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Gradings of finite support. Application to injective objects

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0. Introduction

Let R be a group-graded ring. In this paper we study the relationship between injective objects in R-gr and in R-mod. It is well-known that gr-injectives, i.e. injective objects in R-gr, need not be injective in R-mod. However, in [10], the second author showed that if R has finite support, then gr-injective modules with finite support are injective in R-mod. We generalize this result by showing that the restriction that R have finite support is unnecessary.

Suppose M is a graded R-module with finite support, R also with finite support. In the first section we show that although sometimes one can grade M and R by a finite group, while preserving the homogeneous components of the grading, this is not always possible. Thus the theory of graded rings and modules with finite support does not coincide with the theory of finite group gradings.

In Section 2, the full subcategory C_X of R-gr of graded R-modules with support in $X \subseteq G$ is introduced. We show that the forgetful function $U_X: C_X \to R$-mod, has a right adjoint, and if X is finite, also a left adjoint. We determine necessary and sufficient conditions for C_X to be equivalent to a module category. If X is finite, either C_X is zero or equivalent to be module category, S/I-mod, where S is Quinn's smash product.

Section 3 uses the category C_X to establish the main result, namely that gr-injectives with finite support are injective. As corollaries, we give necessary and sufficient conditions for an injective R-modules (injective indecomposable R-module) to be gradable if G is finite (supp R is finite). Finally, we show that if G is infinite, supp (R) finite and every gr-injective module is injective, then R is left noetherian, thus giving a converse to a result of the second author [10].

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0.1. Preliminaries

All rings are unitary rings and by an R-module we mean a left R-module. Let G be a group, multiplicatively written, with identity element 1. When $R = \bigoplus_{g \in G} R_g$ is a G-graded ring we denote the identity element of R also by 1, hoping that this will not create confusion. The category of R-modules is denoted by ***R-mod*** and the category of graded R-modules is denoted by ***R-gr***. For full detail on these notions we refer to [11].

Recall from loc.cit. that R-gr is a Grothendieck category.

For $M \in R$- gr we define the *λ-suspension of* M to be $M(\lambda) \in R$-gr defined by putting the following gradation on the R-module M, $M(\lambda)_g = M_{g\lambda}$ for $\lambda \in G$, $g \in G$. If $M, N \in R$-gr then an R-linear map $f{:}M \to N$ is said to be a *morphism of degree* g, for $g \in G$, if for all $h \in G$ we have $f(M_h) \subset M_{hg}$. To $\tau \in G$ we associate the exact functor $(-)_\tau{:}R\text{-gr} \to R_1\text{-mod}$, defined by $M \mapsto M_\tau$. To an $N \in R_1$-mod we may associate a graded R-module $\mathrm{Coind}(N)$ constructed as follows:

$$\mathrm{Coind}(N)_g = \{f \in \mathrm{Hom}_{R_1}(R, N),\ f(R_h) = 0 \text{ for } h \neq g^{-1}\},$$

$$\mathrm{Coind}(N) = \bigoplus_{g \in G} \mathrm{Coind}(N)_g, \text{ cf. [10].}$$

This construction defines the coinduced functor $\mathrm{Coind}(-)$. Let T_λ: $R\text{-gr} \to R\text{-gr}$ be the λ-shift functor defined by $T_\lambda(M) = M(\lambda)$, then it is known, [10, Theorem 1.1], that $T_{\lambda^{-1}} \circ \mathrm{Coind}(-)$ is a right adjoint of the functor $(-)_\lambda$.

For an $M \in R$-gr we define the *support of* M to be the set $\mathrm{supp} M) = \{g \in G, M_g \neq 0\}$ and M is said to have finite support if $\mathrm{supp}(M)$ is finite. An $M \in R$-gr is said to be g-faithful for $g \in G$ if $M \neq 0$ and for every graded non-zero submodule N of M we have $N_g \neq 0$. Full detail on the general theory of graded rings and modules, in particular concerning gr-injective modules, may be found in [11]: notation is as in loc.cit.

1. Groups without finite embedding property

For G a finite group, many powerful connections between the properties of a graded module M as an object in R-mod, R-gr and R_1-mod exist; we study when we can use this theory to study modules over graded rings with finite support. An optimistic approach for dealing with the finite support case would consist in trying to change the grading group such that the problem reduces to a problem concerning a gradation by a finite group but without changing the homogeneous components of the ring or module involved. The property that would allow such a change of gradation is the so-called "finite embedding" property.

Definition 1.1. A group G is an *FE-group* if for every finite subset X of G there exists a finite group $(H, *)$ such that $X \subset H$ and for every $x, y \in X$ such that $xy \in X$ we have $x * y = xy$.

Recall that a group G is residually finite if the intersection of its normal subgroups of finite index reduces to $\{1\}$; G is locally residually finite if every finitely generated subgroup of G is residually finite. Examples of locally residually finite groups are: abelian groups, poly-cyclic-by-finite groups, nilpotent groups, solvable groups, free groups, cf. [15].

Proposition 1.2. *A locally residually finite group is an FE-group.*

Proof. Let X be a finite set contained in G. Up to replacing G by the group $\langle X \rangle$ generated by X we may assume that G is residually finite. For any $z \in Y = \{xy^{-1} \mid x, y \in X \text{ and } x \neq y\}$ there exists a normal subgroup N_z of G such that N_z has finite index and $z \notin N_z$. So we obtain $N = \bigcap_{z \in Y} N_z$, a normal subgroup of finite index in G, such that $N \cap Y = \emptyset$. Hence the finite group $H = G/N$ satisfies the conditions, up to identifying X with its image in H.

