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# Non-commutative generalisations of valuations and places

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# Introduction

Mathematics deserve to be cultivated for their own sake, and the theories inapplicable to physics as well as the others.

H. Poincaré, *The value of sciece*

## A short history of valuation theory

Valuations were introduced in 1913 by Josef Kürschák ([45]) to gain understanding in the $p$-adic numbers Hensel had written about. Kürschák's definition was as follows:

**Definition.** *A valuation on a field $k$ is a map $\|\cdot\| : k \to \mathbb{R}$ such that:*

*(V1)* $\|0\| = 0$ *and* $\|a\| > 0$ *if* $a \neq 0$*,*

*(V2) for every* $a \in k$ *one has* $\|1 + a\| \leq 1 + \|a\|$*,*

*(V3) for any* $a, b \in k$ *one has* $\|ab\| = \|a\| \, \|b\|$*,*

*(V4) there is at least one* $a$ *in* $k$ *for which* $0 \neq \|a\| \neq 1$*.*

which corresponds to what is now usually called an absolute value. Valuations have through the years earned a place in modern mathematics, finding applications in many different areas of mathematics, but historically they were first used in number theory. Important results that should be mentioned here include Ostrowski's classification of valuations on $\mathbb{Q}$, several extension theorems, Hasse's local-global principle, and Hensel's celebrated lemma.

In the middle of the interbellum, non-commutative algebras and in particular skewfields began to attract more attention. Hasse showed (in [35]) that maximal orders can be described by localisations at primes contained in the centre. In [36] he showed a local-global principle for central simple algebras by making use of the Brauer group. Ostrowski once more made a great contribution to valuation theory, studying Henselian fieds, ramification of valuations, and defects. Krull allowed for valuations with values in arbitrary ordered groups, whence the term *Krull valuation* which is still sometimes used.

After the second world war, the approximation theorem — which will play an important role in chapter 4 — appeared in [2]. Schilling published what is arguably the most important treatise on valuation theory ([88]) in 1950. In a very short paper, Manis ([56]) generalised the concept of a valuation to general commutative rings[1], which led to the study of semi-valuations (see e.g. [80]). But it is not until the seventies that non-commutative valuation theory starts gaining traction. The concept of a prime was introduced, which has the advantage that it can deal with zero-divisors. However, it is hard to associate good value functions to general primes. For this reason, Van Geel ([95]) restricted attention to orders with a commutative theory of fractional ideals, for which he introduced arithmetical pseudo-valuations.

About a decade later, Dubrovin published two important papers ([22] & [23]) introducing the non-commutative valuation rings that were later named after him. These have been studied quite a lot over the years and many results from commutative valuation theory have counterparts for Dubrovin valuation rings — especially in simple artinian rings which are finite dimensional over their centre. One of the most important results from this thesis is the introduction of arithmetical pseudo-valuations for Dubrovin valuations with non-idempotent Jacobson radicals.

Very recently, two books dedicated to non-commutative valuation theory appeared ([59] & [93]) which will hopefully help to attract more attention to this topic. Let me finish this short history by pointing out that the long history, or at least the first part thereof, can be found in Roquette's excellent write-up [86].

## Generalising valuations

The usual definition of a valuation is:

**Definition.** *A valuation is a surjective map $v$ from a field $k$ to a totally ordered group $\Gamma$ with an additional symbol $\infty$ such that:*

*(V1)* $v(x) = \infty$ *if and only if* $x = 0$*,*

*(V2)* $v(xy) = v(x)v(y)$ *for all* $x$ *and* $y$ *in* $k$*,*

*(V3)* $v(x + y) \geq \min\{v(x), v(y)\}$ *for all* $x$ *and* $y$ *in* $k$*.*

[1] In fact, the term *Manis valuation* also makes the occasional appearance in the literature.

There are three obvious ways to generalise this: one can relax the conditions on $v$ (as for example in the case of quasi-valuations), one can change the domain (as is the case for arithmetic pseudo-valuations cfr. e.g. 2.4), or one can change the co-domain (as in 1.1). However, simply fiddling with one or more of these conditions is not very satisfactory. One of the attractions, at least for the author, of classical valuation theory is the interplay between the more analytical point of view — with valuations and the associated ultrametrics or absolute values — on the one hand and the more algebraic theory — rings with linearly ordered ideals, total subrings etc. — on the other hand.[2]

To any valuation $v$ one can associate the ring of positives $R_v$ and in fact this $R_v$ determines $v$ (up to equivalence, which we will not go into here). This leads to the following alternative definition:

**Definition.** *A valuation ring of a field is a ring $R$ satisfying:*

*(VR1)* $x \notin R$ *implies* $x^{-1} \in R$

*(VR2) for all* $0 \neq x$ *in* $Q$ *one has* $xRx^{-1} = R$.

Here, too, possible generalisations abound. Among those considered in this text are chain rings, Dubrovin valuation rings (cfr. chapter 3 for both of those), total subrings (cfr. section 1.3) etcetera.

Ideally, one should obtain a generalisation combining both definitions e.g. a generalised value function which gives a nice ring of positives which in turn determines a generalised value function (preferably the one we started with, or one that is naturally equivalent to it). We will show that for Dubrovin valuation rings, or at least for the noetherian ones, there is indeed a nice generalised value function. The generalised value function is an *arithmetical pseudo-valuation* here and most of the results are analogous to the classical case. Using these results, we can develop a non-commutative divisor theory, which may lead to Riemann-Roch results in a similar vein as in [103].

## Overview

In the first chapter, the classical concept of a valuation will be decomposed in totality and stability. Stable subrings give rise to partial valuations. These occur naturally in the study of primes, but have not been studied very much. Many of the results in the sections about partial valuations and partial places seem to be technically new, but the proofs are very close to the totally ordered case.

[2] Although admittedly some work has been done on generalised valuations with no (obvious) associated ring theory, e.g. for gauges as in [93] or for valuation-like maps as in [84]. I have a strong suspicion that here, too, some interesting ring theory awaits discovery.

In the second chapter, the existing theory of primes will be outlined and the value function associated to a separated prime will be introduced. Later on, the theory of arithmetical pseudo-valuations will be generalised to invariant primes. The two last sections contain new material from [102].

In the third chapter, we will introduce Dubrovin valuation rings and discuss the pre-existing theory. Afterwards, we will show that, for Dubrovin valuation rings with non-idempotent Jacobson radical, arithmetical pseudo-valuations exist. Here, however, the arithmetical pseudo-valuations will be defined on the set of divisorial ideals instead of the set of fractional ideals. Constructing this arithmetical pseudo-valuation is one of the main new points of this thesis

These results will be used to obtain a divisor theory for bounded Krull orders. It is known that bounded Krull orders localised at height-one prime ideals yield Dubrovin valuation rings with non-idempotent Jacobson radicals, so the results from chapter 3 apply to these localisations. Using this, we obtain a divisor theory for bounded Krull orders. This theory, which is developed in section 4.3, was one perhaps the main goal of [102].

In the last two chapters, alternative approaches to non-commutative valuation theory are investigated. One of the main problems in this area of research is to define a good notion of valuations on matrix rings. In chapter 5, we will generalise the concept of a graded valuation to a groupoid-graded valuation. Since matrix rings are canonically groupoid-graded, this will provide a natural concept of a non-commutative valuation on a matrix ring. The theorems about $G$-valuations and their connection to Dubrovin valuation rings are new results.

In the last chapter, another possible generalisation is briefly studied. The idea here is to use separated exhaustive filtrations as a generalisation of valuations. Some existing results are slightly generalised here, but we will not stray far from the well-beaten path.

In the short appendix A we show that right-ordered value groups of valuation-like functions are bi-ordered. Appendix B is based on [26]. It has little to do with the rest of the thesis, dealing with Verma modules and Appell sets instead of valuations and valuation rings.

## Why non-commutative valuation theory?

Ah, now, that is a hard question. Historically, interest in non-commutative valuation theory comes from an attempt to understand orders but the reason for my interest is slightly different. In classical algebraic geometry, there is a very close correspondence between algebra and geometry. In fact, from a geometric object, i.e. some algebraic variety, one can get an algebraic object by considering its function field. Vice versa, to a field extension with transcendence degree 1, one can associate the abstract Riemann surface with valuations as points (and the trivial valuation as the generic point of the curve).

On the algebraic side, generalising concepts is relatively easy, but the meaning of non-commutative geometry is somewhat unclear. Generalising the translation machinery from the commutative context to a non-commutative one would permit translation of results and insights from the algebraic framework producing some kind of non-commutative geometry.

# Chapter 1

# The building blocks of valuation theory

A valuation is usually defined as a function from (the invertible elements of) a field to a totally ordered group satisfying some axioms. An obvious generalisation can be obtained by replacing the totally ordered group by a partially ordered one. It turns out that, like in the classical case, there is an alternative characterisation in terms of the ring of positives (cfr. 1.1.3). In fact, partial valuations can be obtained by dropping one of the two conditions from the original characterisation of valuation rings (cfr. 1.1.4). By dropping the other condition, total subrings are obtained, which have a natural place in non-commutative valuation theory as well. Total subrings have been studied rather extensively (cfr. e.g. [62]), so we suffice with a short overview. Since the correspondence between valuations and places is so important, we will also introduce partial places and establish a similar correspondence.

Most of the results in the first two sections are very similar to the classical (i.e. totally ordered) context. Van Geel ([95]), who studied partial valuations associated to primes, did some work on partial valuations (which we will encounter in 2.2) and this theory was somewhat expanded in a recent book (cfr. [59]), but not much more has been written about partial valuations.

## 1.1 Partial valuations

**Definition 1.1.1.** *Let $\Gamma$ be a partially ordered group and let $D$ be a skewfield. A* partial valuation *on $D$ is a surjective map $v : D \to \Gamma \cup \{\infty\}$ (where $\infty \geq \gamma$ and $\infty\gamma = \infty = \gamma\infty$ for all $\gamma \in \Gamma$) satisfying:*

*(PV1)* $v(x) = \infty \Leftrightarrow x = 0$,

*(PV2)* $\forall x, y \in D : v(xy) = v(x)v(y)$,

*(PV3)* $\forall x, y, z \in D : v(x) \geq v(z) \leq v(y) \Rightarrow v(x+y) \geq v(z)$.

*If a partial valuation satisfies the stronger*

*(PV4)* $\forall x, y, z \in D : v(x) > v(z) < v(y) \Rightarrow v(x+y) > v(z)$

*we say it is a* strict *partial valuation. If $\Gamma$ is totally ordered, $v$ is called a* valuation.

For any partial valuation, one can define $R_v = \{x \in D \mid v(x) \geq e\}$ where $e = v(1)$ is the neutral element of $\Gamma$. This is a ring since it contains $1$ and is closed under multiplication (PV2) and addition (PV3). Suppose, moreover, that $v(x) \geq e$, then for every $d \in U(D)$ we have $v(dxd^{-1}) = v(d)v(x)v(d^{-1}) \geq v(d)v(d^{-1}) = e$ so $R_v$ is invariant under inner automorphisms.[1]

**Definition 1.1.2.** *If $R$ is a subring of a skewfield $D$, we will call $R$* stable *if it is invariant under inner automorphisms and* total *if $d \notin R$ implies $d^{-1} \in R$ for every $d \in U(D)$.*

**Proposition 1.1.3.** *Suppose $R$ is a subring of a skewfield $D$ which is stable. Then it is $R_v$ for some partial valuation $v$ on $D$.*

*Proof.* Let $R$ be a stable subring of $D$. We write $x \sim y$ if

$$\forall d, d' \in D : dyd' \in R \Leftrightarrow dxd' \in R$$

which is certainly an equivalence relation. Suppose $x \sim x'$ and $y \sim y'$, then for all $d, d'$ in $D$ we have $dxyd' \in R \Leftrightarrow dx'yd' \in R \Leftrightarrow dx'y'd' \in R$, so the equivalence is compatible with the multiplication on $D$. Therefore, $\Gamma = D/\sim$ can be endowed with a canonical multiplication by putting, for all $d, d' \in D$, $\overline{d} \cdot \overline{d'} = \overline{dd'}$. Set $\overline{x} \geq \overline{1}$ if $dd' \in R$ implies $dxd' \in R$ for all $d, d'$ in $D$. This is clearly independent of the chosen representative. Suppose $x, y \in D$ are such that $\overline{x} \geq \overline{1}$ and $\overline{y} \geq \overline{1}$, then for all $d, d' \in D$ such that $dd' \in R$ we also have $dyd' \in R$. But then $(d, yd')$ is a pair with $dyd' \in R$, so $dxyd' \in R$. Combining this with the obvious fact that $\overline{1} \geq \overline{1}$ we find that $\Sigma = \left\{\overline{x} \in \Gamma \mid \overline{x} \geq \overline{1}\right\}$ is a subsemigroup of $\Gamma$. We put a partial order relation on $\Gamma$ by defining, for $x, y \in D \setminus \{0\}$:

$$\overline{x} \geq \overline{y} \quad \Leftrightarrow \quad \overline{xy^{-1}} \in \Sigma$$

and $\overline{0} \geq \overline{x}$ for all $x \in D$. We have to prove that this is indeed a partial order relation. It can easily be verified that $\leq$ is reflexive and transitive, so suppose $\overline{x} \geq \overline{y}$ and $\overline{y} \geq \overline{x}$ for some non-zero $x$ and $y$ in $D$. Then $\overline{xy^{-1}} \in \Sigma \ni \overline{yx^{-1}}$ which means that both $xy^{-1}$ and $yx^{-1}$ are in $R$. Suppose now $dxy^{-1}d' \in R$ for some invertible $d, d' \in D$, then $xy^{-1}d'd \in R$ since $R$ is stable under inner automorphisms and hence $d'd \in R$, i.e. $\overline{xy^{-1}} = \overline{1}$ or $\overline{x} = \overline{y}$. If $\overline{x} \geq \overline{z} \leq \overline{y}$, then $\overline{xz^{-1}} \in \Sigma \ni \overline{yz^{-1}}$ which implies $xz^{-1} \in R \ni yz^{-1}$ so $(x+y)z^{-1} \in R$ hence $\overline{x+y} \geq \overline{z}$, so $v : D \to \Gamma \cup \{\infty\} : x \mapsto \overline{x}$ is a partial valuation. □

[1]With $U(D)$ we denote the set of invertible elements of $D$.

This result warrants the term *partial valuation ring* for a subring of a field which is invariant under inner automorphisms. It is a convenient generalisation of the following classical characterisation of valuation rings, which is due to Schilling (cfr. [88] or many other books on valuation theory):

**Proposition 1.1.4.** *A subring $R$ of a skewfield $D$ is $R_v$ for a valuation $v$ if and only if $R$ is stable and total.*

*Proof.* It suffices to note that $R_v$ is total if and only if $v(x) < 0$ implies $v(x^{-1}) \geq 0$ which is the case if any $\gamma \in \Gamma$ is comparable to zero, hence if $\Gamma$ is totally ordered. $\square$

**Lemma 1.1.5.** *If $R$ is a stable subring of a skewfield $D$ and $v_R$ is the corresponding partial valuation, then $U(R) = \{x \mid v_R(x) = 0\}$.*

*Proof.* Suppose $x \in U(R)$ and suppose $dxd' \in R$ for $d, d' \in U(D)$, then $xd'd \in R$ which implies $d'd \in R$ since $x^{-1} \in R$. From this we can conclude $v_R(x) \leq v_R(1)$ hence $v_R(x) = v_R(1)$. If $v_R(x) = v_R(1)$, then $dxd' \in R$ if and only if $dd' \in R$. Since $1xx^{-1} \in R$, $x^{-1} \in R$ follows. $\square$

**Proposition 1.1.6.** *A stable subring $R$ of a skewfield $D$ which is local induces a strict partial valuation and vice versa.*

*Proof.* If $R$ is local, then $P = R \setminus U(R) = \{x \in D \mid v_R(x) > 0\}$ is an ideal, so it is closed under addition. Hence $v_R$ is strict. On the other hand, if $v_R$ is a strict partial valuation, then $P = R \setminus U(R) = \{x \mid v_R(x) > 0\}$ is closed under addition. Since it is always closed under multiplication and since $v_R(r) \geq 0 < v_R(p)$ implies $v_R(rp) > 0$, we find that $P$ is an ideal hence the unique maximal ideal. $\square$

**Example 1.1.7.** Consider a local ring in a field, say $\mathbb{Z}_{(p)}$ in $\mathbb{Q}$, and consider the formal power series $R = \mathbb{Z}_{(p)}[[X]]$ as a subring of $K = \mathbb{Q}((X))$. Since $K$ is commutative, $R$ is stable and since $R$ is the ring of power series over a local ring, it is again local. This means that $R$ induces a partial valuation $v$ on $K$ which is strict. It is not a valuation because, for example, $1/p$ and $X^{-1}$ have $v$-values which are not comparable.

**Example 1.1.8.** Consider a field $k$ and the field of rational polynomials $k(X)$. Then $R = \left\{\sum_{i=0}^{n} a_i X^i \mid n \in \mathbb{N}, a_1 = 0\right\}$ is a subring of $k(X)$ and since $k(X)$ is commutative, it is obviously stable. One can check quite easily that the associated partial valuation gives the following directed graph of values of monomials:

$$\begin{tikzcd}
\cdots \arrow[r] & v(X^{-4}) \arrow[r] \arrow[drr] & v(X^{-2}) \arrow[r] \arrow[drr] & v(X^{0}) \arrow[r] \arrow[drr] & v(X^{2}) \arrow[r] & v(X^{4}) \arrow[r] & \cdots \\
\cdots \arrow[r] & v(X^{-5}) \arrow[r] \arrow[ur] & v(X^{-3}) \arrow[r] \arrow[ur] & v(X^{-1}) \arrow[r] \arrow[ur] & v(X^{1}) \arrow[r] \arrow[ur] & v(X^{3}) \arrow[r] & \cdots
\end{tikzcd}$$

Consider in the same field the subring $k[X^2]$ of $k(X)$. Then one gets the following graph:

$$\cdots \longrightarrow v(X^{-4}) \rightarrow v(X^{-2}) \longrightarrow v(X^0) \longrightarrow v(X^2) \rightarrow v(X^4) \rightarrow \cdots$$

$$\cdots \longrightarrow v(X^{-5}) \rightarrow v(X^{-3}) \rightarrow v(X^{-1}) \rightarrow v(X^1) \rightarrow v(X^3) \rightarrow \cdots$$

For a prime on a skewfield which extends a valuation on the center, we will show that the orderings as showcased in this last example are essentially the only possibilities: either the ordering is directed (i.e. any two elements have a common upper bound) or it has a number of linearly ordered path-connected components. We will denote the skewfield of fractions of a partial valuation ring as $Q(R_v)$. We define the *path-connected component* of $d$ as

$$\mathrm{pcc}(d) = \{x \in U(D) \mid \exists x_1, ..., x_n \in U(D) : v(d) \gtrless v(x_1) \gtrless \cdots \gtrless v(x_n) \gtrless v(x)\}.$$

**Lemma 1.1.9.** *Suppose $v$ is a partial valuation on a skewfield $D$, then $\mathrm{pcc}(1)$ is the set of units of the field of fractions of $R_v$.*

*Proof.* Take an arbitrary $x \in \mathrm{pcc}(1)$. Then there are $x_1, \ldots, x_n$ with $v(x) \gtrless v(x_1) \gtrless \cdots \gtrless v(x_n) \gtrless 0$. This means $v(x_{n-1}x_n^{-1}) \gtrless 0$, hence either $(x_{n-1}x_n^{-1})$ or its inverse are in $R_v$. By a simple induction argument, we find similarly that either $x_{n-i}x_{n-i+1}^{-1}$ or its inverse is in $R_v$. Since

$$x = xx_1^{-1}x_1x_2^{-1} \cdots x_{n-1}x_n^{-1}x_n,$$

we find for all $i$ that $x$ must be in $Q(R_v)$. Suppose now $z$ is in $Q(R_v)$, then $z = xy^{-1}$ for certain $x$ and $y$ in $R_v$. This means $z \gtrless x \gtrless 0$. □

**Corollary 1.1.10.** *Suppose again that $v$ is a partial valuation on a skewfield $D$, then for every non-zero $x \in D$ we have $\mathrm{pcc}(x) = U(Q(R_v))x = xU(Q(R_v))$.*

*Proof.*

$$\begin{aligned}
\mathrm{pcc}(x) &= \{y | \exists x_0, \ldots, x_n : v(y = x_0) \gtrless v(x_1) \gtrless \cdots \gtrless v(x_n = x)\} \\
&= \left\{y | \exists x_0, \ldots, x_n : v(yx^{-1}) \gtrless v(x_1x^{-1}) \gtrless \cdots \gtrless v(1) = 0\right\} \\
&= \left\{y | yx^{-1} \in \mathrm{pcc}(0) = U(Q(R_v))\right\} = \left\{y | x^{-1}y \in \mathrm{pcc}(0) = U(Q(R_v))\right\} \\
&= U(Q(R_v))x = xU(Q(R_v))
\end{aligned}$$

□

**Corollary 1.1.11.** *If $v : D \to \Gamma \cup \{\infty\}$ is a partial valuation on a skewfield $D$ extending a valuation on $Z(D)$ such that $v(d) \geq 0$ for some non-central $d$, then $\Gamma$ is directed.*

*Proof.* Since $Q(R_v)$ is a stable subskewfield of $D$ which is not contained in $Z(D)$, it must by the Cartan-Brauer-Hua theorem be equal to $D$ itself (cfr. [38]). Two elements $d$ and $d'$ can then be written as $d = pq^{-1}$ and $d' = p'q'^{-1}$ for some $p, p', q, q' \in R_v$. Then $pp'$ is an upper bound for $d$ and $d'$ while $q^{-1}q'^{-1}$ is a lower bound for $d$ and $d'$. □

**Remark 1.1.12.** The fact that $v$ is a partial valuation on the skewfield is not really necessary for this reasoning; we just make use of the fact that the relation $d \leq d' \Leftrightarrow v(d) \leq v(d')$ restricted to $U(D)$ is compatible with the multiplication.

On the set $\{\mathrm{pcc}(x) \mid x \in U(D)\}$, there is a canonical multiplication $\mathrm{pcc}(x) \cdot \mathrm{pcc}(y) = \mathrm{pcc}(xy)$ which is well-defined by corollary 1.1.10. This yields a group $\overline{D}$ of connected components. We find

(1) $\bigcup_{\overline{d} \in \overline{D}} Q(R_v)d = D$

(2) $Q(R_v)dQ(R_v)d' = Q(R_v)dd'$

(3) $\overline{d} \neq \overline{d'} \Rightarrow Q(R_v)\overline{d} \cap Q(R_v)\overline{d'} = \{0\}$

which is tantalizingly close to being a grading[2] on $D$. Properties (1)-(3) may not be enough to define a grading, but they do define a Clifford system.

**Definition 1.1.13.** *Let $G$ be a group. A $G$-*Clifford system[3] *on a ring $R$ is a set $(R_g)_{g \in G}$ of additive subgroups such that:*

*(1) $\sum_{g \in G} R_g = R$,*

*(2) $R_g R_{g'} = R_{gg'}$ for all $g, g' \in G$.*

Clifford systems were introduced by Dade ([19]) in order to better understand and generalise Clifford's theory — whence, obviously, the name. A few papers dealing with Clifford systems have appeared (e.g. [99], [98], [33]), but they have largely been neglected in favour of strong gradings. This is probably due to the very general nature of Clifford systems. Perhaps the more restrictive set of conditions (1)-(3) hits the sweet spot between generality and usefulness? An in-depth study of these objects is required in order to answer this question.

[2] We will consider gradings in some detail in chapter 5; the interested reader may also want to consult [75].

[3] The $G$ will often be omitted, being clear form the context.

## 1.2 Partial places

In this section we will follow the construction of [27], of course adapting everything to our partial and not necessarily commutative case. A skewfield $C$ can be extended with a formal symbol $\infty$ to $\tilde{C} = C \cup \{\infty\}$ where $\infty + c = c + \infty = \infty$ for any $c \in C$ and $c \cdot \infty = \infty \cdot c = \infty$for any $c$ in $C \setminus \{0\}$. The addition $\infty + \infty$ and multiplication $\infty \cdot \infty$ as well as $\infty \cdot 0$ and $0 \cdot \infty$ remain undefined. Let for the remainder of this section $C$ and $D$ be skewfields.

**Definition 1.2.1.** *A* partial $C$-place *of $D$ is a map $\pi : \tilde{D} \to \tilde{C}$ such that:*

*(PP1) If $xy$ and $\pi(x)\pi(y)$ are defined, then $\pi(xy) = \pi(x)\pi(y)$.*

*(PP2) If $x + y$ and $\pi(x) + \pi(y)$ are defined, then $\pi(x + y) = \pi(x) + \pi(y)$.*

*(PP3) $\exists d \in \tilde{D} : \pi(d) = 1$.*

**Lemma 1.2.2.** *Every partial $C$-place $\pi$ satisfies the following conditions:*

- $\pi(1) = 1$, $\pi(0) = 0$, $\pi(\infty) = \infty$.
- *If $\pi(x) + \pi(y)$ (resp. $\pi(x)\pi(y)$) is defined, then so is $x + y$ (resp. $xy$).*
- $\pi(-x) = -\pi(x)$.
- $\pi(x^{-1}) = \pi(x)^{-1}$.

*Proof.* These are all easily adapted from the source mentioned above. □

The following propositions are also not very surprising. Here, too, the proofs are relatively straightforward adaptations from similar proofs in the classical case, but they are perhaps important enough to warrant separate mentioning:

**Proposition 1.2.3.** *Let $R$ be a strict partial valuation ring in a skewfield $D$ and let $\lambda : R \to C$ be a morphism with $\mathrm{Ker}(\lambda) = P$. The mapping*

$$\pi : \tilde{D} \to \tilde{C} : x \mapsto \begin{cases} \infty & \textit{if } x \notin R \\ \lambda(x) & \textit{if } x \in R \end{cases}$$

*is a partial $C$-place of $D$.*

*Proof.* (PP3) is trivially true, so we will restrict our attention to (PP1) and (PP2). Take $x$ and $y$ in $D$. If $\pi(x) + \pi(y)$ is defined, then either $\pi(x) \neq \infty$ or $\pi(y) \neq \infty$ — we assume the former without loss of generality. Then $x \in R$ and if $\pi(y) = \infty$ we have $y \notin R$ so $x + y \notin R$ which implies $\pi(x + y) = \infty$. If $\pi(y) \neq \infty$, then both $x$ and $y$ are in $R$ so $\pi(x + y) = \lambda(x + y)$ which is indeed $\lambda(x) + \lambda(y) = \pi(x) + \pi(y)$ since $\lambda$ is a morphism.

Suppose $x$ and $y$ are such that $\pi(x) = \infty$ and $\pi(y) = \alpha$ for some $0 \neq \alpha \in C$. If $x$ or $y$ is $\infty$, then $\pi(xy) = \infty = \infty \cdot \alpha$. If $x$ nor $y$ equal $\infty$, then $x \notin R$ and $y \in U(R)$ so $xy \notin R$ and consequently $\pi(xy) = \infty = \infty \cdot \alpha$. □

**Example 1.2.4.** Referring back to example 1.1.7 we put $R = \mathbb{Z}_{(p)}[[X]]$ which is a partial valuation ring in $\mathbb{Q}[[X]]$. The canonical morphism $\lambda : R \to \mathbb{F}_p$ obtained by dividing out the unique maximal ideal $P$ induces a partial place $\pi : \mathbb{Q}[[X]] \to \mathbb{F}_p$ sending elements of $R$ to their equivalence classes in $R/P$ and all other elements to $\infty$.

**Proposition 1.2.5.** *Suppose that $D$ is a field and that $\pi$ is a partial $C$-place of $D$, then $\pi^{-1}(C)$ is a strict partial valuation ring on $D$.*

*Proof.* Clearly, $\infty \notin \pi^{-1}(C)$. If $x$ and $y$ are in $\pi^{-1}(C)$, then $\pi(x) + \pi(y)$ and $\pi(x)\pi(y)$ are defined so we have $\pi(x + y) = \pi(x) + \pi(y) \in C$ and $\pi(xy) = \pi(x)\pi(y) \in C$ which implies together with $\pi(1) \in C$ and $\pi(-x) = -\pi(x)$ that $\pi^{-1}(C)$ is a subring of $D$. Stability is obvious due to the commutativity of $D$ and strictness is a simple consequence of the fact that $\pi^{-1}(C)/Ker(\pi|_{\pi^{-1}(C)})$ is a skewfield. □

If $\pi$ is a partial $C$-place, then we can always consider $\pi$ as a partial $C'$-place for some sub-skewfield $C' = C$ such that $C' \subseteq \mathrm{Im}(\pi)$. Considered as such, $\pi$ becomes a surjective partial place. The previous propositions establish a one-one correspondence between surjective partial places on fields and partial valuations.

**Remark 1.2.6.** The commutativity of $D$ is really necessary. Take e.g. the Weyl skewfield (cfr. 6.2.5) $\mathbb{D}_1(k)$ over a field $k$ as a superfield of $k(X)$. The map

$$\pi : \widetilde{\mathbb{D}_1(k)} \to \widetilde{k(X)} : x \mapsto \begin{cases} x & \text{if } x \in k(X) \\ \infty & \text{if } x \notin k(X) \end{cases}$$

is a partial $k(X)$-place of $\mathbb{D}_1(k)$, but $\pi^{-1}(k(X))$ is not a partial valuation ring on $\mathbb{D}_1(k)$ since it is not stable (e.g. $Y^{-1}XY = X + Y^{-1}$ is not in $k(X)$).

**Proposition 1.2.7.** *Let $D$ be a field. Consider a surjective partial $C$-place $\pi$ of $D$ and let $R$ be the associated partial valuation ring. There is an inclusion-preserving one-one correspondence between stable subrings of $R$ and partial valuations on $C$.*

*Proof.* Suppose $S \subseteq R$ is a partial valuation subring of $R$ and suppose $c \in C$. Since $\pi$ is surjective, there is an $r \in R$ with $\pi(r) = c$, so we have $c\pi(S)c^{-1} = \pi(r)\pi(S)\pi(r^{-1}) = \pi(rSr^{-1}) \subseteq \pi(S)$ so $\pi(S)$ is a stable subring, i.e. a partial valuation ring, of $C$.

