Journal of Multivariate Analysis **73**, 82–106 (2000)
doi:10.1006/jmva.1999.1870, available online at http://www.idealibrary.com on IDEAL®

Robustness of Deepest Regression

Stefan Van Aelst[1] and Peter J. Rousseeuw

Department of Mathematics and Computer Science, U.I.A., Universiteitsplein 1, B-2610 Antwerp, Belgium

Received October 12, 1998

In this paper we investigate the robustness properties of the *deepest regression*, a method for linear regression introduced by Rousseeuw and Hubert [6]. We show that the deepest regression functional is Fisher-consistent for the conditional median, and has a breakdown value of $\frac{1}{3}$ in all dimensions. We also derive its influence function, and compare it with sensitivity functions.

AMS 1991 subject classifications: 62F35, 62J05.
Key words and phrases: breakdown value, influence function, regression depth.

1. INTRODUCTION

Let $(\mathbf{x}, y)$ be a random p-dimensional column vector, with distribution H on $\mathbb{R}^p$. We would like to regress the univariate y on the $(p-1)$ dimensional $\mathbf{x}$. For any (potential) fit $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)^t$ we want to verify how well $(\mathbf{x}^t, 1)\,\boldsymbol{\theta}$ approximates y. To measure the quality of a fit, Rousseeuw and Hubert [6] introduced the notion of *regression depth*, which is a counterpart to Tukey's location depth [9].

Definition 1. The regression depth of a fit $\boldsymbol{\theta} \in \mathbb{R}^p$ relative to a given distribution H on $\mathbb{R}^p$, where H is the distribution of the random variable $(\mathbf{x}, y)$, is given by

$$rdepth(\boldsymbol{\theta}, H) = \min_{\mathbf{u}, v} \{H(y-(\mathbf{x}^t, 1)\,\boldsymbol{\theta} > 0 \text{ and } \mathbf{x}^t\mathbf{u} < v) + H(y-(\mathbf{x}^t, 1)\,\boldsymbol{\theta} < 0 \text{ and } \mathbf{x}^t\mathbf{u} > v)\}$$

where the minimum is over all unit vectors $\mathbf{u} \in \mathbb{R}^{p-1}$ and all $v \in \mathbb{R}$ with $H(\mathbf{x}^t\mathbf{u} = v) = 0$.

From this definition it can easily be seen that always $0 \leqslant rdepth(\boldsymbol{\theta}, H) \leqslant 1$. The regression depth of a fit $\boldsymbol{\theta}$ is the minimal amount of probability mass that needs to be passed when tilting $\boldsymbol{\theta}$ in any way until it

[1] Research Assistant with the FWO, Belgium.

0047-259X/00 $35.00

is vertical. In the special case of $p=1$ there is no $\mathbf{x}$, and H is the univariate distribution of y on $\mathbb{R}$. For any $\theta \in \mathbb{R}$ we then have rdepth$(\theta, H) = \min\{H(y>\theta), H(y<\theta)\}$ which is the "rank" of θ when we rank from the outside inwards. For any $p \geqslant 1$, the regression depth of $\boldsymbol{\theta}$ measures how balanced the mass of H is about the linear fit determined by $\boldsymbol{\theta}$.

DEFINITION 2. The deepest regression estimator $T^*(H)$ is defined as the fit $\boldsymbol{\theta}$ with maximal *rdepth*$(\boldsymbol{\theta}, H)$, that is

$$T^*(H) = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\ rdepth(\boldsymbol{\theta}, H). \tag{1}$$

(See Rousseeuw and Hubert [6]). We call T^* a *functional* because its argument is a distribution H on $\mathbb{R}^p$. (For a finite dataset, we apply Definition 2 to the empirical distribution H_n.)

For a distribution H on $\mathbb{R}^1$ the deepest regression (DR) is its median. For H on $\mathbb{R}^p$ with $p>1$, the DR thus generalizes the univariate median to linear regression. The DR is the "most balanced" fit for H. It is a "median-type" regression method, unlike earlier robust methods such as least trimmed squares (Rousseeuw [4]) and S-estimators (Rousseeuw and Yohai [8]) that are "mode-seeking" because they search for a concentrated linear cloud with the majority of the probability mass.

Figure 1a shows a dataset consisting of $n=50$ points generated from a bivariate gaussian distribution H with mean $\boldsymbol{\mu} = (4, 2)^t$, standard deviations $\sigma_1 = 4$ and $\sigma_2 = 3$, and correlation $\rho = 0.8$. Denote the empirical distribution of this dataset by H_n. The fits $\boldsymbol{\theta}_1 = (0.6, 4.6)^t$ and $\boldsymbol{\theta}_2 = (-2, 6)^t$ both have regression depth 1/50 according to Definition 1, and the deepest regression $T^*(H_n) = (0.615, -0.067)^t$ has depth 23/50 which is almost $\frac{1}{2}$. Figure 1a illustrates that lines with high regression depth provide a more

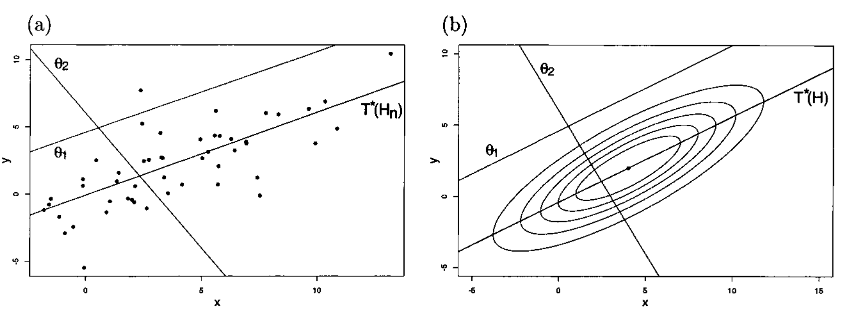

FIG. 1. (a) Dataset consisting of $n=50$ points generated from a bivariate gaussian distribution H on $\mathbb{R}^2$. The lines $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ have regression depth 1/50, and the deepest regression $T^*(H_n)$ has regression depth 23/50. (b) Contours of H. The line $\boldsymbol{\theta}_1$ now has depth 0.0027 and $\boldsymbol{\theta}_2$ has depth 0.05. The deepest regression $T^*(H)$ has regression depth $\frac{1}{2}$.

balanced fit to the data than lines with low depth. This motivates our interest in the properties of the fit T^* with maximal regression depth. Figure 1b shows contours of the corresponding population distribution H, where $T^*(H) = (0.6, -0.4)^t$ has depth exactly $\frac{1}{2}$ while $\boldsymbol{\theta}_1$ has depth 0.0027 and $\boldsymbol{\theta}_2$ has depth 0.05.

The natural setting of deepest regression is a large semiparametric model $\mathscr{H}$ in which the functional form is parametric and the error distribution is nonparametric. Formally, $\mathscr{H}$ consists of all distributions H on $\mathbb{R}^p$ with a strictly positive density such that there exists a $\tilde{\boldsymbol{\theta}} \in \mathbb{R}^p$ with $med_H(y \mid \mathbf{x}) = (\mathbf{x}^t, 1)\, \tilde{\boldsymbol{\theta}}$. Note that this model allows for skewed error distributions and heteroscedasticity. The asymptotic distribution of the deepest regression was obtained by He and Portnoy [3] in simple regression, and by Bai and He [1] in multiple regression.

In this paper we study the robustness properties of the deepest regression functional T^*. Section 2 shows that T^* is Fisher-consistent, and in Section 3 it is shown that T^* has a breakdown value of $\frac{1}{3}$. In Section 4 we derive the influence function of the deepest regression slope and intercept, and compare them with sensitivity functions. The conclusions are formulated in Section 5.

2. FISHER CONSISTENCY

We first define a probability-based distance between fits.

Definition 3. For every H on $\mathbb{R}^p$ and any hyperplanes $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ we define $d_H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = H(A(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2))$, where $A(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \{(\mathbf{x}, y);\, \mathbf{x} \in \mathbb{R}^{p-1}$ and $y \in [(\mathbf{x}^t, 1)\, \boldsymbol{\theta}_1, (\mathbf{x}^t, 1)\, \boldsymbol{\theta}_2]\}$ is the double wedge formed by the hyperplanes $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.

Lemma 1. *For every H on $\mathbb{R}^p$ with density $h > 0$, the function d_H is a metric on $\mathbb{R}^p$.*

Proof. For every $\boldsymbol{\theta} \in \mathbb{R}^p$ it clearly holds that $d_H(\boldsymbol{\theta}, \boldsymbol{\theta}) = 0$. For every $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ we see that $d_H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = d_H(\boldsymbol{\theta}_2, \boldsymbol{\theta}_1)$, and that $d_H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = 0$ implies $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ since $h > 0$. Also the triangle inequality $d_H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_3) \leqslant d_H(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + d_H(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ holds, because for every $\mathbf{x} \in \mathbb{R}^{p-1}$ we have $[(\mathbf{x}^t, 1)\, \boldsymbol{\theta}_1, (\mathbf{x}^t, 1)\boldsymbol{\theta}_3] \subset [(\mathbf{x}^t, 1)\boldsymbol{\theta}_1, (\mathbf{x}^t, 1)\boldsymbol{\theta}_2] \cup [(\mathbf{x}^t, 1)\boldsymbol{\theta}_2, (\mathbf{x}^t, 1)\boldsymbol{\theta}_3]$, hence $A(\boldsymbol{\theta}_1, \boldsymbol{\theta}_3) \subset A(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \cup A(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$. ∎

Lemma 2. *For every $H \in \mathscr{H}$ and any $\boldsymbol{\theta}$ it holds that* $\mathrm{rdepth}(\boldsymbol{\theta}, H) = \frac{1}{2} - d_H(T^*(H), \boldsymbol{\theta})$.

Proof. First note that for every $\boldsymbol{\theta}$ it holds that $d_H(\boldsymbol{\theta}, T^*(H)) \leqslant \frac{1}{2}$. This can be seen as follows. Since $\text{rdepth}(T^*(H), H) = \frac{1}{2}$ the probability mass passed by $T^*(H)$ when tilting it until it is vertical is always exactly $\frac{1}{2}$. If we tilt $T^*(H)$ around the intersection of $T^*(H)$ and $\boldsymbol{\theta}$ so that it passes $\boldsymbol{\theta}$ until it is vertical, then $A(\boldsymbol{\theta}, T^*(H))$ is part of the region passed by $T^*(H)$. Therefore $H(A(\boldsymbol{\theta}, T^*(H))) \leqslant \frac{1}{2}$, hence $d_H(\boldsymbol{\theta}, T^*(H)) \leqslant \frac{1}{2}$. Moreover, if we tilt $\boldsymbol{\theta}$ around this intersection so that it does not pass $T^*(H)$, then the amount of probability mass passed by $\boldsymbol{\theta}$ is exactly $\frac{1}{2} - d_H(T^*(H), \boldsymbol{\theta})$, hence $\text{rdepth}(\boldsymbol{\theta}, H) \leqslant \frac{1}{2} - d_H(T^*(H), \boldsymbol{\theta})$.