We also note that there exist FE-groups which are not locally residually finite (see [17]). Now let G be an FE-group, $R = \bigoplus_{g \in G} R_g$ a G-graded ring of finite support X. Assume that H is a finite group with the properties mentioned in Definition 1.1, then we may view R as an H-graded ring by putting $R_h = 0$ when $h \notin X$ and for $x \in X$ of course R_x is the homogeneous part of degree x. In the case of grading by an FE-group the finite support case may be transformed to a case of a finite group grading. □

An illustration of this is the following.

Corollary 1.3. *Let G be an FE-group, R a G-graded ring of finite support and M an R-module, then the following assertions hold*:

1. *If ${}_RM$ is Noetherian, resp. Artinian, Semisimple of finite length, then so is ${}_{R_1}M$.*
2. *If ${}_RM$ has Krull-resp. Gabriel-dimension then so does ${}_{R_1}M$ and moreover the dimensions are equal.*

Proof. Reduce to the finite case as pointed out above and apply [3] and [13].

Remarks 1.4. (1) Let R and G be as before and let $M \in R$-gr have finite support then $X = \mathrm{supp}(R) \cup \mathrm{supp}(M)$ may be embedded in a finite group and so we may extend results of [9], e.g. Theorem 2.1 of loc.cit., to the case of finite support gradations by FE-groups.

(2) If $\mathrm{supp}(R)$ is finite and $N \in R$-mod then we may embed N in the graded R-module $\mathrm{Coind}_R(N)$, that is exactly $\mathrm{Hom}_{R_1}(R, N)$ when considered as an R-module. This embedding is given by the R-linear map $\varphi : N \to \mathrm{Coind}_R(N)$, with $\varphi(n)(r) = rn$ for $n \in N$, $r \in R$. This will allow to use the Remark in 1 occasionally.

(3) Consider an arbitrary graded ring R and an $M \in R$-gr having finite support $\mathrm{supp}(M) = X$. Clearly, $J = R(\sum_{g \notin XX^{-1}} R_g)R$ is a graded ideal such that $JM = 0$.

The ring $S = R/J$ is G-graded but supp(S) is finite, moreover $M \in S$-gr and the lattices of subobjects of M in R-mod, respectively in S-mod, are the same. Therefore, problems relating the finiteness conditions on M may again be reduced to the situation considered in 1.

If all groups were FE-groups than the theory of finitely supported gradings would be included in the theory of finite group gradings but luckily enough this is not the case. The aim of this section was to arrive at the following example.

Example 1.5. There exist non-FE-groups!

Let Δ be an infinite simple finitely presented group, cf. [4]. If Δ is presented as $\langle S, R \rangle$ with S and R finite then $\Delta = F(S)/R^{F(S)}$, where $F(S)$ is the free group generated by S and $R^{F(S)}$ is the normalizer of the set R in $F(S)$. Let $p: F(S) \to \Delta$ be the canonical projection and put $X = \{p(\alpha), \alpha \in S \text{ or } \alpha \text{ is a subword of an element of } R\}$. Suppose that X is contained in a finite group $(H, *)$ such that $x * y = xy$ for every $x, y \in X$ such that $xy \in X$. If $\pi: F(S) \to H$ is the natural group morphism then $R \subset \operatorname{Ker} \pi$ and therefore $R^{F(S)} \subset \operatorname{Ker} \pi$. Consequently, we arrive at a commutative diagram of group morphisms:

$$\begin{tikzcd}
\mathrm{F(S)} \arrow[r, "\pi"] \arrow[d, "\mathrm{p}"] & \mathrm{H} \\
\Delta \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

Hence $\operatorname{Im} \psi = \Delta / \operatorname{Ker} \psi$, and since $S \subset \operatorname{Im} \psi$ we have $\operatorname{Ker} \psi \neq \Delta$. Therefore, $\operatorname{Ker} \psi = 1$ and $\operatorname{Im} \psi \simeq \Delta$, contradicting the finiteness of φ.

2. The category of X-supported modules

Consider a subset X of G and let $\mathscr{C}_X$ be the full subcategory of R-gr consisting of all graded R-modules with support contained in X. Let $U_X: \mathscr{C}_X \to R$-mod be the forgetful functor.

It is not hard to see that $\mathscr{C}_X$ is a localizing subcategory of the Grothendieck category R-gr, hence $\mathscr{C}_X$ is a Grothendieck category. Moreover, $\mathscr{C}_X$ is closed under direct products and thus we may define the radical s_X on R-gr by putting $s_X(M)$ equal to the "smallest" graded subobject of M such that $M/s_X(M) \in \mathscr{C}_X$, in fact $s_X(M) = \bigcap \{K, K \subset M \text{ and } M/K \in \mathscr{C}_X\} = R(\bigoplus_{g \notin X} M_g)$.

Theorem 2.1. *The following assertions hold:*

(i) *U_X has a right adjoint.*

(ii) *If X is finite then U_X has a left adjoint.*

Proof. The forgetful functor $U: R\text{-gr} \to R\text{-mod}$ has a right adjoint F associating to $N \in R\text{-mod}$, $F(N) = \bigoplus_{x \in G} N^x$, where $\{N^x,\ x \in G\}$ is a family of copies of N and the R-module structure of $F(N)$ is induced by $r_g n^x = (r_g n)^{gx}$ for $g \in G, r_g \in R_g$. On the other hand, if $u: N \to N'$ is R-linear then $F(u)$ is defined by $F(u)(n^x) = (u(n))^x$. Now we define another functor $F_X: R\text{-mod} \to \mathscr{C}_X$, $N \mapsto F_X(N) = \sum\{K,\ K$ a graded submodule of $F(N)$ such that $K \in \mathscr{C}_X\}$, while for $u \in \mathrm{Hom}_R(N, N')$ we take $F_X(u)$ to be the restriction $F(u) | F_X(N)$.