Suppose on the other hand that $S$ is a partial valuation ring on $C$, then by a similar argument as in 1.2.5, $\pi^{-1}(S)$ is a ring which is necessarily stable since $D$ is a field. □

**Proposition 1.2.8.** *Let $D \subseteq D'$ be skewfields and let $R \subseteq R'$ be local stable subrings of $D$ and $D'$. The associated partial valuations $v$ and $v'$ take values in some partially ordered groups $\Gamma$ and $\Gamma'$. The sets of non-invertible elements in $R$ and $R'$ respectively will be denoted by $P$ and $P'$. Furthermore, we use $\pi$ to denote the partial place associated to $R$ and $\pi'$ to denote the partial place associated to $R'$. Then the following are equivalent:*

*(1)* $R' \cap D = R$

*(2) There is an order preserving isomorphism $f : \Delta \to \Gamma$ for some subgroup $\Delta$ of $\Gamma'$ such that $f \circ v_{R'}|_D = v(D)$.*

*(3)* $g : \pi(D) \to \pi'(D') : \pi(d) \mapsto \pi'(d)$ *is injective.*

*Proof.* We show that both (2) and (3) are equivalent to (1). $\underline{(1) \Leftrightarrow (2)}$ Suppose $d, d' \in D$ have the same image under $v_{R'}$ but not under $v_R$. We can assume $v_R(d) \ngeq v_R(d')$, but then $dd'^{-1} \notin R$ while $dd'^{-1} \in R' \cap D$ which is a contradiction. This means that $f : v_{R'}(D) \to v_R(D) : v_{R'}(d) \mapsto v_R(d)$ is well-defined. It can easily be verified that $f$ is an order-preserving isomorphism. If there is an order preserving isomorphism between $v_{R'}(D)$ and $v_R(D)$, then any $d \in D$ is $v_{R'}$-positive if and only if it is $v_R$-positive, or in other words, if and only if it is in $R$. $\underline{(1) \Leftrightarrow (3)}$ Suppose (1) holds, then the kernel of $g$ is $\pi\{d \in D \mid \pi'(d) = 0\} = \pi(P' \cap D) = \pi(P \cap D) = 0$. Suppose now that $Ker(g) = 0$. If $d \in R' \cap D$, then $\pi'(d) \neq \infty$ so $\pi(d) \neq \infty$ and $x \in R$. □

## 1.3 Total subrings

In the previous sections, we have dropped the totality from Schilling's definition of valuation rings. The following natural question then arises: what happens if we retain totality and drop stability instead? The answer turns out to be total subrings. They were introduced by Radó in [83] and were later studied quite extensively by Mathiak, who wrote [62] (cfr. also [61]) which is perhaps the main reference work for total subrings.

**Remark 1.3.1.** The name *total subring* is due to Radó but is not completely standard. Mathiak uses the term *non-invariant valuation ring* while some authors call these rings *valuation rings*, using the term *invariant valuation ring* for what we will call valuation rings.

**Definition 1.3.2.** *Suppose $D$ is a skewfield and $S$ is a totally ordered set, which includes a maximum $\infty$ and at least one other element. A* generalised valuation *is a surjective map $v : D \to S$ satisfying (for all $x, y, z \in D$):*

*(GV1)* $v(x + y) \geq \min\{v(x), v(y)\}$,

*(GV2)* $v(x) \le v(y) \Rightarrow v(xz) \le v(yz)$.

These generalised valuations still retain many properties of valuations, some of which are collected in the following lemma.

**Lemma 1.3.3.** *For a generalised valuation $v : D \to S$ the following hold:*

*(1)* $v(x) = \infty$ *if and only if* $x = 0$*,*

*(2)* $v(x + y) = \min\{v(x), v(y)\}$ *if* $v(x) \ne v(y)$*,*

*(3)* $v(x) = v(-x)$

*Proof.* Easy verifications; proofs are given in [83]. □

Much in the same way as for (partial) valuations, one can associate the ring of positives $R_v = \{d \in D \mid v(d) \ge v(1)\}$ to any generalised valuation $v$ on a skewfield $D$. $P_v = \{d \in D \mid v(d) > 0\}$ is then the unique maximal ideal of $R_v$.

**Proposition 1.3.4.** *Left ideals of $R_v$ are totally ordered.*

*Proof.* See [59] or [62]. □

**Theorem 1.3.5.** *A subring $R$ of a skewfield $D$ is total if and only if $R = R_v$ for some generalised valuation $v$ on $D$.*

*Proof.* See [83]. □

For another characterisation of total subrings, we refer the reader to 2.1.5. It is interesting to compare theorem 1.3.5 with the classical result of 1.1.3. It turns out that the totality of a valuation ring gives the totality of the ordering on the set of values, while the group structure corresponds to stability. The following example of a total subring which is not stable has become standard, being cited by e.g. [83] and [95].

**Example 1.3.6.** Put $\mathbb{R}(t)$ the field of rational functions over the reals with ordering

$$\frac{a_0 t^m + \cdots + a_m}{b_0 t^n + \cdots + b_n} > 0 \quad \Longleftrightarrow \quad a_0 b_0 > 0$$

and put $G$ the group of affine transformations of $\mathbb{R}(t)$, i.e. $G$ is the group of maps

$$g : \mathbb{R}(t) \to \mathbb{R}(t) : x \mapsto ax + b$$

where $a$ and $b$ are in $\mathbb{R}(t)$ and $a$ is non-zero. Consequently, an element $g : x \mapsto ax + b$ of $G$ can be represented by $(a, b)$ with $a, b \in \mathbb{R}(t)$ and $a \ne 0$. The lexicographic ordering

$$(a, b) \ge (a', b') \quad \Leftrightarrow \quad a > a' \text{ or } \big(a = a' \text{ and } b \ge b'\big)$$

turns this into an ordered group. Indeed, suppose that $(a,b) \geq (a',b')$ and let $(s,t)$ be an arbitrary element of $G$, then $(a,b)(s,t) = (as, at+b)$ and $(a',b')(s,t) = (a's, a't+b')$. If $a > a'$ then certainly $as > a's$. If, on the other hand, $a = a'$ and $b \geq b'$, we have $as = a's$ and $at + b \geq a't + b'$ so $\leq$ is compatible with right-multiplication. The argument for compatibility with left-multiplication is similar: in the same context as before we have $(s,t)(a,b) = (sa, sb+t)$ and $(s,t)(a',b') = (sa', sb'+t)$. Again, if $a > a'$ we also have $sa > sa'$. If $a = a'$ and $b \geq b'$, we have $sa = sa'$ and $sb + t > sb' + t$. We will use $e$ to denote the neutral element of $G$.

We can now construct the skew polynomial field

$$K(G,\mathbb{R}(t)) = \{\phi : G \to \mathbb{R}(t) \mid D(\phi) = \{x | \phi(x) \neq 0\} \text{ is well-ordered}\}$$

with addition and multiplication defined as follows:

$$\begin{aligned}(\phi+\psi)(x) &= \phi(x) + \psi(x)\\ (\phi\psi)(x) &= \sum_{\zeta\eta = x} \phi(\zeta)\psi(\eta).\end{aligned}$$

The set $\Delta = \{g \in G \mid \exists r \in \mathbb{R} : g \geq (1,r)\}$ is multiplicatively closed, hence

$$O = \{f \in K(G,\mathbb{R}(t)) \mid D(\phi) \subseteq \Delta\}$$

is multiplicatively closed. Since it is also an additive subgroup of $K(G,\mathbb{R}(t))$, it is a subring. It is clear that, for any $f \in K(G,\mathbb{R}(t))$, $\min D(\phi) > e$ implies $f \in O$. Suppose now $\min D(f) < e$ and $\min D(f^{-1}) < e$. We can pick $x, y \in G$ such that $f(x) = \min D(f)$ and $f^{-1}(y) = \min D(f^{-1})$. As a consequence, we have $ff^{-1}(xy) = \sum_{\zeta\eta=xy} f(\zeta)f^{-1}(\eta) = f(x)f^{-1}(y) \neq 0$. This means $xy \in D(1)$ while $xy < e$. This is impossible, so $f \notin O \Rightarrow f^{-1} \in O$.

For any $g \in G$, we can define

$$f_g : x \mapsto \begin{cases} 1 & \text{if } x = g \\ 0 & \text{if } x \neq g \end{cases}$$

A simple calculation gives $f_{g^{-1}} = f_g^{-1}$ and $f_{gh} = f_g f_h$. We know $f_{(1,-1)} \in O$, and we find

$$\begin{aligned} f_{(t,0)}^{-1} f_{(1,-1)} f_{(t,0)} &= f_{(t^{-1},0)(1,-1)(t,0)} \\ &= f_{(1,-t^{-1})}. \end{aligned}$$

Since $t > r$ for any $r \in \mathbb{R}$, $t^{-1} < r$ for any $r \in \mathbb{R}$. This means $D(f_{(1,-t)}) \notin \Delta$. This means $O$ is not invariant under inner automorphisms and as a result it cannot be a valuation ring.

# Chapter 2

# Primes and associated value functions

Primes were introduced by Van Oystaeyen and Nauwelaerts ([100], [77]; these were only primes with a completely prime ideal) as a non-commutative generalisation of valuation rings, inspired by the fact that a valuation can also be characterised by a ring and a prime ideal in it. The concept of a prime is just about the broadest one that still holds some interest and many other generalisations of valuations, e.g. Dubrovin valuation rings, will turn out to be primes.

One would like to associate some kind of value function to such a prime, but this poses some problems. A general value function does exist, but it takes values only in a partially ordered monoid which, in general, is not even cancellative. A different approach was suggested by Van Geel ([95]): suppose $Q$ is a simple artinian ring with some prime $(R, P)$, instead of a function

$$v : Q \to \Gamma \cup \{\infty\}$$

for some totally ordered group $\Gamma$, one considers an *arithmetical pseudo-valuation*

$$v : \mathcal{F}(R) \to \Gamma \cup \{\infty\}$$

where $\mathcal{F}(R)$ is the set of fractional $R$-ideals (which will be introduced in good time). Unfortunately, for this to work, Van Geel needs a commutative theory of fractional ideals — a rather strong condition.

In this chapter we will show, after introducing the necessary concepts and the general value function, that, in some cases, an arithmetical pseudo-valuation can be defined without assuming the commutativity of fractional ideals. Constructing arithmetical pseudo-valuations in this more general setting was one of the main points of [102] and it will later on (cfr. 3.4) allow for the introduction of arithmetical pseudo-valuations associated to Dubrovin valuation rings.

## 2.1 Generalities about primes

**Definition 2.1.1.** *Let $A$ be a ring. A pair $(R, P)$, with $R$ a subring of $A$ and $P$ a prime ideal of $R$ is called a* prime *in $A$ if*

$$aRb \subseteq P \text{ implies } a \in P \text{ or } b \in P$$

*(for any $a, b \in A$).*

*If, moreover,*

$$ab \in P \text{ implies } a \in P \text{ or } b \in P$$

*(for any $a, b \in A$) holds, then $(R, P)$ is called a* complete prime*.*

If $R$ is a prime, then

$$A^P = \{a \in A \mid aP \subseteq P \text{ and } Pa \subseteq P\}$$

is a subring of $A$ and $(A^P, P)$ is a prime of $A$ which we will call the associated prime. Since many interesting properties of a prime $(R, P)$ hold for the associated prime $(A^P, P)$ as well, it is usually harmless to only consider primes $(R, P)$ for which $R = A^P$, but this is not necessary for us here.

The concept of a prime is a very general one. In fact, for many purposes it is too general and one has to restrict attention to special kinds of primes.

**Definition 2.1.2.** *A prime $(R, P)$ in a ring $A$ is said*[1] *to be*

*(1)* fractional *if for any $a \in A \smallsetminus R$ there are $x, y \in R \smallsetminus \{0\}$, at least one of which is in $P$, with $xay \in R$*

*(2)* localised *if for any $a \in A \smallsetminus R$ there are $x, y \in R \smallsetminus \{0\}$, at least one of which is in $P$, with $xay \in R \smallsetminus P$*

*(3)* separated *if it is localised and for all $r \in R \smallsetminus \{0\}$ there are $a, b \in A$ with $arb \in R \smallsetminus P$.*

Alternatively, one could define a separated prime as a localised prime for which $P_0 = \{p \in P \mid ApA \subseteq P\}$ is zero. For simple rings this is obviously true, so any localised prime in a simple ring is necessarily separated.

**Example 2.1.3.** 1. If $R_v$ is a valuation ring on some field $k$, then $(R_v, P_v)$ is a localised (hence separated) prime. Vice versa, if $k$ is a field and $(R, P)$ is a localised prime in $k$, then $R$ is a valuation ring on $k$ with maximal ideal $P$. This follows from 1.1.4 and 2.1.5.

[1] This terminology comes from [59]. In earlier work, like [95], localised and separated primes were called semi-restricted and restricted, respectively.

2. If $A$ is a ring and $P$ is a prime ideal in $A$, then $(A, P)$ is a prime in $A$ which is always fractional and never localised, unless $P$ is the zero ideal.

3. Let $R$ be a (commutative) Krull domain with fraction field $k$ (see capter 4), let $P$ be a minimal prime ideal of $R$ and let $R_P$ be the localisation of $R$ at $P$. The ring $A = \{a + bX \mid a \in R_P, b \in k\}$ with prime ideal $P = \{a + bX \mid a \in PR_P, b \in k\}$ defines a prime in $k[X]/(X^2)$ which is localised but not separated.

We now recall some basic results about primes.

**Proposition 2.1.4.** *Let $A$ be a ring, $R$ a subring and $P$ a prime ideal of $R$. If for any $a \in A \setminus R$ there are $x, y \in R \setminus \{0\}$ with $xay \in R \setminus P$, then $(R, P)$ is a (necessarily localised) prime of $A$.*

*Proof.* See [59]. □

**Proposition 2.1.5.** *A subring $R$ of a skewfield $D$ is a total subring if and only if there is an ideal $P$ of $R$ such that $(R, P)$ is completely prime.*

*Proof.* See [59]. □

**Proposition 2.1.6.** *If there is some separated prime $(R, P)$ in $A$, then $A$ is a prime ring.*

*Proof.* See [59]. □

**Definition 2.1.7.** *To a prime $(R, P)$ in a ring $A$ we can associate a map $\pi : R \to R/P$ which is called the associated* prime place. *Let $A'$ be a subring of $A$ with some prime place $\pi : R' \to R'/P' = \overline{R'}$. A* $\pi$-pseudo-place *of $A$ is a triple $(R, \psi, \overline{R})$ such that:*

*(1) $R'$ is a subring of $A'$ with $R \cap A' = R'$,*

*(2) $\psi : R \to \overline{R}$ is a ring morphism with $\mathrm{Ker}(\psi) \cap A' = P'$,*

*(3) $\overline{R'}$ is a subring of $\overline{R}$ and $\psi|_{A'} = \pi$.*

*A $\pi$-pseudo-place is called a* $\pi$-pre-place *if $(R, \mathrm{Ker}(\psi))$ is a prime of $A$.*

These definitions will allow us to formulate one of the most important results in the theory of primes: a Chevalley-like extension theorem, proven by Van Geel in [94] (although [59], which also contains a proof, is probably easier to find). If $S$ and $T$ are subsets, we will use the notation $< T >$ for the multiplicative closure of $T \cup \{1\}$ and $ST$ for $\{\sum_i s_i t_i \mid \forall i : s_i \in S, t_i \in T\}$.

**Theorem 2.1.8.** *Let $A$ again be a ring and let $(R', P')$ be a prime in some subring $A'$ with associated prime place $\pi : R' \to R'/P'$. Consider subsets $M \subseteq N \subseteq A$ satisfying*

*(i)* $NA \subseteq P' < N >$ *and* $NP' \subseteq P' < N >$

*(ii)* $P' < N > \cap A' \subseteq P'$

*(iii)* $0 \notin M$ *and for all* $m_1, m_2 \in M$ *there is some* $n \in N$ *such that* $m_1 n m_2 \in N$[2]

*(iv)* $R' \setminus P' \subseteq M$

*(v)* $M \cap P' < N > = \varnothing$

*then there is a $\pi$-pre-place $(R, \psi, \overline{R})$ of $A$.*

*Proof.* See [59].[3] □

Let $A$ be a ring. Suppose $R, R'$ are subrings of $A$ with some respective prime ideals $P$ and $P'$. We can say that $(R, P) \leq (R', P')$ if $R \subseteq R'$ and $P' \cap R = P$.

**Corollary 2.1.9.** *If $(R, P)$ is maximal with respect to $\leq$, then it is a prime.*

*Proof.* See [59]. □

**Remark 2.1.10.**
1. The statement in [59] is a little bit different, since it makes use of so-called dominating pairs. As a corollary of the theorem, it is then shown that dominating pairs are necessarily primes.
2. In the case where $A'$ is a field $k$, $R$ is a valuation ring on $k$ with maximal ideal $P$, and $A$ is a field extension of $k$, this theorem becomes the Chevalley extension theorem for valuations.

## 2.2 Value functions associated to primes in simple rings

Suppose $Q$ is a simple ring and let $(R, P)$ be a prime in $Q$. To any element $x$ of $Q$, one can associate the set $P_x = \{(q, q') \in Q^2 \mid qxq' \in P\}$. This induces an equivalence relation $\sim$ on $Q$ by putting $x_1 \sim x_2$ if $P_{x_1} = P_{x_2}$. Since $Q$ is simple, the fact that $x \sim 0$ implies $P_x = Q \times Q$ yields $x = 0$. The set $\overline{Q} = Q/\sim$ is endowed with a canonical partial ordering — by putting $\overline{x_1} \leq \overline{x_2}$ if and only if $P_{x_1} \subseteq P_{x_2}$ — and multiplication — by putting $\overline{x_1} \cdot \overline{x_2} = \overline{x_1 x_2}$.

[2]This condition means that $M$ is an *m*-system for $N$, which explains why $M$ is called $M$.

[3]But *lasciata ogni speranza voi ch'entrate*; the proof is rather involved and not very intuitive.

**Proposition 2.2.1.** *$\overline{Q}$ is a partially ordered semigroup.*

*Proof.* See [59]. □

It is clear that $\overline{1}$ is the neutral element of $\overline{Q}$ and that $\overline{x}$ is invertible if $x$ is, but $\overline{Q}$ is not cancellative in general. In fact, $\overline{x}$ can never be invertible if $x$ is a zero divisor.

Since $\overline{0} > \overline{x}$ for all non-zero $x$ in $Q$, it makes sense to denote $\overline{0}$ by $\infty$.

**Proposition 2.2.2.** *The map $\phi : Q \to \overline{Q} : x \mapsto \overline{x}$ satisfies the following conditions:*

*(i)* $\phi(x) = \infty$ *if and only if* $x = 0$*,*

*(ii)* $\phi(xx') = \phi(x)\phi(x')$*,*

*(iii) if* $\phi(x) \geq \phi(y) \leq \phi(x')$*, then* $\phi(x + x') \geq \phi(y)$*,*

*(iv) if* $x \in U(Q)$*, then* $\phi(x^{-1}) = \phi(x)^{-1}$*,*

*(v) if* $\phi(x) > \phi(y)$*, then* $\phi(x + y) = \phi(y)$*.*

*Proof.* Some of the statements are proven in [59]. The proofs are anyway not too difficult. □

**Remark 2.2.3.** These definitions still work just as well for non-simple rings, provided that $(R, P)$ is a separated prime. Otherwise, property *(i)* from 2.2.2 fails to hold.

If $Q$ is a skewfield, $\overline{Q}$ is a partially ordered group. In this case, additive notation is traditionally used for the operation on $\overline{Q}$, even if it need not be commutative, but we will avoid this slightly confusing convention for now. For any prime $(R, P)$ in a skewfield $Q$, one can define $O_R = \{q \in Q \mid \phi(q) \geq \overline{1}\}$. Because of 2.2.2, this is clearly a subring of $Q$. There are some alternative characterisations of positive elements which might bear repeating:

**Proposition 2.2.4.** *If $(R, P)$ is a localised prime in a skewfield $Q$, then $O_R = \bigcap_{q \in U(Q)} qRq^{-1}$.*

*Proof.* See [59]. □

This means that for a localised prime $(R, P)$ the value function $\phi$, which *a priori* only depends on $P$, is also only dependent on $R$ which justifies the notation $O_R$. Note that knowing the value function is in general not enough to know the prime $(R, P)$, as the following lemma shows:

**Lemma 2.2.5.** *For any $z \in U(Q)$ and any prime $(R, P)$ in $Q$, $O_R = O_{zRz^{-1}}$ holds.*

*Proof.* Denoting the value function associated to $R$ and $zRz^{-1}$ by $\phi_R$ and $\phi_{zRz^{-1}}$ respectively, we find

$$\begin{aligned}\phi_R(x) \geq \overline{1} &\Leftrightarrow \forall q, q' \in Q : \left(qq' \in P \Rightarrow qxq' \in P\right)\\ &\Leftrightarrow \forall q, q' \in Q : \left(z^{-1}qq'z \in P \Rightarrow z^{-1}qxq'z \in P\right)\\ &\Leftrightarrow \forall q, q' \in Q : \left(qq' \in zPz^{-1} \Rightarrow qxq' \in zPz^{-1}\right)\\ &\Leftrightarrow \phi_{zRz^{-1}}(x) \geq \overline{1}\end{aligned}$$

which had to be shown. □

**Definition 2.2.6.** *A prime $(R, P)$ is called* strict *if the associated partial valuation is a strict partial valuation.*

For a strict localised prime, one can define $\mathfrak{p} = \left\{q \in Q \mid \phi(q) \geq \overline{1}\right\}$, which is the unique maximal ideal of $O_R$.

**Proposition 2.2.7.** *For a strict localised prime $(R, P)$ in a skewfield $Q$, the following hold:*

*(1) $O_R/\mathfrak{p}$ is a skewfield,*

*(2) every left (right) ideal of $O_R$ is a two-sided ideal,*

*(3) if $(O_R, \mathfrak{p})$ is a prime of the skewfield of fractions $Q(O_R)$, then $O_R$ is a valuation ring in $Q(O_R)$.*

*Proof.* See [59]. □

**Proposition 2.2.8.** *Let $(R, P)$ be a strict fractional prime in a simple artinian ring $Q$. If $A$ is any semisimple artinian subring of $Q$ and $A \neq Q$, then $R$ is not contained in $A$.*

*Proof.* Assume $R \subseteq A$ and pick $q \in Q \setminus A$. Since $A$ is noetherian, we may choose $L$ maximal for the property that $Lqy \subseteq A$ for some $y \in A$. Since $A$ is semisimple artinian, we have $A = L \oplus U$ where $U$ is a left ideal and $uqy \notin A$ for every $u \in U$ (otherwise $(L + Ru)qy \subseteq A$ entails $u \in L$ which is a contradiction). There exist $x', y' \in R$ with $0 \neq x'uqyy' \in R \subseteq A$. Since $L$ is maximal for the property that $Lqyy' \subseteq A$, it follows that $x'u \in L$ but $x'u \in U$, so $x'u = 0$ contradicting $x'uayy' \neq 0$. □

A ring is said to be a *Goldie ring* if the set of regular elements satisfies the Ore condition and $S^{-1}R$ is a semisimple Artinian ring.

**Proposition 2.2.9.** *If $(R, P)$ is a strict fractional prime in a simple artinian ring $Q$ and $R$ is a Goldie ring, then $S^{-1}R = Q$ where $S$ is the set of regular elements in $R$.*

*Proof.* Let $r \in S$, then we claim that $r$ is regular in $Q$. Suppose it is not, then $ru = 0$ for some $u \in Q \setminus R$. Since $(R, P)$ is fractional, there exist $x, y \in R$ with $0 \neq xuy \in R$ and since $S$ satisfies the Ore condition, there are $x' \in R$ and $r' \in S$ with $r'x = x'r$ so $r'xuy = x'ruy = 0$ hence $xuy = 0$ which is a contradiction. Since $Q$ is simple Artinian, $r^{-1} \in Q$ so any element of $S$ is invertible in $Q$ and the injection $R \hookrightarrow Q$ extends to an injection $S^{-1}R \hookrightarrow Q$. Since $S^{-1}R$ is semisimple artinian, the preceding proposition implies that $S^{-1}R = Q$. □

**Remark 2.2.10.** If $R$ is a prime Goldie ring and $I$ is an essential left ideal of $R$ then $I$ is generated by the regular elements of $I$. (See [66].)

**Example 2.2.11.** Let $O$ be as in example 1.3.6. We will try to find $O_R$, i.e. the ring of positive elements for the partial valuation associated to $O$ as a prime. Let us first investigate $\phi((1.0))$. When is $(f, f')$ in $\phi((1,0))$? Clearly when

$$\min D(f) \min D(f') > (1, r)$$

for some real number $r$. Put $(a, b) = \min D(f)$ and $(a', b') = \min D(f')$, then $(f, f') \in \phi(1)$ if $aa' > 1$ or $aa' = 1$ and $aa' + b > r$ for some real number $r$. In general, if $f''$ has $\min D(f'') = (c, d)$, then $(f, f') \in \phi(f'')$ if and only if $aa' > c'^{-1}$ or $aa' = c^{-1}$ and $acb' + ad + b > r$ for some real $r$. (Provided that $c$ is positive. If it is not, then $\phi(f'') \geq \phi(1)$ certainly does not hold.) As a consequence, $\phi(f'') \geq 0$ if and only if

$$aa' > 1 \text{ or } \left(aa' = 1 \text{ and } \exists r \in \mathbb{R} : ab' + b > r\right)$$

implies

$$aa' > c^{-1} \text{ or } \left(aa' = c^{-1} \text{ and } \exists r \in \mathbb{R} : acb' + ad + b > r\right).$$

This is clearly the case if $c > 1$ and it is clearly not the case if $c < 1$. If $c = 1$ it is also not the case since the term $ad$ can destroy the desired property. This implies that

$$O_R = \{f \in K(G, \mathbb{R}(t)) \mid \min D(f) = (a, b) \text{ then } a > 1 \text{ or } a = 1 \text{ and } b = 0\}$$

which is a local stable subring of $K(G, \mathbb{R}(t))$ and $U(O_R)$ is the set of those $f$ with $\min D(f) = (1, 0)$.

## 2.3 Invariant primes

The following proposition is a slight generalisation of Schilling's characterisation of valuation rings in skewfields. It provides a motivation for considering invariant primes as a canonical generalisation of valuation rings on skewfields to general simple artinian rings.

**Proposition 2.3.1.** *Let $Q$ be a skewfield. If $R \subseteq Q$ is invariant under inner automorphisms and $R = A^P$ for some prime $(S, P)$, then $R$ is valuation ring.*

*Proof.* Suppose $(R, P)$ is a prime and suppose $qq' \in P$. Obviously, $Rqq' \subseteq P$, but since $R$ is invariant under inner automorphisms we also have $Rqq' = qRq^{-1}qq' = qRq'$ which implies that either $q$ or $q'$ is in $P$. This means that $P$ is completely prime so $R$, as the domain of a complete prime, must be a total subring. A total subring which is stable under inner automorphisms is a valuation ring (cfr. 1.1.4). □

Consider a strict fractional prime $(R, P)$ of a simple artinian ring $A$. We always assume that $R$ is Goldie hence a prime ring and an order of $A$ (by proposition 2.2.9). If $P$ is invariant under inner automorphisms of $A$, we say that $(P, R)$ is an *invariant prime* of $A$.

**Example 2.3.2.** Consider once more the example from 1.3.6 and 2.2.11. If $P$ is the unique maximal ideal of $O_R$, then $(O_R, R)$ is a prime which is invariant. Note that, by 2.3.1, we do not have $R = A^P$ in this case.

For the remainder of this section, we will assume that $(R, P)$ is an invariant prime which is equal to its associated prime.

**Remark 2.3.3.** $R$ is invariant under inner automorphisms of $A$.

*Proof.* Consider $u \in U(A)$. For $p \in P$ we have $uRu^{-1}p = uRu^{-1}puu^{-1}$ and $u^{-1}pu \in P$ so $Ru^{-1}pu \subseteq P$ and $uRu^{-1}p \subseteq uPu^{-1} \subseteq P$. Hence $uRu^{-1}P \subseteq P$ which implies $uRu^{-1} \subseteq R$. A similar reasoning gives $PuRu^{-1} \subseteq R$. □

In general, by a *fractional $R$-ideal* of $A$ we mean an $R$-bimodule $I \subseteq A$ such that $I$ contains a regular element of $R$ and for some $r, s \in R$, $rI \subseteq R$ and $Is \subseteq R$. Observe that we may choose $r$ and $s$ regular since $R$ is an order. We will denote the set of fractional ideals of $R$ by $\mathcal{F}(R)$.

**Lemma 2.3.4.** *The following properties hold:*

*(1) If $u$ is regular and $uI \subseteq R$ then $Iu \subseteq R$ and vice versa. Similarly, $uI \subseteq P$ if and only if $Iu \subseteq P$.*

*(2) If $I, J \in \mathcal{F}(R)$, then $IJ \subseteq P$ if and only if $JI \subseteq P$.*

*(3) If $I, J \in \mathcal{F}(R)$ then $IJ \subseteq R$ implies $JI \subseteq R$ and vice versa. Moreover, if $J \nsubseteq P$ then $I \subseteq R$ and if $P \nsupseteq I \subseteq R$ then $J \subseteq R$.*

*Proof.* (1) If $uI \subseteq R$, then $Iu \subseteq u^{-1}Ru = R$ and if $Iu \subseteq R$, then $uI \subseteq uRu^{-1} = R$. The other case is similar.

(2) If $IJ \subseteq P$ then, since $(R,P)$ is a prime, either $I$ or $J$ is in $P$, say $I \subseteq P$. Since $I$ is an ideal it is left essential so it is , by 2.2.10, generated by regular elements. For every regular element $u \in I$, $uJ \subseteq P$ yields $Ju \subseteq u^{-1}Pu = P$, hence $JI \subseteq P$. The case $I \nsubseteq P$ and $J \subseteq P$ is similar.

(3) Suppose $I, J \in \mathcal{F}(R)$ such that $IJ \subseteq R$. If $I, J \subseteq P$ there is nothing to prove since then $IJ \subseteq P$ and $JI \subseteq P$, so assume $J \nsubseteq P$ ($I \nsubseteq P$ is completely similar). From $PIJ \subseteq P$ we then obtain $PI \subseteq P$ since $(P,R)$ is a prime of $A$, so $I \subseteq A^P = R$. Again, $I$ is generated by regular elements since it is left essential and for $u \in I$ regular $uJ \subseteq R$ gives $Ju \subseteq u^{-1}Ru = R$ hence $JI \subseteq R$.