For $p = 2$ dimensions, take a base point u different from the base point v corresponding to the intersection of $\boldsymbol{\theta}$ and $T^*(H)$, as in Fig. 2. If we tilt $T^*(H)$ at u then we pass exactly $\frac{1}{2}$ of the probability mass. If we tilt $\boldsymbol{\theta}$ at u so that it does not pass $T^*(H)$ then we pass probability mass $\frac{1}{2} + H(C) - H(D)$ where $C = \{(x, y); x \in [min(u, v), max(u, v)] \text{ and } y \in [(x, 1)\, \boldsymbol{\theta}, (x, 1)\, T^*(H)]\}$ and $D = \{(x, y); x \notin [min(u, v), max(u, v)] \text{ and } y \in [(x, 1)\, \boldsymbol{\theta}, (x, 1)\, T^*(H)]\}$. If we tilt $\boldsymbol{\theta}$ at u so that it passes $T^*(H)$ then we pass probability mass $\frac{1}{2} - H(C) + H(D)$. Since $A(\boldsymbol{\theta}, T^*(H)) = C \cup D$, the minimal amount of probability mass passed by $\boldsymbol{\theta}$ when $u \neq v$ is higher than $\frac{1}{2} - d_H(T^*(H), \boldsymbol{\theta})$, hence $\text{rdepth}(\boldsymbol{\theta}, H) = \frac{1}{2} - d_H(T^*(H), \boldsymbol{\theta})$.

For $p = 3$ dimensions, take a base line $U = (u_1, u_2)$ different from the base line $V = (v_1, v_2)$ corresponding to the intersection of $\boldsymbol{\theta}$ and $T^*(H)$. If we tilt $\boldsymbol{\theta}$ at U so that it does not pass $T^*(H)$ then we pass probability mass

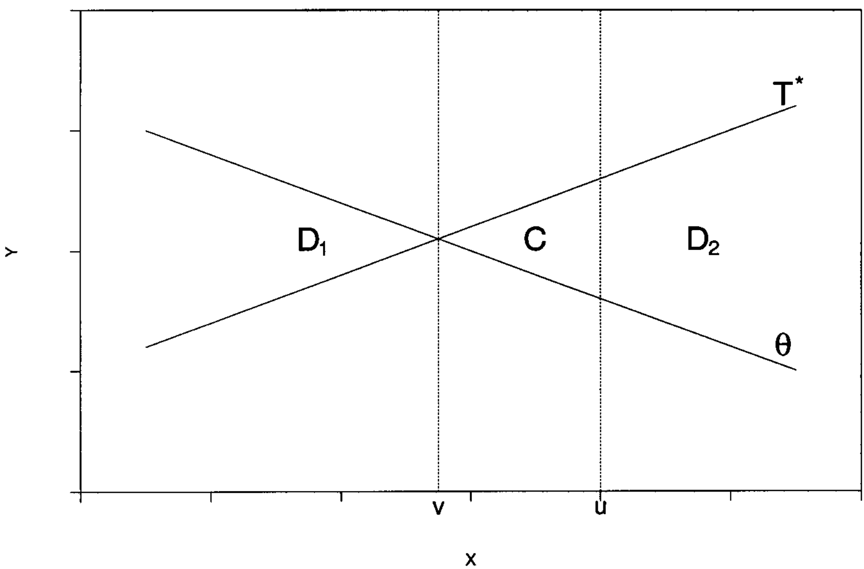

FIG. 2. Example of base points v and $u \neq v$ for $p = 2$ with the corresponding regions C and $D = D_1 \cup D_2$.

$\frac{1}{2}+H(C)-H(D)$ with $C=\{(x, y, z); x\in\mathbb{R}, y\notin[min(v_1x+v_2, u_1x+u_2), max(v_1x+v_2, u_1x+u_2)]$ and $z\in[(x, y, 1)\,\boldsymbol{\theta}, (x, y, 1)\,T^*(H)]\}$ and $D=\{(x, y, z); x\in\mathbb{R}, y\in[min(v_1x+v_2, u_1x+u_2), max(v_1x+v_2, u_1x+u_2)]$ and $z\in[(x, y, 1)\,\boldsymbol{\theta}, (x, y, 1)\,T^*(H)]\}$. If we tilt $\boldsymbol{\theta}$ at U so that it passes $T^*(H)$, then we pass probability mass $\frac{1}{2}-H(C)+H(D)$. Since $A(\boldsymbol{\theta}, T^*(H)) = C\cup D$, the minimal amount of probability mass passed by $\boldsymbol{\theta}$ when $U\neq V$ is higher than $\frac{1}{2}-d_H(T^*(H), \boldsymbol{\theta})$, hence $\text{rdepth}(\boldsymbol{\theta}, H)=\frac{1}{2}-d_H(T^*(H), \boldsymbol{\theta})$. This construction can be generalized for $p>3$.

From this proof it follows that the best base point (in general, the best base hyperplane in $\mathbb{R}^{p-1}$) for tilting a fit $\boldsymbol{\theta}$ is the $\mathbf{x}$-projection of the intersection of $\boldsymbol{\theta}$ with $T^*(H)$, and the direction in which to tilt $\boldsymbol{\theta}$ is such that $\boldsymbol{\theta}$ does not pass $T^*(H)$.

The next theorem shows that the deepest regression $T^*(H)$ is a Fisher-consistent estimator of the conditional median $med_H(y\,|\,\mathbf{x})$ when H belongs to the large semiparametric model $\mathscr{H}$ in which the error distribution is nonparametric.

THEOREM 1 (Fisher-consistency). *For every $H\in\mathscr{H}$ it holds that $T^*(H)=\tilde{\boldsymbol{\theta}}$.*

Proof. The condition $med_H(y\,|\,\mathbf{x})=(\mathbf{x}^t, 1)\,\tilde{\boldsymbol{\theta}}$ implies $rdepth(\tilde{\boldsymbol{\theta}}, H)=\frac{1}{2}$. Since H has a strictly positive density h, for every $\boldsymbol{\theta}\neq\tilde{\boldsymbol{\theta}}$ it holds that $d_H(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})=H(A(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}))>0$. Note that Lemma 2 still holds if we replace $T^*(H)$ by $\tilde{\boldsymbol{\theta}}$, hence we obtain $rdepth(\boldsymbol{\theta}, H)=\frac{1}{2}-d_H(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})<\frac{1}{2}$ for every $\boldsymbol{\theta}\neq\tilde{\boldsymbol{\theta}}$. Therefore, $T^*(H)=\tilde{\boldsymbol{\theta}}$. ▮

From the Fisher consistency in Theorem 1 together with the consistency of the deepest regression T_n^* for $\tilde{\boldsymbol{\theta}}$ shown by Bai and He [1] it follows that $T^*(H_n)=T_n^*(\mathbf{z}_1, ..., \mathbf{z}_n)$ converges to $T^*(H)$ in probability when $\mathbf{z}_1, ..., \mathbf{z}_n$ is i.i.d. according to $H\in\mathscr{H}$. This confirms that $T^*(H)$ is the asymptotic value of T_n^*.

3. BREAKDOWN VALUE

The breakdown value $\varepsilon^*(T, H)$ of any functional T at H is the smallest fraction of the probability mass of H that needs to be replaced to carry T beyond all bounds (see Hampel *et al.* [2]). It is defined by

$$\varepsilon^*(T, H)=\inf\{\varepsilon; \sup_G \|T((1-\varepsilon)\,H+\varepsilon G)-T(H)\|=\infty\} \tag{2}$$

where G is an arbitrary distribution on $\mathbb{R}^p$.

LEMMA 3. *If $H \in \mathscr{H}$ is a distribution on $\mathbb{R}^p$ and there exists a value $0 < \eta < \frac{1}{3}$ and a compact set K with* rdepth$(\boldsymbol{\theta}, H) < \eta$ *for all $\boldsymbol{\theta} \notin K$, then*

$$\varepsilon^*(T^*(H), H) \geqslant \tfrac{1}{3} - \eta.$$

Proof. We will consider contaminated distributions $H_\varepsilon = (1-\varepsilon)\,H + \varepsilon\,G$ where G is any distribution on $\mathbb{R}^p$. The fraction ε is sufficient to cause breakdown only if rdepth$(T^*(H_\varepsilon), H_\varepsilon) \leqslant \varepsilon + \eta$ for some G. Suppose that rdepth$(T^*(H_\varepsilon), H_\varepsilon) > \varepsilon + \eta$ for all G, then we find rdepth$(T^*(H_\varepsilon), H) > \eta$ for all G. Therefore $T^*(H_\varepsilon)$ belongs to K for all G. Since K is compact we have $\sup_G \|T^*(H_\varepsilon) - T^*(H)\| < \infty$ which means that ε is not sufficient to cause breakdown. It follows that

$$\begin{aligned} \varepsilon + \eta &\geqslant \text{rdepth}(T^*(H_\varepsilon), H_\varepsilon) \\ &\geqslant \text{rdepth}(T^*(H), H_\varepsilon) \\ &\geqslant (1-\varepsilon)\,\text{rdepth}(T^*(H), H) \end{aligned}$$

and because rdepth$(T^*(H), H) = \frac{1}{2}$ we obtain $\varepsilon + \eta \geqslant (1-\varepsilon)/2$ hence $\varepsilon \geqslant (1-2\eta)/3 > \frac{1}{3} - \eta$. ∎

DEFINITION 4. Let H be a distribution on $\mathbb{R}^p$. For every $0 < k \leqslant \frac{1}{2}$ the depth region of depth k is defined by $D_k(H) = \{\boldsymbol{\theta};\ \text{rdepth}(\boldsymbol{\theta}, H) \geqslant k\} \subset \mathbb{R}^p$.

LEMMA 4. *For every $H \in \mathscr{H}$ and $0 < k \leqslant \frac{1}{2}$ the depth region $D_k(H)$ is bounded.*

Proof. For any $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ in $\mathbb{R}^p$ denote the euclidian distance between $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ by $d_E(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. Suppose that $\sup\{d_E(\boldsymbol{\theta}, T^*(H));\ \boldsymbol{\theta} \in D_k(H)\} = \infty$. Then there exists a sequence $(\boldsymbol{\theta}^{\mathbf{j}})_j$ in $D_k(H)$ with $d_E(\boldsymbol{\theta}^{\mathbf{j}}, T^*(H)) \xrightarrow{j} \infty$. This implies $d_H(\boldsymbol{\theta}^{\mathbf{j}}, T^*(H)) \xrightarrow{j} \frac{1}{2}$ since we have to pass half of the probability mass to take $\|T^*(H)\|$ to infinity. But because $d_H(\xi, T^*(H)) \leqslant \frac{1}{2} - k$ for every fit ξ in $D_k(H)$, the sequence $(\boldsymbol{\theta}^{\mathbf{j}})_j$ cannot stay in $D_k(H)$. ∎

LEMMA 5. *For every $H \in \mathscr{H}$ and $0 < k \leqslant \frac{1}{2}$ the depth region $D_k(H)$ is closed.*

Proof. Suppose that $\boldsymbol{\theta}$ belongs to the closure of $D_k(H) \subset \mathbb{R}^p$. Then $d_E(\boldsymbol{\theta}, D_k(H)) = 0$, so there exists a sequence $(\boldsymbol{\theta}^{\mathbf{j}})_j$ in $D_k(H)$ with $d_E(\boldsymbol{\theta}, \boldsymbol{\theta}^{\mathbf{j}}) \xrightarrow{j} 0$. Because all metrics on $\mathbb{R}^p$ are topologically equivalent, this implies $d_H(\boldsymbol{\theta}, \boldsymbol{\theta}^{\mathbf{j}}) \xrightarrow{j} 0$. Now

$$\begin{aligned}\text{rdepth}(\boldsymbol{\theta}, H) &= \tfrac{1}{2} - d_H(T^*(H), \boldsymbol{\theta}) \\ &\geqslant \tfrac{1}{2} - d_H(T^*(H), \boldsymbol{\theta}^{\mathbf{j}}) - d_H(\boldsymbol{\theta}^{\mathbf{j}}, \boldsymbol{\theta}) \\ &\xrightarrow{j} \tfrac{1}{2} - d_H(T^*(H), \boldsymbol{\theta}^{\mathbf{j}}) \geqslant k\end{aligned}$$

hence $\boldsymbol{\theta} \in D_k(H)$. ∎

THEOREM 2. *For any dimension $p \geqslant 2$ and any distribution H in $\mathscr{H}$ it holds that*

$$\varepsilon^*(T^*, H) = \tfrac{1}{3}.$$

Proof. Lemmas 4 and 5 show that for every $0 < k < \frac{1}{3}$ there exists a compact set $D_k(H)$ in $\mathbb{R}^p$ with $\text{rdepth}(\boldsymbol{\theta}, H) < k$ for all $\boldsymbol{\theta} \notin D_k(H)$. Therefore it follows from Lemma 3 that $\varepsilon^*(T^*, H) \geqslant \frac{1}{3} - k$ for every $k > 0$, so $\varepsilon^*(T^*, H) \geqslant \frac{1}{3}$.