(i) For $M \in \mathscr{C}_X$ and $N \in R\text{-mod}$ we define:

$\varphi(M,N): \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R\text{-gr}}(M, F_X N))$,

$\psi(M,N): \mathrm{Hom}_{R\text{-gr}}(M, F_X(N)) \to \mathrm{Hom}_R(M,N)$, by putting:

$\varphi(M,N)(u)(m_x) = (u(m_x))^x$, for $u \in \mathrm{Hom}_R(M,N)$, $x \in X$, $m_x \in M_x$, and $\psi(M,N)(v)(m_x) = \pi_x(v(m_x))$ for $v \in \mathrm{Hom}_{R\text{-gr}}(M, F_X(N))$, $x \in X, m_x \in M_x$, where $\pi_x: F_X(N) \to N$ is the projection on the x-homogeneous component. It is not hard to verify that $\psi(M,N) \circ \varphi(M,N) = \mathrm{Id}$ and $\varphi(M,N) \circ \psi(M,N) = \mathrm{Id}$, establishing that F_X is a right adjoint for U_X as desired.

(ii) Assume that X is finite. Note that for any $N \in R\text{-mod}$, $s_X(F(N)) = R(\bigoplus_{g \notin X} N^g)$. Now we define a functor $T_X: R\text{-mod} \to \mathscr{C}_X$ by $T_X(N) = F(N)/s_X(F(N))$ and for $u \in \mathrm{Hom}_R(N, N')$, the graded morphism $T_X(u): T_X(N) \to T_X(N')$ is induced by $F(u)$ (taking into account that $F(u)(s_X(F(N)) \subset s_X(F(N'))$.

Consider $M \in \mathscr{C}_X$ and $N \in R\text{-mod}$ and define: $\delta(N,M): \mathrm{Hom}_{R\text{-gr}}(T_X(N), M) \to \mathrm{Hom}_R(N,M)$, by $\delta(N,M)(u)(n) = u(\alpha(n)$ modulo $s_X(F(N)))$, where $u \in \mathrm{Hom}_{R\text{-gr}}(T_X(N), M)$, $n \in N$ and $\alpha(n) = \sum_{x \in X} n^x$. That $\delta(N,M)(u) \in \mathrm{Hom}_R(N,M)$ is easily verified up to verifying that $\delta(N,M)(u)(r_\tau n) = r_\tau \delta(N,M)(u)(n)$.

The latter will follow from $\alpha(r_\tau n) - r_\tau \alpha(n) \in s_X(F(N))$.

Now

$$\alpha(r_\tau n) - r_\tau \alpha(n) = \sum_{x \in X - \tau X} (r_\tau n)^x - \sum_{g \in \tau X - X} (r_\tau n)^g.$$

Obviously, if $g \in \tau X - X$ then

$$(r_\tau n)^g \in R\left(\bigoplus_{h \notin X} N^h\right) = s_X(F(N)).$$

On the other hand, if $x \in X - \tau X$, putting $y = \tau^{-1} x$, then $\tau y = x \notin \tau X$ and $y \notin X$. Hence $(r_\tau n)^x = r_\tau n^y \in s_X(F(N))$. Define $\eta(N,M): \mathrm{Hom}_R(N,M) \to \mathrm{Hom}_{R\text{-gr}}(T_X(N), M)$ by $\eta(N,M)(v)(\eta^x\text{-modulo } s_X(F(N))) = v(n)_x$, for $n \in N$, $x \in X$. Let us verify that this is well-defined; indeed if $n^x - p^x \in s_X(F(N))$ then $n^x - p^x = \sum_{g \notin X} r_{xg^{-1}} n_g^g$ and $n - p = \sum_{g \notin X} r_{xg^{-1}} n_g$. Then, $v(n)_x - v(p)_x = \sum_{g \notin X} r_{xg^{-1}} v(n_g)_g = 0$. That $\eta(N,M)$ is a graded morphism and R-linear is clear. A straightforward verification, yields that T_X is a left adjoint of U_X. $\square$

Corollary 2.2 (Năstăsescu et al. [12]). *If G is a finite group then F is a left and right adjoint for U. Consequently, if $M \in R$-gr is* gr*-injective then M is injective in R-mod.*

Proof. Put $X = G$ and observe; that $F_X = T_X = F$. Since F is a left and right adjoint it is an exact functor. But if U has an exact left adjoint then it carries injective objects to injectives (see IV.9.5. of [16] or [2] for the latter implication). □

Corollary 2.3. *When X is finite then $\mathscr{C}_X$ has a finitely generated projective generator. In particular, either $\mathscr{C}_X = 0$ or $\mathscr{C}_X$ is equivalent to a module-category.*

Proof. By the theorem T_X is a left adjoint by U_X. It is clear that $T_X(R)$ is projective in $\mathscr{C}_X$ and T_X has an exact right adjoint (see [2] again). We have

$$T_X(R) = F(R)/s_X(F(R)) \cong \bigoplus_{\lambda \in G} R(\lambda)/s_X\left(\bigoplus_{\lambda \in G} R(\lambda)/s_X(R(\lambda))\right).$$

Now we also have

$$s_X(R(\lambda)) = R \sum_{g \notin X} R(\lambda)_g = R \sum_{g \notin X} R_{g\lambda}.$$

