear economics in the rest of his career.⁷⁴

Even though Ramsey and Watson were very helpful, the preface of Sraffa's book left no doubt about the crucial role of Abram Besicovitch:

My greatest debt is to Professor A.S. Besicovitch for invaluable mathematical help over many years (Sraffa 1960: vi).

From the many detailed examples by Kurz & Salvadori (2004, 2008), we learn that Sraffa's preface didn't exaggerate Besicovitch's altruistic contribution. On 21 September 1944 Besicovitch even provided Sraffa with a self-made proof of the existence of a standard commodity.⁷⁵

Jonathan Smith (2011) described the relationship between Sraffa and Trinity College in Cambridge, and stressed the importance of Besicovitch's membership of the same College:

Whether Besicovitch was the best man in Cambridge to attack Sraffa's problems is unclear, though he was certainly able to produce the equations Sraffa wanted. It seems to me that the important factor was that Sraffa had come to know him through Trinity, clearly got on with him; and he was at hand, which made it easier for both parties (Smith 2011: 107).

It is obvious that Besicovitch did not know Perron's articles on matrix algebra, although he surely knew some other work by Perron. Indeed, Perron (1928) himself wrote a paper titled *Über einen Satz von Besicovitsch*, [sic] which simplified a proof that had been given by Besicovitch (1928) in his own paper *On Kakeya's problem and a similar one*. In 1958 the Kakeya-Besicovitch problem was the subject of a film commissioned by the Mathematical Association of America, and with an exposition by Besicovitch.⁷⁶ We should not

⁷⁴ Just like his uncle Bertil Ohlin (Nobel economics 1977) and some other important Swedish economists, Göran Ohlin was well-respected both in the academic networks and in the world of international politics.

⁷⁵ Reproduced and discussed by Salvadori (2011). See also Lippi (2008) and Salvadori (2008). De Vivo (2003: 15-16) mentions that in 1942 Besicovitch had presented to Sraffa an example where his original system of k individual commodities was transformed into a system with k 'compound commodities', but involving some negative elements.

⁷⁶ Besicovitch (1963) gives a good approximation of the script of this film.

underestimate the opportunity cost of Besicovitch's help to Sraffa; he was an important mathematician in his own fields of specialization.⁷⁷

6.4 Besicovitch came close to Perron-Frobenius in 1918

Current textbooks on nonnegative matrices often provide a unified treatment of three topics under one roof: the theory of Markov chains, the Perron-Frobenius theory of nonnegative matrices, and the theorems on diagonal dominance.

For a simple example of the latter, consider the following matrix C :

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$c_{ii} > 0; \quad c_{ij} \leq 0 \quad \text{if } i \neq j$$

$$c_{i1} + c_{i2} + \cdots + c_{in} > 0$$

In every row i , the diagonal element c_{ii} 'dominates' the rest of the row.⁷⁸ Such diagonally dominant matrices appeared in some early papers on electrical equilibria (Lévy 1881), and later also in many articles on pure algebra. The best known example is Minkowski (1900). A rather unknown paper in Russian, containing results on such matrices and implicitly also on their leading principal minors, was written by nobody less than Besicovitch (1918) himself.⁷⁹

I presented an exact copy of Besicovitch's notation and assumptions about the matrix C above. Input-output theory often studies a related situation. Consider a nonnegative input matrix A , with suitably normalized input coefficients, satisfying the so-called Brauer-Solow row sum criterion: *all row sums of A are smaller than one*.⁸⁰ Then $C = I - A$ is diagonally dominant. It is well-known (Takayama 1985, Chapter 4) that the Brauer-Solow criterion is sufficient for the Hawkins-Simon criterion: *all principal minors of $C = I - A$ are positive*. Besicovitch's 1918 paper considers only difference equations and pure matrix algebra, no

⁷⁷ Another revealing example is *Classics on Fractals* (Edgar 1993), a book containing the historic seminal papers on fractal geometry, a topic that had become 'popular' thanks to Benoit Mandelbrot. It turns out that 4 out of 19 papers in this compilation are authored or co-authored by Besicovitch. See Domb (1989) on Hausdorff and Besicovitch as precursors of Mandelbrot. Domb knew Besicovitch personally. See also Burkill (1971) or Taylor (1975) for more biographical details and a list of Besicovitch's publications.

⁷⁸ The matrix C has row dominance. Its transpose has column dominance.

⁷⁹ This Besicovitch article is mentioned in the bibliographical references provided by Ostrowski (1937: 73) and Taussky (1949: 672).