To prove that $\varepsilon^*(T^*, H) \leqslant \frac{1}{3}$ we show that T^* can be made to break down by moving $\frac{1}{3}$ of the probability mass arbitrarily far away. Let us first consider the case $p = 2$. Because of invariance, we may assume w.l.o.g. that $\text{med}_H(x) = 0$. Take $H_\varepsilon = (1-\varepsilon)\, H + \frac{\varepsilon}{2}\Delta_{\mathbf{z}} + \frac{\varepsilon}{2}\Delta_{-\mathbf{z}}$. Denote $A_{v,\boldsymbol{\theta}} = (y < (x, 1)\, \boldsymbol{\theta}$ and $x < v)$, $B_{v,\boldsymbol{\theta}} = (y > (x, 1)\, \boldsymbol{\theta}$ and $x > v)$, $C_{v,\boldsymbol{\theta}} = (y > (x, 1)\, \boldsymbol{\theta}$ and $x < v)$, and $D_{v,\boldsymbol{\theta}} = (y < (x, 1)\, \boldsymbol{\theta}$ and $x > v)$ for some $v \in \mathbb{R}$ and a fit $\boldsymbol{\theta} = (\theta_1, \theta_2)^t$. Figure 3 shows the regions $A_{v,\boldsymbol{\theta}}$, $B_{v,\boldsymbol{\theta}}$, $C_{v,\boldsymbol{\theta}}$ and $D_{v,\boldsymbol{\theta}}$ for a particular v and $\boldsymbol{\theta}$.

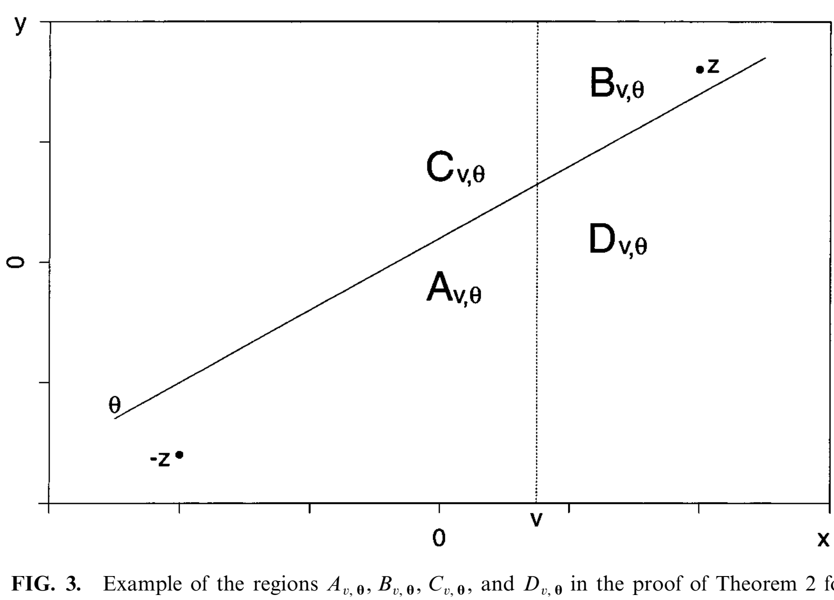

FIG. 3. Example of the regions $A_{v,\boldsymbol{\theta}}$, $B_{v,\boldsymbol{\theta}}$, $C_{v,\boldsymbol{\theta}}$, and $D_{v,\boldsymbol{\theta}}$ in the proof of Theorem 2 for a particular v and $\boldsymbol{\theta}$.

Take $\mathbf{z}=(x, y)$ and denote by $(\theta_{\mathbf{z}}, 0)^t$ the fit through $\mathbf{z}$ and $-\mathbf{z}$. For every $\boldsymbol{\theta}=(\theta_1, \theta_2)^t$, letting $y \to +\infty$ yields $\theta_{\mathbf{z}} > \theta_1$. Moreover, for every $v \in \mathbb{R}$, letting $x \to +\infty$ yields $x > |v|$. For every v and $\boldsymbol{\theta}$ we can thus make $\mathbf{z}$ belong to $B_{v, \boldsymbol{\theta}}$ and $-\mathbf{z}$ belong to $A_{v, \boldsymbol{\theta}}$ as in Fig. 3. It follows that

$$H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}}) = (1-\varepsilon)\,[H(C_{v, \boldsymbol{\theta}}) + H(D_{v, \boldsymbol{\theta}})] \tag{3}$$

$$H_\varepsilon(A_{v, \boldsymbol{\theta}}) + H_\varepsilon(B_{v, \boldsymbol{\theta}}) = \varepsilon + (1-\varepsilon)\,[H(A_{v, \boldsymbol{\theta}}) + H(B_{v, \boldsymbol{\theta}})]. \tag{4}$$

Since $rdepth(T^*(H), H)=\frac{1}{2}$ it holds for every $\boldsymbol{\theta}$ that $\min_v\{H(C_{v, \boldsymbol{\theta}}) + H(D_{v, \boldsymbol{\theta}})\} \leqslant \frac{1}{2}$. From (3) we then obtain $\min_v\{H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}})\} \leqslant (1-\varepsilon)/2$, so the depth of a line $\boldsymbol{\theta}=(\theta_1, \theta_2)$ is at most $(1-\varepsilon)/2$. Now consider $\boldsymbol{\theta}=(\theta_1, 0)^t$ where $\theta_1 \in \mathbb{R}$. Equation (4) yields $\min_v \min_{\theta_1 \uparrow +\infty}\{H_\varepsilon(A_{v, \boldsymbol{\theta}}) + H_\varepsilon(B_{v, \boldsymbol{\theta}})\} = \varepsilon + (1-\varepsilon) \min_v \lim_{\theta_1 \uparrow +\infty}\{H(A_{v, \boldsymbol{\theta}}) + H(B_{v, \boldsymbol{\theta}})\} = \varepsilon$ since $\lim_{\theta_1 \uparrow +\infty} \{H(A_{0, \boldsymbol{\theta}}) + H(B_{0, \boldsymbol{\theta}})\} = 0$. From equation (3) we obtain $\min_v \lim_{\theta_1 \uparrow +\infty} \{H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}})\} = (1-\varepsilon) \min_v \lim_{\theta_1 \uparrow +\infty} \{H(C_{v, \boldsymbol{\theta}}) + H(D_{v, \boldsymbol{\theta}})\} = \lim_{|v| \uparrow +\infty} \lim_{\theta_1 \uparrow +\infty} \{H(C_{v, \boldsymbol{\theta}}) + H(D_{v, \boldsymbol{\theta}})\} = (1-\varepsilon)\ H(x \leqslant 0) = (1-\varepsilon)/2$. So if we take $\varepsilon = \frac{1}{3}$ it holds that $\varepsilon = (1-\varepsilon)/2$ and then T^* breaks down. This construction can easily be generalized for $p > 2$.

Theorem 2 says that the deepest regression does not break down when at least 67% of the data are generated from the semiparametric model $\mathscr{H}$. This result holds in any dimension. Moreover, Theorem 2 illustrates that the deepest regression is different from L^1 regression, which is defined as $L^1(H) = \operatorname{argmin}_{\boldsymbol{\theta}} E_H[\,|y - (\mathbf{x}^t, 1)\,\boldsymbol{\theta}|\,]$. Note that L^1 is another generalization of the univariate median to regression, but with zero breakdown value due to its vulnerability to contaminating distributions G in (2) with long tails in $\mathbf{x}$.

COROLLARY 1. *If T is an estimator with* rdepth$(T(H), H) \geqslant k$ *for any distribution H on $\mathbb{R}^p$ with $H \in \mathscr{H}$, then it holds for any H in $\mathscr{H}$ that*

$$\varepsilon^*(T, H) \geqslant \frac{k}{k+1}.$$

Proof. As in the proof of Lemma 3 we find $\varepsilon + \eta \geqslant (1-\varepsilon)$ rdepth$(T(H), H)$ if there exists a compact set K with rdepth$(\boldsymbol{\theta}, H) < \eta$ for all $\boldsymbol{\theta} \notin K$. This now yields $\varepsilon + \eta \geqslant (1-\varepsilon)\,k$ hence $\varepsilon \geqslant (k-\eta)/(k+1)$. As in the proof of Theorem 2 we obtain $\varepsilon \geqslant k/(k+1)$. ∎

4. INFLUENCE FUNCTION

The influence function (see Hampel *et al.* [2]) of an estimator T at a distribution H measures the effect on T of adding a small mass at $\mathbf{z}=(\mathbf{x}^t, y)$.

If we denote the point mass at $\mathbf{z}$ by $\Delta_{\mathbf{z}}$ and write $H_\varepsilon = (1-\varepsilon)\, H + \varepsilon \Delta_{\mathbf{z}}$ then the influence function is given by

$$IF(\mathbf{z}, T, H) = \lim_{\varepsilon \downarrow 0} \frac{T((1-\varepsilon)\, H + \varepsilon \Delta_{\mathbf{z}}) - T(H)}{\varepsilon} \tag{5}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{T(H_\varepsilon) - T(H)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} T(H_\varepsilon)|_{\varepsilon=0}$$

in all $\mathbf{z}$ where the limit exists. By construction, $IF(\mathbf{z}, T, H) = (IF(\mathbf{z}, T_1, H)^t, IF(\mathbf{z}, T_2, H))^t$ where T_1 is the slope vector and T_2 is the intercept. We consider elliptical distributions $H_{\boldsymbol{\mu}, \Sigma}$ with density

$$h_{\boldsymbol{\mu}, \Sigma}(\mathbf{x}^t, y) = \frac{g(((\mathbf{x}^t, y)^t - \boldsymbol{\mu})^t\, \Sigma^{-1}((\mathbf{x}^t, y)^t - \boldsymbol{\mu}))}{\sqrt{\det(\Sigma)}} \tag{6}$$

with $\boldsymbol{\mu} \in \mathbb{R}^p$ and Σ a positive definite matrix of size p. We assume the function g to have a strictly negative derivative, so that $H_{\boldsymbol{\mu}, \Sigma}$ is unimodal.