Therefore, for any $\lambda \notin X^{-1} = \{x^{-1},\ x \in X\}$ we have $s_X(R(\lambda)) = R(\lambda)$ and $T_X(R) = \bigoplus_{x \in X} P^x$, $P^x = (R/A^x)(x^{-1})$ with $A^x = R(\sum_{g \notin X} R_{gx^{-1}})$ a graded left ideal of R. Since $T_X(R)$ is now obviously finitely generated we only have to verify that $T_X(R)$ is a generator of $\mathscr{C}_X$. Given $M \in \mathscr{C}_X$ and $f: R^{(I)} \to M$ an epimorphism in R-mod for some set I then we obtain the epimorphism $T_X(f): T_X(R)^{(I)} \to T_X(M)$ in $\mathscr{C}_X$, because T_X is right exact and commutes with direct sums. But

$$T_X(M) = F(M)/s_X(F(M)) \cong \bigoplus_{\lambda \in G} M(\lambda)/s_X\left(\bigoplus_{\lambda \in G} M(\lambda)\right) \cong \bigoplus_{\lambda \in G} M(\lambda)/s_X(M(\lambda))$$

and thus $M \in \mathscr{C}_X$ yields that $s_X(M) = 0$ or M is a quotient object of $T_X(M)$ in $\mathscr{C}_X$ hence generated by $T_X(R)$. □.

Remarks 2.4. (i) For the sake of completeness, we recall by [10, Theorem 3.2] that if X is a subset of G such that $G - X$ is finite, then the quotient category R-gr/$\mathscr{C}_X$ is equivalent to a category of modules.

(ii) Of course, if $M \in R$-gr is gr-injective then M need not be injective in general, cf. [11], Example 1.2.6.

The trick of looking for a left adjoint of U cannot be extended beyond the finite group case because of the following.

Proposition 2.5. *The forgetful functor $U: R$-gr $\to R$-mod has a left adjoint if and only if G is finite.*

Proof. When U has a left adjoint then U commutes with arbitrary direct products in the sense that the natural R-morphism $f: U(\prod_{i \in I}^{\mathrm{gr}} M_i) \to \prod_{i \in I} U(M_i)$ is an isomorphism for any set I, and any family of graded R-modules $\{M_i, i \in I\}$, where $\prod_{i \in I}^{\mathrm{gr}} M_i = \bigoplus_{g \in G}(\prod_{i \in I} M_i, g)$. Put $I = G$, $M_i = R(i^{-1})$ for $i \in G$ and $m_i \in M_i$. Then $(m_i)_{i \in I}$ cannot be in Im f when infinitely many different degrees can appear, hence it follows that G is finite. □

In the proof of Corollary 2.3 we constructed a family $(P^x)_{x\in X}$ of projective generators for $\mathscr{C}_X$ in case X if finite.

We now will show that in general the same construction still provides a family of projective generators, hence if $X \subset G$ then we introduce

$$A^x = R\left(\sum_{g\notin X} R_{gx^{-1}}\right)$$

and $P^x = (R/A^x)(x^{-1})$.

Proposition 2.6. (Dăscălescu and Del Rio [1]). *With notation as above, for every $x \in X$, P^x is projective in $\mathscr{C}_X$ and $(P^x)_{x\in X}$ is a family of generators for $\mathscr{C}_X$. In particular, $\bigoplus_{x\in X} P^x$ is a projective generator of $\mathscr{C}_X$.*

Proof. Consider an epimorphism $u: M \to N$ in $\mathscr{C}_X$ together with a graded morphism $f: P^x \to N$. Let $f(\bar{1}) = n \in N_x$ and choose $m \in M_x$ such that $u(m) = n$, then $\varphi: P^x \to M$, $\bar{r} \mapsto rm$ is well-defined because $A^x m = R(\sum_{g\notin X} R_{gx^{-1}} m)$ is in $R(\sum_{g\notin X} M_g) = 0$. Clearly φ is a graded morphism satisfying $u\varphi = f$. For the second statement consider $M \in \mathscr{C}_X$ and suppose that N is a proper subobject of M. Then there exist $x \in X$, $m \in M_x - N_x$ such that the map φ defined above satisfies $\operatorname{Im}\varphi \subset N$ and this completes the proof. □

Corollary 2.7. *We have $\mathscr{C}_X = 0$ if and only if for every $x \in X$ the following relation holds:*

$$\sum_{g\notin X} R_{xg^{-1}} R_{gx^{-1}} = R_1$$

(*convention: the sum over an empty index set is* 0).

Proof. We have $\mathscr{C}_X = 0$ if and only if $P^x = 0$ for every $x \in X$. However, $P^x = 0$ happens only if $A^x = R$ or $(A^x)_1 = R_1$, but the latter is exactly the desired relation.

If ${}_Y\mathscr{C}$ is the category of graded right R-modules having support contained in Y then the foregoing construction may be redone completely and again we arrive at ${}_Y\mathscr{C} = 0$ if and only if for every $y \in Y$, $\sum_{g\notin Y} R_{y^{-1}g} R_{g^{-1}y} = R_1$. From this we obtain the left–right symmetry with respect to $\mathscr{C}_X = 0$. □

Corollary 2.8. *We have $\mathscr{C}_X = 0$ if and only if ${}_{X^{-1}}\mathscr{C} = 0$.*

Drawing some inspiration from [6] we may now use the family $(P^x)_{x\in X}$ in order to find out which sets $X \subset G$ are such that $\mathscr{C}_X$ is equivalent to a category of modules.