□

**Corollary 2.3.5.** *$P$ is the unique maximal ideal of $R$.*

*Proof.* Consider an ideal $I \nsubseteq P$ and a regular element $u$ of $R$ which is in $I$ but not in $P$ (such an element exists since $I$ is generated by regular elements). Then $Ru = RuR \neq R$ so $u^{-1} \notin R$. From $Ru^{-1}RuR = R$ with $RuR \nsubseteq P$ we obtain $Ru^{-1}R \subseteq R$ which is a contradiction. □

**Remark 2.3.6.** In fact, we showed that every regular element of $R \smallsetminus P$ is already invertible in $R$.

**Corollary 2.3.7.** *If $\mathcal{C}(P) = \{x \in R \mid x \mod P$ regular in $R/P\}$ satisfies the Ore condition then it is invertible in $R$, i.e. $Q_P(R) = R$ or $R$ is local and $P$ is the Jacobson radical of $R$.*

*Proof.* If $\mathcal{C}(P)$ is an Ore set in the prime Goldie ring $R$ which is also an order in a simple Artinian ring $A$, then $\mathcal{C}(P)$ consists of regular elements and since $\mathcal{C}(P) \subseteq R \smallsetminus P$ it consists of invertible elements of $R$. Consequently, the localisation of $R$ at $\mathcal{C}(R)$ is equal to $R$. It then follows that $P$ is the Jacobson radical of $R$. □

**Proposition 2.3.8.** *If $\bigcap P^n = 0$ then $\mathcal{C}(P)$ satisfies the Ore condition.*

*Proof.* We claim that $1+P$ consists of units. Indeed, consider $1+p$ with $p \in P$ and assume it is not regular, then $r(1+p) = 0$ for some $0 \neq r \in R$. Then $r = -rp$ yields $r \in \bigcap P^n$ hence $r = 0$ which is a contradiction. If $c \in \mathcal{C}(P)$ then $\overline{c}$ is regular in $R/P$. We have that $P+Rc$ is essential in $R$ since it contains $P$ hence it is generated by regular elements. Since $Ru = RuR$ for regular $u$, it follows that $P+Rc$ is a two-sided ideal of $R$, hence $P+Rc = R$ and $\overline{Rc} = \overline{R}$ i.e. $\overline{c}$ is invertible. Then there is an $\overline{u} \in \overline{R}$ with $\overline{uc} = \overline{1}$ which means $uc \in 1+P$. If $rc = 0$ then $uru^{-1}uc = 0$ which would contradict the fact that all elements of $1+P$ are units. Consequently, $\mathcal{C}(P)$ consists of $R$-regular elements. For every $r \in R$ and $c \in \mathcal{C}(P)$ we have $cr = crc^{-1}c = r'c$ which gives the left Ore condition and also $rc = cc^{-1}rc = cr'$ which gives the right Ore condition. Therefore $\mathcal{C}(P)$ is an Ore set. □

**Corollary 2.3.9.** *If $\bigcap P^n = 0$ then $\mathcal{C}(P)$ is invertible in $R$ and $R$ is local with Jacobson radical $P$.*

**Corollary 2.3.10.** *$R/P$ is a skewfield.*

*Proof.* If $\overline{a} \in R/P$ is not invertible then it is not regular (cfr. the proof of proposition 2.3.8), say $\overline{sa} = 0$. Let $\overline{a} = a \mod P$ and $\overline{s} = s \mod P$, then $sa \in P$ implies $(Rs+P)a \subseteq P$. Furthermore, $Rs+P$ is two-sided and it contains $P$ strictly so $Rs + P = R$. This means that $Ra \subseteq P$ so $\overline{a} = 0$. □

**Proposition 2.3.11.** *Under assumptions as before, the left $R$-ideals are totally ordered and every finitely generated left $R$-ideal is generated by one regular element.*

*Proof.* By remarks 2.2.10 and 2.3.3, left $R$-ideals are $R$-ideals. Suppose $xy \in P$ with either $x$ or $y$ regular in $A$. We suppose without loss of generality that $x$ is regular, so it is invertible in $A$. We find $xRy = xRx^{-1}xy = Rxy \subseteq P$ so since $(R, P)$ is prime, $x$ or $y$ must be in $P$. Consider now $x$ regular (hence invertible) in $A \smallsetminus R$. Since $x \notin R$, there must be a $p \in P$ with $xp \notin P$ (or $px \notin P$ in which case we argue similarly). Then we have $Rx^{-1}xp \subseteq P$ so $x^{-1} \in P$ since it is $A$-regular. Consider now a finitely generated left ideal $I$ in $R$. By [66], it is generated by $R$-regular elements so it is generated by a finite number of $R$-regular elements say $I = Ru_1 + \cdots + Ru_n$. Since $R$ is Goldie, every $R$-regular element is $A$-regular, so by the preceding statements either $u_1u_2^{-1}$ or $u_2u_1^{-1}$ must be in $R$. Suppose the latter (again, in the other case we argue similarly), then $Ru_2 = Ru_2u_1^{-1}u_1 \subseteq Ru_1$ which means that $Ru_1 + Ru_2 = Ru_1$. By induction we find that every finitely generated left ideal is principal and in fact even principal for a regular element. This in turn implies that the finitely generated left ideals are totally ordered by inclusion. Suppose now that $I$ and $J$ are left $R$-ideals with $J \nsubseteq I$. There must be a regular $x \in J \smallsetminus I$ and for every $y \in I$ we have either $yx^{-1} \in R$ which would imply $y \in xR \subseteq J$ or $xy^{-1} \in R$ but this is contradictory since it implies $x \in Ry \subseteq I$. □

## 2.4 Arithmetical pseudo-valuations associated to invariant primes

An *arithmetical pseudo-valuation* (or *apv* for short) on $R$ as before is a function $v : \mathcal{F}(R) \to \Gamma$ for some partially ordered semigroup $\Gamma$ such that:

(APV1) $v(IJ) = v(I) + v(J)$;

(APV2) $v(I + J) \geq \min\{v(I), v(J)\}$;

(APV3) $v(R) = 0$;

(APV4) $I \subseteq J$ implies $v(I) \geq v(J)$.

For more information about arithmetical pseudo-valuations, we refer to [59] and [95]. In this section, we will assume that $(R, P)$ is an invariant prime. Note that we will use + for the operation on $\Gamma$ even if it need not be commutative. Similarly, we use $0$ for the neutral element. This notation is in concordance with the usage in e.g. [59] and [95].

**Theorem 2.4.1.** *For any $(R, P)$ there is an arithmetical pseudo-valuation $v : \mathcal{F}(R) \to \Gamma$, where $\Gamma$ is a totally ordered semigroup, such that*

$$P = \{a \in A \mid v(RaR) > 0\} \quad \text{and} \quad R = \{a \in A \mid v(RaR) \geq 0\}.$$

*Proof.* Observe that for any $I, J \in \mathcal{F}(R)$ we have $IJ \in \mathcal{F}(R)$ and $I+J \in \mathcal{F}(R)$, moreover for every $a \in A$ we have $RaR \in \mathcal{F}(R)$. Indeed, if $a \in A$ then there is a regular $u \in R$ such that $ua \in R$ since $R$ is an order, then $RuaR = RuRaR \subseteq RaR$ and as an $R$-ideal, $RuaR$ contains a regular element of $R$. If $I$ and $J$ are in $\mathcal{F}(R)$ then $IJ$ contains a regular element and if $uI \subseteq R$ and $vJ \subseteq R$ for regular $u$ and $v$ then $Jv \subseteq R$ so $uIJv \subseteq R$ whence $vuIJ \subseteq vRv^{-1} = R$ with $vu$ regular. For $I+J$ we have $vu(I+J) \subseteq R+vuJ$ with $vuJ = vuv^{-1}vJ \subseteq R$ since $vuv^{-1} \in R$.

For any $I \in \mathcal{F}(R)$ we define $v(I) = (P : I) = \{a \in A \mid aI \subseteq P\}$ and since $RaRI \subseteq P$ if and only if $IRaR \subseteq P$ this is also equal to $v(I) = \{a \in A \mid Ia \subseteq P\}$. Note that $v(I) \neq \{0\}$ because $uI \subseteq R$ for some regular $u \in R$, hence $0 \neq Pu \subseteq v(I)$. We also have $v(R) = P$. Put $\Gamma = \{v(I) \mid I \in \mathcal{F}(R)\}$ and define a partial order $\leq$ by

$$v(I) \leq v(J) \quad \Leftrightarrow \quad v(I) \subseteq v(J).$$

Note that if $I \subseteq J$ then $v(I) \geq v(J)$. We claim that $\Gamma$ is in fact totally ordered. Indeed, if $I, J \in \mathcal{F}(R)$ such that $v(I) \nsubseteq v(J)$ and $v(J) \nsubseteq v(I)$ then there is an $a \in A$ with $aI \subseteq P$ but $aJ \nsubseteq P$ and a $b \in A$ with $bJ \subseteq P$ but $bI \nsubseteq P$. Since $P$ is prime, $aJbI \nsubseteq P$ but $RbIaJ \subseteq RbPJ \subseteq RbJ \subseteq P$ yields $RaJbI \subseteq P$ which is a contradiction in view of lemma 2.3.4.

We can define a (not necessarily commutative) operation + on $\Gamma$ by putting $v(I) + v(J) = v(IJ)$. The unit for this operation is $v(R)$. We now verify that + is well-defined. Suppose $v(I) = v(I')$ and $v(J) = v(J')$ and consider $x \in v(IJ)$, then $RxRIJ \subseteq P$ so $RxRI \subseteq v(J) = v(J')$ or $RxRIJ' \subseteq P$. By the same lemma as before, $IJ'RxR \subseteq P$ follows hence $J'RxR \subseteq v(I) = v(I')$ i.e. $I'J'RxR \subseteq P$ which implies $x \in v(I'J')$ and consequently $v(IJ) \subseteq v(I'J')$. The other inclusion can be obtained by the same argument if the roles of $I, J$ and $I', J'$ are interchanged.

We now check that this operation is compatible with $\leq$. Take some $v(I) \geq v(J)$ and consider $v(HI)$ and $v(HJ)$. If $q \in v(HJ)$ then $qHJ \subseteq P$ so $qH \subseteq v(J) \subseteq v(I)$ which implies $qHI \subseteq P$ so $q \in v(HI)$. To prove that $\leq$ is also stable under right multiplication, we consider $q \in v(JH)$. Then $qJH \subseteq P$ or equivalently

$JHq \subseteq P$. By lemma 2.3.4 $HqJ \subseteq P$ follows so $Hq \subseteq v(J) \subseteq v(I)$ hence $IHq \subseteq P$ i.e. $q \in v(IH)$.

If $v(I) \leq v(J)$ then $aI \subseteq P$ yields $a(I+J) \subseteq P$ since $aJ \subseteq P$, so $v(I+J) \supseteq v(I) = \min\{v(I), v(J)\}$. Together with the preceding, this implies that $v$ is an arithmetical pseudo-valuation. The only thing left to prove is that

$$R = \{a \in A \mid v(RaR) \geq 0\} \quad \text{and} \quad P = \{a \in A \mid v(RaR) > 0\}.$$

Suppose $v(RaR) > 0 = v(R) = P$, then there is some $x \in v(RaR) \setminus P$. Now $xRaR \subseteq P$ gives $a \in P$, so $\{a \in A \mid v(RaR) > 0\} \subseteq P$. If $p \in P$, then $v(RpR) \supseteq R \supsetneq P$, hence $p \in \{a \in A \mid v(RaR) > 0\}$ so $P = \{a \in A \mid v(RaR) > 0\}$. If $a \in A$ is such that $v(RaR) = 0$ and $p \in P$ then $v(RaRRpR) = v(RaR) + v(RpR) = v(RpR) > 0$ so $RaRRpR \subseteq P$ and therefore $ap \in P$ which implies $a \in R$ since $R = A^P$. On the other hand, if $r \in R$ then $PRrR \subseteq P$. Since $RrR$ is generated by regular elements, it follows that $r \in \sum Ru_iR$ for a finite set of regular $u_i$. Consequently, since $\Gamma$ is totally ordered, $v(RrR) = v(Ru_iR)$ where $v(Ru_iR)$ has the minimal value among these regular elements. If $v(RrR) < 0$ then $v(P) \leq v(PRrR)$ since $PRrR \subseteq P$ and then

$$v(P) \leq v(PRrR) = v(P) + v(RrR) \leq v(P) \tag{2.1}$$

since $v(RrR) < 0$. This means that all $\leq$ in 2.1 are actually equalities and in fact $v(P) = v(P) + v(Ru_iR) = v(PRu_iR)$ so if $aPRu_iR \subseteq P$ then also $aP \subseteq P$. By choosing $a = u_i^{-1}$ we find $u_i^{-1}P \subseteq P$. In a similar fashion we find $Pu_i^{-1} \subseteq P$ and consequently $u_i^{-1} \in A^P = R$ so $v(RrR) = v(Ru_iR) = v(R) = 0$ which contradicts $v(RrR) < 0$. Consequently $R = \{a \in A \mid v(RaR) \geq 0\}$. $\square$

**Proposition 2.4.2.** *With $R, P$ and $A$ as before, $\Gamma$ is a group if and only if for any fractional $R$-ideal $I$ there is a nonzero $y \in R$ with $yI \subseteq R$ but $yI \nsubseteq P$.*

*Proof.* If $\Gamma$ is a group and $I \in \mathcal{F}(R)$ then for some $J \in \mathcal{F}(R)$ we have $v(I) + v(J) = 0$ i.e. $v(IJ) = v(JI) = v(R)$. Consequently, $IJP \subseteq P \supseteq PIJ$ so $IJ \subseteq A^P = R$. Since $aIJ \subseteq P$ iff $aR \subseteq P$ we have $IJ \nsubseteq P$. Then we can choose a $y \in J$ with $Iy \subseteq R$ but $Iy \nsubseteq P$ which implies $RIRy \subseteq R$ and $RIRy \nsubseteq P$.

Suppose now that there is some $y$ with $yI \subseteq R$ but $yI \nsubseteq P$. For any $x \in v(RyRI)$ we have $RxRRyRI \subseteq P$ which implies $RxR \subseteq P$ and consequently $x \in v(R)$. From $RyRI \subseteq R$ we can deduce $v(R) \subseteq v(RyRI)$ hence $v(R) = v(RyRI)$ which means that $v(RyR)$ is the inverse of $v(I)$. $\square$

Note that the second part of the proof of the preceding theorem guarantees that every $v(I)$ is also $v(RaR)$ for some $a \in A$.

**Lemma 2.4.3.** *If $\bigcap P^n = 0$ and $\Gamma$ is a group then $R$ is a Dubrovin valuation ring (cfr. 3.2.1 for a definition of Dubrovin valuation rings).*

*Proof.* By applying corollary 2.3.9 one finds that $R/P$ is prime Goldie with invertible regular elements, i.e. it is a simple artinian ring. Consider $q \in A \smallsetminus R$. There exists some $y \in A$ with $RyRRqR \subseteq R$ but $RyRRqR \nsubseteq P$. Then there exists a $z \in RyR$ with $zq \in R \smallsetminus P$ and since $RqRRyR \subseteq R$ but $RqRRyR \nsubseteq P$ we can use a similar construction to find an element $z'$ with $qz' \in R \smallsetminus P$. □

**Theorem 2.4.4.** *If $\Gamma$ is a group and $\bigcap P^n = 0$, then $R$ is a valuation ring and $A$ is a skewfield.*

*Proof.* If $a \in R \smallsetminus P$ then $P + Ra$ is essential, two-sided and contains $P$ so it is equal to $R$. Then, for some $r \in R$ and $p \in P$, we have $1 = p + ra$. We have already seen (cfr. proof of proposition 2.3.8) that $1 + P$ consists of units, so $ra$ is a unit hence $a$ is a unit. If $q \in A \smallsetminus R$ there is some $y$ with $yq \in R \smallsetminus P$, so $yq$ is a unit of $R$ hence $q$ is a unit of $A$. Finally, if some $p \in P$ were not invertible, then $Ap \subseteq P$ since no element in $Ap$ is a unit. Then we would have $A(RpR) \subseteq P$, but this would contain some regular $u$ which is invertible in $A$ and $Au \subseteq P$ would give a contradiction. This implies that $A$ is a skewfield and $R$ is an invariant Dubrovin valuation ring on $A$, so it must be a valuation ring. □

Recall that a partially ordered group $G$ is called *archimedean* if (for all $a, b \in G$)

$$\forall n \in \mathbb{Z} : a^n < b \quad \Longrightarrow \quad a = e$$

where $e$ is the neutral element of $G$.

**Corollary 2.4.5.** *If $\Gamma$ is an archimedean group, then $R$ is a valuation ring.*

*Proof.* In view of the preceding proposition we only have to show that $\bigcap P^n = 0$. Suppose it is not, then $I = \bigcap P^n$ is a nonzero ideal. Pick $0 \neq b \in I$, then $RbR \subseteq I$ is a fractional ideal, hence there exists an ideal $J \in \mathcal{F}(R)$ with $v(J) + v(RbR) = 0$. Then $v(P^n) + v(RbR) \leq 0$ for any $n$, so $nv(P) + v(RbR) \leq 0$. However, putting $v(RbR) = \gamma$, there must be some $n$ with $nv(p) > -\gamma$ which is a contradiction. □

**Proposition 2.4.6.** *Let $R$ be any order in a simple artinian ring $A$ and suppose that $v : \mathcal{F}(R) \to \Gamma$ is an apv which takes values in a totally ordered semigroup $\Gamma$. Then:*

*(1) $P = \{a \in A \mid v(RaR) > 0\}$ defines a prime $(P, A^P)$ for which*

$$\{a \in A \mid v(RaR) \geq 0\} \subseteq A^P.$$

*(2) if $v(I) = \{a \in A \mid aI \subseteq P\}$ and $\Gamma$ is a group, then the inclusion from (1) is an equality.*

*Proof.* (1) For $a, b \in P$ we have $v(R(a+b)R) \geq \min\{v(RaR), v(RbR)\}$ which is strictly positive, so $a + b \in P$. Clearly, $P$ is an ideal in $A^P$ and $R \subseteq A^P$ since $v(R) = 0$ and we have $v(RrRRpR) = v(RpR) > 0$ for all $r \in R$ and $p \in P$. If $a, a' \in A$ are such that $aA^P a' \subseteq P$ then $aRa' \subseteq P$ hence $v(RaRa'R) > 0$. From $v(RaR) + v(Ra'R) > 0$ it follows that either $v(RaR) > 0$ or $v(Ra'R) > 0$, i.e. either $a \in P$ or $a' \in P$. If $v(RaR) \geq 0$ for some $a \in A$ then for all $p \in P$ we have $v(RaRpR) = v(RaR) + v(Rp) > 0$ and $v(RpRaR) = v(RpR) + v(RaR) > 0$ so $a \in A^P$.

(2) Consider $a \in A^P$. $RaR$ is invertible in $\mathcal{F}(R)$ so there is some $J \in \mathcal{F}(R)$ with $v(RaR) + v(J) = 0 = v(J) + v(RaR)$, hence $v(JaR) = v(RaJ) = 0$. If $v(RaR) < 0$ then $v(J) > 0$ or in other words $J \subseteq P$. But then $a \in A^P$ would give $RaRJ \subseteq P$ which implies $v(RaRJ) > 0$ which is a contradiction. Therefore $v(RaR) \geq 0$ and $A^P = \{a \in A \mid v(RaR) \geq 0\}$. $\square$

# Chapter 3

# Dubrovin valuation rings

Generalising valuations to a non-commutative context poses a few basic problems. Firstly, in some sense, the right non-commutative counterpart to fields are simple artinian rings, but these may contain zero-divisor — something valuations cannot handle.[1] Secondly, even for skewfields, good extensions do not exist in general.[2] To deal with these problems, Dubrovin introduced what he called non-commutative valuation rings in two seminal papers: [22] and [23]. In [22], he proved a number of equivalent characterisations of these non-commutative valuation rings and studied the ideal theory of such rings. In [23], he proved some extension results, the most important of which is probably that if a skewfield $D$ is finite dimensional over its centre, every central valuation extends to a non-commutative valuation ring on $D$. Non-commutative valuation rings — under the name of Dubrovin valuation rings — have been studied quite a bit since then by, a.o., Gräter, Morandi, Brungs and Wadsworth. For general theory about Dubrovin valuation rings, we refer the reader to [59] or [58].

A problem which has garnered attention since the introduction of Dubrovin valuation rings is defining a good notion of value functions to associate to these rings; after all, one of the nicest things about commutative valuation rings is the existence of both a completely ring theoretical definition and a more analytic one in terms of value functions. Various attempts have been made to introduce value functions, cfr. e.g. [69] or [29].

We will give an overview of the main results concerning Dubrovin valuation rings, including some characterisations and some of the more important results. We will then follow the outline from Van Geel's work and introduce arithmetical pseudo-valuations in much the same way as in 2.4. For this we will need to restrict our attention to Dubrovin valuations rings $R$ with $J(R)^2 \neq J(R)$. These rings can be considered as the right non-commutative counterpart of classical

[1] For this reason, some authors — in particular Mahdavi Hezavehi — have considered matrix-valuations (see e.g. [52] or [16]) but the results remain somewhat unsatisfactory.

[2] Although they do in some important cases, cfr. e.g. [68] or [3].

valuation rings, as they will allow us to study divisor theory for bounded Krull orders (cfr. chapter 4). The exposition in this chapter is along the lines of [59], while most of the new results (which are contained in section 3.4) come from [102].

## 3.1 Chain rings and $n$-chain rings

**Definition 3.1.1.** *A subring $S$ of some ring $R$ is called a left $n$-*chain ring *in $R$ if for all $a_0, ..., a_{n+1}$ in $S$ there is some $i$ with*

$$a_i \in \sum_{j \neq i} a_j S.$$

*A right $n$-chain ring is defined similarly and if $S$ is both a left $n$-chain ring and a right $n$-chain ring it is called an $n$-chain ring. If $n = 1$, we say $S$ is a chain ring.*

Note that if $S$ is an $n$-chain ring of $R$ and $S \subseteq R' \subseteq R$, then $S$ is an $n$-chain ring of $R'$. Since the ideals of a valuation ring are linearly ordered, chain rings can be considered as generalisations of valuation rings. They have been studied in some detail, mainly by Brungs and Törner (cfr. e.g. [10], [4]), but very little has been written about $n$-chain rings for arbitrary $n$. In fact, apart from Dubrovin's original paper ([22]) and some books dealing with Dubrovin valuation rings ([59] & [58]), nothing seems to have been published about them.

Let now $S$ be an $n$-chain ring in a simple artinian ring $Q$ and suppose $S$-ideals of $Q$ to be linearly ordered. In this case, we can mimic a construction by Morandi ([69]). Define the stabiliser $\mathrm{st}(S) = \left\{q \in Q \mid qSq^{-1} = S\right\}$ and put $\Gamma_S = \mathrm{st}(S)/U(S)$. $\Gamma_S$ can be ordered in a canonical way by $\overline{x} \geq \overline{y}$ if $xS \subseteq yS$ (where $x$ and $y$ are in $\mathrm{st}(S)$). Since $S$-ideals are linearly ordered, $\Gamma_S$ must also be linearly ordered.

**Proposition 3.1.2.** *Suppose $S$-ideals of $Q$ are linearly ordered. If for all $q \neq 0$ in $Q$ there is an $s_q$ in $\mathrm{st}(S)$ with $SqS = s_qS$, then there is a map $v : Q \to \Gamma$ to a totally ordered group $\Gamma$ satisfying (for all $q, q' \in Q$):*

*(i)* $v(q - q') \geq \min\{v(q), v(q')\}$,

*(ii)* $v(qq') \geq v(q)v(q')$,

*(iii)* $v(\left\{q \in U(Q) \mid v(q) = v(q^{-1})^{-1}\right\}) = \Gamma$,

*(iv)* $\Gamma \simeq \Gamma_S$,

*(v)* $S = \{q \in Q \mid v(q) \geq 0\}$.

*Proof.* The proof is exactly the same as for Dubrovin valuation rings, which was done by Morandi in the aforementioned [69]. □

It might be of interest to mention the special case of chain rings in skewfields. Brungs and Törner introduced in [10] a partial valuation on a chain ring $S$ by putting $W = \{aS \mid a \neq 0\}$, a map $\tilde{x} : W \to W : aS \mapsto xaS$ for any $x \neq 0$, and $\tilde{H}(S) = \{\tilde{x} \mid x \neq 0\}$ the group of those maps. This group is partially ordered by

$$\tilde{x} \geq \tilde{y} \Leftrightarrow xaS \subseteq yaS$$

leading to a canonical partial valuation $v : Q \to \tilde{H}(S)$. The group of positives for $v$ can be described in a very nice way:

**Proposition 3.1.3.** *If $S$ is a chain ring in a skewfield and $v$ is the associated partial valuation, then*

$$\{q \in Q \mid v(q) \geq v(1)\} = \bigcap_{0 \neq s \in S} sRs^{-1}.$$

*Proof.* See [10], □

Compare this result with proposition 2.2.4 where the intersection is taken over all $q$ in $U(Q)$. Finally, we can not but mention the existence of an interesting structure theorem for chain rings. Since the statement is quite involved and not necessary for us, we will not state it and instead suffice with pointing the interested reader to [28].

**Remark 3.1.4.** [28] and various other sources use the term serial ring instead of chain ring.

## 3.2 The basics of Dubrovin valuation rings

The usual definition of a Dubrovin valuation ring is as follows:

**Definition 3.2.1.** *Let $Q$ be a simple artinian ring. A* Dubrovin valuation ring *on $Q$ is a subring $R$ with a prime ideal $M$ such that:*

*(DV1) $R/M$ is simple artinian,*

*(DV2) for all $q \in Q$ there are $r, r' \in R$ such that $rq, qr' \in R \setminus M$.*

There are quite a few alternative characterisations, but in order to even state them, we will need some more terminology.

Suppose that $Q$ and $Q'$ are simple artinian rings. We can extend them both by a symbol $\infty$ which satisfies the rules $x + \infty = \infty = \infty + x$ (for any $x$) and

$x \cdot \infty = \infty = \infty \cdot x$ (for any invertible $x$). Consider a surjective map $f : Q \to Q'$ such that $f(1) = 1$, $f(xy) = f(x)f(y)$ and $f(x+y) = f(x)+f(y)$ provided the right-hand terms are defined. We call such a map a *left point* if for any $q$ with $f(q) = \infty$ there is some $r \in Q$ such that $f(r) \neq \infty$ and $0 \neq f(rq) \neq \infty$. We define *right points* in a similar way. If $f$ is both a left and right point, it is called a *point*.

If $R$ is a Dubrovin valuation ring, one can consider the canonical map $\pi : R \to R/M$. $R/M$ considered as a finitely generated module over itself is a direct sum of a finite number of simple $R/M$ modules. This finite number will be denoted by $d(R/M)$.

Recall that an order in a simple artinian ring is called *Bezout* if any finitely generated left or right ideal is cyclic. If every finitely generated ideal of a ring $R$ is projective as an $R$ module, $R$ is said to be *semi-hereditary*. Now we are ready to state the alternative characterisations theorem:

**Theorem 3.2.2.** *For a subring of a simple artinian ring $Q$, the following are equivalent:*

*(1) $R$ is a Dubrovin valuation ring,*

*(2) there is some simple artinian $Q'$ and a point $f : Q \to Q'$ such that $\{q \in Q \mid f(q) \neq \infty\} = R$,*

*(3) $R$ is a local Bezout order,*

*(4) $R$ is a local semi-hereditary order,*

*(5) $R$ is a local $d(R/M)$-chain ring.*

*Proof.* See [59], [58] or [22]. □

**Remark 3.2.3.** 1. Any Dubrovin valuation ring $R$ on a simple artinian $Q$ is the domain of a localised prime, so $R$ determines $M$ uniquely and vice versa. In fact, $M$ must necessarily be the Jacobson radical of $R$ (cfr. [59]). Consequently, $1 + P$ consists of units.

2. Given a Dubrovin valuation ring $R$, it can easily be checked that

$$f : Q \cup \{\infty\} \to R/M \cup \{\infty\} : q \mapsto \begin{cases} \overline{q} & \text{if } q \in R \\ \infty & \text{if } q \notin R \end{cases}$$

is the (two-sided) point associated to $R$.

It is well-known that any simple artinian ring is isomorphic to $M_n(D)$ for some $n$ and some skewfield $D$ (cfr. e.g. [67]).

**Proposition 3.2.4.** *If $R$ is a Dubrovin valuation ring on $Q \simeq M_n(D)$, then*

*(i)* $M_m(R)$ *is a Dubrovin valuation ring of* $M_m(Q)$*,*

*(ii)* $eRe$ *is a Dubrovin valuation ring for any idempotent* $e$*,*

*(iii)* $R = qM_n(S)q^{-1}$ *for some* $q \in U(Q)$ *and some Dubrovin valuation ring* $S$ *of* $D$*,*

*so the class of Dubrovin valuation rings is Morita invariant.*

*Proof.* See [59], [58] or [22]. □

**Example 3.2.5.** The following is perhaps the easiest non-trivial example of a Dubrovin valuation ring: let $\mathbb{H}$ denote the Hamilton quaternions over $\mathbb{Q}$, let $\mathbb{Z}_p$ be the $p$-valuation ring for some prime number $p$ and put $R = \mathbb{Z}_p + \mathbb{Z}_p i + \mathbb{Z}_p j + \mathbb{Z}_p k$ where $\{1, i, j, k\}$ is the usual basis of $\mathbb{H}$. We will denote the $p$-valuation by $v_p$ and the Jacobson radical of $R$, which is $p\mathbb{Z}_p + p\mathbb{Z}_p i + p\mathbb{Z}_p j + p\mathbb{Z}_p k$ by $P$. Suppose $x = a + bi + cj + dk$ is an element of $\mathbb{H} \setminus R$ and put $n = \min\{v_p(a), v_p(b), v_p(c), v_p(d)\}$. Multiplying $x$ with $p^n$ yields an element of $R$ which is not in $P$, so $(R, P)$ satisfies $(DV2)$.

## 3.3 Some ideal theory for Dubrovin valuation rings

We would like to introduce arithmetical pseudo-valuations associated to some class of Dubrovin valuation rings, but in order to do so we will have to study the ideal theory of Dubrovin valuation rings a little bit. We will strive to keep this section as concise as possible and will therefore not go into any details whatever. Fix for the remainder of this section a Dubrovin valuation ring $R$ in a simple artinian ring $Q$. The following proposition will be crucial for us:

**Proposition 3.3.1.** *The set of $R$ ideals is totally ordered.*

*Proof.* See [59]. □

Slightly less important but still very interesting is the following:

**Proposition 3.3.2.** *Any $R$-bimodule in $Q$ is in fact an $R$-ideal.*

*Proof.* See [59]. □

We will now do some arithmetic using the $R$-ideals of $Q$, in much the same vein as in [85]. Fix for an $R$-ideal $I$ the following notation:

$$\begin{aligned} O_l(I) &= \{q \in Q \mid qI \subseteq I\} \\ O_r(I) &= \{q \in Q \mid Iq \subseteq I\} \\ I^{-1} &= \{q \in Q \mid qIq \subseteq I\}. \end{aligned}$$

For overrings $S$ and $T$ of $R$, we introduce further notation

$$(T : I)_l = \{q \in Q \mid qI \subseteq T\} \qquad (S : I)_r = \{q \in Q \mid Iq \subseteq S\}$$

and finally we put

$${}^*I = (O_l(I) : (O_l : I)_r)_l \qquad I^* = (O_r(I) : (O_r : I)_l)_r.$$

**Lemma 3.3.3.** *With notation as before we have*

*(1)* $I^* = (I^{-1})^{-1} = {}^*I = \{Sc \mid c \in U(Q), I \subseteq Sc\}$,

*(2)* $(I^*)^* = I^*$,

*(3)* $(I^{-1})^* = I^{-1}$.

*Proof.* See e.g. [59]. ☐

**Definition 3.3.4.** *An $R$-ideal $I$ of $Q$ is called* divisorial *if $I^* = I$. The set of divisorial $R$-ideals will be denoted by $\mathcal{D}(R)$.*

**Proposition 3.3.5.** *The set $\mathcal{D}(R)$ is a groupoid*[3] *with a multiplication defined by $I \circ J = (IJ)^*$ if $O_r(I) = O_l(J)$.*

*Proof.* This is an easy verification. (Or see [59].) ☐

As an immediate consequence, the set

$$\mathcal{D}_S(R) = \{I \in \mathcal{D}(R) \mid O_l(I) = S = O_r(I)\}$$

is a group for every overring $S$ of $R$. Recall that an ideal of a ring $R$ is called a *Goldie prime ideal* if $R/P$ is a prime Goldie ring, i.e. if $R/P$ has a simple artinian ring of fractions. The maximal length of a chain

$$P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$$

of Goldie prime ideals is called the *rank* of $R$. Note that a rank $1$ Dubrovin valuation ring in a simple artinian ring $A$ is a maximal subring of $A$. A pair $P_1 \subsetneq P_2$ of Goldie primes with no further Goldie primes in between is called a

[3] For groupoids see chapter 5.