For a column vector $(\mathbf{x}, y)$ with distribution $H_{\boldsymbol{\mu}, \Sigma}$ the Choleski decomposition of Σ yields a nonsingular lower triangular matrix M of the form

$$M = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{v}^t & c \end{pmatrix} \tag{7}$$

with $MM^t = \Sigma$, hence $\det(A) \neq 0 \neq c$. Then the transformed vector $(\tilde{\mathbf{x}}^t, \tilde{y})^t = M^{-1}((\mathbf{x}^t, y)^t - \boldsymbol{\mu})$ is distributed according to $H_{\mathbf{0}, I}$ where I is the identity matrix. Equivalently we have $(\mathbf{x}^t, y)^t = M(\tilde{\mathbf{x}}^t, \tilde{y})^t + (\boldsymbol{\mu}_1^t, \mu_2)^t$ with $\boldsymbol{\mu} = (\boldsymbol{\mu}_1^t, \mu_2)^t$. We can therefore write $\mathbf{x}$ and y as

$$\mathbf{x} = A\tilde{\mathbf{x}} + \boldsymbol{\mu}_1 \tag{8}$$

$$y = \mathbf{v}^t \tilde{\mathbf{x}} + cy + \mu_2. \tag{9}$$

Here (8) is an *affine transformation* of $\mathbf{x}$. We can write (9) as a *scale transformation* $y \to cy$, followed by a so-called *regression transformation* $y \to y + (\mathbf{x}^t, 1)(v^t, \mu_2)^t$.

Suppose that T is a regression, scale, and affine equivariant functional. In order to compute the influence function of T at a distribution $H_{\boldsymbol{\mu}, \Sigma}$ as in (6), it then suffices to know the influence function of T at $H_{\mathbf{0}, I}$. This is shown in the following result.

PROPOSITION 1. *Let T be a regression, scale, and affine equivariant functional. Then its influence function at an elliptical distribution $H_{\boldsymbol{\mu}, \Sigma}$ is* *completely determined by its influence function at the corresponding spherical distribution* $H_{\mathbf{0}, I}$ *through the equations*

$$IF((\mathbf{x}, y), T_1, H_{\boldsymbol{\mu}, \Sigma}) = c\, A^{-t}\, IF((\tilde{\mathbf{x}}, \tilde{y}), T_1, H_{\mathbf{0}, I}) + A^{-t}\mathbf{v}$$

$$IF((\mathbf{x}, y), T_2, H_{\boldsymbol{\mu}, \Sigma}) = c\, IF((\tilde{\mathbf{x}}, \tilde{y}), T_2, H_{\mathbf{0}, I}) - c\, \boldsymbol{\mu}_1^t\, A^{-t} IF((\tilde{\mathbf{x}}, \tilde{y}), T_1, H_{\mathbf{0}, I}) - \boldsymbol{\mu}_1^t\, A^{-t}\mathbf{v} + \boldsymbol{\mu}_2.$$

The proof is given in the Appendix.

In $p = 2$ dimensions we will derive the influence function of the deepest regression $T^* = (T_1^*, T_2^*)^t$ where T_1^* is its slope and T_2^* is its intercept. Since the deepest regression is regression, scale, and affine equivariant (see Rousseeuw and Hubert [6]), it suffices to derive the influence function at spherical distributions $H = H_{\mathbf{0}, I}$. (By the equivariances, $T_1^*(H_{\mathbf{0}, I}) = 0$ and $T_2^*(H_{\mathbf{0}, I}) = 0$.) The influence function for elliptical distributions then follows from Proposition 1.

THEOREM 3. (a) *The influence function of the deepest regression at* $H = H_{\mathbf{0}, I}$ *is*

$$IF((x, y), T_1^*, H) = sgn(x)\, sgn(y) \times \left(\frac{I(G(|x|) \leqslant 2G(+\infty)/3)}{4[G(+\infty) - G(|x|)]} + \frac{I(G(|x|) \geqslant 2G(+\infty)/3)}{[2G(+\infty) - G(|x|)]} \right)$$

$$IF((x, y), T_2^*, H) = \frac{sgn(y)}{2h_Y(0)} \left(\frac{I(H_{X|Y}(|x|\ |0) \leqslant \frac{2}{3})}{H_{X|Y}(|x|\ |0)} + \frac{I(H_{X|Y}(|x|\ |0) \geqslant \frac{2}{3})}{2(2H_{X|Y}(|x|\ |0) - 1)} \right)$$

with $G(t) = \int_0^{t^2} g(u)\, du$ *and where* $h_Y(0)$ *is the marginal density of* Y *in* 0 *and* $H_{X|Y}(|x|\ |0)$ *is the conditional cdf of* X *given* Y *in* $|x|$ *given* $y = 0$.

(b) *For the bivariate standard gaussian distribution* $H = N_2(\mathbf{0}, I)$ *we have*

$$IF((x, y), T_1^*, H) = \frac{sgn(x)\, sgn(y)}{2\phi(0)} \left(\frac{I(\phi(x) \geqslant \phi(0)/3)}{4\phi(x)} + \frac{I(\phi(x) < \phi(0)/3)}{\phi(0) + \phi(x)} \right)$$

$$IF((x, y), T_2^*, H) = \frac{sgn(y)}{2\phi(0)} \left(\frac{I(|x| \leqslant \Phi^{-1}(\frac{2}{3}))}{\Phi(|x|)} + \frac{I(|x| > \Phi^{-1}(\frac{2}{3}))}{2(2\Phi(|x|) - 1)} \right)$$

with ϕ *the density and* Φ *the cdf of the univariate gaussian distribution* $N(0, 1)$.

This proof is given in the Appendix. Figure 4a shows the influence function of the DR slope at the bivariate gaussian distribution $H = N_2(\mathbf{0}, I)$,

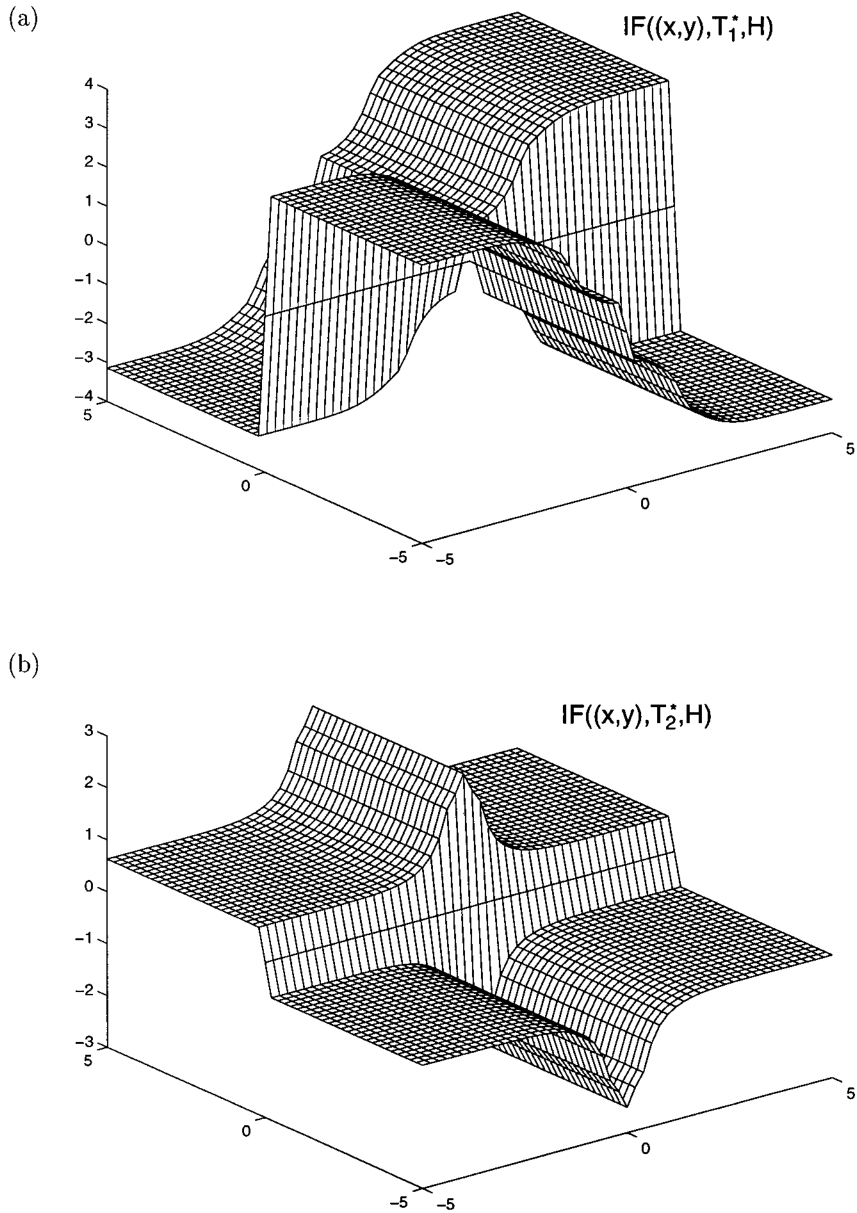

FIG. 4. (a) Influence function of the deepest regression slope T_1^*; (b) Influence function of the deepest regression intercept T_2^*.

and Figure 4b gives that of the intercept at $N_2(\mathbf{0}, I)$. Note that both influence functions are bounded, meaning that T^* is B-robust in the terminology of (Hampel *et al.* [2]).

In simple regression He and Portnoy [3] prove $n^{1/2}$ consistency of T^*, with a limiting distribution that is slightly different from a gaussian distribution. He and Pornoy performed simulations to compute the variance of the limiting distribution and gave approximate asymptotic efficiencies of T^* relative to L^1. For bivariate gaussian data they obtained 87% for the slope and 82% for the intercept, which shows that the robustness of T^* does not cost too much efficiency compared to L^1.

When drawing a sample $\mathbf{z}_1, ..., \mathbf{z}_n$ from H, most estimators satisfy

$$T_n(\mathbf{z}_1, ..., \mathbf{z}_n) = T(H) + \frac{1}{n} \sum_{i=1}^{n} IF(\mathbf{z}_i, T, H) + o_H(n^{-1/2}) \tag{10}$$

where $o_H(n^{-1/2})$ stands for a remainder term R_n such that $n^{1/2} R_n \to 0$ in probability. Then the central limit theorem implies that T_n is asymptotically normal with asymptotic variance

$$V(T, H) = \mathrm{Var}_H [IF(\mathbf{z}, T, H)] = \int IF(\mathbf{z}, T, H)^2 \, dH(\mathbf{z}) \tag{11}$$

(see Hampel *et al.* [2], p. 85). Since T^* is not asymptotically normal the expression (10) cannot hold, hence the deepest regression is not linear in its influence function. Moreover, applying expression (11) to the slope T_1^* would yield a relative efficiency of 75% instead of 87%. This discrepancy can be explained as follows. The limiting distribution of T_1^* has shorter tails than the normal distribution (Stephen Portnoy, personal communication). Therefore, (11) overestimates the asymptotic variance in this case, which leads to an underestimation of the efficiency. This is an interesting example of an estimator whose distribution converges at the $n^{1/2}$ rate but whose asymptotic variance cannot be computed using expression (11).