Theorem 2.9. *Consider $X \subset G$; R a G-graded ring. The following assertions are equivalent:*

(i) *$\mathscr{C}_X$ is equivalent to a category of modules.*

(ii) *There is a finite subset F of X such that $\bigoplus_{x\in F} P^x$ is a non-zero generator of $\mathscr{C}_X$.*

(iii) *There is a finite subset* F *of* X *such that for every* $z \in X$, $R_1 = \sum_{g \notin X} R_{xg^{-1}} R_{gz^{-1}} + \sum_{g \in F} R_{xg^{-1}} R_{gz^{-1}}$., *and there is* $Z \in X$ *with* $\sum_{g \notin X} R_{Zg^{-1}} R_{gZ^{-1}} \neq R_1$

In case (iii) *holds then* (*for that* F *appearing in* (iii)) *we have that* $\bigoplus_{x \in F} P^x$ *is a generator of* $\mathscr{C}_X$.

Proof. (i) $\Rightarrow$ (ii): If $\mathscr{C}_X$ is equivalent to a category of modules then it has a non-zero projective small generator, U say. As $\mathscr{C}_X$ is a Grothendieck category U must be finitely generated in $\mathscr{C}_X$ and hence in R-gr. On the other hand, $\bigoplus_{x \in X} P^x$ is a generator of $\mathscr{C}_X$, thus there exists a set I and a graded epimorphism $\varphi : (\bigoplus_{x \in X} P^x)^{(I)} \to U$. Since U is finitely generated we may find a finite subset F of X and a graded epimorphism $\varphi' : (\bigoplus_{x \in F} P^x)^{(I)} \to U$, as claimed.

(ii) $\Rightarrow$ (i): When $\mathscr{C}_X$ has a non-zero finitely generated projective generator, $\bigoplus_{x \in F} P^x$, we may apply Mitchell's theorem and conclude that $\mathscr{C}_X$ is equivalent to $End_{R\text{-gr}}(\bigoplus_{x \in F} P^x)$-mod.

(ii) $\Leftrightarrow$ (iii): Note that the second inequality in (iii) is equivalent to $\mathscr{C}_X \neq 0$ because of Corollary 2.7. Assuming this, then $\bigoplus_{x \in F} P^x$ is a generator for $\mathscr{C}_X$ if and only if for every $z \in X$ there is a surjective graded morphism $\varphi : (\bigoplus_{x \in F} P^x)^n \to P^z$, for some $n \in \mathbb{N}$. It is easily seen that we have an isomorphism of abelian groups:

$\mathrm{Hom}_{R-\mathrm{gr}}(P^x, P^z) \simeq R_{xz^{-1}} / A^z_{xz^{-1}}$. Hence, up to identifying $(\bigoplus_{x \in F} P^x)^n$ and $\bigoplus_{x \in F} (P^x)^n$ we may find for every $x \in F$ some $r_{x1}, \ldots, r_{xn} \in R_{xz^{-1}}$ such that for $b = (b_x)_{x \in F}$, where $b_x = (\bar{a}_{x1}, \ldots, \bar{a}_{xn})$ with $a_{xi} \in R$ and $\bar{a}_{xi}$ denoting the class modulo A^x, we have

$$\varphi(b) = \sum_{x \in F} \sum_{i=1}^{n} \bar{a}_{xi} \bar{r}_{xi}.$$

Clearly, φ is surjective if and only if we may find $\sum_{x \in F} \sum_{i=1}^{n} \bar{a}_{xi} \bar{r}_{xi} = 1$, or $1 - \sum_{x \in F} \sum_{i=1}^{n} a_{xi} r_{xi} \in (A^z)_1$, what means exactly that

$$1 \in \sum_{x \in F} R_{zx^{-1}} R_{xz^{-1}} + \sum_{g \notin X} R_{zg^{-1}} R_{gz^{-1}}.$$

Finally when the latter relation holds for every $x \in X$ there exists $n \in \mathbb{N}$ and $(a_{xi})_{x \in F, i=1,\ldots,n}$ such that the corresponding φ is surjective. $\square$

Corollary 2.10 (Dăscălescu and Del Rio [1]). (1) $\mathscr{C}_X$ *is equivalent to a category of modules if and only if* ${}_{X^{-1}}\mathscr{C}$ *is.*

(2) *When* X *is finite then either* $\mathscr{C}_X$ *is zero or* $\mathscr{C}_X$ *is equivalent to a category of modules.*

The final point of this section is devoted to the study of $\mathscr{C}_X$ for a finite set X such that $\mathscr{C}_X \neq 0$ and G is infinite. In this case $\mathscr{C}_X$ is the module category of $\mathrm{End}_{R\text{-gr}}(\bigoplus_{x \in F} P^x)$ and we aim to describe the latter ring in some more detail by using the smash product as given by Quinn [14].

We denote by $M_G(R)$ the set of row and column finite matrices of type $G \times G$ with entries from R. If $A \subset G$ then p_A is the matrix having 1 in the (a, a)-position for all $a \in A$ and 0 elsewhere; if $g \in G$ then we write p_g for $p_{\{g\}}$.

To $r \in R$ we associate the matrix $\tilde{r}$ having $r_{xy^{-1}}$ in the (x, y)-position for any $x, y \in G$. In this way we obtain a subring $\tilde{R} = \{\tilde{r}, r \in R\}$ in $M_G(R)$ and $\tilde{R}$ is isomorphic to R. The smash product of R and G is denoted by $\tilde{R} \# G^*$ and it is the subring of $M_G(R)$ generated by $\tilde{R}$ and $\{p_g, g \in G\}$. When G is infinite, $\tilde{R} \# G^*$ is a free left and right $\tilde{R}$-module with basis $\{p_G\} \cup \{p_g, g \in G\}$.

Let us write S for $\tilde{R} \# G^*$. We know that R-gr is isomorphic to the full localizing subcategory $\mathscr{C}^* = \{M \in \text{S-mod},\ M = \sum_{g \in G} p_g M\}$ of S-mod. Explicitly: an $M \in R$-gr may be viewed as an S-module by putting: $\tilde{r}m = rm, p_g m = m_g$, for any $m \in M$, $r \in R$ and $g \in G$. Conversely if $M \in \mathscr{C}^*$ then the R-module structure of M is obtained by restriction of scalars via $R \to \tilde{R} \to S$ with gradation defined by $M_g = p_g M$ (for detail see also [9]).