*prime segment*. Prime segments have been studied in some generality (cfr. [8]) and have been classified for Dubrovin valuation rings (cfr. [7]), but are not very important for us. We will suffice with stating a theorem indicating which groups can occur as the group of divisorial ideals for some rank one Dubrovin valuation ring.

**Proposition 3.3.6.** *For a rank one Dubrovin valuation ring $R$, one of the following holds:*

*(i) There is no ideal $0 \subsetneq I \subsetneq J(R)$ and $\mathcal{D}(R)$ is the trivial group.*

*(ii) There is a (non-Goldie) prime ideal $0 \subsetneq P \subsetneq J(R)$ and $\mathcal{D}(R) \simeq (\mathbb{Z}, +)$ is generated by $P^*$.*

*(iii) For every $r \in J(R) \setminus \{0\}$ there is an ideal $I$ with $a \in I$ and $\bigcap_{n \in \mathbb{N}} I^n = 0$. In this case we have*

$$\mathcal{D}(R) \simeq \begin{cases} (\mathbb{Z}, +) & \text{if } J(R)^2 \subsetneq J(R) \\ (\mathbb{R}, +) & \text{if } J(R)^2 = J(R) \end{cases}.$$

This suggests that our definition of rank is the right one, since the valuations of rank one in the classical sense are those with value groups isomorphic to a subgroup of $(\mathbb{R}, +)$.

## 3.4 Arithmetical pseudo-valuations on Dubrovin valuation rings

For primes containing an order with commutative semigroup of fractional ideals, Van Geel ([95]) introduced arithmetical pseudo-valuations, but this condition is very strong and reduces the applicability in practice to maximal orders and Dubrovin valuation rings in finite dimensional central simple algebras. For Dubrovin valuation rings on infinite dimensional central simple algebras the semigroup $\mathcal{F}(R)$ need not be commutative. In this section, which is based on [102], we will show that this condition can be relaxed so as to obtain arithmetical pseudo-valuations for more general Dubrovin valuation rings.

Throughout, the Jacobson radical of a Dubrovin valuation ring $R$ will be denoted by $P$.

**Proposition 3.4.1.** *For a noetherian Dubrovin valuation ring $R$ we have for all $I, J \in \mathcal{F}(R)$ that $IJ \subseteq P$ iff $JI \subseteq P$.*

*Proof.* From $IJ \subseteq P$ it follows that either $I \subseteq P$ or $J \subseteq P$, since $(R, P)$ is a localised prime. Assume without loss of generality $I \subseteq P$. Since $R$-ideals are linearly ordered (3.3.1), if $JI \nsubseteq P$ then $P \subsetneq JI$ so we have $R \subseteq JI \subseteq JP \subseteq J$ hence $J = RJ \subseteq JIJ \subseteq JP \subseteq J$ which gives $J = JP$. Since $R$ is an order, there is some regular $u \in R$ with $uJ \subseteq R$ and since $R$ is noetherian $uJ = \sum a_i R$ for a finite set of $a_i$'s in $uJ$. Then also $J = \sum u^{-1} a_i R$, so $J$ is a finitely generated $R$-submodule of $A$. By Nakayama's lemma $J$ must be zero, which is a contradiction. □

**Corollary 3.4.2.** *If $R$ is a noetherian Dubrovin valuation ring then there is an arithmetical pseudo-valuation*

$$v : \mathcal{F}(R) \to \Gamma : I \mapsto (P : I) = \{a \in A \mid aI \subseteq P\}$$

*for some totally ordered group $\Gamma$. Furthermore, $P = \{a \in A \mid v(RaR) > 0\}$ and $R = \{a \in A \mid v(RaR) \geq 0\}$.*

*Proof.* Using the preceding proposition instead of 2.3.4 we can repeat the proof of theorem 2.4.1. The only thing we need to prove is that the $\Gamma$ which said theorem provides is a group, so consider $I \in \mathcal{F}(R)$. By a similar argument as in the proof of proposition 3.4.1 it is finitely generated as a left $R$-ideal of $A$. Since a Dubrovin valuation ring is a Bezout order (3.2.2), it is cyclic. In fact $I = Ru$ for some regular $u$ and thus $RuR = Ru$. Since $R$ is a Dubrovin valuation ring, there is some $a \in A$ with $ua \in R \setminus P$. Then $Rua \subseteq R$, $Rua \nsubseteq P$ and $Ia \subseteq R$, $Ia \nsubseteq P$. in the same way one obtains a $b$ such that $bI \subseteq R$ but $bI \nsubseteq P$. We can now repeat the last part of the proof of proposition 2.4.2 to conclude that $\Gamma$ is a group. □

**Remark 3.4.3.** If $R$ is a Dubrovin valuation ring where

$$v : \mathcal{F}(R) \to \Gamma : I \mapsto \{a \in A \mid aI \subseteq P\}$$

is a non-trivial arithmetical pseudo-valuation with values in a totally ordered group, then $IJ \subseteq P$ if and only if $JI \subseteq P$. Indeed, suppose $IJ \subseteq P$ and $JI \nsubseteq P$ then, as in proposition 3.4.1, we find $JP = J$ but then $v(P) = 0$ which is impossible.

If $R$ is non-noetherian, then $P = P^2$ is possible in which case no nice apv can exist since otherwise $v(P) = 2v(P)$ which would imply $v(P) = 0$. If we exclude this slightly pathological case, a nice apv does exist.

**Proposition 3.4.4.** *Let $R$ be a Dubrovin valuation ring with $\bigcap P^n = 0$, then there is an apv as before.*

*Proof.* If $I, J \in \mathcal{F}(R)$ with $IJ \subseteq P$ but $JI \nsubseteq P$. The same argument as in proposition 3.4.1 leads to $J = JP$ so $J = JP^n$ for any $n$. There is some regular $u \in R$ with $uJ \subseteq P$ hence $uJ = uJP^n \subseteq P^{n+1}$. But then $uJ = 0$ which implies

$J = 0$ and this is a contradiction. Now we can proceed as in corollary 3.4.2 to find an apv with values in a semigroup.

The only thing we need to prove is that $\Gamma$ is a group. Lemma 1.5.4 in [59] says that $P = Rp = pR$ for some regular $p \in P$. Since $P$ is principal as a left $R$-ideal, lemma 1.5.6 in the same source gives $PP^{-1} = R = P^{-1}P$ (recall that $P^{-1} = \{a \in A \mid PaP \subseteq P\}$). Consider now a fractional $R$-ideal $I$. Clearly, $(R:I)I \subseteq R$. Suppose we also have $(R:I)I \subseteq P$, then $P^{-1}(R:I)I \subseteq R$ hence $P^{-1}(R:I) \subseteq (R:I)$ so $P^{-1}(R:I)I \subseteq P$. This means $(R:I)I \subseteq P^2$ and by repeating this process we find $(R:I)I \subseteq P^n$ for any $n$, but $(R:I)I \subseteq \bigcap P^n = 0$ which is a contradiction. Therefore, $(R:I)I \subseteq R$ but $(R:I)I \nsubseteq P$, so there exists an $a \in (R:I)$ such that $aI \subseteq R$ but $aI \nsubseteq P$. □

**Example 3.4.5.** Consider $(R, P)$ as in example 3.2.5. The maximal ideal of $R$ is $P = P_p + P_p i + P_p j + P_p k$ where $P_p$ is the maximal ideal of $\mathbb{Z}_p$ and $P^n$ is just $P_p^n + P_p^n i + P_p^n j + P_p^n k$. This immediately yields $\bigcap P^n = 0$, so there exists an apv for this Dubrovin valuation ring; it is simply the map $v : \mathcal{F} \to \mathbb{Z}$ which sends $P^n$ to $n$. This definition makes sense, since we will show later (cfr. 3.4.8) that all fractional ideals are of the form $P^n$.

Perhaps more interesting than this simple example if the following, which demonstrates the use of our construction to obtain apvs on more exotic Dubrovin valuation rings.

**Example 3.4.6.** Let $Q$ be a simple Artinian ring, let $\sigma \in \mathrm{Aut}(Q)$, and put $Q[X,\sigma]$ the skew polynomial ring over $Q$. $Q[X,\sigma]$ has a maximal ideal $P = XQ[X,\sigma]$. Put $T$ the localisation of $Q[X,\sigma]$ at $P$. For $t = (\sum a_i x^i)(\sum b_i x^i)^{-1}$ arbitrary in $T$, we can define $f(t) = a_0 b_0^{-1}$. This gives a map $\phi : T \to Q : t \mapsto f(t)$. It has been shown in [107] that an order $R$ of $Q$ is a Dubrovin valuation ring if and only if $\tilde{R} = \phi^{-1}(R)$ is a Dubrovin valuation ring of $T$ and that $J(\tilde{R}) = J(R) + J(T)$. It is clear that if $R$ is a Dubrovin valuation ring on $Q$ with $\bigcap J(R)^n = 0$ we also have $\bigcap J(\tilde{R}) = 0$. Therefore an apv exists, but $R$ is not finite dimensional over its centre.

The following characterises noetherian Dubrovin valuation rings within the class of rank one Dubrovin valuation rings. The result may be known but we found no reference for it in the literature.

**Proposition 3.4.7.** *For a Dubrovin valuation ring $R$ on a simple Artinian ring $A$ the following are equivalent:*

*(1) $R$ is noetherian.*

*(2) $R$ has rank $1$ and $P \neq P^2$.*

*(3) $R$ has rank $1$ and $\bigcap P^n = 0$.*

*Proof.* $(1) \Rightarrow (2)$ If $R$ is noetherian then all ideals (and $R$-ideals of $A$) are principal, so $P \neq P^2$. Suppose $0 \neq Q$ is another prime ideal in $P$. Let $P = Rp$, then $Q = Ip$ for some non-trivial ideal $I$ of $R$. $Q = IP$ yields $I \subseteq Q$ since $Q$ is prime and $P \nsubseteq Q$. Hence $Q = IP \subseteq QP \subseteq Q$ implies $Q = QP$ which implies $Q = 0$ by Nakayama's lemma. $(2) \Rightarrow (3)$ If $P \neq P^2$, then $\bigcap P^n \neq P$ which, by Lemma 1.5.15 in [59], gives $\bigcap P^n = 0$. $(3) \Rightarrow (1)$ Since $\bigcap P^n = 0$, $P \neq P^2$. Since $R$ is rank $1$, $R = O_l(I) = O_r(I)$ for any $R$-ideal $I$. By proposition 1.5.8 in [59], it follows that if $I$ is not principal, then $II^{-1} = P$ and $P = P^2$ which is a contradiction. $\square$

Recall that an order is an *Asano order* if every ideal $I \neq 0$ of $R$ is invertible. If $R$ is an Asano order satisfying the ascending chain condition on ideals, then $\mathcal{F}(R)$ is the Abelian group generated by maximal ideals and every maximal ideal is a minimal non-zero prime ideal (see e.g. [66]).

A semi-local order $R$ in a simple Artinian $A$ is a noetherian Asano order if and only if it is a principal ideal ring. If $R$ is a Dubrovin valuation ring of $A$ then $R$ is a maximal order if and only if $\mathrm{rk}(R) = 1$ and $R$ is Asano if and only if it is a principal ideal ring, so a noetherian Dubrovin valuation ring is a noetherian maximal order and an Asano order, i.e. a principal ideal ring.

**Proposition 3.4.8.** *If $R$ is a noetherian Dubrovin valuation ring then the corresponding apv takes values in $\mathbb{Z}$.*

*Proof.* Since $R$ is a noetherian Asano order, $\mathcal{F}(R)$ is generated by the maximal ideals of $R$, but since $P$ is the unique maximal ideal and the value group is necessarily torsion-free, we have $\mathcal{F}(R) = \mathbb{Z}$. $\square$

# Chapter 4

# A divisor theory for bounded Krull orders

Divisor theory is an important tool in classical algebraic geometry. Therefore, when trying to develop a non-commutative algebraic geometry, it is natural to look for a suitable analogon of divisor theory in the non-commutative world. In the commutative case, the proper context for developing divisor theory are Krull domains. These are, by definition, integral domains $R$ satisfying

(1) $R_{\mathfrak{p}}$ is a discrete valuation ring for all $\mathfrak{p} \in X^1(R)$

(2) $R = \bigcap_{\mathfrak{p} \in X^1(R)} R_{\mathfrak{p}}$

(3) Any $r \in R \setminus \{0\}$ is contained in $R_{\mathfrak{p}}$ for but finitely many $\mathfrak{p} \in X^1(R)$

where $X^1(R)$ is the set of height one prime ideals of $R$. In [57], Marubayashi gave a generalisation of this concept by replacing the discrete valuation rings with noetherian local Asano orders and the localisations with noetherian essential overrings of $R$.[1] This definition was again generalised by Chamarie (cfr. [12]) to the concept of non-commutative Krull orders as we will use it[2]. When comparing the definition above to 4.2.2, it becomes clear that Chamarie's definition is a logical choice.[3]

We will use the arithmetical pseudo-valuations associated to Dubrovin valuation rings with non-idempotent Jacobson radical, as established in chapter 3, together with results from [59] to develop a divisor theory for bounded Krull domains. We will, in particular, be able to prove some useful approximation

[1] A similar approach was followed in [48].

[2] Although we will be mainly interested in the case of *bounded* Krull domains, where both definitions coincide.

[3] It may be of interest to also mention the concept of $\Omega$-Krull rings, see [41].

theorems which are quite similar to the commutative case. The first two sections of this chapter are based on the exposition in [59], while the last section contains mostly material from [102].

## 4.1 Bounded Krull orders...

**Definition 4.1.1.** *Let $R$ be a ring and let $E$ be an injective left $R$-module. A collection $\mathcal{F}$ of left ideals of $R$ is called a* left Gabriel topology *on $R$ if $\mathrm{Hom}(R/I, E) = 0$ for all $\in \mathcal{F}$ and*

*(1) $Ir^{-1} = \{x \in R \mid xr \in I\} \in \mathcal{F}$ for all $I \in \mathcal{F}$ and $r \in R$,*

*(2) if $I$ is a left ideal of $R$ such that $Ix^{-1} \in \mathcal{F}$ for all $x$ in some $J \in \mathcal{F}$, then $I \in \mathcal{F}$.*

Mutatis mutandis right Gabriel topologies are defined. If $\mathcal{F}$ is a Gabriel topology, it follows immediately that $\mathcal{F}$ is closed under multiplication and that it is a filter. In fact, in e.g. [92] Gabriel topologies are introduced as filters satisfying (2). The interested reader is referred to either [92] or [32] for more information on Gabriel topologies.

Let $R$ now be a subring of some simple artinian $Q$. Consider

$$\mathcal{F}_R = \left\{ I \mid I \text{ left ideal} : (R : Ix^{-1})_r = R \text{ for all } x \in R \right\}.$$

This $\mathcal{F}_R$ is a left Gabriel topology known as the *canonical* left Gabriel topology. If $I$ is an ideal, we put $\overline{I}^\tau = \{r \in R \mid \exists F \in \mathcal{F}_{\mathcal{R}} : Fr \subseteq I\}$. We call $\overline{I}^\tau$ the $\tau$*-closure* of $I$ and we say that $I$ is $\tau$*-closed* if $I = \overline{I}^\tau$.

**Definition 4.1.2.** *A maximal order is called a* Krull order *if it satisfies the ascending chain condition on $\tau$-closed left ideals.*

**Example 4.1.3.** Clearly, any noetherian maximal order is a Krull order, but the converse does not hold in general. Consider for example $R = k[(X_i)_{i \in \mathcal{I}}]$ where $k$ is a field and $\mathcal{I}$ is an infinite set. Then an Ore extension $R[X; \sigma, \delta]$ will be a non-noetherian Krull order.

It is perhaps of interest to compare this definition to the one given by Marubayashi in [57]. To avoid confusion, we will call a Krull order in the sense of Marubayashi a *Marubayashi order*. A Marubayashi order, then, is a prime Goldie ring $A$ with two collections $(R_i)_{i \in \mathcal{I}}$ and $(S_j)_{j \in \mathcal{J}}$ of essential overrings of $A$ satisfying:

(1) $A = (\bigcap_i R_i) \cap (\bigcap_j S_j)$

(2) $R_i$ is noetherian and local Asano for all $i$

(3) $S_j$ is noetherian and simple for all $j$

(4) $|\mathcal{J}| < \infty$

(5) if $a \in A$ is regular, then $aR_i = R_i$ for all but finitely many $i$.

Definitions for the terminology may be found in loc. cit.

Recall that two orders $R$ and $R'$ in a simple artinian ring $Q$ are said to be equivalent if there are some invertible $c, c', d, d' \in Q$ such that $cRc' \subseteq R'$ and $dR'd' \subseteq R$.

**Proposition 4.1.4.** *Any maximal order equivalent to a Krull order is itself a Krull order.*

*Proof.* See [12]. □

**Proposition 4.1.5.** *Any Ore extension $R[X, \sigma]$ of a Krull order $R$ is again a Krull order.*

*Proof.* See [12]. □

**Remark 4.1.6.** As Chamarie points out in [12], there are Krull orders which are not Marubayashi orders. His argument is as follows: by [18], a simple noetherian integral domain $R$ has global dimension $\leq 2$ if and only if every maximal order equivalent to $M_n(R)$ is simple. Since e.g. the Weyl algebra $\mathbb{A}_n(\mathbb{C})$ with $n > 2$ is a simple noetherian integral domain with global dimension $> 2$, there are maximal orders equivalent to $M_n(R)$ which are not simple. By the preceding proposition, these are Krull orders which cannot be Marubayashi orders since they are not simple.

The following proposition, which is also due to Chamarie, relates the two definitions in a convenient manner:

**Proposition 4.1.7.** *An order $R$ in a simple artinian ring $Q$ is a Marubayashi order if and only if:*

*(i) $R$ is a Krull order*

*(ii) $R_0 = \{q \in Q \mid \exists I$ two-sided ideal of $R : qI \subseteq R\}$ is noetherian*

*(iii) for any non-zero two-sided ideal $I$ of $R$, one has $IR_0 = R_0 I = R_0$.*

*Proof.* See [12]. □

**Definition 4.1.8.** *A ring is called* left bounded *if any essential left ideal contains a non-zero two-sided ideal.*

Once more, the concept of a right bounded ring is defined analogously. It has been shown by Chamarie ([12]) that a Krull order is left bounded if and only if it is right bounded, so we will simply talk about bounded Krull orders. If a Krull order $R$ is bounded, then $R_0 = Q$ and consequently $R$ is a Marubayashi order. Remark 4.1.6 gives some examples of non-bounded Krull orders.

## 4.2 ...are the right context...

In this section we will study divisorial ideals in Krull orders and their localisation. We will follow the exposition from [59] although many results are from [12]. Some additional results from [102] are also included.

**Proposition 4.2.1.** *Consider a Krull order $R$. Let $I = {}^*(P_1^{e_1} \cdots P_n^{e_n})$ be a divisorial ideal for some maximal divisorial ideals $P_i$. Then $R$ can be localised at $I$, $R_I$ is a bounded principal ideal ring, and*

*(i)* $R_I = \bigcap_i R_{P_i}$;

*(ii)* $J(R_I) = \bigcap P_i R_I$.

*Proof.* See [59]. □

The following theorem is crucial. It shows that bounded Krull orders and Dubrovin valuation rings are the proper counterparts for the Krull domains and discrete valuation rings of the commutative case.

**Theorem 4.2.2.** *Let $R$ be a Krull order in some simple artinian $Q$. Then:*

*(1)* $R = R_0 \cap \left(\bigcap_{P \in X^1(R)} R_P\right)$,

*(2) for any $P \in X^1(R)$, $R_P$ is a rank one Dubrovin valuation ring with $J(R_P)^2 \neq J(R_P)$,*

*(3) any regular $c \in R$ is in $U(R_P)$ for all but finitely many $P \in X^1(R)$.*

*Proof.* See [59]. □

In the case where $R$ is a bounded Krull order, $R_0 = Q$ which yields $R = \bigcap_{P \in X^1(R)} R_P$ — in others words: bounded Krull orders are a proper context for non-commutative divisor theory. We will now show that the definite article in the title of this section is well-chosen. To this end we repeat a result from [59] which is a converse of sorts to the previous theorem:

**Proposition 4.2.3.** *If $R$ is an order in some simple artinian $Q$ such that* *(i)* $R = \bigcap_i R_{\mathcal{O}_i}$ *for regular Ore sets* $\mathcal{O}_i$

*(ii)* $R_{\mathcal{O}_i}$ *is a local principal ideal ring*

*(iii) any regular* $c \in R$ *is in* $U(R_{\mathcal{O}_i})$ *for all but finitely many* $i$

*then* $R$ *is a bounded Krull order.*

*Proof.* See [59]. □

**Theorem 4.2.4.** *Let* $R$ *be a prime noetherian ring and an order in a simple artinian* $Q$*. Suppose every minimal non-zero prime ideal is localiseable,* $R_P$ *is a Dubrovin valuation ring for every* $P \in X^1(R)$ *and* $R = \bigcap R_P$*, then* $R$ *is a bounded Krull order.*

*Proof.* Since $R$ is noetherian, every $R_P$ is noetherian too and since it is a Dubrovin valuation ring it must be an Asano order hence a principal ideal ring. Since every $R_P$ is a maximal order, so is $R$. As a noetherian maximal order, $R$ is a Krull order and by theorem 4.2.2 every regular element is a non-unit in only finitely many of the $R_P$'s (for $P$ maximal divisorial, i.e. $P \in X^1(R)$). The theorem now follows from the preceding proposition. □

## 4.3 ...for non-commutative divisor theory.

The following example shows that, in a very simple case, we can readily introduce divisors. Starting from this idea, we will develop some divisor theory and, in particular, prove some approximation theorems.

**Example 4.3.1.** Consider the simple artinian ring $M_n(\mathbb{Q})$ [4]. The subring $M_n(\mathbb{Z})$ is a Krull order in $M_n(\mathbb{Q})$ with minimal prime ideals of the form $pM_n(\mathbb{Z})$ for some prime number $p$. The value function associated to $pM_n(\mathbb{Z})$ is the map

$$v_p : M_n(\mathbb{Q}) \to \mathbb{Z} : (a_{i,j})_{i,j} \mapsto \min_{i,j} \{V_p(a_{i,j})\}$$

where $V_p$ is the valuation on $\mathbb{Q}$ associated to $p$. In this way, we can associate to any element $x$ of $M_n(\mathbb{Q})$ the sum $\sum_p v_p(x)p$. Clearly, this sum has only finitely many non-zero factors.

Let, for the remainder of this section, $Q$ be a simple Artinian ring and let $R$ be a bounded Krull order in $Q$. We want to develop some divisor theory in this context. The localisation of $R$ at a rank 1 prime ideal $P$ is a Dubrovin valuation ring with $J(R_P)^2 \neq J(R_P)$, so we can associate an arithmetical pseudo-valuation $v_P$ to it.

[4] Or indeed $M_n(Q)$ for $Q$ the field of fractions of a Krull domain $Z$; the reasoning is essentially the same.

**Definition 4.3.2.** *A* divisor *of a bounded Krull order $R$ is an element in the free Abelian group $\bigoplus_{P \in X^1(R)} \mathbb{Z}P$. To any $I \in \mathcal{F}(R)$ we can associate the divisor $\operatorname{div}(I) = \sum v_P(I_P)P$ where $I_P = R_P I$. The set of divisors of $R$ will de denoted by $\operatorname{Div}(R)$.*

This definition is justified by the following proposition (which is also given, with a slightly different proof, in [66]):

**Proposition 4.3.3.** *Suppose $R_P$ is noetherian. If $I$ is an $R$-ideal of $A$, then $I_P$ is an $R_P$-ideal of $A$.*

*Proof.* Let $u$ be regular in $R$ with $uI \subseteq R$, then $RuRI \subseteq R$ and $RuR = Ru'$ for some regular $u' \in R$. Then $R_P u' I$ is the localisation of $Ru'I$ and it is an ideal of $R_P$. $R_P u'$ is the localisation of $Ru'$ so it is also an ideal of $R_P$, hence $R_P u' I = R_P u' R_P I = R_P u' I_P$ is an ideal of $R_P$. Now $U' I_P \subseteq R_P$, i.e. $I_P$ is an $R_P$-ideal of $A$. □

Observe that, since any regular element is a non-unit in only finitely many localisations, $\operatorname{div}(I)$ contains only finitely many non-zero terms. Moreover, $\operatorname{div}(I) \leq \operatorname{div}(J)$ if and only if $v_P(I) \leq v_P(J)$ for all $P \in X^1(R)$. By putting $I^* = \bigcap_{P \in X^1(R)} I_P$ we find $v_P(I) = v_P(I^*)$. We can consider $\operatorname{div} : \mathcal{D}(R) \to \operatorname{Div}(R)$, which is a group morphism of Abelian groups. It reverses the ordering in the sense that $I \subseteq J$ yields $\operatorname{div}(J) | \operatorname{div}(I)$. We will now prove some approximation theorems. Maury (cfr. [64]) proved the first approximation result in the sense of (A) hereafter. However, we will generalise the results in a somewhat more elaborate way as used by Van Geel in [95] for rings with a commutative (semi)group of fractional deals.

By theorem 4.2.2 and proposition 3.4.7, all $R_P$ are noetherian. In view of 4.3.3, $I_P$ is an $R_P$-ideal so it makes sense to define $v_P(x) = v((RxR)_P)$ for an element $x$ of $R$.

**Lemma 4.3.4.** *Let $I$ be a fractional ideal. For any $v = v_P$, the set $\{v(x) \mid x \in I\}$ has a minimum which is equal to $v(I)$.*

*Proof.* It is clear that $v(x) \geq v(I)$ for any $x \in I$, so suppose there is some $y$ with $v(I) \leq v(y) \leq v(x)$ for all $x \in I$. Any $z \in v(y)$ must be in $v(RxR)$, so $yRxR \subseteq P$ for all $x \in I$ which implies $yI \subseteq P$, hence $y \in v(I)$ which means $v(I) = v(y)$. □

**Lemma 4.3.5.** *For any fractional ideal $I$, $v(I) = 0$ for almost all $v$.*

*Proof.* For any $a \in Q$ there is some regular $c \in R$ with $ca \in R$. It is known that any regular element of a bounded Krull order is invertible in all but finitely many localisations $R_P$, so $v(c) = 0$ for almost all $v$ and consequently $v(a) \geq 0$

for almost all $v$. By the preceding lemma, there is for any $v$ some $x_v$ with $v(I) = \min\{v(x) \mid x \in I\} = v(x_v) = v((Rx_vR)^*)$. Since $R$ is a Krull order, the ascending chain condition holds on divisorial ideals, so $\{(Rx_vR)^* \mid v\}$ has but finitely many elements, $(Rx_{v_1}R)^*, ..., (Rx_{v_n}R)^*$ say. But every divisorial ideal contains a regular element, which is almost always invertible, hence $v(Rx_{v_i}R) = 0$ for all but finitely many $v$ and all $i \in \{1, ..., n\}$ whence $v(I) = 0$ for almost all $v$. □

We will denote by (A) the following approximation property:

*Let $v_1, ..., v_t$ be a finite number of arithmetical pseudo-valuations associated to rank $1$ prime ideals in $R$, let $n_1, ..., n_t$ be integers, and let $a_1, ..., a_t$ be elements of $Q$. Then there exists some $x \in Q$ such that $v_i(x - a_i) \geq n_i$ for $i = 1, ..., t$ and $v(x) \geq 0$ for any $v \notin \{v_1, ..., v_t\}$.*

**Lemma 4.3.6.** *If (A) holds, we can pick $x$ in such a way that $v_i(x - a_i) = n_i$.*

*Proof.* By (A), we can pick $z$ such that $v_i(z - a_i) > n_i$. Since $v$ is surjective and because of 4.3.4 we can also find $z_i$ with $v_i(z_i) = n_i$. Then, once more by (A), we can find $z'$ with $v_i(z' - z_i) > n_i$ from which we can deduce

$$v_i(z') = v_i((z' - z_i) + z_i) = n_i$$

and

$$v_i(z + z' - a_i) = n_i.$$

Therefore, $z + z'$ is the $x$ we were looking for. □

**Lemma 4.3.7.** *Suppose again that (A) holds. For any $v_1, ..., v_t$ and $n_2, ..., n_t \in \mathbb{N}$ there is a regular $c \in R$ with $v_1(c) = 0$, $v_i(c) \geq n_i$ for $i = 2, ..., t$ and $v(c) = 0$ for any $v \notin \{v_1, ..., v_t\}$.*

*Proof.* We can certainly find some $x$ with $v_1(x) = 0$, $v_i(x) = n_i$ for $i = 2, .., t$ and $v(x) \geq 0$ for any $v \notin \{v_1, ..., v_t\}$. Since $v(x) \geq 0$ for any $v$, $x$ is in $R$. Every $v_i$ comes from a Dubrovin valuation ring obtained by localising $R$ at a minimal non-zero prime ideal which we will call $P_i$. $RxR$ is an ideal with $v_1(RxR) = v_1(x) = 0$, so $RxR \nsubseteq P_1$. Since these ideals are generated by their regular elements, we can find some regular $c$ in $RxR \setminus P_1$. Clearly, $v(c) \geq v(x)$ for any $v$, but $v(c) = 0$ – otherwise $c \in P_1$ would hold – which means that it satisfies all the conditions from the statement. □

**Lemma 4.3.8.** *Assume the same setting as before. For every $z \in Q$ and every $v$ there is some regular right invariant $r_v$ with $v(r_v) = v(z)$ and $v'(r_v) \geq v'(z)$ for every other $v'$.*

*Proof.* Consider the fractional ideal $R_v z R_v$. This is equal to $R_v r'_v R_v$ for some regular $r'_v$. Clearly $v(z) = v(r'_v)$. Since $r'_v \in R_v z R_v$ there are some $a_i, b_i \in R_v$ with $r'_v = \sum_{i=0}^n a_i z b_i$. Since $a_i, b_i \in R_v$, we have $v(a_i), v(b_i) \geq 0$. There are only finitely many $v'$, say $v_2, ..., v_t$, for which there is some $i$ with $v'(a) < 0$ or $v'(b) < 0$. Take $c$ regular with $v(c) = 0$ and $v_j(c) \geq 2\max_{j,i}\{-v_j(a_i), -v_i(a_j)\}$ for $j = 2, ..., t$. By our choice of $c$, we have $v'(ca_i) \geq 0$ and $v'(cb_i) \geq 0$ for all $v'$, which implies $ca_i \in R \ni cb_i$. The $r_v$ we are looking for is $cr'_v$. Indeed: $v(r_v) = v(cr'_v) = v(r'_v) = v(z)$ and for any $v'$ we have $v'(r_v)$. □

**Lemma 4.3.9.** *In the same context as before, we can find some regular right invariant element $r$ with $v_1(r) = n_1$ and $v_i(r) \geq n_i$.*

*Proof.* We know we can find some $z$ with $v_1(z) = n_1$ and $v_i(z) \geq 0$ for the other $i$. By the previous lemma we find the desired $r$. □

We now will consider systems of equations

$$\begin{aligned} y_1 &= x_1 a_{11} + \cdots + x_n a_{1n} + b_1 \\ &\vdots \\ y_m &= x_1 a_{m1} + \cdots + x_n a_{mn} + b_m \end{aligned}$$

for certain $a_{ij}$ and $b_i$ in $Q$. A *local solution with respect to $R_v$* for such a system of equations is a set of elements $x_i \in R_v$ such that all $y_j$ are in $R_v$ as well. A *global solution* is a set of $x_i \in R$ such that all $y_j$ are in $R$ too.