Whereas the influence function is defined on the population distribution, we also want to compare it with a finite-sample version. For this we compute the *averaged permutation-stylized sensitivity function*, defined by (Rousseeuw *et al.* [5]) as follows. For any estimator T_n the sensitivity function measures the (standardized) effect of adding an observation at $\mathbf{z} = (x, y)$ to the sample $Z_n = \{\mathbf{z}_1, ..., \mathbf{z}_n\}$, i.e.

$$SF_n(\mathbf{z}, T, Z_n) = n(T_{n+1}(\mathbf{z}_1, ..., \mathbf{z}_n, \mathbf{z}) - T_n(\mathbf{z}_1, ..., \mathbf{z}_n)). \tag{12}$$

The resulting sensitivity function strongly depends on the actual sample Z_n, but we alleviate this effect by using a permutation-stylized dataset

$Z_n(\pi)=\{(x_i^s, x_{\pi(i)}^s); i=1, ..., n\}$ where $x_i^s=\Phi^{-1}(\frac{i}{n+1})$ and where π is a permutation on $\{1, ..., n\}$. This stylized sample gives a better approximation to the population distribution $N_2(\mathbf{0}, I)$ than a random sample does, and has the advantage that the marginal distributions are symmetric with zero median. The effect of the particular permutation π is tempered by averaging the sensitivity function over a collection of random permutations, leading to

$$APSF_n(\mathbf{z})=\underset{\pi}{\text{average}}\ SF_n(\mathbf{z}, T, Z_n(\pi)). \tag{13}$$

A further refinement we applied is the following. For each random permutation π in (13) we also consider the permutations $\rho\pi$, $\pi\rho$, and $\rho\pi\rho$, where $\rho(i)=n-i+1$ is the permutation which reverses the order. So, for each generated π we use the four datasets

$$\begin{aligned} Z_n(\pi)&=\{(x_i^s, x_{\pi(i)}^s); i=1, ..., n\}\\ Z_n(\rho\pi)&=\{(x_i^s, -x_{\pi(i)}^s); i=1, ..., n\}\\ Z_n(\pi\rho)&=\{(-x_i^s, x_{\pi(i)}^s); i=1, ..., n\}\\ Z_n(\rho\pi\rho)&=\{(-x_i^s, -x_{\pi(i)}^s); i=1, ..., n\} \end{aligned} \tag{14}$$

Due to the symmetry properties of the slope T_1^* it holds that

$$\begin{aligned} SF_n((x, y), T_1^*, Z_n(\rho\pi))&=-SF_n((x, -y), T_1^*, Z_n(\pi))\\ SF_n((x, y), T_1^*, Z_n(\pi\rho))&=-SF_n((-x, y), T_1^*, Z_n(\pi))\\ SF_n((x, y), T_1^*, Z_n(\rho\pi\rho))&=SF_n((-x, -y), T_1^*, Z_n(\pi)) \end{aligned} \tag{15}$$

Therefore, if for each generated π we use all four permutations $\{\pi, \rho\pi, \pi\rho, \rho\pi\rho\}$ in (13) the resulting $APSF_n$ will have the same symmetry properties. This implies that we only need to compute (12) for grid points (x, y) in the first quadrant, so we get four permutations at the computational cost of one, and (13) becomes a smoother surface.

Figure 5a shows the sensitivity surface of the deepest regression slope and Fig. 5b that of the deepest regression intercept, both for $n=20$. These sensitivity functions were obtained by generating $m=1200$ random permutations π and for grid points in the first quadrant. The results in the other quadrants follow by symmetry. Note that these sensitivity surfaces are very similar to the asymptotic influence functions in Fig. 4, which means that the robustness interpretation of the influence function of T^* already applies to small sample sizes.

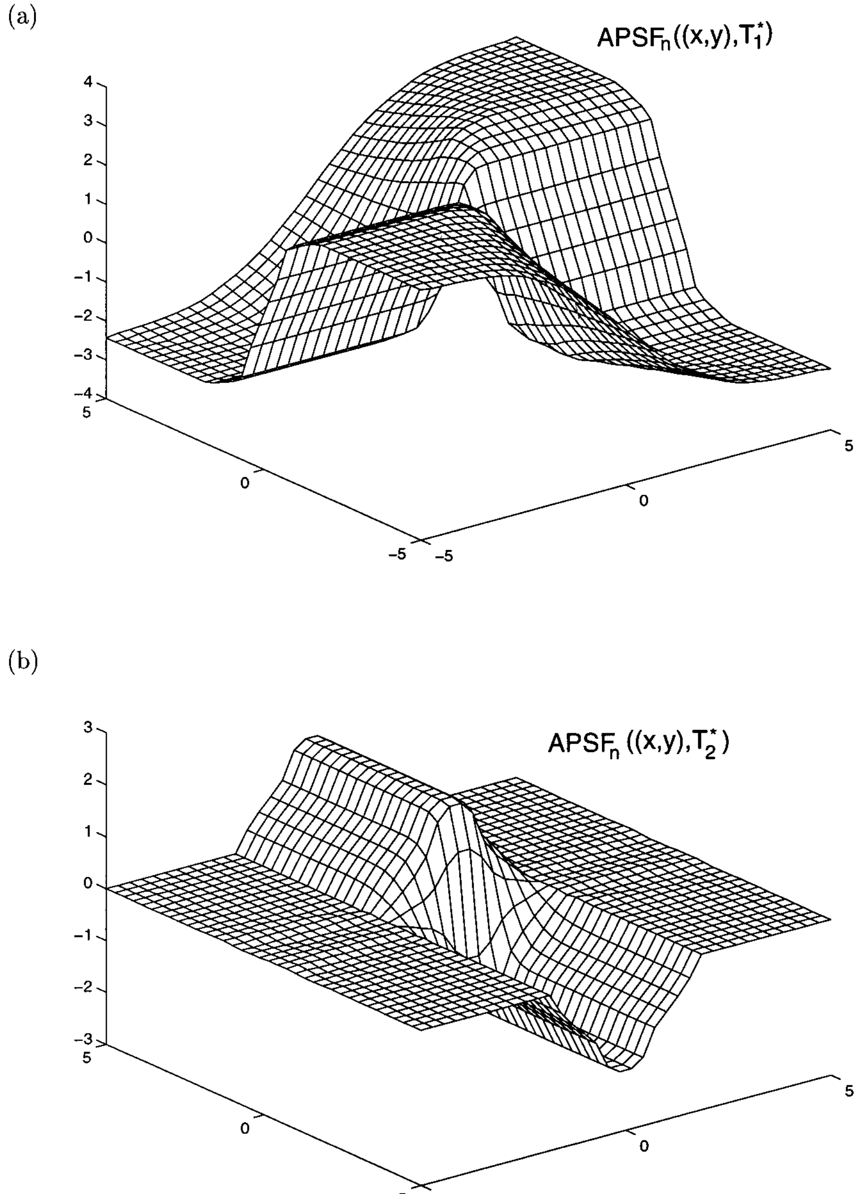

FIG. 5. (a) Averaged permutation-stylized sensitivity function $APSF_n$ of the deepest regression slope T_1^* for $n=20$; (b) $APSF_n$ of the deepest regression intercept T_2^* for $n=20$.

5. CONCLUSIONS

We have shown that the deepest regression DR is a Fisher-consistent estimator of $med(y \mid \mathbf{x})$ and has a positive breakdown value of $\frac{1}{3}$ when the good data come from the natural semiparametric model. Note that no moment conditions are required (for instance, all the results in this paper also hold for the bivariate Cauchy distribution; see expression (8) in [7]). Therefore the DR is a robust generalization of the median. The L^1 estimator also generalizes the median but its breakdown value is zero because L^1 is not robust to leverage points, whereas DR can resist vertical outliers as well as leverage points. In [6] it is shown that DR inherits a monotone equivariance property from the univariate median, which L^1 doesn't have. The influence functions of the DR slope and intercept are piecewise smooth and bounded, meaning that an outlier cannot affect DR too much, and the corresponding sensitivity functions show that this already holds for small sample sizes.

APPENDIX

Proof of Proposition 1. Since $(\mathbf{x}^t, y)^t = M(\tilde{\mathbf{x}}^t, \tilde{y})^t + \boldsymbol{\mu} = ((A\tilde{\mathbf{x}} + \boldsymbol{\mu}_1)^t, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t$, we have $T((\mathbf{x}^t, 1, y)^t) = T(((A\tilde{\mathbf{x}} + \boldsymbol{\mu}_1)^t, 1, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t)$. This can be rewritten as $T(((A\tilde{\mathbf{x}} + \boldsymbol{\mu}_1)^t, 1, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t) = T(((B(\tilde{\mathbf{x}}^t, 1)^t)^t, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t)$ with

$$B = \begin{pmatrix} A & \boldsymbol{\mu}_1 \\ \mathbf{0}^t & 1 \end{pmatrix}.$$

Affine equivariance of T yields $T(((B(\tilde{\mathbf{x}}^t, 1)^t)^t, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t) = B^{-t}T((\tilde{\mathbf{x}}^t, 1, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t)$ where

$$B^{-t} = \begin{pmatrix} A^{-t} & \mathbf{0} \\ -\boldsymbol{\mu}_1^t A^{-t} & 1 \end{pmatrix}.$$

From regression equivariance we obtain $T((\tilde{\mathbf{x}}^t, 1, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t) = T((\tilde{\mathbf{x}}^t, 1, c\tilde{y})^t) + (\mathbf{v}^t, \mu_2)^t$ and scale equivariance yields $T((\tilde{\mathbf{x}}^t, 1, c\tilde{y})^t) = cT((\tilde{\mathbf{x}}^t, 1, \tilde{y})^t)$. Combining these three results gives $T(((B(\tilde{\mathbf{x}}^t, 1)^t)^t, c\tilde{y} + \mathbf{v}^t\tilde{\mathbf{x}} + \mu_2)^t) = B^{-t}[cT((\tilde{\mathbf{x}}^t, 1, \tilde{y})^t) + (\mathbf{v}^t, \mu_2)^t]$ or

$$T_1((\mathbf{x}^t, y)^t) = cA^{-t}T_1((\tilde{\mathbf{x}}^t, \tilde{y})^t) + A^{-t}\mathbf{v}$$

$$T_2((\mathbf{x}^t, y)^t) = cT_2((\tilde{\mathbf{x}}^t, \tilde{y})^t) - c\boldsymbol{\mu}_1^t A^{-t}T_1((\tilde{\mathbf{x}}^t, \tilde{y})^t) - \boldsymbol{\mu}_1^t A^{-t}\mathbf{v} + \mu_2.$$

Together with expression (5) for the influence function this yields the result in Proposition 1. ∎

Proof of Theorem 3. (a) First we derive the influence function of the DR slope T_1^*. Since H is spherically symmetric, placing a mass ε in $(-x, -y)$ has the same effect on the slope T_1^* as placing this mass in (x, y). Therefore $IF((x, y), T_1^*, H) = IF((-x, -y), T_1^*, H)$ wherever it exists, hence

$$IF((x, y), T_1^*, H) = \frac{1}{2} IF((x, y), T_1^*, H) + \frac{1}{2} IF((-x, -y), T_1^*, H)$$
$$= \lim_{\varepsilon \downarrow 0} \frac{T_1^*(H_\varepsilon) - T_1^*(H)}{\varepsilon}$$

where $H_\varepsilon = (1-\varepsilon) H + (\varepsilon/2) \Delta_{(x, y)} + (\varepsilon/2) \Delta_{(-x, -y)}$. By symmetry, $T^*(H_\varepsilon)$ passes through the origin. Now consider all $\boldsymbol{\theta} = (b, 0)$ with $b \in \mathbb{R}$ and take $A_{v, \boldsymbol{\theta}}$, $B_{v, \boldsymbol{\theta}}$, $C_{v, \boldsymbol{\theta}}$ and $D_{v, \boldsymbol{\theta}}$ as in the proof of Theorem 2. We then find that $T_1^*(H_\varepsilon)$ maximizes $\min_v \{H_\varepsilon(A_{v, \boldsymbol{\theta}}) + H_\varepsilon(B_{v, \boldsymbol{\theta}}), H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}})\}$ where the minimum is over all v with $H_\varepsilon(x = v) = 0$, hence $|v| \neq x$ and by symmetry it suffices to take $v \geqslant 0$. Note that by spherical symmetry of H it holds that $H(B_{v, \boldsymbol{\theta}}) = \frac{1}{2} - H(C_{v, \boldsymbol{\theta}})$ and $H(D_{v, \boldsymbol{\theta}}) = \frac{1}{2} - H(A_{v, \boldsymbol{\theta}})$ for all v and $\boldsymbol{\theta} = (b, 0)$. Take $x > 0$ and $y > 0$.