The foregoing isomorphism of categories may be restricted to $\mathscr{C}_X$ and we then obtain a category isomorphism between $\mathscr{C}_X$ and $\{M \in \mathscr{C}^*, M = p_X M\} = \{M \in \text{S-mod}, p_X M = M\} = \{M \in \text{S-mod}, p_{G-X} M = 0\} = \{M \in \text{S-mod}, \text{IM} = 0\}$, where I is the ideal generated in S by $p_{G-X}, I = Sp_{G-X}S$. This leads to an isomorphism between $\mathscr{C}_X$ and S/I-mod. It is this point of view that is particularly useful when studying injective objects in $\mathscr{C}_X$. Indeed, the composed functor

$$\underbrace{S/I\text{-mod}} \xrightarrow{\simeq} \underbrace{\mathscr{C}_X} \xrightarrow{U_X} \underbrace{R\text{-mod}}$$

coincides exactly with the restriction of scalars functor, $\text{Res}: S/I \to R$-mod. The latter has left adjoint $S/I \otimes_R -$ and this allows us to use the classical result concerning right adjoint functor and injective objects.

Proposition 2.11. *Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring and consider a finite subset X of G. The following assertions are equivalent:*

(i) *Every injective object of $\mathscr{C}_X$ is injective in* R-mod.
(ii) *S/I is a flat right R-module.*
(iii) *I is a pure right R-submodule of S.*

Proof. (i)$\leftrightarrow$(ii): A consequence of the well-known result concerning right adjoint functors and injective objects (cf. [2]).

(ii)$\leftrightarrow$(iii): A consequence of Proposition 11.1 of [16], considering the exact sequence of right R-modules: $0 \to I \to S \to S/I \to 0$ and using the fact that S is a free R-module. $\square$

It follows from the proposition, in the particular case that R is a (Von Neumann) regular ring that injectives in $\mathscr{C}_X$ are injective in R-mod. This observation may be extended to gr-regular rings; recall that a graded ring R is gr-regular if for any homogeneous $r \in R$ there exists an $a \in R$ such that $r = rar$. Clearly R is gr-regular if and only if every graded R-module is gr-flat; hence we arrive at:

Corollary 2.12. *If R is* gr*-regular then every injective object in $\mathscr{C}_X$ is injective in* R-mod.

Proof. Since $p_{G-X}(S/I) = 0$ we have $p_X(S/I) = S/I$ and S/I is a graded R-module hence it is gr-flat and therefore flat as an R-module.

Now if $M \in R$-gr has finite support X and M is gr-injective then M is an injective object of $\mathscr{C}_X$. The previous observations learn that in certain particular cases the gr-injectivity of M will entail the injectivity of M in R-mod. □

However, in general (see Example 3.2(ii) later) there may be a considerable difference between the property of being injective in $\mathscr{C}_X$ and being injective in R-mod.

3. Finitely supported graded injectives

Recall from [10] the following:

Theorem 3.1. *Let $R = \bigoplus_{g \in G} R_g$ be a graded ring of finite support and let $M \in R$-gr be a gr-injective of finite support, them M is injective as an R-module.*

How far can we relax the conditions on M and R in the theorem before gr-injective and injective become different properties for graded modules? If we assume R has infinite support but $M \in R$-gr has finite support is it then true that gr-injectivity of M implies injectivity? In fact, the mere existence of gr-injective M with finite support is now not entirely trivial.

Example 3.2. (i) Let R_0 be a commutative semisimple ring having at least two types of simple modules (e.g. one can take $R_0 = k \times k$, k a field). Let $(R_n)_{n \geq 1}$ be a family of simple R_0-modules and let N be a simple R_0-module not isomorphic to any of the R_n, $n \geq 1$. The abelian group $R = \bigoplus_{n \geq 0} R_n$ may be made into a graded ring of type $\mathbb{Z}$ by putting $R_{-n} = 0$ for $n > 0$; $r_n r_m = 0$ for any n, $m > 0$ and $r_0 r_n = r_n r_0$ defined by the R_0-module structure of R_n.

Consider $M = \mathrm{Coind}(N)$. Then M is gr-injective because N is R_0-injective and $\mathrm{Coind}(-)$ has an exact left adjoint. Since for every $n \neq 0$ we have $M_n = \mathrm{Hom}_{R_0}(R_{-n}, N) = 0$ it follows that M has support $\{0\}$ and this is finite indeed!

(ii) With notation as before let S be a simple R_0-module such that an infinite number of the R_n are isomorphic to S. Concentrating S in degree zero yields a graded R-module S with $r_n s = 0$ for all $n > 0$, $s \in S$. Now consider the morphism of graded R-modules:

$v(S): S \to \mathrm{Coind(S)}$, given by $v(S)(s)(a) = a_0 s$ for any $s \in S$ and $a \in R$. In view of Proposition 1.1 of [10] $v(S)$ is essential and obviously, since S is graded-simple $v(S)$ is also injective. It follows that $\mathrm{Coind}(S)$ is the injective hull of S in R-gr and it has infinite support. This establishes an example of a graded module of finite support such that an essential extension of it (in R-gr) has infinite support. This example also provides an example of an injective object of $\mathscr{C}_{\{0\}}$ that is not gr-injective and not injective in R-mod.

When considering a graded ring R with infinite support, we want to give reasonable conditions for a gr-injective object to have infinite support automatically.