**Lemma 4.3.10.** *A system of equations as described above has a global solution if and only if it has local solutions with respect to any $R_v$.*

*Proof.* It is quite clear that any global solution immediately entails a local solution with respect to any $R_v$: if $v$ is such that $v(a_{ij}) \geq 0$ and $v(b_i) \geq 0$, then any $n$-tuple $x_1, ..., x_n$ with $v(x_i) \geq 0$ for all $i$ is a local solution.

Suppose now that $x_i$ is a local solution with respect to $v$ and suppose that there are $x'_i$ such that $v(x_i - x'_i) + v(a_{ki}) \geq 0$. We have now

$$\begin{aligned} v(\textstyle\sum(x_i - x'_i)a_{ki}) &\geq \min_i v((x_i - x'_i)a_{ki}) \\ &\geq \min_i (v(x_i - x'_i) + v(a_{ki})) \\ &\geq 0 \end{aligned}$$

which, since $\sum x_i a_{ki} \in R_v$, implies $\sum x'_i a_{ki} \in R_v$. Hence the $n$-tuple $x'_1, ..., x'_n$ is also a local solution with respect to $v$.

Suppose now that there is a local solution for every $v$. It is clear that we can choose the same solution for all but finitely many $v$'s, so we can just consider a finite set of local solutions $x_{i1}, ..., x_{is}$ with respect to the respective pseudo-valuations $v_1, ..., v_s$. Define $n_{ti} = \max_k(a_{ki})$ for any $1 \leq t \leq s$. By the approximation property (A), we can find $x_1, ..., x_n$ with $v_t(x_i - x_{it}) \geq -n_{ti}$ and $v(x_i) \geq 0$ for any $v \notin \{v_1, ..., v_s\}$. These $x_i$ are a local solution for every $v$, hence a global solution. □

**Lemma 4.3.11.** *Let $x_1a_1 + \cdots + x_na_n = b$ be an equation such that if there is some $i$ with $v(a_i) < 0$ for a certain $v$, then there is some regular right invariant $a_k$ with $v(a_k) = \min_i v(a_i)$. Then a global solution exists (for this equation) if and only if $v(b) \geq v(a_i)$ for all $v$.*

*Proof.* If there is a global solution for the equation at hand, then certainly $v(x_i) \geq 0$ for all $v$ and all $i$. Consequently,

$$\begin{aligned} v(b) &\geq \min_i v(x_i a_i) \\ &\geq \min_i v(a_i) \end{aligned}$$

which implies that the condition is necessary.

We will now show that it is also sufficient. Suppose $v(b) \geq \min_i v(a_i)$. If the right-hand side is greater than or equal to zero, then $v(b) \geq 0$ so a local solution with respect to $v$ exists by the same argument as in the beginning of the previous lemma. If the right-hand side is smaller than zero, we find $\min_i(v(a_i)) = v(a_k)$ where we can choose $a_k$ to be regular and right invariant. Since $a_k$ is regular and $Q$ is simple Artinian, $a_k$ is invertible in $Q$. Consider now the equation $x_k = -x_1a_1a_k^{-1} - \cdots - x_{k-1}a_{k-1}a_k^{-1} - x_{k+1}a_{k+1}a_k^{-1} - \cdots - x_sa_sa_k^{-1} - ba_k^{-1}$. Since $a_k$ is right-invariant, $Ra_k^{-1}a_kR = Ra_k^{-1}Ra_kR$ so $-v(a_k) = v(a_k^{-1})$. This in turn implies $v(ba_k^{-1}) \geq 0$, but then $x_k = ba_k^{-1}$ and $x_i = 0$ for $i \neq k$ gives a local solution with respect to $v$. By the previous lemma, a global solution must exist. □

**Proposition 4.3.12.** *There is a $1-1$ correspondence between divisorial ideals of $R$ and divisors.*

*Proof.* Consider a divisorial ideal $I$ with $v(I) = \gamma_v$. We already know that $\gamma_v = 0$ for all but finitely many $v$, so let $v_1, ..., v_s$ be the set of arithmetical pseudo-valuations for which $\gamma_{v_i} \neq 0$. Since $v(I) = \min\{v(r) \mid r \in I\}$, there are $z_i \in I$ with $v_i(z_i) = \gamma_i$ and $v(z_i) \geq 0$ for all other $v$. By a previous lemma, one can also find regular right-$R_{P_i}$-invariant $a_i$ with $v_i(a_i) = v_i(z_i) = \gamma_i$, $v_j(a_i) \geq \gamma_j$, $v(a_i) \geq 0$ for all other $v$, and with $a_i \in Rz_iR$. Consider now the equation $x_1a_1 + \cdots + x_sa_s = b$. By our choice of $a_i$, there is a global solution if and only if $v_i(b) \geq \gamma_i$. Hence the set of global solutions $\{b \mid \forall v : v(b) \geq \gamma_v\}$ is a subset of $I = I^*$. The other inclusion holds by definition, so divisorial ideals are uniquely determined by their associated divisor.

Suppose we have a divisor $\delta$. Then there is some $z$ with $v(z) \geq \mathrm{ord}_v(\delta)$ if $\mathrm{ord}_v(\delta) \neq 0$ and $v'(z) \geq 0$ otherwise. Moreover, this $z$ can be chosen to be regular and $R_{P_u}$-right-invariant for some fixed $u$. Define

$$V = \{v \mid v(z) \neq 0 \text{ or } \mathrm{ord}_v(\delta) \neq 0\}.$$

This is a finite set, so there is some $x$ with $v(x) = \mathrm{ord}_v(\delta)$ for all $v \in V$ and $v'(x) \geq 0$ for all other $v'$. Put $I$ the ideal generated by $z$ and $x$, then $v(I^*) = \mathrm{ord}_v(\delta)$, but since $v(I^*) = \min\{v(y) \mid y \in I^*\}$ we have $v(I^*) = \mathrm{ord}_v(\delta)$ for all $v$, which shows that every divisor is associated to some divisorial ideal $I^*$. □

Since similar results have been obtained (cfr. [95]) for rings with a commutative group of fractional ideals and in view of the simple example 4.3.1, the objection might be raised that perhaps no new examples exist. This, however, is not the case, as the following example shows.

**Example 4.3.13.** Let $R$ be the ring $\mathbb{F}_p[X][Y,\phi]$ where $\phi$ is the automorphism $\phi : \mathbb{F}_p[X] \to \mathbb{F}_p[X]$ defined by $\phi(X) = X+1$. Now, $\mathbb{F}_p[X]$ is a Krull order and, by [12], Ore extensions of Krull orders are again Krull orders. Consequently, $R$ is a Krull order. In fact, since it is a prime PI-ring, it is also bounded (cfr. [66]). We will show that $RXRYR \neq RYRXR$. Suppose, for the sake of contradiction, that $RXRYR = RYRXR$ and call this ideal $I$. We have that $YX = (X+1)Y = XY + Y$, so $Y = YX - XY \in I$. Moreover, we can also conclude that, for any polynomial $p(X)$, there is some polynomial $p'(X)$ with $Yp(X) = p'(X)Y$. Consequently, for every polynomial $p(X,Y)$ there is a polynomial $p'(X,Y)$ with $Yp(X,Y) = p'(X,Y)Y$.

We find $Y = \sum_{p,q} pXqY$ and consequently $1 = \sum_{p,q} pXq$ for certain $p, q \in \mathbb{F}_p[X][Y,\phi]$. We can write

$$p = p_n Y^n + \cdots + p_0 \quad \text{and} \quad q = q_m Y^m + \cdots + q_0$$

where the $p_i$ and $q_i$ are elements of $\mathbb{F}_p[X]$ with $p_n$ and $q_m$ different from zero. Then we find $1 = p_n Y^n X q_m Y^m + r$ where $r$ is of lower $Y$-degree than $p_n Y^n X q_m Y^m$. But then $p_n Y^n X q_m Y^m$ must be zero, which implies that $p_n$ or $q_m$ must be zero — but this is a contradiction. Therefore, $p$ and $q$ must be in $\mathbb{F}_p[X]$ whence $1 \in \mathbb{F}_p[X]X\mathbb{F}_p[X]$. This, too, is impossible, hence $RXRYR \neq RYRXR$ which implies that $\mathcal{F}(R)$ is not abelian. The group of divisorial ideals, on the other hand, is commutative.

# Chapter 5

# Groupoid valuation rings

Valuations on fields, or equivalently valuation subrings of fields, are both interesting and useful tools, finding applications in number theory, algebraic geometry and many more subjects of commutative algebra. Many generalisations of valuations and of valuation rings to simple Artinian rings have been proposed. These include localised primes, where rings with prime ideals satisfying certain conditions are used as generalised valuation rings, arithmetical pseudo-valuations, where value functions are defined on certain sets of ideals instead of on elements, and Dubrovin valuation rings, which are a special kind of localised primes with very nice properties.

Since any simple Artinian ring is isomorphic to a matrix ring over a skewfield and since matrices are groupoid rings, it is natural to look for generalised valuations in groupoid rings and general groupoid graded rings. Kelarev ([43], [44]) was the first to study groupoid graded rings. He managed to generalise many theorems from the classical group graded case, relating properties — like semi-local, PI-ness, etc. — of the ring $R$ to the rings $R_e$ for $e \in G_0$. Further contributions to the theory of groupoid graded rings include work by Lundström ([51]), Öinert ([81]) and a Clifford-like theorem ([50]). Groupoid-graded rings have also been studied as special cases of semigroup-graded rings. Important work here has been done by a.o. Clase, Jespers and Okniński (e.g. [13], [42], [40]).

For a groupoid $G$, we will define $G$-skewfields as the proper analogon of fields and construct some examples. We will then give suitable generalisations for stability and totality in groupoid-graded rings. This will lead to Theorem 5.3.9, which describes the correspondence between $G$-valuation rings and $G$-valuations. As will be seen in the last section, there is an interesting link between $G$-valuations and Dubrovin valuation rings. In fact, a subring of $M_n(k)$ where $k$ is a field is a Dubrovin valuation ring if and only if it is a groupoid-valuation ring for a suitable groupoid.

In the first two sections of this chapter we will introduce some necessary concepts and basic results. Many of these results have been known in some form for some time already and some others are relatively straightforward generalisations of analogous results from the group-graded case. The last two sections contain new work and provide a link with Dubrovin valuations. This chapter is based on [105].

## 5.1 Groupoid graded rings

By a groupoid one usually means a category wherein all morphisms are isomorphisms, but the original definition (as given by Brandt in [6]) of a *groupoid* is a set $G$ with a unitary operation $\cdot^{-1} : G \to G$ and a partial function $\cdot \star \cdot : G \times G \to G$ such that (for $a, b$ and $c$ in $G$)

(1) if $a \star b = c$ holds, then every one of these elements is uniquely determined by the other two;

(2) $(a \star b) \star c = a \star (b \star c)$ if all terms involved exist;

(3) $a \star a^{-1}$ and $a^{-1} \star a$ exist for all $a \in G$;

(4) $a^{-1} \star a \star b = b$ (resp. $b \star a \star a^{-1} = b$) if $a \star b$ (resp. $b \star a$) exists;

(5) for every two idempotents $e$ and $e'$ in $G$, there is some $g \in G$ with $gg^{-1} = e$ and $g^{-1}g = e'$;

although (1) is a consequence of (2)-(4). The $\star$ is usually left out to lighten notation. Brandt's definition differs from the categorical one only because of (5) (and the fact that his definition only deals with small categories). We will use the term groupoid for a set with an operation satisfying (1)-(4) and call it *connected* if it also satisfies (5). If for $g, g' \in G$ there is some $h$ with $gh = g'$ we say that $g$ and $g'$ are *connected* and that $h$ *connects* them. Connectedness is an equivalence relation and equivalence classes with respect to connectedness are called *connected components*.

**Remark 5.1.1.** Every groupoid $G$ can be embedded in a semigroup by adding a formal symbol $0$, i.e. $\tilde{G} = G \cup \{0\}$, and by putting

$$\cdot \circ \cdot : \tilde{G} \times \tilde{G} \to \tilde{G} : (x, y) \mapsto \begin{cases} x \star y & \text{if } x \star y \text{ is defined} \\ 0 & \text{if } x \star y \text{ is not defined} \end{cases}$$

If $G$ is connected, $\tilde{G}$ will be a completely $0$-simple inverse semigroup. (See e.g. [46] for terminology concerning semigroups.)

For every element $g$ of a groupoid $G$ the elements $\mathbf{s}(g) = gg^{-1}$ and $\mathbf{t}(g) = g^{-1}g$ are idempotents called the *source* and the *target* of $g$.[1] A multiplication $gg'$ of two elements $g, g' \in G$ exists if and only if $\mathbf{t}(g) = \mathbf{s}(g')$.

A ring $R$ is said to be $G$-graded if there are abelian subgroups $(R_g)_{g\in G}$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_{g'} \subseteq R_{gg'}$ if $gg'$ exists while $R_g R_{g'} = 0$ otherwise. An additive subgroup $S$ of a $G$-graded ring $R$ is called $G$-graded if $S = \bigoplus_{g\in G}(S \cap R_g)$. In this case, we will shorten $S \cap R_g$ to $S_g$. The elements of $H = \bigcup_{g\in G} R_g$ are the *homogeneous* elements of $R$. A subring (resp. ideal) of $R$ is a $G$-graded subring (resp. $G$-graded ideal) if and only if it is generated by homogeneous elements. Therefore, we will also use the term *homogeneous ideal* for a $G$-graded ideal. If $I$ is a homogeneous ideal of $R$, then $R/I$ inherits a canonical $G$-grading by $R/I = \bigoplus_{g\in G}(R_g + I)/I$. For any subset $S$ of a $G$-graded ring we define the *support* of $S$ as $\sup(S) = \{g \in G \mid R_g \cap S \neq 0\}$.

**Example 5.1.2.** The matrix ring $M_n(R)$ (over some ring $R$) is the classical example of a groupoid graded ring. Let $G$ be the groupoid obtained by defining a multiplication on the set $\{1, ...n\} \times \{1, ..., n\}$ by $(i,j)(j,k) = (i,k)$ and $(i,j)(k,l)$ undefined if $j \neq k$. Putting $M_n(R)_{(i,j)} = RE_{i,j}$, where $E_{i,j}$ is the matrix with a one on place $(i,j)$ and zeroes everywhere else, yields a $G$-grading of $M_n(R)$.

In fact, this is an example of a *groupoid ring*. For any ring $R$ and any groupoid $G$, the groupoid ring $R[G]$ is constructed by endowing the set

$$R[G] = \{f : G \to R \mid \#\{g \in G \mid f(g) \neq 0\} < \infty\},$$

with a sum and a multiplication as follows:

$$(f + f')(g) = f(g) + f'(g), \quad (ff')(g) = \sum_{g'g''=g} f(g')f'(g'').$$

Note that these operations are well-defined because $f$ and $f'$ have finite support. In a similar fashion as for group rings it can be checked that they define a ring structure on $R[G]$. This ring is $G$-graded by putting $R[G]_g = \{f : G \to R \mid \forall g' \neq g : f(g') = 0\}$.

From now on we will suppose that $G$ is a groupoid and $R$ is a $G$-graded ring. We will denote the set of idempotent elements of $G$ by $G_0$ and we define $R_0$ as $\bigoplus_{g\in G_0} R_g$. This set is called the *principal component* of $R$ and contains $1$. Let $1 = \sum_{e\in G} 1_e$ be the homogeneous decomposition of $1$. In fact, $1_g = 0$ for every $g \in G \setminus G_0$ as was shown e.g. in [51]. Let $G'$ be the subgroupoid defined by $G' = \left\{g \in G \mid 1_{\mathbf{s}(g)} \neq 0 \neq 1_{\mathbf{t}(g)}\right\}$.

[1] In his original paper, Brandt uses the terms left-unit for the source and right-unit for the target. Because of this, the source and target of an element $g$ of $G$ are sometimes denoted by $l(g)$ and $r(g)$ respectively.

**Proposition 5.1.3.** *If $R = \bigoplus_{g \in G} R_g$ is a $G$-grading on $R$, then $R = \bigoplus_{g' \in G'} R_{g'}$, i.e. $R$ is already $G'$-graded.*

*Proof.* See [51]. $\square$

It is not hard to verify that $G'$ has only finitely many idempotent elements so, by the preceding proposition, we can without loss of generality assume that all groupoids considered will have only finitely many idempotents. In fact we can even assume that $1_e \neq 0$ for all $e \in G_0$.

**Proposition 5.1.4.** *If $G$ is a groupoid and $R$ is a $G$-graded ring, then the following elementary properties hold:*

*(1) $R_e$ is a ring for any idempotent $e$ of $G$.*

*(2) If $I$ is a $G$-ideal of $R$, then $I_e$ is an ideal of $R_e$ for every idempotent $e$.*

*(3) $R_g$ is a left $R_{\mathbf{s}(g)}$, right $R_{\mathbf{t}(g)}$-module.*

*(4) $G$ is a group if and only if there is some invertible homogeneous element.*

*Proof.* Since the product of two distinct idempotents $e$ and $e'$ of $G$ is always undefined, we have $r = r1 = \sum_{e \in G_0} r1_e = r1_e$ for all $r \in R_e$. So we find that $1_e$ is the unit of $R_e$. Moreover, $R_e$ is by definition closed under addition and, since $e$ is an idempotent, it is also closed under multiplication. For (3) it suffices to note that the map

$$(\cdot\,\cdot) : R_{\mathbf{s}(g)} \times R_g \to R_g : (x, y) \mapsto xy$$

defines a left $R_{\mathbf{s}(g)}$-multiplication on $R_g$, the right $R_{\mathbf{t}(g)}$-multiplication being defined analogously. (2) is a special case of (3) in disguise where $I_e \subseteq R_e$. To prove (4), note that, since $ee'$ is undefined for idempotents $e \neq e'$, any homogeneous element $h \in R_g$ must be a zero divisor if there is some unit $e \neq \mathbf{t}(h)$ or $e \neq \mathbf{s}(h)$. If $G$ is a group with unique unit $e$, then $1 \in R_e$ is homogeneous and invertible. $\square$

Let $R$ be a $G$-graded ring for some groupoid $G$. If $R_g R_{g'} = R_{gg'}$ whenever $gg'$ exists we say that $R$ is *strongly* $G$-graded. This is equivalent with $1_{\mathbf{s}(g)} \in R_g R_{g^{-1}}$ (or $1_{\mathbf{t}(g)} \in R_{g^{-1}} R_g$) for all $g$. Indeed, suppose $1_{\mathbf{s}(g)} \in R_g R_{g^{-1}}$ for all $g$, then

$$R_{gg'} \subseteq R_g R_{g^{-1}} R_{gg'} \subseteq R_g R_{g'}.$$

The other implication is immediate.

**Proposition 5.1.5.** *Let $R$ be a strongly $G$-graded ring. The homogeneous ideals of $R$ are in 1-1 correspondence with ideals*

$$I_{e_1} \subseteq R_{e_1}, ..., I_{e_n} \subseteq R_{e_n}$$

*where the $e_i$ are representatives of the connected components of $G$.*

*Proof.* Since $R$ is strongly graded, we must have that

$$I_{gg'} = R_g R_{g^{-1}} I_{gg'} \subseteq R_g I_{g'} \subseteq I_{gg'}$$

if $gg'$ is defined, so any two homogeneous ideals of $R$ restricting to the same ideals on $R_{e_1}, ..., R_{e_n}$ must be equal. On the other hand, if $I_{e_1} \subseteq R_{e_1}, ..., I_{e_n} \subseteq R_{e_n}$ are ideals in their respective rings, then we can define $I_g = R_{gg'} I_e$ where $g'$ is the element connecting $g$ and $e$. $I = \bigoplus_{g \in G} I_g$ is then a homogeneous ideal of $R$. □

We say a homogeneous ideal is $G$-maximal if the only strictly larger homogeneous ideal is $R$ itself. As an immediate consequence of the preceding proposition, the $G$-maximal ideals of a strongly $G$-graded $R$ are those corresponding to a maximal ideal in one of the connected components and to $R_g$ for any $g$ not in that component. Therefore, the intersection of the $G$-maximal ideals – which we call the $G$-Jacobson radical – is the homogeneous ideal corresponding to the Jacobson radical in every connected component.

We write, for any $a \in R$,

$$\mathbf{t}(a) = \sum_{\substack{e \in G_0 \\ a 1_e \neq 0}} 1_e \quad \text{and} \quad \mathbf{s}(a) = \sum_{\substack{e \in G_0 \\ 1_e a \neq 0}} 1_e.$$

A $G$-inverse of $a$ is an element $b$ satisfying

$$\mathbf{s}(a) = ab = \mathbf{t}(b) \quad \text{and} \quad \mathbf{s}(b) = ba = \mathbf{t}(a).$$

If $a$ has a $G$-inverse, we say that it is $G$-invertible and a $G$-graded ring for which every non-zero homogeneous element is $G$-invertible is called a *$G$-skewfield*. Notice that the grading on a $G$-skewfield is necessarily strong. We will use the notation $a^{-1}$ for the $G$-inverse of $a$, but one should keep in mind that the $G$-inverse of $a$ may exist even if $a$ is not invertible in $R$. We will denote by $R^*$ the set of $G$-invertible elements of a $G$-graded ring $R$, while the set of invertible elements will be denoted by $U(R)$.

**Proposition 5.1.6.** *If $G$ is a groupoid and $R$ is a $G$-graded ring, then:*

*(1) The $G$-inverse of $a$, if it exists, is unique.*

*(2) The $G$-inverse of a homogeneous element, if it exists, is homogeneous.*

*(3) If $a$ is invertible in $R$, say $ba = ab = 1$, then $b$ is the $G$-inverse of $a$.*

*(4) If $R$ is a $G$-skewfield, then $R_e$ is a skewfield for any idempotent $e$.*

*Proof.* If $b$ and $b'$ are $G$-inverses of $a$, then

$$b = b\mathbf{t}(b) = b\mathbf{s}(a) = bab' = \mathbf{s}(a)b' = b'ab' = b'\mathbf{t}(b') = b'.$$

Suppose $a \in R_h$ is homogeneous and let $a^{-1} = \sum_{g\in S\subseteq G} b_g$. For all $g \neq h^{-1}$ we have that $aa^{-1} = \mathbf{s}(a)$ implies $ab_g = 0$ and $a^{-1}a = \mathbf{t}(a)$ implies $b_g a = 0$ . Therefore, $b_{h^{-1}}$ is a $G$-inverse and by (1) it must be unique. If $a$ is invertible with inverse $b$, then $1_e a \neq 0$ for all $e \in G_0$, which establishes that $\mathbf{s}(a) = 1$. Similarly, we find $\mathbf{t}(a) = 1$ and by symmetry the same holds for $b$. Consequently, $a$ and $b$ are each others $G$-inverses. Since for any $a \in R_g$ we have $a^{-1} \in R_{g^{-1}}$ (4) follows. □

If $G$ is a groupoid and $S$ is subgroupoid containing all idempotents, then one can construct a factor groupoid $G/S = G/\sim$ where

$$g \sim h \quad \Leftrightarrow \quad \exists g_1, \dots, g_n \in G, s_0, \dots, s_n \in S : \prod_{i=1}^{n} g_i = g, s_0 \prod_{i=1}^{n} g_i s_i = h.$$

Since $S_e$ contains all idempotents, $\mathbf{t}(g)$ and $\mathbf{s}(g)$ must be in $S$ for all $g$ so we have reflexivity of $\sim$. If $x \sim y$, then we have $x_i$ and $s_i$ with $x_1 \cdots x_n = x$ and $s_0 x_1 s_1 \cdots x_n s_n = y$, so

$$x = (s_0^{-1} y s_n^{-1} x_n^{-1} \cdots x_2^{-1} s_1^{-1}) \cdots (s_{n-1}^{-1} x_{n-1}^{-1} \cdots x_1^{-1} s_0^{-1} y s_n^{-1}).$$

Note that between two successive $y$'s in this expression, we always find the term $s_n^{-1} x_n^{-1} \cdots s_1^{-1} x_1^{-1} s_0^{-1} = y^{-1}$ with some extra $s_i^{-1}$ in between. This means that we have written $x$ as a product $t_0 y_1 t_1 \cdots y_m t_m$ where $y_1 \cdots y_m = y$ and for some $t_0, \dots, t_m$ in $S$, i. e. $\sim$ is symmetric. Suppose, finally, that $x \sim y$ and $y \sim z$, then we have decompositions $y = s_0 x_1 \cdots x_n s_n$ and $z = t_0 y_1 \cdots y_m t_m$ for certain $s_i, x_i, t_i$ and $y_i$. Consequently,

$$z = t_0 \left(s_0 x_1 s_1 \cdots x_n s_n y_m^{-1} \cdots y_2^{-1}\right) \cdots t_{m-1} \left(y_{m-1}^{-1} \cdots y_1^{-1} s_0 x_1 s_1 \cdots x_n s_n\right) t_m.$$

Every time $y_m^{-1} \cdots y_1^{-1} = y^{-1}$ (with some $t_i$ in between) occurs, it is preceded by $s_0 x_1 s_1 \cdots x_n s_n = y$. This gives the desired decomposition of $z$ as a product $u_0 x_1' u_1 \cdots x_l' u_l$ with $x_1' \cdots x_l' = x$. By definition, $\sim$ is compatible with the multiplication on $G$, so $G/S$ is a well-defined groupoid.

## 5.2 Skew twisted groupoid rings

We have given the definition of a groupoid ring in example 5.1.2. Just like in the group-graded case, this example can be generalised by giving an abstract construction of skew twisted groupoid rings, using a straightforward modification of the approach in e.g. [75]. Let $R$ be a ring, let $G$ be a groupoid (which we still suppose to have only finitely many idempotents) and let $\alpha : G \times G \to U(R)$ and $\sigma : G \to \mathrm{Aut}(R)$ be functions which satisfy, for $a, b, c \in G$, $r \in R$ and $e \in G_0$, the following conditions:

(i) $\sigma(a)(\sigma(b)(r)) = \alpha(a,b)\sigma(ab)(r)\alpha(a,b)^{-1}$

(ii) $\alpha(a,b)\alpha(ab,c) = \sigma(a)(\alpha(b,c))\alpha(a,bc)$

(iii) $\alpha(a,e) = 1_R = \alpha(1,e)$

if all terms involved exist. In classical terminology, $\sigma$ would be called a weak action of $G$ on $R$ and $\alpha$ would be called a $\sigma$-cocycle.

**Proposition 5.2.1.** *The free left $R$-module $R^\sigma_\alpha[G]$ with basis $G$ and multiplication defined by:*

$$(ra)(sb) = \begin{cases} r\sigma(a)(s)\alpha(a,b)ab & \text{if } ab \text{ exists} \\ 0 & \text{if } ab \text{ does not exist} \end{cases}$$

*and distributivity is a strongly $G$-graded ring. If $x \in U(R)$, then $xg \in R^\sigma_\alpha[G]^*$ for any $g \in G$.*

*Proof.* First we have to establish associativity of the multiplication, so take $r, s, t \in R$ and $a, b, c \in G$. If $abc$ does not exist there is nothing to prove since all terms are zero, so suppose $abc$ does exist. Then we have:

$$\begin{aligned}
(ra)((sb)(tc)) &= (ra)(s\sigma(b)(t)\alpha(b,c)bc) \\
&= r\sigma(a)(s\sigma(b)(t)\alpha(b,c))\alpha(a,bc)abc \\
&= r\sigma(a)(s)\sigma(a)(\sigma(b)(t))\sigma(a)(\alpha(b,c))\alpha(a,bc)abc \\
&\overset{(i)}{=} r\sigma(a)(s)\alpha(a,b)\sigma(ab)(t)\alpha(a,b)^{-1}\sigma(a)(\alpha(b,c))\alpha(a,bc)abc \\
&\overset{(ii)}{=} r\sigma(a)(s)\alpha(a,b)\sigma(ab)(t)\alpha(a,b)^{-1}\alpha(a,b)\alpha(ab,c)abc \\
&= r\sigma(a)(s)\alpha(a,b)\sigma(ab)(t)\alpha(ab,c)abc \\
&= (r\sigma(a)(s)\alpha(a,b)ab)(ct) \\
&= ((ra)(sb))(ct)
\end{aligned}$$

Clearly, $\sum_{g \in G_0} 1_R g$ is the identity for the multiplication and distributivity holds by construction. The fact that $R^\sigma_\alpha[G] = \bigoplus_{g \in G} Rg$ is also an immediate consequence of the definition and the same is true for $(Ra)(Rb) = R(ab)$ — provided, of course, that $ab$ exists. We have, for all $g \in G$, that $gg^{-1} = \alpha(g, g^{-1})l(g)$ while $g^{-1}g = \alpha(g^{-1}, g)r(g)$ and by $(ii)$ and $(iii)$ we also have that

$$\begin{aligned}\sigma(g^{-1})\alpha(g, g^{-1}) &= \sigma(g^{-1})\alpha(g, g^{-1})\alpha(g^{-1}, gg^{-1}) \\ &= \alpha(g^{-1}, g)\alpha(g^{-1}g, g^{-1}) = \alpha(g^{-1}, g)\end{aligned}$$

so $\alpha(g, g^{-1})^{-1}g$ is the inverse for $g^{-1}$. If $r \in U(R)$ and if $g$ is arbitrary in $G$, then we can choose $s \in R$ such that $\sigma(g^{-1})(s) = r^{-1}$. Then

$$\begin{aligned}rg^{-1}s\alpha(g, g^{-1})^{-1}g &= r\sigma(g^{-1})(s\alpha(g, g^{-1})^{-1})\alpha(g^{-1}, g)g^{-1}g \\ &= rr^{-1}\sigma(g^{-1})(\alpha(g, g^{-1})^{-1})\alpha(g^{-1}, g)g^{-1}g \\ &= g^{-1}g\end{aligned}$$

while

$$\begin{aligned}s\alpha(g, g^{-1})^{-1}grg^{-1} &= s\alpha(g, g^{-1})\sigma(g)(r)\alpha(g, g^{-1})gg^{-1} \\ &= s\alpha(g, g^{-1})\sigma(g)(\sigma(g^{-1})(s^{-1}))\alpha(g, g^{-1})gg^{-1} \\ &= s\alpha(g, g^{-1})\alpha(g, g^{-1})s^{-1}\alpha(g, g^{-1})^{-1}\alpha(g, g^{-1})gg^{-1} \\ &= gg^{-1}\end{aligned}$$

which proves $rg$ to be $G$-invertible. □

In the classical case, any crossed product, i.e. a graded ring with an invertible element in every $R_g$, is of this form for some $\alpha$ and $\sigma$ (cfr. [75]), but there is no hope for such a theorem here, in view of the following example.