(i) First consider a line $\boldsymbol{\theta}$ through the origin with slope $b \leqslant 0$, as in Fig. 6a. For $0 \leqslant v < x$ we find

$$\begin{aligned} H_\varepsilon(A_{v, \boldsymbol{\theta}}) + H_\varepsilon(B_{v, \boldsymbol{\theta}}) &= \varepsilon/2 + (1-\varepsilon) H(A_{v, \boldsymbol{\theta}}) + \varepsilon/2 + (1-\varepsilon) H(B_{v, \boldsymbol{\theta}}) \\ &= \varepsilon + (1-\varepsilon)[H(A_{v, \boldsymbol{\theta}}) + \tfrac{1}{2} - H(C_{v, \boldsymbol{\theta}})] \\ &= (1+\varepsilon)/2 + (1-\varepsilon) k(v, b) \end{aligned}$$

and

$$\begin{aligned} H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}}) &= (1-\varepsilon) H(C_{v, \boldsymbol{\theta}}) + (1-\varepsilon) H(D_{v, \boldsymbol{\theta}}) \\ &= (1-\varepsilon)[H(C_{v, \boldsymbol{\theta}}) + \tfrac{1}{2} - H(A_{v, \boldsymbol{\theta}})] \\ &= (1-\varepsilon)/2 - (1-\varepsilon) k(v, b) \end{aligned}$$

with $k(v, b) = H(A_{v, \boldsymbol{\theta}}) - H(C_{v, \boldsymbol{\theta}}) = H(Y - bX < 0 \text{ and } X < v) - H(Y - bX > 0 \text{ and } X < v)$. For $b \leqslant 0$ the function $k(v, b)$ is positive and decreasing in v. Therefore

$$\begin{aligned} \min_v \{H_\varepsilon(A_{v, \boldsymbol{\theta}}) + H_\varepsilon(B_{v, \boldsymbol{\theta}}), H_\varepsilon(C_{v, \boldsymbol{\theta}}) + H_\varepsilon(D_{v, \boldsymbol{\theta}})\} &\leqslant (1-\varepsilon)[1/2 - k(0, b)] \\ &\leqslant (1-\varepsilon)/2. \end{aligned}$$

Therefore, any fit $\boldsymbol{\theta} = (b, 0)$ with $b \leqslant 0$ has depth at most $(1-\varepsilon)/2$.

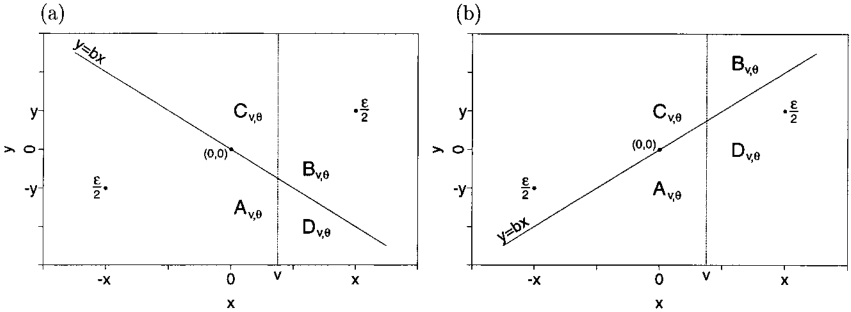

FIG. 6. Example of the regions $A_{v,\boldsymbol{\theta}}$, $B_{v,\boldsymbol{\theta}}$, $C_{v,\boldsymbol{\theta}}$, and $D_{v,\boldsymbol{\theta}}$ for (a) some $b<0$ and $0 \leqslant v < x$ and (b) for some $b > y/x$ and $0 \leqslant v < x$.

(ii) Now consider a line $\boldsymbol{\theta}$ through the origin with slope $b > y/x$, as in Figure 6b. For $0 \leqslant v < x$ we now obtain in the same way as for (i) that

$$H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}) = (1-\varepsilon)\,H(A_{v,\boldsymbol{\theta}}) + (1-\varepsilon)\,H(B_{v,\boldsymbol{\theta}})$$
$$= (1-\varepsilon)/2 + (1-\varepsilon)\,k(v,b)$$

and

$$H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}}) = \varepsilon/2 + (1-\varepsilon)\,H(C_{v,\boldsymbol{\theta}}) + \varepsilon/2 + H(D_{v,\boldsymbol{\theta}})$$
$$= (1+\varepsilon)/2 - (1-\varepsilon)\,k(v,b),$$

but now $k(v, b)$ is negative and increasing in v. Therefore

$$\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}),\, H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}})\} \leqslant (1-\varepsilon)[1/2 + k(0,b)]$$
$$\leqslant (1-\varepsilon)/2.$$

Hence also any fit $\boldsymbol{\theta} = (b, 0)$ with $b > y/x$ has depth at most $(1-\varepsilon)/2$.

(iii) If $b = y/x$ then for $0 \leqslant v$ we find

$$H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}) = \varepsilon/2 + (1-\varepsilon)[H(A_{v,\boldsymbol{\theta}}) + H(B_{v,\boldsymbol{\theta}})]$$
$$= \tfrac{1}{2} + (1-\varepsilon)\,k(v,b)$$

and

$$H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}}) = \varepsilon/2 + (1-\varepsilon)[H(C_{v,\boldsymbol{\theta}}) + H(D_{v,\boldsymbol{\theta}})]$$
$$= \tfrac{1}{2} - (1-\varepsilon)\,k(v,b)$$

where $k(v, b)$ is negative and increasing in v. Therefore

$$\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\}$$

$$\leqslant \tfrac{1}{2}+(1-\varepsilon)\,k(0, y/x)<\tfrac{1}{2}$$

for $x, y>0$.

(iv) Finally, consider a line $\boldsymbol{\theta}$ through the origin with slope $0<b<y/x$. The function $k(v, b)$ is now negative and increasing in v. For $v>x$ we obtain

$$H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}})=\varepsilon/2+(1-\varepsilon)\,H(A_{v,\boldsymbol{\theta}})+(1-\varepsilon)\,H(B_{v,\boldsymbol{\theta}})$$
$$=\tfrac{1}{2}+(1-\varepsilon)\,k(v, b)$$

and

$$H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})=\varepsilon/2+(1-\varepsilon)\,H(C_{v,\boldsymbol{\theta}})+(1-\varepsilon)\,H(D_{v,\boldsymbol{\theta}})$$
$$=\tfrac{1}{2}-(1-\varepsilon)\,k(v, b),$$

so $\min_{v>x}\{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\}=\lim_{v\to x}(H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}))=\tfrac{1}{2}+(1-\varepsilon)\,k(x, b)$. For $0\leqslant v<x$ we obtain

$$H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}})=\varepsilon/2+(1-\varepsilon)\,H(A_{v,\boldsymbol{\theta}})+\varepsilon/2+(1-\varepsilon)\,H(B_{v,\boldsymbol{\theta}})$$
$$=(1+\varepsilon)/2+(1-\varepsilon)\,k(v, b)$$

and

$$H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})=(1-\varepsilon)\,H(C_{v,\boldsymbol{\theta}})+(1-\varepsilon)\,H(D_{v,\boldsymbol{\theta}})$$
$$=(1-\varepsilon)/2-(1-\varepsilon)\,k(v, b),$$

so $\min_{0\leqslant v<x}\{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\}=\min\{\lim_{v\to 0}(H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}})), \lim_{v\to x}(H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}))\}=\min\{\tfrac{1}{2}+\varepsilon/2+(1-\varepsilon)\,k(0, b), \tfrac{1}{2}-\varepsilon/2-(1-\varepsilon)\,k(x, b)\}$. Therefore, the depth of a fit $\boldsymbol{\theta}=(b, 0)$ with $0<b<y/x$ is

$$\min\{\tfrac{1}{2}+(1-\varepsilon)\,k(x, b), \tfrac{1}{2}+\varepsilon/2+(1-\varepsilon)\,k(0, b), \tfrac{1}{2}-\varepsilon/2-(1-\varepsilon)\,k(x, b)\}.$$

Since H is spherically symmetric, integration yields $k(0, b)=H(Y-bX<0$ and $X<0)-H(Y-bX>0$ and $X<0)=\int_{-\infty}^{0}\int_{-bx}^{bx} g(x^2+y^2)\,dx\,dy=-\arctan(b)\int_0^{+\infty} g(r^2)\,dr^2=-G(+\infty)\arctan(b)$ and

$$\lim_{\varepsilon\downarrow 0}\frac{G(x)\arctan(b)}{H(Y-bX<0 \text{ and } 0<X<x)-H(Y-bX>0 \text{ and } 0<X<x)}=1$$

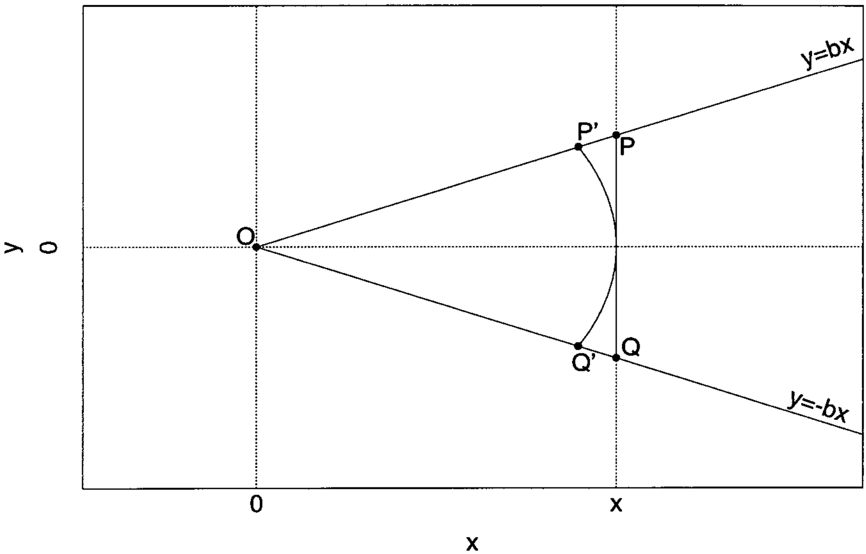

FIG. 7. Example of a triangle OPQ which contains probability mass $H(Y-bX<0$ and $0<X<x)-H(Y-bX>0$ and $0<X<x)$ and its approximation OP'Q' which contains mass $G(x)\arctan(b)$.

which means that for ε small enough we can approximate the probability mass in triangle OPQ shown in Fig. 7 by the mass in the circle segment OP'Q' of Fig. 7.

Using these two results, we obtain $k(x,b)\approx[G(x)-G(+\infty)]\arctan b$ for ε sufficiently small, and the depth of $\boldsymbol{\theta}=(b,0)$ is therefore

$$\begin{aligned}\min\{&\tfrac{1}{2}-(1-\varepsilon)[G(+\infty)-G(x)]\arctan b,\\ &\tfrac{1}{2}+\varepsilon/2-(1-\varepsilon)\,G(+\infty)\arctan b,\\ &\tfrac{1}{2}-\varepsilon/2+(1-\varepsilon)[G(+\infty)-G(x)]\arctan b\}.\end{aligned}\tag{16}$$

To find the fit with maximal depth, we have to maximize (16) in function of b.