Proposition 3.3. (cf. [10]). *Let $M \in R$-gr be gr-injective and g-faithful for some $g \in G$, then M_g is an injective R_1-module and $M \simeq \mathrm{Coind}(M_g)(g^{-1})$ as graded modules.*

Lemma 3.4. *If $M \in R$-gr has finite support then there exists a $g \in \mathrm{supp}(M)$ and a graded submodule M' in M such that M' is g-faithful.*

Proof. We proceed by induction on the cardinality of $\mathrm{supp}(M)$, noting that the claim holds when $\mathrm{supp}(M)$ consists of one element only.

Assume that the result holds when $|\mathrm{supp}(M)| < n$ and consider an $M \in R$-gr with $|\mathrm{supp}(M)| = n$. Choose $g \in \mathrm{supp}(\mathrm{M})$; if M is g-faithful we were lucky and stop the proof, otherwise there is a non-zero graded submodule N of M such that $N_g = 0$. Then $|\mathrm{supp}(N)| < |\mathrm{supp}(M)|$ and the induction hypothesis applied to N yields the result. □

Corollary 3.5. *If $M \in R$-gr is gr-injective and $\mathrm{supp}(M)$ is finite then there exists a graded submodule Q of M such that Q is gr-injective and Q is g-faithful for some $g \in G$.*

In particular, every injective indecomposable object of finite support of R-gr is g-faithful for some $g \in G$.

Proposition 3.6. *Let R be a graded ring with infinite support and assume that either one of the following assumptions holds:*

(i) *The set $A = \{g \in G,\ R_g$ contains a regular element of $R\}$ is infinite.*

(ii) *The set $B = \{g \in G,\ R_g$ is a generator in R_1-mod$\}$ is infinite.*

Then every injective object of R-gr has infinite support.

Proof. Consider a gr-injective $M \in R$-gr.

(i) M is gr-divisible, i.e. for any regular homogeneous element $r_g \in R_g$ and any $m \in h(M)$ there exists a homogeneous $y \in M$ such that $r_g y = m$. Choose $m \neq 0$ in M_τ. The solution y of $r_g y = m$ is non-zero and it belongs to $M_{g^{-1}\tau}$, thus $\mathrm{supp}(M) \supset A^{-1}\tau$, where A is as above, and consequently $\mathrm{supp}(M)$ is then infinite too.

(ii) If M has finite support then Corollary 3.5 entails that there is a graded R-submodule Q in M such that Q is τ-faithful for some $\tau \in G$ and Q having finite support.

Then $Q \simeq \mathrm{Coind}(Q_\tau)(\tau^{-1})$ in R-gr and thus $\mathrm{Coind}(Q_\tau)$ has finite support. However, as abelian groups $\mathrm{Coind}(Q_\tau)_{g^{-1}} \cong \mathrm{Hom}_{R_1}(R_g, Q_\tau)$. Since $t_{R_1}(R_g) = R_g^* R_g = R_1$ there exist $f_1, \dots, f_n \in R_g^*$ and $x_1, \dots, x_n \in R_g$ such that $1 = \sum_{i=1}^n f_i(x_i)$. Choose $m_\tau \in Q_\tau, m_\tau \neq 0$ and $u \in \mathrm{Hom}_{R_1}(R_1, Q_\tau)$ is defined by $u(r) = rm_\tau$. If $uf_i = 0$ for all i then we have:

$(uf_i)(x_i) = f_i(x_i)m_\tau = 0$, hence $\sum_{i=1}^u f_i(x_i)\, m_\tau = 0$ or $m_\tau = 0$, contradiction. Therefore, $uf_i \neq 0$ for some i and so it follows that $\mathrm{Hom}_{R_1}(R_g, Q_\tau) \neq 0$ or $\mathrm{Coind}(Q_\tau)_{g^{-1}} \neq 0$.

All of this leads to the fact that M could not have finite support. □

For $M \in R$-gr put $\mathscr{F}(M) = \{N, N$ a graded subobject of M such that N is g-faithful for some $g \in G\}$.

Proposition 3.7. *If $M \in R$-gr has finite support then there is a finite direct sum of elements of $\mathscr{F}(M)$ which is essential as an R-module in M.*

Proof. Let $\mathscr{F}(M) = \{N_i \mid i \in I\}$ and $\mathscr{A} = \{J \mid J \subseteq I$ and the sum $\sum_{i \in J} N_i$ is direct$\}$. Then the inclusion makes $\mathscr{A}$ an inductive ordered set, so using Zorn's lemma we get a maximal element J of $\mathscr{A}$. If $S = \sum_{i \in J} N_i$ is not gr-essential in M, then there is $0 \neq N \leq_{\mathrm{gr}} M$ with $S \cap N = 0$. By Lemma 3.4 we can find an $i \in I$ with $N_i \subseteq N$, hence $J \cup \{i\} \in \mathscr{A}$, which is a contradiction. Therefore, S is graded essential, hence also essential as an R-module. Since the direct sum of g-faithful graded modules is g-faithful the result follows because $\mathrm{supp}(M)$ is finite. □

Lemma 3.8. *Let $Q \in R$-gr be gr-injective of finite support and g-faithful for some $g \in G$, then Q is injective in R-mod.*

Proof. From Proposition 3.3 it follows that Q_g is injective in R_1-mod and $Q = \mathrm{Coind}(Q_g)(g^{-1})$. Now we know:

$$\mathrm{Coind}(Q_g)_x = \{f \in \mathrm{Hom}_{R_1}(R, Q_g), f(R_\lambda) = 0 \text{ for } \lambda \neq x^{-1}\}.$$