**Example 5.2.2.** Take $G$ the same groupoid as in example 5.1.2. Let $R$ be a subring of a field $k$ and let $I$ be a non-trivial ideal of $R$. Put $I^{-1} = \{x \in k \mid xI \subseteq R\}$. Then

$$Q = \begin{pmatrix} R & I & I^2 & \cdots & I^{n-1} \\ I^{-1} & R & I & \cdots & I^{n-2} \\ I^{-2} & I^{-1} & R & \cdots & I^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I^{-n+1} & I^{-n+2} & I^{-n+3} & \cdots & R \end{pmatrix}$$

defines a $G$-crossed product where not all $Q_g$ are isomorphic.[2]

If $S$ is any set, we can define a groupoid structure on $S \times S$ in a similar way as in example 5.1.2: we put $(s, s') \star (t, t') = (s, t')$ if $s' = t$ and we say it is undefined otherwise. We denote this structure by $\Delta_S$.

[2] Note the similarity between this example and *fragments* as defined in [78].

**Remark 5.2.3.** Every ideal of $R^\sigma_\alpha[\Delta_S]$ is homogeneous, regardless of $S$ and $R$, for if $\sum r_{(s,t)}(s,t)$ is in an ideal $I$, then so is

$$(\sigma(a,a)(r_{a,b}\alpha((a,b),(b,b)))\alpha((a,a),(a,b)))^{-1}(a,a) \\ \left(\sum r_{(s,t)}(s,t)\right)1_{(b,b)}(b,b) = 1_{(a,b)}(a,b).$$

The following theorem is very similar to a theorem of Munn (cfr. [71]), although he restricts attention to finite groupoids $G$.

**Proposition 5.2.4.** *Any groupoid ring $R[G]$ is isomorphic to a subring of $R[\Delta_G]$.*

*Proof.* Define for any $x = \sum x_g g$ in $R[G]$ an element $f(x)$ in $R[\Delta_G]$ by putting, for any $g, g' \in G$, $f(x)_{g,g'} = 0$ if there is no $h$ with $g = hg'$ and $f(x)_{g,g'} = x_{gg'^{-1}}$ otherwise. We can now define a map

$$f : R[G] \to R[\Delta_G] : x \mapsto f(x).$$

It is immediate from the definition that $f$ preserves sums. Suppose now that $a = \sum a_g g$ and $b = \sum b_g g$ are in $R[G]$, then $f(a)_{g,g'} = a_{gg'^{-1}}$ and $f(b)_{g,g'} = b_{gg'^{-1}}$ if $gg'^{-1}$ exists and zero otherwise. Consequently,

$$(f(a)f(b))_{g,g'} = \sum_s f(a)_{g,s} f(b)_{s,g'} = \sum_s a_{gs^{-1}} b_{sg'^{-1}}$$

where the summation index $s$ runs over all those elements of $G$ for which all terms involved exist. On the other hand, $ab = \sum_g \sum_{g'g''=g} a_{g'}b_{g''}g$ so $f(ab)_{g,g'}$ is either zero or $\sum_{ss'=gg'^{-1}} a_s b_{s'}$. By renaming the summation index we find that this is indeed equal to $(f(a)f(b))_{g,g'}$, so $f$ is a ring morphism, which is clearly injective. □

**Remark 5.2.5.** If $G$ is a subgroup of $\mathrm{Aut}(R)$, $\sigma$ is the canonical injection and $\alpha : G \times G \to R^*$ is a $\sigma$-cocycle, then this proposition still holds; it suffices to redefine $f(x)_{g,g'}$ as $\alpha(gg'^{-1}, g')x_{gg'^{-1}}$. Whether a similar proposition is true for arbitrary skew twisted groupoid rings remains unclear.

**Example 5.2.6.** Let $G$ be the $\mathbb{Z}/2\mathbb{Z}$-groupoid on two elements, i.e. the category with objects $a$ and $b$ and morphisms $e_0, e_1 : a \to a$, $f_0, f_1 : b \to b$, $g_0, h_0 : a \to b$ and $g_1, h_1 : b \to a$ where $e_0$ and $f_0$ are the identity morphisms, $g_0^{-1} = g_1$ and $h_0^{-1} = h_1$. In this case $|G| = 8$ and $f$ is the map which sends an element

$x = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 f_0 + \alpha_3 f_1 + \alpha_4 g_0 + \alpha_5 g_1 + \alpha_6 h_0 + \alpha_7 h_1$ to the matrix

$$f(x) = \begin{array}{c} \\ e_0 \\ e_1 \\ f_0 \\ f_1 \\ g_0 \\ g_1 \\ h_0 \\ h_1 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_0 & e_1 & f_0 & f_1 & g_0 & g_1 & h_0 & h_1 \end{array} \\ \begin{bmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 & \alpha_4 & 0 & \alpha_6 \\ \alpha_1 & \alpha_0 & 0 & 0 & 0 & \alpha_6 & 0 & \alpha_4 \\ 0 & 0 & \alpha_2 & \alpha_3 & \alpha_5 & 0 & \alpha_7 & 0 \\ 0 & 0 & \alpha_3 & \alpha_2 & \alpha_7 & 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_4 & \alpha_6 & \alpha_2 & 0 & \alpha_3 & 0 \\ \alpha_5 & \alpha_7 & 0 & 0 & 0 & \alpha_0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_6 & \alpha_4 & \alpha_3 & 0 & \alpha_2 & 0 \\ \alpha_7 & \alpha_5 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_0 \end{bmatrix} \end{array}.$$

Notice that the coefficients of the elements in $G_0$ appear on the diagonal and that zeroes occur symmetrically.

If $G$ is finite, then $R[\Delta_G]$ is a matrix ring over $R$. In this case, it makes sense to define the determinant of an element of $R[G]$ as the determinant of its associated matrix in $R[\Delta_G]$. If $R$ is commutative, this allows us to compute inverses in $R[G]$ by computing inverses in a matrix ring, as the following proposition shows.

**Proposition 5.2.7.** *If $G$ is finite and $R$ is commutative, then $f(R[G])$ is closed under inverses.*

*Proof.* This is an immediate consequence of the Cayley-Hamilton theorem [3]. Indeed, if $A$ is a matrix over a commutative ring we know $A(\sum c_i A^i) = I$ for some coefficients $c_i$ in $R$. If $A$ is in $\mathrm{Im}(f)$, then so are all $c_i A^i$ and consequently $A^{-1} \in \mathrm{Im}(f)$. □

## 5.3 $G$-valuations

Let $R$ be a $G$-graded subring of a $G$-skewfield $Q$. If for every homogeneous $h \in Q$ we have either $h \in R$ or $h^{-1} \in R$, then we say that $R$ is *$G$-total*. This is the canonical generalisation of totality and gives rise to somewhat similar results. Note that if $R$ is a $G$-total subring of the $G$-skewfield $Q$, then $R_e$ is a total subring of the skewfield $Q_e$ for any idempotent $e \in G$. In particular, any $G$-total subring of a $G$-skewfield contains $1_e$ for all idempotents $e$.

**Proposition 5.3.1.** *Suppose $R$ is a $G$-total subring of the $G$-skewfield $Q$. If $I$ and $J$ are homogeneous left (resp. right) ideals, then $I_g \subseteq J_g$ or $J_g \subseteq I_g$. Moreover, if $I_g \subsetneq J_g$ then $I_{g'} \subseteq J_{g'}$ for any $g'$ with the same right (resp. left) unit as $g$.*[4]

[3] This very old result is not due to Cayley or Hamilton, but rather Frobenius (cfr. [30])

[4] This is a rare case where I knew the proof of the proposition before I knew the statement! Indeed, the proof is basically the same as in the commutative case, but it took me quite some time to realise that the final $\subseteq$ was not a $\subsetneq$.

*Proof.* Suppose $I$ and $J$ are homogeneous and $J_g \nsubseteq I_g$, so there exists some non-zero $h \in J_g \smallsetminus I_g$. Suppose $\mathbf{t}(g') = \mathbf{t}(g)$, and let $h' \neq 0$ be in $I_{g'}$. This means that $hh'^{-1}$ and $h'h^{-1}$ are defined and at least one of these is in $R$. If $hh'^{-1}$ is in $R$, then $hh'^{-1}h'$ is in $I \cap R_g = I_g$ which is a contradiction, so $h'h^{-1}$ must be in $R$ and consequently $h'h^{-1}h$ is in $J \cap R_{g'} = J_{g'}$. The other case is similar. □

**Corollary 5.3.2.** *If $R$ is a $G$-total subring of the $G$-skewfield $Q$, then any left (resp. right) ideal generated by homogeneous elements $h_1, ..., h_n$ with the same target (resp. source) is cyclic.*

**Proposition 5.3.3.** *Let $R$ be a $G$-total subring of a $G$-skewfield $Q$, and put $M$ the (homogeneous) ideal generated by the set of homogeneous elements which are not $G$-invertible in $R$. Then $R/M$ is a $G$-skewfield and $M$ is maximal for the property that it contains no $1_{e \in G_0}$.*

*Proof.* If $\overline{x} \neq \overline{0}$ is some homogeneous element of $R/M$, then $x = h + p$ where $h$ is a non-zero homogeneous element of $R \smallsetminus M$ and $p \in M$. Let $p = h_1 + \cdots + h_n$ be the homogeneous decomposition of $p$. Then $h^{-1}$ is also in $R \smallsetminus M$ and $\overline{x}\overline{h^{-1}} = \overline{hh^{-1} + ph^{-1}} = \overline{1_{\mathbf{s}(h)}} = 1_{\mathbf{s}(\overline{x})}$ since $ph^{-1}$ must be in $M$ – otherwise some $h_i h^{-1}$ is not in $M$ and we would have $hh_i^{-1} \in R$ and consequently $h^{-1}hh_i^{-1} = h_i^{-1} \in R$, which contradicts $h_i \in M$. Analogously, we find $\overline{h^{-1}}\overline{x} = 1_{\mathbf{t}(\overline{x})}$ which implies that $R/M$ is a $G$-skewfield. If $M'$ is an ideal which contains $M$ strictly, then there is some $x \in M' \smallsetminus M$ so $x$ is $G$-invertible in $R$ and consequently there $xx^{-1}$ is in $M$, which implies that $1_e \in M$ for some $e \in G_0$. □

A $G$-graded subring $R$ of a $G$-skewfield $Q$ is called *$G$-stable* if $hR_{\mathbf{t}(h)}h^{-1} = R_{\mathbf{s}(h)}$ for any homogeneous $h$. This implies that $R_e$ is stable for all $e \in G_0$. In particular, if $R$ is a $G$-total $G$-stable subring of the $G$-skewfield $Q$, then $R_e$ is a valuation ring for every $e$ in $G_0$.

**Proposition 5.3.4.** *Let $R$ be a $G$-stable subring of the $G$-skewfield $Q$. Any right (resp. left) homogeneous ideal of $R$ is a left (resp. right) ideal.*

*Proof.* Let $I$ be a right $G$-ideal of $R$, let $h \in I \cap Q_g$ be homogeneous and pick $r \in R \cap Q_{g'}$ arbitrary. If $g'g$ does not exist $rh = 0 \in I$ follows, so suppose $g'g$ does exist. Because of $G$-stability of $R$, $h^{-1}rh$ is in $R$ – whether it is zero or not. If $h^{-1}r$ exists, we have $hh^{-1}rh = rh \in I$ since $I$ is a right $G$-ideal. If $h^{-1}r$ does not exist, $hh^{-1}rh = 0$ so it is again in $I$. This proves the claim for right $G$-ideals; the reasoning for left $G$-ideals is similar. □

**Example 5.3.5.** Let $k$ be a field and let $R_v$ be a valuation ring in $k$ with unique maximal ideal $\mathfrak{m}_v$. Consider the subring

$$R = \begin{bmatrix} R_v & k \\ 0 & R_v \end{bmatrix}$$

of the $\Delta_2$-skewfield $Q = M_2(k)$. $R$ is a $G$-total (and $G$-stable) subring of $Q$, but the homogeneous ideals are not totally ordered since

$$M_1 = \begin{bmatrix} \mathfrak{m}_v & k \\ 0 & R_v \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} R_v & k \\ 0 & \mathfrak{m}_v \end{bmatrix}$$

are incomparable.

**Definition 5.3.6.** *Let $G$ be a groupoid and let $Q$ be a $G$-skewfield. A (partial) $G$-valuation on $Q$ is a surjective map $v : Q \to \Gamma \cup \{\infty\}$ for some (partially) ordered groupoid $\Gamma$ (and $\infty > \gamma$ for all $\gamma \in \Gamma$) satisfying:*

*(1) $v(x) = \infty \Leftrightarrow x = 0$,*

*(2) $v(x+y) \geq v(z)$ if $v(y) \geq v(z) \leq v(x)$,*

*(3) $v(hh') = v(h)v(h')$ for homogeneous $h, h'$ and if $hh'$ is defined.*

For the theory of (partially) ordered groupoids we refer the interested reader to [46].

If $v : Q \to \Gamma \cup \{\infty\}$ is a $G$-valuation, then we let $R_v$ be the ring generated by homogeneous elements $h$ with $v(h) \geq v(\mathbf{t}(h))$. Note that, since $G_0$ is a finite set, $1 \in R$ follows. Since $R_v$ is generated by homogeneous elements, it inherits a $G$-grading from $Q$.

**Proposition 5.3.7.** *For any $G$-valuation $v : Q \to \Gamma \cup \{\infty\}$ the ring $R_v$ is $G$-stable and $G$-total.*

*Proof.* Suppose $h$ is a homogeneous element of $Q$ and suppose $v(h) < v(\mathbf{t}(h))$. Then

$$v(h^{-1}) = v(\mathbf{t}(h))v(h^{-1}) > v(h)v(h^{-1}) = v(\mathbf{s}(h)) = v(\mathbf{t}(h^{-1}))$$

showing that $h^{-1} \in R_v$. To show $G$-stability, pick some homogeneous element $h \in Q_g$ and suppose that $r \in R_{r(h)}$, then $v(hrh^{-1}) \geq v(h)v(1_{\mathbf{t}(h)})v(h^{-1}) = v(1_{\mathbf{s}(h)})$. Since $\mathbf{s}(h)$ is the target of $hrh^{-1}$, this shows that $hrh^{-1} \in R \cap Q_{\mathbf{s}(h)} = R_{\mathbf{s}(h)}$. Similarly, if $r'$ is in $R_{\mathbf{s}(h)}$, it follows that $h^{-1}r'h$ is in $R_{\mathbf{t}(h)}$, so $r' \in hR_{\mathbf{t}(h)}h^{-1}$ which establishes the $G$-stability of $R_v$. □

**Corollary 5.3.8.** *For any $G$-valuation $v : Q \to \Gamma \cup \{\infty\}$ and any homogeneous $h \in Q$, we have either $v(h) \geq v(\mathbf{t}(h))$ or $v(h) \leq v(\mathbf{t}(h))$.*

**Proposition 5.3.9.** *Any $G$-stable, $G$-total subring $R$ of a $G$-skewfield $Q$ is $R_v$ for some ordered groupoid $\Gamma$ and some partial $G$-valuation $v : Q \to \Gamma \cup \{\infty\}$.*

*Proof.* $H(Q)^*$ is a groupoid for the multiplication and $H(R)^*$ is a subgroupoid containing all $1_e$ for $e \in G_0$, so there is a quotient groupoid $\Omega = H(Q)^*/H(R)^*$. We can define an ordering on $\Omega$ by

$$\overline{x} \geq \overline{y} \Leftrightarrow \exists r_0, ..., r_n \in H(R), y_1, ..., y_n \in H(Q)^* : \prod_{i=1}^{n} y_i = y, r_0 \prod_{i=1}^{n} y_i r_i = x.$$

We show that this is a well-defined relation. Since for $\overline{x} = \overline{z}$ there are $r_i \in H(R)^*$ with $x_1 \cdots x_n = x$ and $r_0 x_1 r_1 \cdots x_n r_n = z$. If $\overline{x} \geq \overline{y}$, we have $s_0, ..., s_m \in H(R)$ with $x = s_0 y_1 s_1 \cdots y_m s_m$ and $y_1 \cdots y_m = y$. Consequently

$$z = r_0(s_0 y_1 s_1 \cdots y_m s_m x_n^{-1} \cdots x_2^{-1}) r_1 \cdots (x_{n-1}^{-1} \cdots x_1^{-1} s_0 y_1 s_1 \cdots y_m s_m) r_n.$$

This shows that $\overline{z} \geq \overline{y}$. Of course, one can use a similar reasoning to prove $\overline{x} \geq \overline{y'}$ if $\overline{y} = \overline{y'}$. Moreover, $\leq$ is reflexive and transitive by similar arguments as used in the construction of the factor groupoid. To prove that $\geq$ is antisymmetric, assume that $\overline{x} \geq \overline{y} \geq \overline{x}$. Then we have $x_1 \cdots x_n = x$, $y_1 \cdots y_m = y$, $r_0 x_1 r_1 \cdots x_n r_n = y$ and $s_0 y_1 s_1 \cdots y_m s_m = x$ for some $x_i, y_i \in Q$ and $r_i, s_i \in R$. Therefore,

$$y = r_0(s_0 y_1 s_1 \cdots y_m s_m x_n^{-1} \cdots x_2^{-1}) r_1 \cdots (x_{n-1}^{-1} \cdots x_1^{-1} s_0 y_1 s_1 \cdots y_m s_m).$$

Every $x_n^{-1} \cdots x_1^{-1}$ (with some $r_i$ in between) occurring in this decomposition can be cancelled out against $s_0 y_1 s_1 \cdots y_m s_m$ if we choose a good decomposition. This leaves us with $s_0 y_1 \cdots y_m s_m$. Depending on which decomposition we choose for that term, we can make it either into $x$ or into $y$. If we choose $x$, we have shown that $\overline{y} = \overline{x}$ and therefore that $\leq$ is antisymmetric. It is obvious that $\leq$ is compatible with multiplication in $\Omega$. If $\mathbf{t}(x) = \mathbf{t}(y)$ for some homogeneous $x$ and $y$, then either $xy^{-1} \in R$ or $yx^{-1} \in R$. Suppose the former holds, then $\overline{x} = \overline{xy^{-1}}\overline{y}$, so $\overline{x} \geq \overline{y}$. If the latter holds, then $\overline{y} \geq \overline{x}$ follows by a similar reasoning. Note that, for $h, h' \in Q_g$, we have $\overline{h + h'} \geq \min\{\overline{h}, \overline{h'}\}$. Indeed, since $h$ and $h'$ are in the same component, they have the same target and consequently we have either $hh'^{-1} \in R$ or $h'h^{-1} \in R$. Suppose without loss of generality $hh'^{-1} \in R$, then $h + h' = (hh'^{-1} + 1)h'$ hence $\overline{h + h'}$ is larger than $\overline{h'}$.

Note that the ordering on $\Omega$ induces an ordering on $G$ by putting $g \leq g'$ if and only if $1_g \leq 1_{g'}$. Define a pre-order on $Q$ by putting, for $q = \sum_g q_g g$ and $q' = \sum_g q'_g g$ in $Q$,

$$q \leq q' \quad \Leftrightarrow \quad \left( \forall g \in G : \overline{q_g g} > \overline{q'_g g} \Rightarrow \exists g' > g : \overline{q_{g'} g'} > \overline{q'_{g'} g'} \right).$$

This pre-order is compatible with multiplication, since this holds for the ordering on $\Omega$. Consequently, we can endow $\Gamma = Q^*/\sim$, where $x \sim y$ iff $x \leq y$ and $y \leq x$, with a multiplication defined as $\overline{q}\overline{q'} = \overline{qq'}$ if $\overline{qq'} \neq 0$ and $\overline{q}\overline{q'}$ undefined otherwise. Define the map

$$v : Q \to \Gamma \cup \{\infty\} : x \mapsto \begin{cases} \overline{x} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

We have to check that $v(x+y) \geq v(z)$ if $\underline{v(x) \geq v(z)} \leq v(y)$, but since, for $x = \sum x_g g$ and $x' = \sum x'_g g$ in $Q^*$, we have $\overline{(x_g + x'_g)g} \geq \min\left\{\overline{x_g g}, \overline{x'_g g}\right\}$ for all $g$, this follows.

It is clear that $R$ is a subset of $R_v$ and we know that $1_e \in R$ for any idempotent $e \in G_0$. Suppose now that $v(h) > v(\mathbf{t}(h))$ for some homogeneous $h$. If $h$ is not in $R$, then $h^{-1}$ is in $R$, whence $v(h^{-1}) \geq v(\mathbf{s}(h^{-1})) = v(\mathbf{t}(h))$, leading to $v(h^{-1}h) > v(\mathbf{t}(h))$ which is impossible, so $h \in R$. If $v(h) = v(\mathbf{t}(h))$, then $h \sim 1_{\mathbf{t}(h)}$, so we have $g_1, \ldots, g_n \in H(Q)^*$ and $s_0, \ldots, s_n \in H(R)^*$ with $\prod g_i = h$ and $s_0 \prod g_i s_i = 1_{\mathbf{t}(h)}$. Due to the fact that $R$ is $G$-stable, we can rewrite this as $1_{\mathbf{t}(h)} = \prod g_i s'_0 \cdots s'_{n-1} s_n$ for some $s'_i \in H(R)^*$. This implies $h = 1_{\mathbf{t}(h)} s_n^{-1} s'^{-1}_{n-1} \cdots s'^{-1}_0 \in H(R)^*$. $\square$

**Example 5.3.10.**

1. Let us first consider an example of the simplest kind: the groupoid valuation ring
$$\begin{pmatrix} R_v & R_v \\ R_v & R_v \end{pmatrix} \text{ contained in the } \Delta_2\text{-skewfield } \begin{pmatrix} k & k \\ k & k \end{pmatrix}$$
for a valuation ring $R_v$ with maximal ideal $P$ in a field $k$. In this case, $\Omega$ is simply a group. In fact, $\Omega \simeq \Gamma \simeq R_v/P$ and the associated value function is
$$v : Q \to \Gamma \cup \{\infty\} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \min\{v(a), v(b), v(c), v(d)\}.$$
2. Matters get a bit more complicated if we consider the situation from example 5.3.5. In this case we find that e.g. $\overline{1_{1,1}}$ and $\overline{1_{2,2}}$ are incomparable in $\Omega$, while $\overline{1_{1,2}}$ is larger and $\overline{1_{2,1}}$ is smaller than both. Let $a$ and $b$ be in $k$ with $v(a) > v(b)$. Then we have e.g.
$$v\left(\begin{pmatrix} b & a \\ b & b \end{pmatrix}\right) \geq v\left(\begin{pmatrix} a & b \\ a & a \end{pmatrix}\right) \text{ while e.g. } v\left(\begin{pmatrix} a & b \\ b & b \end{pmatrix}\right) \text{ and } v\left(\begin{pmatrix} b & b \\ b & a \end{pmatrix}\right)$$
are incomparable.

**Corollary 5.3.11.** *In the same context as 5.3.9, we have*
$$\left\{h \in H(R) \mid h^{-1} \notin R\right\} = \{h \in H(R) \mid v(h) > v(\mathbf{t}(h))\}.$$

*Proof.* Take a homogeneous $h$ with $v(h) > v(\mathbf{t}(h))$, then
$$v(h^{-1}) = v(\mathbf{t}(h))v(h^{-1}) < v(h)v(h^{-1}) = v(\mathbf{s}(h)) = v(\mathbf{t}(h^{-1}))$$
so $h^{-1} \notin R$. On the other hand, if $h \in H(R)$ and $h^{-1} \notin R$, then $v(h^{-1}) < v(\mathbf{t}(h^{-1})) = v(\mathbf{s}(h))$ so
$$v(\mathbf{t}(h)) = v(h^{-1})v(h) < v(\mathbf{s}(h))v(h) = v(h).$$
$\square$

Some of the theory developed here has been studied for $G$ a group, e.g. in [73] and [47].

## 5.4 A connection with Dubrovin valuation rings

Dubrovin valuation rings are generally not equipped with nice value functions. Van Geel defined (cfr. [95]) a very general kind of value functions, but these take values in partially ordered groups. Morandi (cfr. [69]), Ferreira & Wadsworth (cfr. [29]), and Van Oystaeyen & Verhulst (cfr. [102]) all introduced value functions with values in totally ordered groups for different classes of Dubrovin valuation rings. In this section, we will show that Dubrovin valuation rings in matrix rings over fields are $\Delta_n$-valuation rings.

**Proposition 5.4.1.** *If $Q = k^\sigma_\alpha[\Delta_n]$ is a skew twisted matrix ring over a skewfield $k$ and $R$ is a $\Delta_n$-stable, $\Delta_n$-total subring containing all $1_k\delta_{ij}$, then $R$ is a Dubrovin valuation ring.*

*Proof.* Let $I$ be an ideal in $R$, which by remark 5.2.3 is homogeneous. By 5.3.3, the ideal $M$ generated by homogeneous elements of $R$ which are not in $R^*$ is the unique maximal ideal which does not contain $1_{\delta_{ii}}$ for any $i$. Consider some ideal $I$ which contains some $1_{\delta_{ii}}$. Since $R$ contains all $1_{\delta_{ij}}$, it follows that $\delta_{jj} \in I$ for any $j$. Therefore $I$ must be $R$, so $M$ is maximal. We have $R/M \simeq \bigoplus_{\delta \in \Delta_n} R_\delta / M_\delta$, so $R$ is simple Artinian.

Let $v$ be the $G$-valuation as constructed in 5.3.9. If $\sum a_\delta \delta$ is not in $R$, then there is some $\delta$ with $v(a_\delta \delta) < 0$ minimal hence $(a_\delta \delta)^{-1}$ is in $R$. Furthermore,

$$\begin{aligned} v(a(a_\delta \delta^{-1})) &= \min\left\{ v(a_\gamma \gamma) v((a_\delta \delta)^{-1}) \mid \gamma \in \Delta_n \right\} \\ &= v(a_\delta \delta) v((a_\delta \delta)^{-1}) = v(1_{l(\delta)}) \end{aligned}$$

which implies that $a(a_\delta \delta)^{-1}$ is in $R \setminus M$. In a similar fashion we find that $(a_\delta \delta)^{-1} s \in R \setminus M$, so $R$ is a Dubrovin valuation ring. ☐

**Lemma 5.4.2.** *If $Q \simeq M_n(k)$ for a field $k$ and $R$ is a Dubrovin valuation ring on $Q$, then $R$ is a $\Delta_n$-stable, $\Delta_n$-total subring of $Q$.*

*Proof.* It is known (cfr. [59]) that $R = q M_n(S) q^{-1}$ for some $q \in Q^*$ and some Dubrovin valuation ring $S$ of $k$. We will show that $M_n(S)$ is $\Delta$-stable and $\Delta$-total, from which our claim follows immediately. $M_n(S)$ contains $1_\delta$ for all $\delta \in \Delta_n$ and since $k$ is a field, $S$ is a valuation ring. If then $a_\delta \delta \notin M_n(S)$, we have $a_\delta \notin S$ so $a_\delta^{-1} \in S$ and $(a_\delta \delta)^{-1} = a_\delta \delta^{-1} \in S$ and $M_n(S)$ is $\Delta_n$-total. If $a = \sum a_\epsilon \epsilon$ is in $R$, then for any homogeneous $b_\delta \delta$ we have $b_\delta \delta a b_\delta^{-1} \delta^{-1} = b_\delta a_{r(\delta)} b_\delta^{-1} r(\delta)$ which is in $M_n(S)$ since $S$ is stable. ☐

Unfortunately, if $Q = M_n(D)$ is a matrix ring over a skewfield, the restriction of a Dubrovin valuation ring $R$ in $Q$ to $D$ might be a Dubrovin valuation ring which is neither total nor stable. In this case, $R$ will obviously not be $\Delta_n$-stable or $\Delta_n$-total. Suppose $Q$ is some simple Artinian ring which is finite dimensional over its centre, then there exists a splitting field $k$ with $Q \otimes k \simeq M_n(k)$ such that $k$ is a finite extension of $Z(Q)$.

**Lemma 5.4.3.** *If $R$ is a Dubrovin valuation ring in $Q$, then $R \otimes k$ is a Dubrovin valuation ring in $Q \otimes k$.*

*Proof.* Let $M$ be the unique maximal ideal of $R$. Since $Q$ and $k$ are both finite dimensional and simple over $Z(Q)$, $Q \otimes k$ is also finite dimensional and simple over $Z(Q)$. Clearly, $M \otimes k$ is a maximal ideal of $R \otimes k$. If $q \otimes c$ is in $Q \otimes k \setminus R \otimes k$, then $q$ is in $Q \setminus R$, so there is some $m \in M$ with $qm \in R \setminus M$. Then $(q \otimes c)(m \otimes 1) = qm \otimes c \in R \otimes k \setminus M \otimes k$. Similarly, $(m' \otimes 1)(q \otimes c) \in R \otimes k \setminus M \otimes k$ □

**Corollary 5.4.4.** *If $R$ is a Dubrovin valuation ring on a simple Artinian $Q = M_n(\Delta)$ with $\Delta$ finite dimensional over $k = Z(Q)$, then there is some finite extension $k'$ of $k$, some valuation $v$ on $k'$ and some invertible $q \in Q \otimes k'$ such that $R = qM_n(R_v)q^{-1}$.*

The $v$ in this proposition is obviously an extension of the valuation associated to $R \cap Z(Q)$.

If $R^\sigma_\alpha[G]$ is a skew twisted groupoid ring and $I$ is an additive subgroup of $R$, we say that $I$ is $\sigma$-invariant if $\sigma(g)(I) = I$ for all $g \in G$.

**Proposition 5.4.5.** *Suppose that $Q^\sigma_\alpha[G]$ is a skew twisted groupoid ring and that $R$ is a $\sigma$-invariant subring of $Q$. Define for a $\sigma$-invariant ideal $I$ of $R$ and $R^\sigma_\alpha[G]$ ideal $I' = \{\sum x_g g \mid \forall g \in G : x_g \in I\}$. If $I$ satisfies (D2), then so does $I'$.*

*Proof.* Since $I$ is a $\sigma$-invariant ideal of $R$, $I'$ is an ideal of $R^\sigma_\alpha[G]$. Assume that $I$ is an ideal of $R$ which satisfies (D2). Suppose there is some $x = \sum x_g g \in Q^\sigma_\alpha[G] \setminus R^\sigma_\alpha[G]$. Take such an $x_g \notin R$. For some appropriate $r_{r(g)} \in R$ we have $x_g r_{r(g)} \in R \setminus I$. Since $R$ is $\sigma$-stable, $\sigma(r(g))^{-1}(r_{r(g)})$ is in $R$. Moreover,

$$(x_g g)(\sigma(r(g))^{-1}(r_{r(g)})r(g)) = x_g r_{r(g)} g$$

which is in $R^\sigma_\alpha[G] \setminus I'$. By a similar reasoning we find a $y$ with $yx \in R^\sigma_\alpha[G] \setminus I'$ so $I'$ satisfies (D2). □

**Remark 5.4.6.** If $G$ is finite and connected, $I$ is $\sigma$-invariant and $R/I$ is simple Artinian, then by a similar reasoning as in 5.4.1 we find that $R^\sigma_\alpha[G]/I'$ is simple Artinian as well.