⁸⁰ Takayama (1985: 364).

economics. But a modern reader can easily recognize the implicit result that was contained in the second part of Besicovitch's paper: 'Brauer-Solow implies Hawkins-Simon'.

Why did Besicovitch, a former student of Markov and author of a paper on diagonally dominant matrices, overlook Perron-Frobenius? When answering this question, we must see Besicovitch's paper in the right context of 1918. Today many textbooks on Markov probability theory heavily rely on Perron-Frobenius mathematics, but Markov himself developed his pioneering results independently.⁸¹ Today dominant diagonals often appear in the same chapter as Perron-Frobenius, but in the early decades of the twentieth century Perron-Frobenius and diagonal dominance were not well-known textbook topics. Quite the contrary: Taussky (1949) showed how many journal articles 'invented' the same theorems on diagonal dominance again and again, often unnoticed by journal editors or referees.⁸²

To explain this ignorance, Schneider (1977) distinguished between 'inward matrix theory' and 'outward matrix theory'. Inward theory looks downward and inward to the minors of a matrix, its diagonals, its eigenvalues, eigenvectors, etc. Outward theory of matrices looks 'outward and upward to those great societies of groups and algebras of which they are members' (Schneider 1977: 205). In the first half of the twentieth century, abstract algebra (outward theory) started to flourish and became a high-status subject. Inward matrix theory looked old-fashioned and less prestigious, and in the first decades after Perron-Frobenius few significant new mathematical results were published. In this context, it becomes less bizarre that even the mathematicians in linear economics overlooked the old Perron-Frobenius results.

6.5. Sraffa and the Econometric Society: 'Che barba'

Sraffa didn't consult his economic colleagues in Cambridge, and neither did he use the mathematical knowledge of the specialists of the *Econometric Society*. In point of fact, his relation to this Society was rather special.

In a letter dated 18 February 1938, Ragnar Frisch, the editor of *Econometrica* himself, asked Sraffa whether he had correctly received the journal for the last years, and whether he wished

⁸¹ Markov's doctoral students include Besicovitch and Romanovsky. The latter published influential texts on Markov's probability theory, he overlooked Perron-Frobenius until 1933, but afterwards included long paragraphs on Frobenius' results in some of his later work; see Hawkins (2008: 702-704).

⁸² Based upon their economic research of the early 1940s, both Jacob Mosak (1944: 49-51) and Lloyd Metzler (1945, 1950) published some original results on matrices and their principal minors. They didn't refer to Perron-Frobenius, but their results can be incorporated in a unified treatment of 'Frobenius theorems, dominant diagonal matrices, and applications': see Chapter 4 in Takayama (1985).

to remain a member of the *Econometric Society*. Such a personal letter suggested that more than routine administration was involved. In his handwritten unsigned draft of 22 February 1938, Sraffa wrote that Keynes informed him that the name Sraffa was on a black-list circulated by Alfred Cowles (sponsor of the *Econometric Society*). Therefore Sraffa announced he wanted to resign membership of the *Econometric Society* and expected that the Treasurer would refund the subscription for the year 1938 which Sraffa had sent to him on 4 February.⁸³

According to Bjerkholt (1998: 55) four candidates had passed the nomination for the prestigious title of *Fellow of the Econometric Society* in 1938 (Lange, Leontief, Stamp and Yntema), but Sraffa was disapproved of by Cowles. It is not clear what happened next. If Sraffa effectively resigned, then it is strange that in the following decades, until the early 1960s, the name Sraffa remained on the lists of members that were regularly published in *Econometrica*.

Note that in 1931 Sraffa had attended the very first *European Meeting of the Econometric Society* at the University of Lausanne. There was a lot of last minute planning of the program. François Divisia couldn't meet his logistic commitments, due to family reasons. At short notice, Hans Staehle and Ragnar Frisch had to step in, and worked nearly the whole night, because the evening before the start of the Lausanne conference, there was still no program (Bjerkholt 1998: 45). Only 15 different economists appeared on stage in this elite gathering, including Piero Sraffa, who gave the only international conference presentation of his entire career, on the evening of Wednesday 22 September 1931, with a paper *Un économiste mathématicien du 18me siècle: le général Lloyd*, a paper on the (slightly) mathematical monetary theory of Henry Lloyd (1771).