First suppose that $\frac{1}{2}-(1-\varepsilon)[G(+\infty)-G(x)]\arctan b$ is the minimum of (16). This can only happen under the conditions

$$\frac{\varepsilon}{4(1-\varepsilon)[G(+\infty)-G(x)]}\leqslant\arctan(b)\leqslant\frac{\varepsilon}{2(1-\varepsilon)\,G(x)}$$

i.e. for $G(x)\leqslant(2/3)\,G(+\infty)$, and then $\frac{1}{2}-(1-\varepsilon)[G(+\infty)-G(x)]\arctan b=\lim_{v\downarrow x}(H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}))$ is maximal if $\boldsymbol{\theta}=(b,0)$ is minimal, i.e. if $\arctan(b)=\varepsilon/\{4\,(1-\varepsilon)[G(+\infty)-G(x)]\}$ with corresponding depth $\frac{1}{2}-\varepsilon/4$.

Secondly, $\frac{1}{2}+\varepsilon/2-(1-\varepsilon)\, G(+\infty)\arctan b$ is the minimum of (16) under the conditions

$$\arctan(b)\geqslant\frac{\varepsilon}{2\,(1-\varepsilon)\,G(x)}\qquad\text{and}\qquad\arctan(b)\geqslant\frac{\varepsilon}{(1-\varepsilon)[2G(+\infty)-G(x)]}$$

and $\frac{1}{2}+\varepsilon/2-(1-\varepsilon)\,G(+\infty)\arctan b=H_\varepsilon(A_{0,\boldsymbol{\theta}})+H_\varepsilon(B_{0,\boldsymbol{\theta}})$ is maximal if $\boldsymbol{\theta}=(b,0)$ is minimal. Therefore, if $G(x)\geqslant(2/3)\,G(+\infty)$ then $\frac{1}{2}+\varepsilon/2-(1-\varepsilon)\,G(+\infty)\arctan b$ is maximal iff $\arctan(b)=\varepsilon/\{(1-\varepsilon)[2G(+\infty)-G(x)]\}$ with corresponding depth $\frac{1}{2}+\varepsilon/2-\varepsilon G(+\infty)/[2G(+\infty)-G(x)]$. If $G(x)\leqslant(2/3)\,G(+\infty)$ then $\frac{1}{2}+\varepsilon/2-(1-\varepsilon)\,G(+\infty)\arctan b$ is maximal iff $\arctan(b)=\varepsilon/\{2\,(1-\varepsilon)\,G(x)\}$ with corresponding depth $\frac{1}{2}+\varepsilon/2-\varepsilon G(+\infty)/G(x)\leqslant\frac{1}{2}-\varepsilon/4$.

Thirdly, $\frac{1}{2}-\varepsilon/2+(1-\varepsilon)[G(+\infty)-G(x)]\arctan b$ is the minimum of (16) under the conditions

$$\arctan(b)\leqslant\frac{\varepsilon}{4\,(1-\varepsilon)[G(+\infty)-G(x)]}$$

and

$$\arctan(b)\leqslant\frac{\varepsilon}{(1-\varepsilon)[2G(+\infty)-G(x)]}$$

and $\frac{1}{2}-\varepsilon/2+(1-\varepsilon)[G(+\infty)-G(x)]\arctan b=\lim_{v\uparrow x}(H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}))$ is maximal if $\boldsymbol{\theta}=(b,0)$ is maximal. Therefore, if $G(x)\geqslant(2/3)\,G(+\infty)$ then $\frac{1}{2}-\varepsilon/2+(1-\varepsilon)[G(+\infty)-G(x)]\arctan b$ is maximal for $\arctan(b)=\varepsilon\{(1-\varepsilon)[2G(+\infty)-G(x)]\}$ with corresponding depth $\frac{1}{2}-\varepsilon/2+\varepsilon[G(+\infty)-G(x)]/[2G(+\infty)-G(x)]=\frac{1}{2}+\varepsilon/2-\varepsilon G(+\infty)/[2G(+\infty)-G(x)]$. If $G(x)\leqslant\phi(0)/3$ then $\frac{1}{2}-\varepsilon/2+(1-\varepsilon)[G(+\infty)-G(x)]\arctan b$ is maximal for $\arctan(b)=\varepsilon/\{4\,(1-\varepsilon)[G(+\infty)-G(x)]\}$ with corresponding depth $\frac{1}{2}-\varepsilon/4$.

It is easy to see that the maximal depth in (iv) is higher than the maximal depth in (i) and (ii). Since $\lim_{\varepsilon\to 0}(\frac{1}{2}-\varepsilon/4)=\frac{1}{2}$ and $\lim_{\varepsilon\to 0}[\frac{1}{2}+\varepsilon/2-\varepsilon G(+\infty)/(2G(+\infty)-G(x))]=\frac{1}{2}$ it follows that for ε sufficiently small the maximal depth in (iv) is also higher than the maximal depth in (iii). So for ε sufficiently small and $G(x)\leqslant(2/3)\,G(+\infty)$ we find $\lim_{\varepsilon\downarrow 0}(T_1^*/\tan[\varepsilon/\{4\,(1-\varepsilon)(G(+\infty)-G(x))\}])=1$ and for $G(x)\geqslant(2/3)\,G(+\infty)$ we find $\lim_{\varepsilon\downarrow 0}(T_1^*/\tan[\varepsilon/\{(1-\varepsilon)(2G(+\infty)-G(x))\}])=1$. Thus for $G(x)\leqslant(2/3)\,G(+\infty)$ the influence function becomes

$$IF((x,y),T_1^*,H)=\frac{\partial}{\partial\varepsilon}\tan\left(\frac{\varepsilon}{4\,(1-\varepsilon)[G(+\infty)-G(x)]}\right)\Bigg|_{\varepsilon=0}$$

$$=\frac{1}{4[G(+\infty)-G(x)]}$$

and for $G(x) \geqslant (2/3)\, G(+\infty)$ it becomes

$$IF((x, y), T_1^*, H) = \frac{\partial}{\partial \varepsilon} \tan\left(\frac{\varepsilon}{(1-\varepsilon)[2G(+\infty) - G(x)]}\right)\Bigg|_{\varepsilon=0}$$

$$= \frac{1}{[2G(+\infty) - G(x)]}.$$

By symmetry we obtain the results for (x, y) in the other quadrants.

Let us now derive the influence function of the DR intercept T_2^*. Since H is spherically symmetric, placing a mass ε in $(-x, y)$ has the same effect on the intercept T_2^* as placing this mass in (x, y). Therefore $IF((x, y), T_2^*, H) = IF((-x, y), T_2^*, H)$ wherever it exists, hence

$$IF((x, y), T_2^*, H) = \frac{1}{2} IF((x, y), T_2^*, H) + \frac{1}{2} IF((-x, y), T_2^*, H)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{T_2^*(H_\varepsilon) - T_2^*(H)}{\varepsilon}$$

where now $H_\varepsilon = (1-\varepsilon)\, H + (\varepsilon/2)\, \Delta_{(x, y)} + (\varepsilon/2)\, \Delta_{(-x, y)}$. By symmetry, $T^*(H_\varepsilon)$ is horizontal. Considering all $\boldsymbol{\theta} = (0, a)$ with $a \in \mathbb{R}$, we now find that $T_2^*(H_\varepsilon)$ maximizes $\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}})\}$ where the minimum is over all v with $H_\varepsilon(x = v) = 0$, hence $|v| \neq x$ and by symmetry it suffices to take $v \geqslant 0$. Take $x > 0$ and $y > 0$.

(i) First consider a horizontal line $\boldsymbol{\theta} = (0, a)$ with intercept $a \leqslant 0$ as in Figure 8a. For $v > x$ we find

$$H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}) = (1-\varepsilon)\, H(A_{v,\boldsymbol{\theta}}) + (1-\varepsilon)\, H(B_{v,\boldsymbol{\theta}})$$

$$= (1-\varepsilon)\, H(v, a) + (1-\varepsilon)[1 - H_X(v)$$

$$- H_Y(a) + H(v, a)],$$

which is minimal if $v \to \infty$. Therefore $\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}})\} \leqslant \lim_{v \uparrow +\infty} (H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}})) = (1-\varepsilon)\ H_Y(a) \leqslant (1-\varepsilon)/2$ since $H_Y(a) \leqslant \frac{1}{2}$ for $a \leqslant 0$. Therefore, any fit $\boldsymbol{\theta} = (0, a)$ with $a \leqslant 0$ has depth at most $(1-\varepsilon)/2$.

(ii) Now consider a horizontal line $\boldsymbol{\theta} = (0, a)$ with intercept $a > y$ as in Fig. 8b. For $v > x$ we have

$$H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}}) = (1-\varepsilon)\, H(C_{v,\boldsymbol{\theta}}) + (1-\varepsilon)\, H(D_{v,\boldsymbol{\theta}})$$

$$= (1-\varepsilon)[H_X(v) - H(v, a)] + (1-\varepsilon)[H_Y(a) - H(v, a)],$$

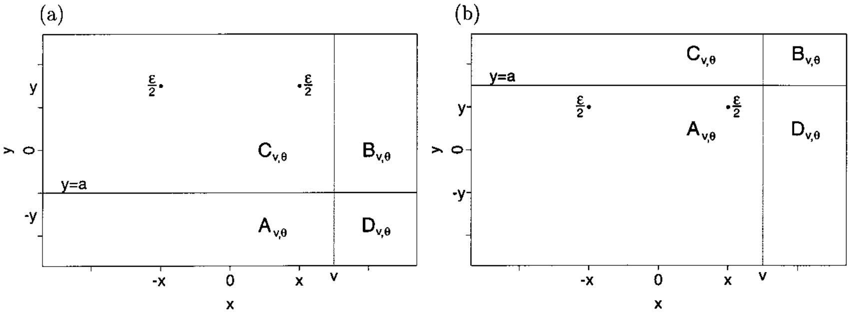

FIG. 8. Example of the regions $A_{v,\boldsymbol{\theta}}$, $B_{v,\boldsymbol{\theta}}$, $C_{v,\boldsymbol{\theta}}$, and $D_{v,\boldsymbol{\theta}}$ for (a) some $a<0$ and $v>x$ and (b) for some $a>y$ and $v>x$.

which is minimal if $v \to \infty$. Thus $\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}),\ H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\} \leqslant \lim_{v\uparrow+\infty}(H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}})) = (1-\varepsilon)[1-H_Y(a)] < (1-\varepsilon)/2$ since $H_Y(a)>\frac{1}{2}$ for $a>y$. Hence also any fit $\boldsymbol{\theta}=(0,a)$ with $a>y$ has depth at most $(1-\varepsilon)/2$.

(iii) If $a=y$, analogous calculations yield $\min_v \{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}),\ H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\} = \varepsilon/2+(1-\varepsilon)[1-H_Y(y)] < \frac{1}{2}$ for $y>0$.