# Chapter 6

# Filtrations associated to pseudo-valuations

There is a very close link between between valuation theory on the one hand and filtrations on fields on the other hand. In fact, as we will show later, to any valuation on a field one can associate a separated exhaustive strictly increasing filtration. This suggests that filtrations could be used to generalise results from classical valuation theory to a more general (non-commutative) context. This idea was explored a.o. in [1]. There are also connections with topics like Auslander regular rings, micro-localisations, and Zariski rings, for which we refer the interested reader to [49]. After an introductory section to explain the terminology, we will give some new and tantalising generalisations of results by Willaert and Makar-Limanov, but much work still remains to be done.

## 6.1 Basic concepts

**Definition 6.1.1.** *Let $\Gamma$ be a partially ordered group.*[1] *A* filtration *on a ring $R$ is a series $(F_\gamma R)_{\gamma\in\Gamma}$ of subgroups satisfying (for $\gamma,\delta\in\Gamma$):*

*(1)* $\gamma\le\delta \Longrightarrow F_\gamma R\subseteq F_\delta R$,

*(2)* $F_\gamma R F_\delta R\subseteq F_{\gamma\delta}R$,

*(3)* $1\in F_e R$ *if $e$ is the neutral element of $\Gamma$.*

*If the stronger*

[1] It is possible to define profiltrations if $\Gamma$ is only partially right-ordered (cfr. [17]), but then condition (2) must be replaced by the more unwieldy $F_\gamma R F_\delta R\subseteq \bigcup_{\epsilon\le\delta} F_{\gamma\epsilon}R$.

*(2') $F_\gamma R F_\delta R = F_{\gamma\delta}R$*

*holds, it is called a* strong *filtration.*

Filtered modules can be defined in a similar fashion, keeping in mind that the filtration should be compatible with the one on the base ring, but we do not need them here. We refer the interested reader to [74] for more on filtered rings and modules.

An important concept in the theory of filtered rings is the *associated graded ring*, which is introduced as follows: suppose $FR$ is a $\Gamma$-filtration on some ring $R$. For any $\gamma$ in $\Gamma$, we can define $F_\gamma^- = \bigcup_{\delta<\gamma} F_\delta R$. The associated graded ring is then

$$G(R) = \bigoplus_{\gamma\in\Gamma} F_\gamma R / F_\gamma^- R$$

which is by definition $\Gamma$-graded. To make sure that this is indeed a ring, one has to use the principal symbol map $\sigma : R \to G(R) : r \mapsto \overline{r}$ where $\overline{r}$ is the equivalence class of $r$ in $G(R)$. For any two homogeneous elements $g_1 = \sigma(r_1)$ of degree $d_1$ and $g_2 = \sigma(r_2)$ of degree $d_2$, the multiplication $g_1 g_2$ is defined as the equivalence class of $r_1 r_2$ in $F_{d_1 d_2}/F_{d_1 d_2}^-$. By bilinear extension, this gives a multiplication on all of $G(R)$. Notice that $\sigma(r_1 r_2) = \sigma(r_1)\sigma(r_2)$ if $\sigma(r_1 r_2) \neq 0$.

We will use the following terminology:

- A filtration is called *exhaustive* if $\bigcup_{\gamma\in\Gamma} F_\gamma R = R$.
- A filtration is called *separated* if $\bigcap_{\gamma\in\Gamma} F_\gamma R = 0$.
- A filtration is called *nice* if it is both separated and exhaustive and the associated principal symbol map is surjective.

It is worth noticing that $F_0 R$ is a subring and that $F_\gamma R$ is an $F_0 R$-ideal if $\gamma < e$ where $e$ is the neutral element of $\Gamma$.

**Example 6.1.2.** 1. If $\mathfrak{g}$ is a Lie-algebra over $k$ with universal enveloping algebra $U$, then $U$ is filtered by the degree. By Poincaré-Birkhoff-Witt (cfr. any handbook on Lie-algebras, e.g. [89]), the associated graded ring is simply a polynomial ring.

2. If $R$ is any ring and $I$ is an ideal of $R$, then $F_{-n}R = I^n$, $F_n R = 0$ for $n \geq 0$ defines a $\mathbb{Z}$-filtration on $R$. It is called the *$I$-adic filtration*.

3. If $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is a graded ring, then it has a canonical $\Gamma$-filtration by $F_\gamma R = \bigoplus_{\delta \geq \gamma^{-1}} F_\delta R$.

**Definition 6.1.3.** *Let $R$ be a ring and let $\Gamma$ be a totally ordered group. A* pseudo-valuation *on $R$ is a surjective map $v : R \to \Gamma \cup \{\infty\}$ such that:*

*1.* $v(r) = \infty$ *if and only if* $r = 0$*,*

*2.* $v(rr') \geq v(r)v(r')$*,*

*3.* $v(r - r') \geq \min\{v(r), v(r')\}$*.*

*Sometimes (as in [59], where these objects are called value functions) one more condition is added:*

*4.* $v(\mathrm{st}(v)) = \Gamma$

*where* $\mathrm{st}(v) = \{r \in U(R) \mid v(r^{-1}) = -v(r)\}$*. Pseudo-valuations of this kind are called* regular*, after the terminology of [14]. If* $\Gamma \simeq \mathbb{Z}$*, we say that the pseudo-valuation is* discrete*.*

Pseudo-valuations were introduced by Mahler ([53] & [54]) in the thirties. They were studied, if at all, mainly from a topological point of view (e.g. in [14]). Two pseudo-valuations are said to be equivalent of they induce the same topology. However, we will not explore the topological side of pseudo-valuation theory here.

**Proposition 6.1.4.** *A pseudo-valuation* $v : R \to \Gamma$ *satisfies the following (for all* $s \in \mathrm{st}(v)$ *and* $x, y \in R$*):*

*(1)* $v(x) = v(-x)$*.*

*(2)* $v(x + y) = \min\{v(x), v(y)\}$ *if* $v(x) \neq v(y)$*.*

*(3)* $v(sx) = v(s)v(x)$ *and* $v(xs) = v(x)v(s)$*.*

*Proof.* (1) and (2) can be found in [59] or [53]. For (3) it suffices to note that $v(sy) \geq v(s)v(s^{-1}sy) \geq v(sy)$. □

Any pseudo-valuation $v : R \to \Gamma$ induces a $\Gamma$-filtration on $R$ by putting

$$F^v_\gamma R = \{r \in R \mid v(r) \geq \gamma^{-1}\}.$$

On the other hand, if $FR$ is a nice $\Gamma$-filtration for some totally ordered $\Gamma$, then $\sigma$ is a pseudo-valuation (provided it is surjective). This correspondence between pseudo-valuations and filtrations is the subject of current investigations.

**Example 6.1.5.** 1. If $R$ is a (commutative) PID and $I$ is an ideal, then $I = (r)$ for some $r \in R$ and the localisation of $R$ at all prime ideals which do not contain $I$ is a subring $S$ of the field of fractions $K(R)$. Any element $x$ of $K(R)$ can be expressed as $ab^{-1}$ for $a, b$ coprime in $S$ and

$$v : K(R) \to \mathbb{Z} : ab^{-1} \mapsto m_r(a) - m_r(b),$$

where $m_r(t)$ is the multiplicity of $r$ as a divisor of $t$, defines a pseudo-valuation on $K(R)$. By taking the completion with respect to this pseudo-valuation, one gets a counterpart to the $n$-adic numbers (cfr. e.g. [91]).

2. If $\phi : \mathcal{F}(R) \to \mathbb{Z}$ is an arithmetical pseudo-valuation on some ring $R$, then $v : R \to \mathbb{Z} : x \mapsto \phi(RxR)$ is a discrete pseudo-valuation. In particular, if $R$ is a noetherian Dubrovin valuation in a simple artinian ring, then by theorem 3.4.8 it is the ring of positives for such a pseudo-valuation.

The following theorem, which can be found in [3], gives a characterisation of the filtrations coming from valuations in terms of the associated graded ring.

**Theorem 6.1.6.** *For a separated filtration $F_nQ$ on an artinian ring $Q$, the following are equivalent:*

*(i) $G(Q)$ is a domain.*

*(ii) $Q$ is a skewfield and $G(Q)$ is a graded skewfield.*

*(iii) $Q$ is a skewfield and $F_0Q$ is a valuation ring on $Q$ and $G(Q)$ is strongly $\Gamma_s$-graded for the subgroup $\Gamma_s = \{\gamma \in \Gamma \mid G(Q)_\gamma \neq 0\}$.*

*Proof.* Cfr. loc. cit. □

**Proposition 6.1.7.** *For a pseudo-valuation $v$ on a ring $Q$ which takes values in $\mathbb{Z}$ and has the induced filtration $FQ$, the following are equivalent:*

*(i) $v$ is regular,*

*(ii) there is an $x \in F_{-1}Q$ with $x^{-1} \in F_1Q$,*

*(iii) the filtration is strong.*

*Proof.* (ii) ⇔ (iii) That (ii) is a consequence of (iii) is clear since if $F_nQF_mQ = F_{n+m}Q$ for $n, m \geq -1$, then there are some $a \in F_{-1}Q$, $b \in F_1Q$ with $ab = 1$. So suppose there is some $x \in F_{-1}Q$ with $x^{-1} \in F_1Q$. We certainly have $F_{-1}QF_1Q = F_0Q$. We will now show that $F_nQ = (F_1Q)^n$ for $n \geq 1$. For any $a \in F_nQ \setminus F_{n-1}Q$ we have $aF_{-1}Q \subseteq F_{n-1}Q$. Therefore $ax \in F_{n-1}Q$. By induction, $F_{n-1}Q = (F_1Q)^{n-1}$ and since $x^{-1} \in F_1Q$ we find $a \in F_1Q^n$. By definition of a filtration $F_{-1}^2Q \subseteq F_{-2}Q \subseteq F_{-1}Q$. Multiplying on the left with $F_1Q$ yields $F_{-1}Q = F_1QF_{-2}Q$ and consequently $F_{-1}^2Q = F_{-2}Q$.

(i) ⇔ (ii) If $v$ is regular, there is an $x$ with $v(x^{-1}) = v(x)^{-1} = 1$. If $x \in F_{-1}Q$ with $v(x^{-1}) = 1$, then $v(x^n)^{-1} = n = v(x^n)^{-1}$. □

**Example 6.1.8.** 1. Consider $R = M_2(\mathbb{Z}_p) \subseteq M_2(\mathbb{Q})$ for some prime number $p$. If we put $R = F_0Q$, then several choices are possible for $F_{-1}Q$. Pick for example $F_{-1}Q = M_2(p\mathbb{Z}_p)$, then $F_1Q$ is still not determined. Here we can pick, say

$$F_1Q = \begin{bmatrix} \frac{1}{p}\mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \frac{1}{p}\mathbb{Z}_p \end{bmatrix}.$$

There is clearly some $x$ in $F_{-1}Q$ with $x^{-1}$ in $F_1Q$, so the filtration (and therefore also the pseudo-valuation) is now completely fixed.

2. The following construction is due to Cohn ([14]). Consider the field $F = \mathbb{Q}((X_i)_{i\in\mathbb{N}\smallsetminus\{0\}})$ and some fixed prime number $p$. Let $S$ be the subring generated by

   (1) $\frac{r}{s}$ (for $r, s \in \mathbb{Z}$ with $\gcd(r,s) = 1$) such that $p$ does not divide $s$,

   (2) $f^{-1}$ for a polynomial $f$ with all coefficients as in (1) but not all divisible by $p$,

   (3) $p^n X_n^i$ for all $i, n \in \mathbb{N} \smallsetminus \{0\}$.

   As Cohn has shown (cfr. loc. cit.), there is for every $f \in F$ a unique $n \in \mathbb{Z}$ with $p^n f \in S$. This defines a nice filtration on $F$ and consequently a pseudo-valuation. Cohn showed that the topology induced by this pseudo-valuation cannot be induced by a regular pseudo-valuation.

One last theorem worth mentioning, especially because of the possible generalisations, is also due to Cohn (cfr. loc. cit. once more) and gives another very nice characterisation of regular pseudo-valuations:

**Theorem 6.1.9.** *A pseudo-valuation $v : k \to \mathbb{R}$ on a field $k$ is equivalent to a regular pseudo-valuation if and only if there is a collection of valuations $(v_i)_{i\in\mathcal{I}}$ such that*

$$v(x) = \inf_{i\in\mathcal{I}} \{v_i(x)\}.$$

*Proof.* See once more loc. cit. although the valuations are written exponentially there. Consequently, this $\inf$ appears as a $\sup$. □

## 6.2 Pseudo-valuations compatible with a valuation

Willaert ([106]) was the first to introduce a compatibility criterion for valuations, allowing him to describe and classify many valuations on the Weyl skewfield. He defined compatibility only with respect to the Bernstein filtration (or equivalently with the Bernstein valuation), but some of his results hold true in much greater generality.

Suppose $Q$ is a skewfield and let $v$ and $w$ be pseudo-valuations on $Q$. Then $v$ and $w$ induce respective filtrations $F_\gamma^v R$ and $F_\gamma^w R$, associated graded rings $G_v(Q)$ and $G_w(Q)$, and principal symbol maps $\sigma_v$ and $\sigma_w$. Moreover, $v$ induces a filtration on $G_w(R)_0$ by:

$$F_\gamma(G_w(R)_0) = (F_\gamma(Q) \cap F_0^w(Q))/(F_\gamma(Q) \cap F_0^{w-}Q).$$

This filtration is exhaustive, but not necessarily separated. The following lemma, which is a straightforward adaptation from [106], gives a condition under which it is separated.

**Lemma 6.2.1.** *Suppose $w$ is a valuation and $\Gamma$ is archimedean. $F_\gamma(G_w(R)_0)$ is separated if and only if:*

$$\sigma_w(x) = 1 \quad \Rightarrow \quad v(x) \leq 0.^2$$

*Proof.* Suppose $\sigma_w(x)$ is non-zero and in $\bigcap_\gamma F^v_\gamma(G_w(Q)_0)$, then there is some $y$ with $\sigma_w(x) = \sigma_w(y)$ and $v(y) > -v(x^{-1})$. Then $v(yx^{-1}) > 0$ while $\sigma_w(yx^{-1}) = 1$. If, on the other hand, $x \in G_w(Q)_0$ with $\sigma_w(x) = 1$ and $v(x) > 0$, then $1 = 1 - x^n + x^n$ must be in $F^v_\gamma(G_w(R)_0)$ for all $\gamma$ because $\Gamma$ is archimedean. □

**Remark 6.2.2.** An exhaustive filtration on a simple artinian ring $Q$ which is not the constant filtration $F_nQ = Q$ is necessarily separated, since $\bigcap_{\gamma\geq 1} F_{\gamma^{-1}}Q$ is an ideal of $\bigcup_\gamma F_\gamma Q$. As a consequence, $\bigcap_\gamma F_\gamma(G_w(Q)_0)$ is either $0$ or all of $G_w(Q)_0$.

**Definition 6.2.3.** *If $\bigcap_\gamma F_\gamma(G_w(Q)_0) = 0$, we say that $v$ is* compatible *with $w$.*

Let $R = \bigoplus R_\gamma$ be a $\Gamma$-graded ring with a (skew)field of fractions $D$ obtained by localising at a set of homogeneous elements which satisfy the Ore conditions. Then $D$ inherits a canonical $\Gamma$-grading from $R$. This grading also induces a pseudo-valuation $w$ on $D$.

**Proposition 6.2.4.** *Assume that $w$ is a valuation and $\Gamma$ is Archimedean. Any valuation $v$ on $D$ which is compatible with $w$ is then determined by its value on homogeneous components of $R$.*

*Proof.* It is clear that $v$ is determined by its values on $R$. Consider $r = h_1 + h_2$ for some homogeneous $h_1, h_2$ in $R$. Assume without loss of generality that $\sigma_w(h_2) < \sigma_w(h_1) = \sigma_w(r)$. From $\sigma_w(r/h_1) = 1$ we can then conclude that $v(r) \leq v(h_1)$ and consequently $v(r) = \min\{v(h_1), v(h_2)\}$. □

**Example 6.2.5.** One of the most important examples will be the Weyl algebra and the associated Weyl skewfield. For a (skew)field $k$, the $n$-th Weyl algebra over $k$ is defined as:

$$\mathbb{A}_n(k) = k < X_i, Y_i \mid i = 1, ..., n > /(X_iY_i - Y_iX_i - 1, X_iX_j - X_jX_i, Y_iY_j - Y_jY_i).$$

Intuitively, one should think of $Y_i$ as the operator of partial differentiation with respect to $X_i$. We refer the interested reader to [21] for more on Weyl algebras.

[2] Here it is justified to write $0$ for the neutral element of $\Gamma$, since any archimedean ordered group is a subgroup of $\mathbb{R}$, hence abelian.

A monograph on the Weyl algebra[3] has appeared ([97]), but seems to be very rare. Weyl algebras have Ore localisations and the skewfields of fractions are denoted by $\mathbb{D}_n(k)$. We will mainly be interested in $\mathbb{D}_1(k)$. On the Weyl algebra, there is a canonical filtration by putting $F_n^B\mathbb{A}_1(k)$ the $k$-vector space generated by monomials $X^iY^j$ with $i+j \le n$. Since this filtration is usually called the Bernstein filtration, we have used a superindex $B$; we will similarly use $\sigma_B$ for the associated principal symbol map. Following [106], we will classify discrete valuations compatible with the valuation induced by the Bernstein filtration.

In the Weyl algebra, we have $XY-YX=1$ which yields $v(XY-YX)=v(0)$ so $v(X)+v(Y)\le 0$. If $v(X)<v(Y)$, then $v(X)=V(X+Y)$. Since $[X,X+Y]=1$, we may replace $Y$ by $X+Y$ so we can assume $v(X)=v(Y)<0$. If the valuation of $F=\sum a_iX^iY^{-i}$ is larger than the valuation of $G=\sum a_i(XY^{-1})^i$, then $v(F/G)>0$ while $\sigma_B(F/G)=1$ which contradicts the compatibility of $v$ with the Bernstein filtration. If $E=\sum a_iX^iY^{n-i}$ is a homogeneous element of degree $n$, then $EY^{-n}=F$ hence $v(E)=v(G)+nv(Y)$. Consequently, $v$ is determined by the values it takes on $k(XY^{-1})$. Moreover, $v(XY^{-1})=0$ so $v$ restricted to $k(XY^{-1})$ is either the trivial valuation — in which case $v$ is equivalent to the Bernstein valuation — or it measures the multiplicity of some non-zero $\alpha\in k$. In the latter case, $v$ is determined by $\alpha$, $v(X)$ and $v(XY^{-1}-\alpha)$. It is shown in [106] that — for any $\alpha\in k$, $v(X)\in\mathbb{Z}$ and $v(XY^{-1}-\alpha)\in\mathbb{Z}$ — this does indeed yield a discrete valuation provided that $v(XY^{-1}-\alpha)\le -v(X)$.

## 6.3 Valuations and nilpotent Lie brackets

The valuations on the Weyl skewfields have been studied in depth, for example in [90], [55], [104], and [106]. That these studies have been so fruitful is in no small part due to the fact that $\mathbb{A}_n(k)$ is graded and that the commutator lowers the degree, i.e. $\deg([a,b])<\min\{\deg(a),\deg(b)\}$. This little factoid makes the following trick work:

**Lemma 6.3.1.** *Let $R$ be a ring with skewfield of fractions $D$ and let $v$ be a valuation on $D$. If $r\in R$ is such that*

$$[\cdot,r]:R\to R:x\mapsto xr-rx$$

*is nilpotent, then $v(r)\in Z(\Gamma)$.*

*Proof.* Let $r$ be as in the statement and suppose $v(rx)\ne v(xr)$ for some $x\in R$. It is harmless to assume $v(rx)<v(xr)$. Then $v([\cdot,r](x))=v(rx)$ and by induction $v([\cdot,r]^n(x))=v(r^nx)$ for every $n$. But this is in contradiction with the fact that, for some $m$, $[x,r]^m=0$ and consequently $v([x,r]^m)=\infty$. Therefore $v(rx)=v(xr)$ for all $x$ in $R$, but since $D$ is the skewfield of fractions of $R$ we have $v(r)\in Z(\Gamma)$. □

[3] The definite particle indicates that $n=1$.

By the following lemma, we get one dimension for free when investigating commutativity of some valuation.

**Lemma 6.3.2.** *Let $R$ be a ring with skewfield of fractions $D$, let $v : D \to \Gamma$ be a valuation on $D$, and let $R'$ be a subring with skewfield of fractions $Q'$. Suppose furthermore that $GKdim(R') = GKdim(R) - 1$ and that $v(R') \subseteq Z(\Gamma)$. Then $v$ is abelian on $Q$.*

*Proof.* Suppose $r \notin Z(\Gamma)$, then $r^n \notin Z(\Gamma)$ for all $n$. For any $s \in R$ we then have that

$$\sum \alpha_{ij} r^i s^j = 0$$

for some coefficients $\alpha_{ij}$ in $R'$, so there must be two different monomials in this sum with the same valuation. This implies that $v(s)^l$ is in the subgroup of $\Gamma$ generated by $Z(\Gamma)$ and $v(r)$, ergo some power of $v(s)$ commutes with some power of $v(r)$, but then $v(s)$ and $v(r)$ must also commute. □

This is essentially what Makar-Limanov used in [55] to show that every valuation on $\mathbb{D}_n(k)$ is abelian (see below). Shtipel'man had already proven this theorem ([90]), but he had to rely on more computational methods which are intrinsic to the Weyl skewfields.

The following examples give the two most extreme possibilities: in the first case all valuations are abelian, in the second case, none are.

**Example 6.3.3.**

1. In the Weyl algebra $R = \mathbb{A}_1(k)$, the generator $X$ has a nilpotent Lie bracket.[4] Put $R' = k[X]$, then the Gelfand-Kirillov dimension of $R'$ is the Gelfand-Kirillov dimension of $R$ minus one, so every valuation on $\mathbb{D}_1(k)$ is abelian.

2. Let $G$ be a non-abelian linearly ordered group. Construct $K(G, \mathbb{R}(t))$ as in example 1.3.6. It is proven in [55] that this skewfield admits no non-trivial abelian valuations.

[4] See [21] for a classification of elements of the Weyl algebra for which this is true.

# Appendices

# Appendix A

# There is no such thing as a right valuation

The title might seem a bit harsh, because the term does pop up in the literature every now and then (e.g. in [9]), but not in the sense we will use it.

**Definition.** *A* partially right ordered monoid *or* promonoid *is a monoid $M$ endowed with a partial ordering $\leq$ such that*

$$a \leq b \quad \Rightarrow \quad ac \leq bc$$

*for all $a, b, c$ in $M$. A* right value map *is a surjective monoid morphism from some monoid $M$ to a promonoid $M'$.*

Right ordered groups (see e.g. [17] or [63]) and partially ordered monoids (see e.g. [31]) have been studied a bit, but promonoids have been mostly neglected. Of course, plomonoids and left value maps can be defined analogously. If a right value map is also a left value map we will call it a value map. The nuetral element in $M$ will be denoted $e$. The following proposition is a generalisation of 1.1.3.

**Proposition.** *If $M$ is a monoid and $M'$ is a submonoid, then $M'$ induces a canonical right value map $\phi$ on $M$ such that $M_\phi = \{m \in M \mid \phi(m) \geq \phi(e)\} \subseteq M'$.*

*Proof.* For any $x, y \in M$, set $x \sim y$ if and only if

$$axb \in M' \quad \Leftrightarrow \quad ayb \in M'.$$

This equivalence is compatible with multiplication — as can be easily checked — so we obtain a quotient monoid $\overline{M} = M/\sim$. The equivalence class of $x \in M$ in $\overline{M}$ will be denoted by $\overline{x}$. Consider $S = \{\overline{m} \in M \mid ab \in M' \Rightarrow amb \in M'\}$.

Clearly, $S$ is closed under products and contains $e$ so it is a submonoid of $\overline{M}$. Put $\overline{x} \geq \overline{y}$ if and only if $\overline{x} = \overline{sy}$ for some $\overline{s} \in S$. This is clearly a reflexive and transitive relation. Suppose $\overline{x} \geq \overline{yx}$, then there are $\overline{s}, \overline{s'} \in S$ with $\overline{x} = \overline{sy}$ and $\overline{y} = \overline{s'x}$. Suppose $axb \in M'$, then $as'xb \in M'$ so $ayb \in M'$. Similarly, if $ayb \in M'$ then $asyb \in M'$ so $axb \in M'$. This means that $x \sim y$ so $\overline{x} = \overline{y}$ and consequently $\leq$ is a partial ordering. Hence the projection $\phi : M \to \overline{M}$ is a value map. If $\phi(x) \geq \phi(e)$ we find $x = exe \in M'$ so $M_\phi \subseteq M'$. □

**Remark.** If $M$ is a group and $M'$ is a normal subgroup, then $x$ and $y$ in $M$ are equivalent in the sense of the preceding proposition if and only if

$$\begin{aligned} axb \in M' \Leftrightarrow ayb \in M' \Leftrightarrow bax \in M' \Leftrightarrow bay \in M' \\ \Leftrightarrow M'x = M'y \end{aligned}$$

which means that they are equivalent in the sense of the classical group quotient.

Note that by the same reasoning one can also construct a left value map. Moreover, one can reverse the ordering on $\overline{M}$ to obtain another right respectively left value map, $\psi$ say, where $M^\psi = \{m \in M \mid \psi(m) \leq \psi(e)\} \subseteq M'$.

**Proposition.** *Let $M$ be a monoid, $\phi$ a right-value map and $M_\phi$ the set of positive elements. Then:*

*(1) $m \in U(M_\phi)$ implies $\phi(m) = \phi(e)$,*

*(2) $\{m \in M \mid \phi(m) > 0\}$ is multiplicatively closed,*

*(3) $M' = M_\phi$ if and only if $ab \in M'$ implies $amb \in M'$ for every $m \in M'$,*

*(4) $\phi$ is a (two sided) value map if and only if $M_\phi x = xM_\phi$ for any $x \in M$.*

*Proof.* (1) If $x \in U(M_\phi)$ then $\phi(e) = \phi(x^{-1})\phi(x) \geq \phi(x) \geq \phi(e)$.

(2) Let $\phi(m) > \phi(e)$ and $\phi(l) \geq \phi(e)$, then $\phi(lm) \geq \phi(m) > 0$. On the other hand $\phi(ml) \geq \phi(m) > 0$.

(3) If $m \in M'$ implies that $amb \in M'$ if $ab \in M'$, then $\phi(m) \in S$ hence $\phi(m) \geq \phi(e)$ so $m \in M_\phi$. The other inclusion is always true. Similarly, if $\phi(m) \geq \phi(e)$ for all $m \in M'$ then $ab \in M'$ implies $amb \in M'$.

(4) If $M_\phi x = xM_\phi$ for all $x \in M$ then $\{y \in M \mid \phi(y) \geq \phi(x)\} = M_\phi x = xM_\phi$ so for all $y \in M_\phi x$ there is some $m \in M_\phi$ with $\overline{y} = \overline{xm}$. As a consequence, we have $\overline{ay} = \overline{axm}$ for every $a \in M$ which implies that the ordering on $\overline{M}$ is two-sided. If $\overline{M}$ is bi-ordered, then an element $m$ is in $M_\phi x$ if an only if $\phi(m)$ is larger than $\phi(x)$ if and only if $m \in xM_\phi$ which had to be shown.

□

**Proposition.** *Suppose $G$ is a right cancellative monoid. Any submonoid $M$ with $U(M) = e$ induces a partial right ordering $\leq_M$ on $G$ such that $\{g \in G \mid g \geq_M e\} = M$.*

*Proof.* If $M$ is such a submonoid then we can say $a \leq_M b$ if $ma = b$ for some $m \in M$. Since $e \in M$, this is a reflexive relation and since $M$ is closed under multiplication it is transitive. Suppose $a \leq_M b \leq_M a$, then there are $m$ and $m'$ in $M$ with $mb = a$ and $m'a = b$ so $a = mm'a$ which implies that $m$ is invertible since $G$ is right cancellative. But since $U(M) = e$, $m$ must be $e$ hence $a = b$. This ordering is preserved under right multiplication, for if $a \leq_M b$, then $a = mb$ for some $m \in M$ but then $ac = mbc$ hence $ac \leq_M bc$. If $g \geq_M e$ then there is some $m \in M$ with $g = me = m$ so $g \in M$ and for any $m \in M$ we clearly have $m \geq_M e$. □

If $Q$ is a simple Artinian ring and $S$ is a subring with $U(S) = \{e\}$ then it induces a partial right order on $Q$. Moreover, since $S$ is closed under sums, we have that $x \geq 0 \leq y$ implies $x + y \geq 0$ which in turn means $a \geq b \leq c \Rightarrow a + c \geq b$. A right value map with this property will be called a *provaluation*. But the condition $U(S) = \{e\}$ is far too strong.

**Lemma.** *Let $R$ be a ring without zero-divisors and let $S$ be a subring. $S$ is the ring of positives for a provaluation iff $U(S)$ is the kernel for some monoid morphism $\phi : R \setminus \{0\} \to M$. This morphism is then the associated provaluation.*

*Proof.* If $U(S)$ is the kernel of $\phi$ then $\overline{S}$, the image of $S \setminus \{0\}$ under $\phi$, is a submonoid of $R \setminus \{0\} / \ker(\phi) \simeq M$ for which $U(\overline{S}) = \{\overline{1}\}$, so it induces an ordering on $T$ such that $\{t \in T \mid t \geq 1\} = \overline{S}$. This implies that $\phi$ is in fact a provaluation, since $\phi(x) \geq \phi(1) \leq \phi(y)$ implies $\overline{x}, \overline{y} \in \overline{S}$ hence $x, y \in S$. Then we have $x + y \in S$ and consequently $\phi(x + y) \geq \phi(1)$.

If, on the other hand, $S$ is the ring of positives for some provaluation $\phi$, then $U(S) = \ker(\phi)$, which proves the claim. □

**Corollary.** *If $D$ is a skewfield, then a subring $S$ is the ring of positives for some provaluation iff $U(S)$ is a normal subgroup of $U(D)$.*

*Proof.* If $U(S)$ is a normal subgroup, then $\pi : U(D) \to U(D)/U(S)$ is a monoid (even group) morphism with the required kernel. On the other hand, if $S$ is a subring of positives, then $U(S)$ is the kernel of a map $U(D) \to M$. This monoid morphism is necessarily a group morphism, so $U(S)$ is normal. □

**Corollary.** *A strict provaluation is a partial valuation.*

*Proof.* Put $S = \{x \mid \phi(x) \geq 0\}$. Since $\phi$ is strict, $P = \{x \mid \phi(x) > 0\}$ is the unique maximal ideal. If $s \in S$ then either $s$ or $1 + s \in U(S)$ in which case

$1 + xsx^{-1} = x(1+s)x^{-1} \in U(S)$, so $xSx^{-1} \in S$. Therefore, $S$ is stable, hence it induces a partial valuation. $\square$

So if $v$ is a provaluation which takes values in a totally right-ordered group, then $v$ takes values in a bi-ordered group, hence it is a valuation as the title of this appendix suggests.