Frisch (1970: 152) held fond memories of this 1931 Lausanne conference: 'We, the Lausanne people, were indeed so enthusiastic all of us about the new venture, and so eager to give and take, that we had hardly time to eat when we sat together at lunch or at dinner with all our

⁸³ The letter by Frisch and the handwritten draft of a reply by Sraffa are conserved in the Sraffa Papers, File C 101. My reference numbers follow the catalogue prepared by Jonathan Smith, Archivist and Modern Manuscript Cataloguer, Trinity College, Cambridge. I am very grateful to him for his friendly and competent help in the Wren Library. I also owe thanks to John Eatwell, the literary executor of the Sraffa papers. Olav Bjerkholt recently confirmed me that the original Sraffa letter is now conserved in the Ragnar Frisch Correspondence, Manuscripts Collection, National Library of Norway, Oslo.

notes floating around on the table to the despair of the waiters.’

From the report by Hans Staehle (1933: 79), it seems that Sraffa’s paper drew no questions from the audience. Sraffa himself didn’t share the general enthusiasm. Many weeks of his *Cambridge Pocket Diary 1930-1931* are blank, but for that special day of 22 September 1931, Sraffa filled in two words: ‘*Che barba*’ (an Italian expression indicating that something is boring).

No further details on Sraffa’s *che barba* are available, but he never repeated the conference experiment again. Maybe Sraffa had expected some helpful comments on his own paper, and probably he didn’t like to concentrate on the many other conference papers that used heavy mathematics or econometrics. Moreover, he preferred to do his own research in isolation and secrecy, and preferred to discuss his mathematical problems with a few mathematical colleagues in Cambridge, not with the many mathematicians or mathematical economists that joined the *Econometric Society*.

7. Epilogue

After the Second World War, Perron-Frobenius mathematics finally infiltrated mainstream linear economic theory. Anglo-Saxon economists could learn it from an *Economic Journal* article by Richard Goodwin (1950), Japanese economists from Takuma Yasui (1948) or Michio Morishima (1950). According to Takayama (1985: 366), citing Morishima (1955: 1), the rediscoveries were independent. Maybe Yasui's work should have been given more attention (Weintraub, 1987).

In the early 1950s *Econometrica* published various articles that disseminated the mathematical and economic properties of nonnegative matrices, starting with a brilliant paper by Solow (1952) on the structure of linear models. Solow referred to some relevant sources in German, but not to a special issue of the *Mathematische Zeitschrift*, in honour of Perron's 70th birthday. In this issue the German mathematician Helmut Wielandt (1950) had provided a drastically simplified proof of Perron-Frobenius, which is used in the majority of the textbooks ever since. Israel Nathan Herstein in Chicago immediately translated Wielandt's text into English, in a discussion paper for the *Cowles Commission for Research in Economics*. Herstein (1952) brought Wielandt's proof and his own English translation under the attention of the economists in his *Econometrica* comment on Solow. One year later, together with his Cowles colleague Gérard Debreu, Herstein then published their often cited and reprinted *Econometrica* paper on nonnegative matrices (Debreu & Herstein 1953). From 1952 or 1953 on, attentive readers of *Econometrica* should have been aware of Perron-Frobenius.

Independently of what had happened in Western economic journals, in 1953 Felix Gantmacher published his comprehensive textbook on the theory of matrices in Russian, based on lectures he had given in the Soviet Union for many years. After the very influential English translation (Gantmacher 1959) became available, the Perron-Frobenius theorems finally became common knowledge to many scholars who applied matrix theory in widely varying fields.

Whatever the order of discovery or rediscovery of Perron-Frobenius after the Second World war, a debate about the first use of Perron-Frobenius mathematics is redundant, because we now know that Maurice Potron applied Perron-Frobenius as early as 1911, in the core arguments of many of his economic writings. Potron's discovery of the connection between Perron-Frobenius and linear models of the input-output type should be held in high esteem,

especially in the light of a comparison with the other mathematicians who overlooked this connection in their seminal writings on linear economics.

It is tempting to claim that the other pioneers of the 1910s and 1920s (Charasoff, Bray, Remak) or Sraffa's 'mathematical friends' (Ramsey, Watson, Besicovitch) missed some 'open goal opportunities', to use available tools of mathematics that, from today's viewpoint, seem simple to apply. But were Charasoff, Bray, Remak, Ramsey, Besicovitch, Watson, etc. all suffering from scientific blindness or absent-mindedness? And what about other disciplines (probability theory, demography, etc.) where Perron-Frobenius entered in the same slow way? Or, for example, what about the slow infiltration of linear inequalities and convex sets in game theory?