Since Q has finite support only a finite number of the components $\mathrm{Coind}(C_g)_x$ can be non-zero. Therefore we arrive at $\mathrm{Coind}(Q_g) = \mathrm{Hom}_{R_1}(R, Q_g)$ which is injective in R-mod hence Q is injective in R-mod too. □

Theorem 3.9. *If $M \in R$-gr is gr-injective and having finite support then M is injective as an R-module.*

Proof. Proposition 3.7 provides us with $M_1 \oplus \cdots \oplus M_n$ essential in M, with $M_i \in \mathscr{F}(M)$, $i = 1, \ldots, n$. Let $E^g(M_1), \ldots, E^g(M_n)$ be the injective hulls in R-gr and put:

$$E = E^g(M_1 \oplus \cdots \oplus M_n) = E^g(M_1) \oplus \cdots \oplus E^g(M_n).$$

This inclusion, $i: M_1 \oplus \cdots \oplus M_n \to M$ yields a graded morphism $f: E \to M$ such that $fu = i$, where $u: M_1 \oplus \cdots \oplus M_n \to E$ is the canonical inclusion. Since u is essential f is injective. Obviously, f is essential. However, $f(E)$ is gr-injective and essential in M, hence $f(E) = M$ and f is a graded isomorphism.

On the other hand, $E^g(M_j)$ has finite support and it is gr-injective, moreover it is in $\mathscr{F}(M)$ because it is an essential extension of an object in $\mathscr{F}(M)$. By Lemma 3.8 it follows that M is injective in R-mod and then it also follows that M is injective in R-mod. □

As a by-product of the proof of the foregoing theorem we actually obtain a structure result for gr-injectives of finite support.

Corollary 3.10. *Let $M \in R$-gr be a gr-injective having finite support then there exist $g_1, \dots, g_n$ in $\operatorname{supp}(M)$ and $N_1, \dots, N_s$ injective R_1-modules such that: $M \cong \bigoplus_{i=1}^{s} \operatorname{Coind}(N_i)(g_i^{-1})$.*

Observation 3.11. As in the proof of Theorem 3.9 we obtain for $M \in R$-gr having finite support and $M_1 \oplus \cdots \oplus M_n$ being a direct sum of modules of $\mathscr{F}(M)$ which is gr-essential in M, that $E^g(M) = E^g(M_1) \oplus \cdots \oplus E^g(M_n)$. Since all $M_i \in \mathscr{F}(M)$ we arrive at $E^g(M) = \bigoplus_{i=1}^{n} \operatorname{Coind}(N_i)(g_i^{-1})$ for some $g_i \in \operatorname{supp}(M)$, $i = 1, \dots, n$. In particular: $E^g(M) = \operatorname{Coind}(N)$ for some R_1-module N (as R-modules).

Now it becomes possible to obtain information about the gradability of injective modules over a graded ring in case the grading group is finite.

Corollary 3.12. *Let G be finite, R a G-graded ring.*

For an injective R-module the following assertions are equivalent:

(i) *M can be endowed with a gradation of type G*

(ii) *M is the injective hull (in R-mod) of a graded R-module*

(iii) *There exists an injective R_1-module N such that $M \simeq \operatorname{Coind}(N) = \operatorname{Hom}_{R_1}(R, N)$ as R-modules.*

(*Note that* (ii) $\Leftrightarrow$ (iii) *is exactly Corollary* 1.20 *of* [5]).

Corollary 3.13. *Let R be graded by arbitrary G but having finite support and let M be an injective indecomposable R-module, then the following assertions are equivalent:*

(i) *M can be made into a G-graded R-module*

(ii) *There exists an injective indecomposable R_1-module N such that $M \simeq \operatorname{Coind}(N) = \operatorname{Hom}_{R_1}(R, N)$ as R-modules.*

Proof. (ii) $\Rightarrow$ (i): Clear.

(i) $\Rightarrow$ (ii): It suffices to observe that the gradation on M makes M into a gr-injective indecomposable R-module, hence if the homogeneous $m \neq 0$ is chosen in M then M is an essential extension of Rm and the latter has finite support. Therefore, M has finite support too and we may apply Corollary 3.10.

Finally, we show that in Theorem 3.1 we cannot drop the condition for M to have finite support. Let R be a non-Noetherian ring trivially graded by in infinite G. Consider a family or injective R-modules $\{M_{(g)}, g \in G\}$ such that $M = \bigoplus_{g \in G} M_{(g)}$ is not injective. Then it is clear that M may be made into a graded R-module (in the obvious way) and it is gr-injective but not injective.

In this observation the fact that R is not left Noetherian was crucial. A result from [10] states that for a graded ring R having finite support and being left Noetherian it is true that any gr-injective is injective. Let us conclude this paper by establishing a converse for this assertion. $\square$

Theorem 3.14. *Het G be an infinite and R a G-graded ring such that any R-module can be embedded in a graded R-module (for instance if* $\mathrm{supp}(R)$ *is finite this property is satisfied). If every* gr-*injective left* R-module *is injective, then R is left Noetherian.*

Proof. Assume that every gr-injective left R-module is injective and R is not left Noetherian. Since any left R-module can be embedded in a graded R-module, there exists a graded R-module E, which is a gr-injective (hence injective) cogenerator. Since G is infinite and R is not Noetherian, then $E^{(G)}$ is not injective. Since U is exact, then F preserves injectivity and hence $F(G)$ is gr-injective. But $E \in R$-gr, so $F(G)$ is isomorphic to $\bigoplus_{g \in G} E(g)$ in R-gr. By assumption $UF(E) \simeq E^G$ is injective, and this yields to a contradiction. □

Remark 3.15. Graded rings with the property that any module can be embedded in a graded one have been studied in [7].

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