(iv) Finally, consider a horizontal line $\boldsymbol{\theta}=(0,a)$ with intercept $0<a<y$. For $v>x$ we obtain

$$\begin{aligned} H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}) &= (1-\varepsilon)(H(A_{v,\boldsymbol{\theta}})+H(B_{v,\boldsymbol{\theta}})) \\ &= (1-\varepsilon)[1+2H(v,a)-H_X(v)-H_Y(a)] \end{aligned}$$

and

$$\begin{aligned} H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}) &= \varepsilon+(1-\varepsilon)(H(C_{v,\boldsymbol{\theta}})+H(D_{v,\boldsymbol{\theta}})) \\ &= \varepsilon+(1-\varepsilon)[H_X(v)+H_Y(a)-2H(v,a)]. \end{aligned}$$

Therefore $\min_{v>x}\{H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}})\} = \min\{(1-\varepsilon)\lim_{v\to x}(H(A_{v,\boldsymbol{\theta}})+H(B_{v,\boldsymbol{\theta}})),\ \varepsilon+(1-\varepsilon)\ \lim_{v\uparrow+\infty}(H(C_{v,\boldsymbol{\theta}})+H(D_{v,\boldsymbol{\theta}}))\} = \min\{(1-\varepsilon)[1+2H(x,a)-H_X(x)-H_Y(a)], \varepsilon+(1-\varepsilon)[1-H_Y(a)]\}$. For $0\leqslant v<x$ we obtain

$$H_\varepsilon(A_{v,\boldsymbol{\theta}})+H_\varepsilon(B_{v,\boldsymbol{\theta}}) = \varepsilon/2+(1-\varepsilon)[1+2H(v,a)-H_X(v)-H_Y(a)]$$

and

$$H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}) = \varepsilon/2+(1-\varepsilon)[H_X(v)+H_Y(a)-2H(v,a)].$$

Since $v \geqslant 0$ and $a > 0$ we have $\min_{0 \leqslant v < x} \{H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}), H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}})\} = \lim_{v \to x} (H_\varepsilon(C_{v,\boldsymbol{\theta}}) + H_\varepsilon(D_{v,\boldsymbol{\theta}})) = \varepsilon/2 + (1-\varepsilon)[H_X(x) + H_Y(a) - 2H(x,a)]$. Therefore, the depth of a fit $\boldsymbol{\theta} = (0, a)$ with $0 < a < y$ is

$$\min\{(1-\varepsilon)[1 + 2H(x,a) - H_X(x) - H_Y(a)], \varepsilon + (1-\varepsilon)[1 - H_Y(a)],$$
$$\varepsilon/2 + (1-\varepsilon)[H_X(x) + H_Y(a) - 2H(x,a)]\}.$$

Since H is spherically symmetric we find

$$\lim_{\varepsilon \downarrow 0} \frac{2H(x,a) - H_X(x)}{H_{X|Y}(x|0)[2H_Y(a) - 1]} = 1$$

which means that for ε small enough we can approximate the probability mass $\int_{-\infty}^{x} \int_{-a}^{a} h(x,y)\,dx\,dy$ by $\int_{-\infty}^{x} \int_{-a}^{a} h_{X|Y}(x|0)\,h_Y(y)\,dy$. For ε sufficiently small the depth of $\boldsymbol{\theta} = (0, a)$ is therefore

$$\begin{aligned} \min\{&(1-\varepsilon)[1 - H_Y(a) + H_{X|Y}(x|0)[2H_Y(a) - 1]], \\ &\varepsilon + (1-\varepsilon)[1 - H_Y(a)], \\ &\varepsilon/2 + (1-\varepsilon)[H_Y(a) - H_{X|Y}(x|0)[2H_Y(a) - 1]]\}. \end{aligned} \tag{17}$$

To find the line with maximal depth we have to maximize (17) in function of a.

First suppose that $(1-\varepsilon)[1 - H_Y(a) + H_{X|Y}(x|0)[2H_Y(a) - 1]]$ is the minimum of (17). This can only happen when both conditions

$$H_Y(a) \leqslant \frac{1}{2} + \frac{\varepsilon}{2(1-\varepsilon)\,H_{X|Y}(x|0)}$$

and

$$H_Y(a) \leqslant \frac{1}{2} + \frac{\varepsilon}{4(1-\varepsilon)[2H_{X|Y}(x|0) - 1]}$$

hold, and $(1-\varepsilon)[1 - H_Y(a) + H_{X|Y}(x|0)[2H_Y(a) - 1]] = \lim_{v \downarrow x}(H_\varepsilon(A_{v,\boldsymbol{\theta}}) + H_\varepsilon(B_{v,\boldsymbol{\theta}}))$ is maximal if $\boldsymbol{\theta} = (0, a)$ is maximal. Therefore, if $H_{X|Y}(x|0) \leqslant 2/3$ then $(1-\varepsilon)[1 - H_Y(a) + H_{X|Y}(x|0)[2H_Y(a) - 1]]$ is maximal for $H_Y(a) = \frac{1}{2} + \varepsilon/\{2(1-\varepsilon)\,H_{X|Y}(x|0)\}$ with corresponding depth $\frac{1}{2} + \varepsilon/2 - \varepsilon/(2H_{X|Y}(x|0))$. If $H_{X|Y}(x|0) \geqslant 2/3$ then $(1-\varepsilon)[1 - H_Y(a) + H_{X|Y}(x|0)[2H_Y(a) - 1]]$ is maximal for $H_Y(a) = \frac{1}{2} + \varepsilon/\{4(1-\varepsilon)[2H_{X|Y}(x|0) - 1]\}$ with corresponding depth $\frac{1}{2} - \varepsilon/4$.

Secondly, $\varepsilon+(1-\varepsilon)[1-H_Y(a)]$ is the minimum of (17) under the conditions

$$H_Y(a)\geqslant\frac{1}{2}+\frac{\varepsilon}{2(1-\varepsilon)\,H_{X|Y}(x|0)}$$

and

$$H_Y(a)\geqslant\frac{1}{2}+\frac{\varepsilon}{4(1-\varepsilon)[1-H_{X|Y}(x|0)]}$$

and $\varepsilon+(1-\varepsilon)[1-H_Y(a)]=\lim_{v\uparrow+\infty}(H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}))$ is maximal if $\boldsymbol{\theta}=(0,a)$ is minimal. Therefore, if $H_{X|Y}(x|0)\leqslant 2/3$ then $\varepsilon+(1-\varepsilon)[1-H_Y(a)]$ is maximal iff $H_Y(a)=\frac{1}{2}+\varepsilon/\{2(1-\varepsilon)\,H_{X|Y}(x|0)\}$ with corresponding depth $\frac{1}{2}+\varepsilon/2-\varepsilon/(2H_{X|Y}(x|0))$. If $H_{X|Y}(x|0)\geqslant 2/3$ then $\varepsilon+(1-\varepsilon)[1-H_Y(a)]$ is maximal iff $H_Y(a)=\frac{1}{2}+\varepsilon/\{4(1-\varepsilon)[1-H_{X|Y}(x|0)]\}$ with corresponding depth $\frac{1}{2}+\varepsilon/2-\varepsilon\,[4(1-H_{X|Y}(x|0))]\leqslant\frac{1}{2}-\varepsilon/4$.

Thirdly, $\varepsilon/2+(1-\varepsilon)[H_Y(a)-H_{X|Y}(x|0)[2H_Y(a)-1]]$ is the minimum of (17) under the conditions

$$\frac{1}{2}+\frac{\varepsilon}{4(1-\varepsilon)[2H_{X|Y}(x|0)-1]}\leqslant H_Y(a)\leqslant\frac{1}{2}+\frac{\varepsilon}{4(1-\varepsilon)[1-H_{X|Y}(x|0)]}$$

i.e. $H_{X|Y}(x|0)\geqslant 2/3$. Then $\varepsilon/2+(1-\varepsilon)[H_Y(a)-H_{X|Y}(x|0)[2H_Y(a)-1]]=\lim_{v\downarrow x}(H_\varepsilon(C_{v,\boldsymbol{\theta}})+H_\varepsilon(D_{v,\boldsymbol{\theta}}))$ is maximal if $\boldsymbol{\theta}=(0,a)$ is minimal, i.e. if $H_Y(a)=\frac{1}{2}+\varepsilon/\{4(1-\varepsilon)[2H_{X|Y}(x|0)-1]\}$ with corresponding depth $\frac{1}{2}-\varepsilon/4$.

It can easily be seen that the maximal depth in (iv) is higher than those in (i) and (ii). Since $\lim_{\varepsilon\to 0}[\frac{1}{2}+\varepsilon/2-\varepsilon/(2H_{X|Y}(x|0))]=\frac{1}{2}$ and $\lim_{\varepsilon\to 0}(\frac{1}{2}-\varepsilon/4)=\frac{1}{2}$ it follows that for ε sufficiently small the maximal depth in (iv) is also higher than the maximal depth in (iii). So for ε sufficiently small and $H_{X|Y}(x|0)\leqslant 2/3$ we have $\lim_{\varepsilon\downarrow 0}[T_2^*/H_Y^{-1}(\frac{1}{2}+\varepsilon/\{2(1-\varepsilon)\,H_{X|Y}(x|0)\})]=1$. For $H_{X|Y}(x|0)\geqslant 2/3$ we have $\lim_{\varepsilon\downarrow 0}[T_2^*/H_Y^{-1}(\frac{1}{2}+\varepsilon/\{4(1-\varepsilon)(2H_{X|Y}(x|0)-1)\})]=1$. Hence, for $H_{X|Y}(x|0)\leqslant 2/3$ the influence function becomes

$$IF((x,y),T_2^*,H)=\frac{\partial}{\partial\varepsilon}\,H_Y^{-1}\left(\frac{1}{2}+\frac{\varepsilon}{2(1-\varepsilon)\,H_{X|Y}(x|0)}\right)\Bigg|_{\varepsilon=0}$$

$$=\frac{1}{2h_Y(0)\,H_{X|Y}(x|0)}$$

and for $H_{X|Y}(x|0) \geqslant 2/3$ it becomes

$$IF((x, y), T_2^*, H) = \frac{\partial}{\partial \varepsilon} H_Y^{-1}\left(\frac{1}{2} + \frac{\varepsilon}{4(1-\varepsilon)[2H_{X|Y}(x|0)-1]}\right)\Bigg|_{\varepsilon=0}$$

$$= \frac{1}{4h_Y(0)(2H_{X|Y}(x|0)-1)}.$$

We obtain the results for (x, y) in the other quadrants by symmetry.

(b) For $H = N_2(\mathbf{0}, I)$ the influence function of the slope T_1^* follows from (a) with $G(t) = \int_0^{t^2} g(u)\, du = 1/(2\pi) \int_0^{t^2} \exp(-u/2)\, du = 2\phi(0)[\phi(0) - \phi(t)]$. The influence function of the intercept T_2^* for $H = N_2(\mathbf{0}, I)$ follows from (a) with $h_Y(0) = \phi(0)$ and $H_{X|Y}(|x|\,|0) = \Phi(|x|)$.

REFERENCES

1. Z. Bai and X. He, Asymptotic Distributions of the Maximal Depth Estimators for Regression and Multivariate Location, *Ann. Statist.*, to appear.
2. F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel, "Robust Statistics: The Approach Based on Influence Functions," John Wiley, New York, 1986.
3. X. He and S. Portnoy, Asymptotics of the deepest line, *in* "Applied Statistical Science III: Nonparametric Statistics and Related Topics" (S. E. Ahmed, M. Ahsanullah, and B. K. Sinha, Eds.), pp. 71–81, Nova Science Publishers Inc, New York, 1998.
4. P. J. Rousseeuw, Least median of squares regression, *J. Amer. Statist. Assoc.* **79** (1984), 871–880.
5. P. J. Rousseeuw, C. Croux, and O. Hössjer, Sensitivity functions and numerical analysis of the repeated median slope, *Computational Statistics* **10** (1995), 71–90.
6. P. J. Rousseeuw and M. Hubert, Regression depth, *J. Amer. Statist. Assoc.* **94** (1999), 388–402.
7. P. J. Rousseeuw, S. Van Aelst, and M. Hubert, Rejoinder to the discussion of regression depth, *J. Amer. Statist. Assoc.* **94** (1999), 419–433.
8. P. J. Rousseeuw and V. J. Yohai, Robust regression by means of S-estimators, *in* "Robust and Nonlinear Time Series Analysis," Lecture Notes in Statistics No. 26 (J. Franke, W. Härdle, and R. D. Martin, Eds.), pp. 256–272, Springer, New York, 1984.
9. J. W. Tukey, Mathematics and the picturing of data, *Proc. Int. Congress Math.*, Vancouver, **2** (1975), 523–531.