Tinne Kjeldsen (2001) described the analogous situation in the history of game theory. She drew attention to von Neumann's 1953 remarks on the relation between his 1928 minimax theorem and the theory of linear inequalities and convex sets. From today's elementary textbooks on game theory, this connection seems simple and straightforward. But for more than a decade von Neumann himself didn't see the connection, until Oskar Morgenstern in 1941 showed him a 1938 paper by Jean Ville (for more details, see Kjeldsen, 2001: 58, 65). The lesson from this and related stories is clear:

It is common and tempting fallacy to view the later steps in a mathematical evolution as much more obvious and cogent after the fact than they were beforehand (von Neumann 1953: 125).

A similar remark can be made with respect to the use of Perron-Frobenius, the more so because the 'old-fashioned' matrix theorems of Perron and Frobenius were less widely known than the more prestigious new results of abstract algebra. We should see a scholar's work in its context. Results, connections or applications that seem evident today, were not obvious to the original pioneers.

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Abstract

In the period 1907-1912 the German 'pure mathematicians' Oskar Perron and Georg Frobenius developed the fundamental results of the theory of nonnegative matrices. Today Perron-Frobenius mathematics enjoys wide applications in many fields, for example in economics, probability theory, demography and even in Google's ranking algorithm. In linear economic models of the Leontief-Sraffa type it is often the crucial tool to solve many mathematical economic problems. My paper concentrates on the history of Perron-Frobenius in linear economics, and some related stories.

In the 1910s and 1920s, several pioneering publications in linear economics could have benefited from applying Perron-Frobenius results, but failed to do so, even the economic publications authored by the mathematicians Georg Charasoff, Hubert Bray and Robert Remak. Either they didn't know Perron-Frobenius, or they didn't realize its usefulness. The only exception was the French Jesuit mathematician Maurice Potron, who used Perron-Frobenius mathematics in the core of his economic model, in many of his writings, as early as 1911. He constructed a sort of disaggregated open input-output system, formulated duality theorems between his quantity system and his price system, and anticipated the Hawkins-Simon conditions. Potron's economic or mathematical contemporaries didn't recognize his originality.

A general treatment of Charasoff's economic system needs Perron-Frobenius mathematics, especially Perron's Limit Lemma. Although some of Charasoff's mathematical interests (irreducibility, continued fractions) were close to those of Perron or Frobenius, the theory of nonnegative matrices is never explicitly used in Charasoff's work. It is doubtful whether Charasoff knew the relevant matrix theorems. Probably he just assumed that the properties of his numerical examples with three commodities also hold in the general case with n commodities.

Frobenius had been Remak's doctoral supervisor in 1911. After a forgotten non-mathematical paper in 1918, on the repayment of the national debt, Remak presented his mathematical system of 'superposed prices' in 1929, twelve years after Frobenius' death. With suitable units of measurement, Remak's system can be handled by Perron-Frobenius tools. However, Remak failed to normalize his units, and provided lengthy proofs of his own. Moreover, he spent most of his mathematical efforts on freak systems in which the most important commodities have zero prices. A few years earlier, in 1922, Bray also had overlooked Perron-

Frobenius in a mathematically similar model that studied Cournot's equations of currency exchange.

Contrary to Dorfman's well-known article on Leontief's Nobel Prize in 1973, I provide archival evidence that Leontief knew Remak's results already in the early 1930s, before he submitted a paper containing ideas of input-output theory to Keynes for the Economic Journal in 1933. Keynes quickly rejected Leontief's paper; a few months later Leontief submitted it to Frisch for Econometrica. Frisch formulated a lot of critical remarks on Leontief's first and revised version in 1933-34. In the light of this criticism, it is highly probable that Leontief simplified and linearized his mathematics, and a few years later he finally started publishing his Nobel Prize winning empirical and theoretical results in American journals.

Just like Leontief, Sraffa started related research in the late 1920s. He didn't discuss his mathematical problems with competent economic colleagues in Cambridge, nor with the specialists of the Econometric Society, but preferred mathematical help from three non-economists: Ramsey, Watson and especially Besicovitch. I suggest that Besicovitch in his early mathematical research in Russia 'came close' to Perron-Frobenius results, but it is well-known that he didn't know Perron-Frobenius, and tried to invent his own proofs for Sraffa in the 1940s.

In the first half of the twentieth century, abstract algebra started to flourish and became a more prestigious and widely researched subject than the 'old-fashioned' Perron-Frobenius matrices. In this context, it is less surprising that for many decades even the mathematicians (except Potron) overlooked the usefulness of Perron-Frobenius in linear economics. Results, connections or applications that seem evident after the fact, were not obvious to the original pioneers.

Keywords: *Perron-Frobenius, Charasoff, Potron, Bray, Remak, Leontief, Sraffa, nonnegative matrices, input-output analysis.*