# Appendix B

# Appell sets and Verma modules for $\mathfrak{sl}(2)$

## Introduction

Mathematical analysis and classical representation theory have always gained from a mutual infusion of ideas and techniques. The former often provides concrete examples which may illustrate and initiate more abstract notions which are studied and generalised in the latter. The present paper is written with this philosophy in mind, hereby drawing inspiration from two particular problems arising in harmonic analysis on $\mathbb{R}^m$. First of all, there exist quite a few function theories which are centred around a set of operators which realize a copy of the simple Lie algebra $\mathfrak{sl}(2)$ under the commutator bracket. Classical harmonic analysis itself, for example, is centred around the Laplace operator $\Delta_m$, acting as an endomorphism on $\mathbb{C}[x_1, \ldots, x_m]$. The associated Lie algebra $\mathfrak{sl}(2)$ is then given by

$$\mathfrak{sl}(2) \cong \mathrm{Alg}\,(X, Y, H) \cong \mathrm{Alg}\left(\frac{1}{2}\Delta_m, -\frac{1}{2}|\underline{x}|^2, -\mathbb{E}_x - \frac{m}{2}\right)\ , \tag{B.1}$$

where $|\underline{x}|^2$ is the squared norm of $\underline{x} \in \mathbb{R}^m$ and $\mathbb{E}_x = \sum x_j \partial_{x_j}$ stands for the Euler operator on $\mathbb{R}^m$. A celebrated result in harmonic analysis, due to R. Howe [37], describes this Lie algebra as the so-called dual partner of the orthogonal group $\mathrm{SO}(m)$, acting on polynomials in $\mathbb{C}[x_1, \ldots, x_m]$ through the regular representation, leading to a multiplicity-free decomposition of this space in terms of Verma modules for $\mathfrak{sl}(2)$. Also, in super analysis (a function theory in which both commuting and anti-commuting variables are taken into account, see e.g. [20]) and the theory of Dunkl operators (in which the rotational symmetry is reduced to a finite subgroup, see e.g. [24]), a key ingredient is the existence of a subalgebra $\mathcal{A} \cong \mathfrak{sl}(2)$ inside the full endomorphism algebra acting on the polynomials (or smooth functions in general). This algebra is then used to define generalisations of the classical Hermite polynomials, which lead to an expression for the Fourier integral as an exponential operator $\exp(A)$, with $A \in \mathcal{A}$.

A second motivation comes from the theory of Appell sequences. Classically, these are defined in terms of a complex variable $z$, as sets of holomorphic polynomials $\{P_k(z) \mid k \in \mathbb{Z}^+\}$ satisfying the relation $P_k'(z) = kP_{k-1}(z)$, where $\deg P_k(z) = k$ and $P_0 \neq 0$. Denoting the derivation operator by means of $P = \partial_z$, one can also interpret these Appell sequences as representations for the Heisenberg-Weyl algebra $\mathfrak{h}_1$, provided there exists an associated raising operator $M$ satisfying $MP_k(z) = P_{k+1}(z)$ for all $k \in \mathbb{Z}^+$. Formally, this operator thus performs the integration within the Appell sequence. One can indeed verify that these defining relations imply that $P$ and $M$ satisfy the canonical commutation relation $[P, M] = 1 \in \mathbb{C} = \mathfrak{z}(\mathfrak{h}_1)$. Notable examples are the Hermite polynomials, but also other orthogonal polynomials such as Bernouilli and Euler polynomials. More generally, we say:

**Definition.** *A representation for the Heisenberg algebra (denoted by $\mathfrak{h}_1$) is an algebra morphism $\rho_V : \mathfrak{h}_1 \to \mathrm{End}(\mathbb{V})$. If there exists a vector $v_0$ in $\ker \rho_V(P)$, we will from now on refer to the set $\mathbb{A} = \{\rho_V^k(M)[1] \mid k \in \mathbb{Z}^+\}$ as an Appell sequence related to the operators $(\rho_V(P), \rho_V(M))$.*

In the classical situation, the vector space $\mathbb{V} = \mathrm{span}(P_k(z))_k$ is a polynomial family, with $\rho_V(P)$ a differential operator and $\rho_V(M)$ a conjugation thereof (a formal integration operator). In this paper, we will consider generalisations of Appell sequences, for which $\mathbb{V}$ may be a more abstract representation space. Note that the representation $\mathbb{V}$ must be infinite-dimensional, which follows from the fact that $\mathrm{tr}([\rho_V(P), \rho_V(M)]) = 0$, as it is the trace of a commutator. This means that finite-dimensional representations must necessarily satisfy $\dim(\mathbb{V}) = 1$. This suggests looking for generalisations of Appell sequences in canonical infinite-dimensional modules for $\mathfrak{sl}(2)$, i.e. Verma modules.

Recently, the problem of constructing analogues of complex Appell sequences in multivariate analysis has gained new interest, see a.o. [11]. These sequences are defined as polynomial sets $\mathfrak{V} = \mathrm{span}(P_k(\underline{x}))_k$ containing scalar-valued (resp. Clifford algebra-valued) null solutions for the Laplace or Dirac operator in the variable $\underline{x} \in \mathbb{R}^m$, for which the lowering operator $P$ is a differential operator belonging to the Clifford-Weyl algebra $\mathcal{W}_m^C$, defined by $\mathcal{W}_m^C = \mathrm{Alg}\left(x_i; \partial_{x_j}\right) \otimes \mathbb{C}_m$ (with $\mathbb{C}_m$ the universal Clifford algebra in $m$ dimensions). It turns out that some of these Appell sequences can be related to the branching problem for certain irreducible representations for the spin group, the (inductive) construction of orthonormal bases for these spaces and generalisations of the classical Fueter theorem, see e.g. [25]. This has given rise to Gegenbauer and Jacobi polynomials in harmonic (resp. Clifford) analysis, for which the formal integration operator was obtained in terms of an operator containing fractions. The results form this section come from [26].

## Appell sets in Verma modules for $\mathfrak{sl}(2)$

Adopting the classical definition (which can be given for any Lie algebra $\mathfrak{g}$) to our case of interest, we have the following:

**Definition.** *Let $\lambda \in \mathbb{C}$ be a complex number, and consider the vector space $V_\lambda = \mathbb{C}v_\lambda$ on which $\mathfrak{sl}(2)$ acts as $H[v_\lambda] = \lambda v_\lambda$ and $X[v_\lambda] = 0, Y[v_\lambda] = 0$. One may then define the highest-weight[1] Verma module $M_\lambda = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathcal{U}(\mathfrak{b}^+)} V_\lambda$, where $\mathfrak{n}^- = \mathbb{C}Y$ and $\mathfrak{b}^+ = \mathbb{C}H \oplus \mathbb{C}X$, with highest weight $\lambda \in \mathbb{C}$.*

In view of the PBW-theorem for the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$, the weight space decomposition is given by $M_\lambda = \mathrm{span}_\mathbb{C}\left(Y^k \otimes v_\lambda \mid k \in \mathbb{Z}^+\right)$. The following fact about Verma modules is well-known:

**Proposition.** *The Verma modules $M_\lambda$ are irreducible for $\lambda \notin \mathbb{Z}^+$.*

*Proof.* See e.g. [34]. □

The conditions on $\lambda$ ensuring that $M_\lambda$ is irreducible are very convenient, in the sense that we will only be able to introduce generalisations of Appell sequences as algebra morphisms $\sigma_\lambda$ from $\mathfrak{h}_1$ into $\mathrm{End}(M_\lambda)$ under the same conditions on $\lambda$ as in the proposition above (which means that the action of $\sigma_\lambda(P)$ and $\sigma_\lambda(M)$ will only be well-defined for $\lambda \in \mathbb{C} \setminus \mathbb{Z}^+$). In what follows, we consider the subring $R = \mathcal{U}(\mathfrak{b}^-) \subset \mathcal{U}(\mathfrak{sl}(2))$, where we use the notation $\mathfrak{b}^- = \mathbb{C}H \oplus \mathbb{C}Y$ for the (other) Borel subalgebra (compare with $\mathfrak{b}^+$).

**Remark.** Fractions will play a crucial role throughout this paper, and we therefore define them as $A/B = AB^{-1}$, where the inversion always appears at the right hand side. This is important, because in general $A$ and $B$ will belong to a non-commutative ring.

**Definition.** *For all $\alpha \in \mathbb{C}$, we define $S_\alpha^- = \left\langle (H + \alpha + 2j) \mid j \in \mathbb{Z}^- \right\rangle \subset R$ as the set which is multiplicatively generated by the elements between brackets.*

It is then easily verified that $S_\alpha^- \subset R$ satisfies the right Ore condition for arbitrary $\alpha \in \mathbb{C}$, see e.g. [15]. This condition is needed whenever one wants to consider the right ring of fractions $R(S^{-1})$ with respect to a multiplicatively closed subset $S \subseteq R$. Indeed, for arbitrary $\xi \in R$ and $\sigma \in S_\alpha^-$ one has that $\xi S_\alpha^- \cap \sigma R \neq \varnothing$. Indeed, in view of the PBW-theorem for $\mathcal{U}(\mathfrak{sl}(2))$, it suffices to consider an element $\xi$ of the form $\xi = Y^a H^b$ (hereby omitting the tensor product symbols and with $a, b \in \mathbb{Z}^+$). For $\sigma = H + \alpha + 2j$ (with $j \in \mathbb{Z}^-$), one then clearly has that $(H + \alpha + 2j)Y^a H^b = Y^a H^b (H + \alpha + 2(j - a))$, since $2(j - a) \in \mathbb{Z}^-$. We can thus define the localisation w.r.t. the set $S_\alpha^-$, which will be denoted by means of $R_\alpha^- = R\left((S_\alpha^-)^{-1}\right)$.

[1] In this paper, we have chosen to work with highest-weight Verma modules $M_\lambda$, but the construction for lowest-weight modules is similar.

**Remark.** Since $(H+\alpha+2j)Y^aH^bX^c = Y^aH^bX^c(H+\alpha+2(j+c-a))$, it is clear that one can also consider the localisation of the full enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ with respect to the (enlarged) subset $S_\alpha = \langle (H+\alpha+2j) : j \in \mathbb{Z}\rangle$, where $j \in \mathbb{Z}$ is now an arbitrary integer. We therefore also introduce the notation $\mathcal{U}_\alpha(\mathfrak{sl}(2)) = \mathcal{U}(\mathfrak{sl}(2))(S_\alpha^{-1})$. Obviously, for all $\alpha \in \mathbb{C}$ one has that $R_\alpha^- \subset \mathcal{U}_\alpha(\mathfrak{sl}(2))$. Note that these are all inside the skew field over $\mathcal{U}(\mathfrak{sl}(2))$.

The motivation for considering this particular localisation $R_\alpha^-$, instead of just $\mathcal{U}_\alpha(\mathfrak{sl}(2))$, comes from the fact that the action of the latter localisation will not always be well-defined on the Verma modules we would like to consider for practical purposes. In full generality, one has the following:

**Proposition.** *Whenever $(\lambda+\alpha) \notin 2\mathbb{Z}^+$, the action of $R_\alpha^-$ is well-defined on irreducible Verma modules $M_\lambda$.*

*Proof.* To prove that the action on the localisation w.r.t. $S_\alpha^-$ is well-defined, we first of all note that the elements in $S_\alpha^-$ act as a constant on the weight spaces. It then suffices to verify that for all integers $k \geq 0$ and $j \leq 0$ one has that $(H+\alpha+2j)[Y^k \otimes v_\lambda] = (\alpha+\lambda-2k+2j)Y^k \otimes v_\lambda \neq 0$. This is indeed guaranteed whenever $\alpha+\lambda \neq 2(k+|j|) \in 2\mathbb{Z}^+$. □

**Corollary.** *The action of $R_\lambda^-$ is well-defined on irreducible Verma modules $M_\lambda$ (i.e. for arbitrary $\lambda \in \mathbb{C}$ such that $\lambda$ is not a positive integer).*

**Corollary.** *For arbitrary $\alpha \in \mathbb{C}$, the action of $\mathcal{U}_\alpha(\mathfrak{sl}(2))$ is well-defined on irreducible Verma modules $M_\lambda$ whenever $\alpha+\lambda \notin 2\mathbb{Z}$.*

**Remark.** Note that the action of $\mathcal{U}_\lambda(\mathfrak{sl}(2))$ on $M_\lambda$ is not always well-defined, in view of the fact that e.g. for $\lambda = l \in \mathbb{Z}^-$ and $j = k-l$ we get that $(H+\lambda+2j)[Y^k \otimes v_\lambda] = 0$.

Let us then prove the main result of this section:

**Theorem.** *Suppose $M_\lambda$ is an irreducible Verma module for the algebra $\mathfrak{sl}(2)$, which means that $\lambda \notin \mathbb{Z}^+$. One can then define an action of $\mathfrak{h}_1$ on $M_\lambda$, by means of the algebra morphism*

$$\sigma_\lambda : \mathfrak{h}_1 \to \mathrm{End}(M_\lambda) : (P,M) \mapsto \left(X, \frac{2Y}{H+\lambda}\right) .$$

*Note that the operator $\sigma_\lambda(M)$ actually belongs to the localisation $R_\lambda^-$, which means that its (repeated) action on $M_\lambda$ is well-defined.*

**Remark.** Recalling the notation from the introduction, we thus have that the basis for $M_\lambda$ defined by $\mathbb{A} = \{\sigma_\lambda^k(M)[1 \otimes v_\lambda] : k \in \mathbb{Z}^+\}$ defines an Appell sequence.

*Proof.* It suffices to verify that the action of $\sigma_\lambda(P)$ and $\sigma_\lambda(M)$ on the module $M_\lambda$ satisfies the Heisenberg relation $[\sigma_\lambda(P),\sigma_\lambda(M)] = 1$. For that purpose, we note that for all $k > 0$, we have:

$$
\begin{aligned}
[\sigma_\lambda(P),\sigma_\lambda(M)](Y^k \otimes v_\lambda) &= \left(XY\frac{2}{H+\lambda} - Y\frac{2}{H+\lambda}X\right)(Y^k \otimes v_\lambda)\\
&= Y^k \otimes v_\lambda.
\end{aligned}
$$

For $k = 0$, the statement is trivial, which proves the theorem. □

Invoking the definition $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$, we then have:

**Definition.** *Suppose $M_\lambda$ is a highest-weight Verma module for $\mathfrak{sl}(2)$, with $\lambda \in \mathbb{C}\setminus\mathbb{Z}^+$. The monomial basis for $M_\lambda$ is given by the weight vectors*

$$
v_\lambda(k) = \frac{1}{(\lambda)_k}Y^k \otimes v_\lambda \qquad (k \in \mathbb{Z}^+).
$$

Note that the embedding of $\mathfrak{h}_1$ into $\mathcal{U}_\lambda(\mathfrak{sl}(2))$ also gives rise to another Appell sequence. To see this, we will calculate the commutator $[\sigma_\lambda(P),\sigma_\lambda(M)]$ inside the localisation and then investigate its action on Verma modules $M_\mu$ (with $\mu \in \mathbb{C}$ arbitrary). In view of the fact that $(H+\lambda)X = X(H+\lambda+2)$, we get:

$$
[\sigma_\lambda(P),\sigma_\lambda(M)] = \frac{2H}{H+\lambda} + 2Y\left[X,\frac{1}{H+\lambda}\right] = 2\frac{H(H+\lambda+2)+2YX}{(H+\lambda)(H+\lambda+2)}\ .
$$

When acting on an arbitrary weight space in the module $M_\mu$, we thus get:

$$
[\sigma_\lambda(P),\sigma_\lambda(M)]Y^k \otimes v_\mu = 2\frac{(\mu-2k)(\mu+\lambda-2k+2)+2k(\mu+1-k)}{(\mu+\lambda-2k)(\mu+\lambda-2k+2)}Y^k \otimes v_\mu.
$$

It is then easily verified that the Appell condition is verified (i.e. the constant in front of the weight vector is equal to $1$) for $\mu \in \{\lambda, 2-\lambda\}$. This means that we have now obtained the following (somewhat stronger) result:

**Corollary.** *Consider an complex number $\lambda \in \mathbb{C}\setminus\mathbb{Z}$. One can then define*

$$
\sigma_\lambda : \mathfrak{h}_1 \to \mathrm{End}(M_\mu) : (P,M) \mapsto \left(X, \frac{2Y}{H+\lambda}\right)\ ,
$$

*for $\mu \in \{\lambda, 2-\lambda\}$. Both Verma modules $M_\lambda$ and $M_{2-\lambda}$ then become Appell sequences for the operators $\sigma_\lambda(P)$ and $\sigma_\lambda(M)$. Note that the latter operator belongs to $R_\lambda^-$, which means that its action is always well-defined.*

Note that we imposed the condition $\lambda \notin \mathbb{Z}$ in the corollary above, to ensure that both $\lambda, \mu = 2-\lambda \in \mathbb{C}\setminus\mathbb{Z}^+$. Let us then consider a few examples:

1. Consider the classical realisation for $\mathfrak{sl}(2)$ in harmonic analysis on $\mathbb{R}^m$, see (B.1). It is then clear that we can start from an arbitrary harmonic function $f_\alpha(\underline{x})$ on an open subset $\Omega \subset \mathbb{R}^m$ which is homogeneous of degree $\alpha \in \mathbb{C}$. This function then plays the role of a highest weight vector for which $\lambda = -\alpha - \frac{m}{2} \notin \mathbb{Z}^-$, leading to the Appell sequence

$$\mathbb{A}_\lambda = \left\{ \frac{|\underline{x}|^{2k} f_\alpha(\underline{x})}{2^k \left(\alpha + \frac{m}{2}\right)\left(\alpha + \frac{m}{2} + 1\right)\ldots\left(\alpha + \frac{m}{2} + k - 1\right)} \right\},$$

for the lowering operator $P = \frac{1}{2}\Delta_x$.

In case $\alpha + \frac{m}{2} \notin \mathbb{Z}$, we can also consider an Appell sequence starting from $M_\mu$, with $\mu = 2 - \lambda$. In the context of harmonic analysis, there is a well-known realisation for the highest weight vector for $M_\lambda$ in terms of the Kelvin inversion:

$$\mathcal{J}_0 : f_\alpha(\underline{x}) \mapsto \frac{1}{|\underline{x}|^{m-2}} f\left(\frac{\underline{x}}{|\underline{x}|^2}\right) = |\underline{x}|^{2-m-2\alpha} f_\alpha(\underline{x}).$$

This gives rise to the Appel sequence

$$\mathbb{A}_\mu = \left\{ \frac{(-1)^k |\underline{x}|^{2-m-2\alpha+2k} f_\alpha(\underline{x})}{2^k \left(\alpha + \frac{m}{2} - 2\right)\left(\alpha + \frac{m}{2} - 3\right)\ldots\left(\alpha + \frac{m}{2} - k - 1\right)} \right\}.$$

2. In [25], we have obtained the harmonic (resp. monogenic) Gegenbauer polynomials through the knowledge of a particular subalgebra of the Weyl algebra $\mathcal{W}_m$ (resp. $\mathcal{W}_m^C$), for all $m \geq 3$ given by

$$\mathfrak{sl}(2) \cong \mathrm{Alg}\left(-\partial_{x_m}, x_m(2\mathbb{E}_x + m - 2) - r^2\partial_{x_m}, -2\mathbb{E}_x - (m-2)\right).$$

It is then clear that the polynomial set

$$\mathcal{G}_{2-m} = \left\{ (x_m(2\mathbb{E}_x + m - 2) - r^2\partial_{x_m})^k[1] \mid k \in \mathbb{N} \right\}$$

can be considered as a highest-weight Verma module $M_\lambda$ with highest weight vector $1 \in \mathbb{C}$, for $\lambda = -(m-2)$.

## Hermite bases in Verma modules for $\mathfrak{sl}(2)$

One can now develop a general framework to define special polynomials (e.g. Hermite polynomials). Traditionally, such polynomials can be defined through an explicit formula of the form $S_k(z) = \sum_{j=0}^k c_{j,k}(S) z^j$, with $z \in \mathbb{C}$ and $c_{j,k}(S)$ a certain coefficient that determines the special function under consideration. We will generalise this picture, hereby using the following idea: instead of using a complex variable $z$, we will use the operator $\sigma_\lambda(Y)$ which creates the monomial basis for an arbitrary (fixed) Verma module $M_\lambda$, hereby fixing the realization $\mathfrak{sl}(2) = \mathrm{Alg}(X, Y, H)$. For example, the Hermite basis for the Verma module $M_\lambda$ is then defined through the repeated action of the following operators in $\mathcal{U}_\lambda(\mathfrak{sl}(2))$:

$$\sigma_\lambda^{(h)}(P) = X \quad \text{and} \quad \sigma_\lambda^{(h)}(M) = \sigma_\lambda(M - P) = \frac{2Y}{H + \lambda} - X.$$

Note that we have added a superscript $(h)$ to indicate that these generate the Hermite basis, corresponding to the *probabilists' Hermite polynomials*, as opposed to the *physicists' Hermite polynomials* which would require adding a factor 2 to the term $\sigma_\lambda(M)$. The raising operator can also be defined as

$$\sigma_\lambda^{(h)}(M) = -\exp\left(\frac{1}{2}\sigma_\lambda^2(M)\right)\sigma_\lambda(P)\exp\left(-\frac{1}{2}\sigma_\lambda^2(M)\right)\,, \tag{B.2}$$

where the exponential is defined through its formal Taylor expansion.

**Definition.** *Suppose $M_\lambda$ is a Verma module, with $\lambda \in \mathbb{C} \setminus \mathbb{Z}^+$ and weight spaces $Y^k \otimes v_\lambda$ ($k \in \mathbb{Z}^+$). The Hermite basis for $M_\lambda$ is then given by the following set of vectors:*

$$v_\lambda^{(h)}(k) = \left(\frac{2Y}{H + \lambda} - X\right)^k [1 \otimes v_\lambda] \qquad (k \in \mathbb{Z}^+).$$

*As a result of expression (B.2), this can also be written as follows:*

$$v_\lambda^{(h)}(k) = (-1)^k \exp\left(\frac{1}{2}\sigma_\lambda^2(M)\right)\sigma_\lambda^k(P)\exp\left(-\frac{1}{2}\sigma_\lambda^2(M)\right)[1 \otimes v_\lambda].$$

The fact that this defines a basis follows from the following proposition, the essence of which is encoded in a technical lemma:

**Lemma.** *For all $k \in \mathbb{Z}^+$, we have the following expansion of binomial type when acting on the highest weight vector $1 \otimes v_\lambda$:*

$$\begin{aligned}(\sigma_\lambda(M) - \sigma_\lambda(P))^{2k} &= \sum_{j=0}^{k} \frac{(-1)^j k!}{2^j j!(k-2j)!}\sigma_\lambda^{2k-2j}(M)\\ (\sigma_\lambda(M) - \sigma_\lambda(P))^{2k+1} &= \sum_{j=0}^{k} \frac{(-1)^j k!}{2^j j!(k-2j)!}\sigma_\lambda^{1+2k-2j}(M)\end{aligned}$$

*Proof.* The theorem can easily be proved by induction, taking into account that factors $\sigma_\lambda(P)$ may safely be ignored once they are at the right-hand side (in view of the fact that the expression is meant to act on the highest weight vector $1 \otimes v_\lambda \in M_\lambda$). □

**Proposition.** *The explicit expression for the Hermite basis vectors for a Verma module $M_\lambda$ in terms of the monomial basis, is given by:*

$$v_\lambda^{(h)}(k) = \sum_{j=0}^{\kappa} \frac{(-1)^j k!}{2^j j!(k-2j)!(\lambda)_{k-2j}} Y^{k-2j} \otimes v_\lambda,$$

*hereby introducing the integer $\kappa = \lfloor \frac{k}{2} \rfloor \in \mathbb{Z}^+$.*

*Proof.* This immediately follows from the previous lemma, hereby making use of the fact that $\sigma_\lambda(M)$ generates the monomial basis for $M_\lambda$. □

The classical Hermite polynomial $H_k(x)$ in a real variable $x \in \mathbb{R}$, as in the probabilistic normalisation, corresponds to the case where monomial basis vectors $v_\lambda(k) \in M_\lambda$ are identified with monomials $x^k$. Note that the Hermite basis vectors satisfy the following recurrence relations:

$$\begin{aligned} v_\lambda^{(h)}(k+1) &= \frac{1}{(\lambda-k)} Y v_\lambda^{(h)}(k) - X v_\lambda^{(h)}(k) \\ &= \frac{1}{(\lambda-k)} Y v_\lambda^{(h)}(k) - k v_\lambda^{(h)}(k-1) \ , \end{aligned}$$

where we explicitly made use of the fact that the Hermite basis vectors define an Appell sequence. This gives then rise to the following eigenvalue problem for the linear operator $\mathcal{L}_\lambda \in \mathcal{U}_\lambda(\mathfrak{sl}(2))$, which is the equivalent of the Hermite equation in the classical context:

$$\mathcal{L}_\lambda v_\lambda^{(h)}(k) = \left( X^2 - \frac{2Y}{H+\lambda} \right) v_\lambda^{(h)}(k) = -k v_\lambda^{(h)}(k).$$

# Future prospects

The end of this text in no way indicates the end of the research. The theory of partial valuations and partial places is unfortunately still underdeveloped and there can be little doubt that are many statements about valuations can be generalised to this context, quite possibly without too much difficulties. The geometric meaning of partial valuations should also be investigated. There is a possible role for partial valuations in interpreting certain quantum effects; this, too, deserves further study.

The restricted Clifford systems as introduced in 1.1 promise to offer an interesting new perspective in primes extending valuation rings. By our results, these can be separated in two categories: those with an associated value function taking values in a directed group and those consisting of copies of the field stitched together in some way. How exactly this stitching works is still unclear at the moment, but it will probably involve something similar to crossed products. Probably, the restricted Clifford systems will determine this stitching. Clifford systems are epimorphic images of strongly graded rings (cfr. [72]), which suggests that there is a close link with graded valuation theory as well.

In chapter 5 we have introduced some groupoid-graded valuation theory, but there is certainly a lot of work yet to be done here. Extensions of groupoid valuations should be studied in detail. Some kind of divisor theory for (sufficiently well-behaved) groupoid valuations as well as a ramification theory can probably be introduced and studied.

Another topic that still wants exploration is that of crystalline graded (and possibly even crystalline groupoid graded) valuation theory. Crystalline gradings were introduced in [76] as a formal setting for a.o. generalised Weyl algebras and generalised Clifford algebras. Over the years, they have been studied in some detail (cfr. besides [76] e.g. [82] or [79]) but, as far as I am aware, no valuation theory has been introduced in this setting.

In a related topic, one could study (crystalline groupoid) graded pseudo-valuations and their associated filtrations. The following question, for example, seems quite natural: when is a subring of a (crystalline groupoid) graded skewfield the ring of positives for a (crystalline groupoid) graded pseudo-valuation? Seeing that a classification of filtrations associated to pseudo-valuations is — even for very nice rings — rather difficult, this question is probably very hard. Profiltrations and some concept of groupoid-filtrations should also not be ignored.

There are also many questions left open for noetherian Dubrovin valuation rings, their associated arithmetical pseudo-valuations, and the possible geometry of bounded Krull orders. The most obvious one is of course whether some kind of Riemann-Roch theorem holds in this setting. To answer this question, one has to introduce, for a divisor $d$,

$$\mathcal{L}(d) = \{f \in \mathcal{F}(R) \mid \operatorname{div}(f) + d \geq 0\}$$

and, as in [104], consider the quotient $\mathcal{L}(d)/\mathcal{L}(d')$ for some other divisor $d'$ which divides $d$. Its dimension should then correspond to the degree of the divisor $d - d'$. Of course, stating a tentative version of a statement is easy — actually proving it is a different matter.

Something which has barely been touched upon in this thesis is the topological side of valuation theory. Gabriel topologies on Dubrovin valuation rings have been classified (cfr. [60]) and Janesch ([39]) used $V$-topologies to show the non-existence of certain extensions of valuations to Dubrovin valuation rings, but topologies on the space of Dubrovin valuation rings which are localisations of some bounded Krull domain have yet to be investigated. This would be a natural non-commutative analogon of the abstract Riemann surfaces from classical algebraic geometry.

It is also important to construct more examples of (noetherian) Dubrovin valuations, or at least to investigate their (non-)existence in certain classes of skewfields. It would be very interesting, for example, to know whether non-trivial (noetherian) Dubrovin valuation rings exist on the Weyl skewfields.

For other non-commutative generalisations of valuations, there is still more work to be done. Topologies associated to groupoid graded valuation rings and topologies on the space of such generalised valuations for a given simple Artinian ring should be studied. Since we have shown the existence of a link between the world of Dubrovin valuation rings and that of groupoid graded valuations, there is no doubt a topological link as well, but what this could be remains for the moment unclear to me.

Let me end with one final suggestions for future work: a categorification of valuation theory. Category theory has, by now, earned a prominent place in modern mathematics and has proven to be a useful tool with a wide range of applications. It would therefore be both interesting and convenient could some kind of valuation theory be defined on a purely categorical level. To realise this, one should first have a notion of ordered and pre-ordered categories. It is probably necessary to restrict attention to abelian (or at least additive) categories in order to generalise the condition $v(x+y) \geq \min\{v(x), v(y)\}$, but in that context it should be possible to define orderings by using generalisations of the concept of positives cones. After all, a category is nothing more than a semigroupoid with neutral elements — a monoidoid, if you want — and ordered semigroups have been studied extensively (cfr. e.g. [31], [5]). By extending results from this theory to the more general categorical setting, orderings on and positive cones of categories could be introduced. Another possibility is the use of 2-categories, where the ordering corresponds to the 2-morphisms.

# Nederlandse samenvatting

Klassieke valuatietheorie speelt een belangrijke rol in algebraïsche meetkunde, algebraïsche getaltheorie etc., maar het is niet onmiddellijk duidelijk wat de correcte veralgemening naar een niet-commutatieve context is. De voor de hand liggende tegenhanger voor het commutatieve lichaam is de simpele artinse ring, maar dergelijke ringen bevatten nuldelers, hetgeen het bestaan van valuaties onmogelijk maakt. In deze thesis worden een aantal natuurlijke veralgemeningen bestudeerd.

In het eerste hoofdstuk wordt het concept *valuatie* in zijn hoofdbestanddelen ontbonden: enerzijds de totale deelringen en anderzijds de partiële valuaties. Aangezien totale deelringen al tamelijk uitvoerig bestudeerd zijn, concentreren wij ons op de partiële valuaties. Partiële plaatsen worden ook ingevoerd en in verband gebracht met partiële valuaties. Enkele resultaten uit de klassieke valuatietheorie worden veralgemeend naar deze nieuwe (niet-commutatieve) context.

In het tweede hoofdstuk worden priemen bestudeerd. Het was al lang bekend dat aan elke lokale priem $(R, P)$ in een simpele artinse ring $Q$ een partiële valuatie $v$ kan worden geassocieerd, maar de ring van $v$-positieve elementen is in het algemeen een strikte deelring van $R$. Voor het geval van invariante priemen, i.e. priemen die stabiel zijn onder inwendige automorfismen van $Q$, hebben we een aritmetische pseudo-valuaties geconstrueerd waarvoor de ring van positieven wel degelijk terug $R$ is. Deze resultaten veralgemenen gelijkaardige stellingen van Van Geel en Van Oystaeyen en kunnen op hun beurt ongetwijfeld nog veralgemeend worden; het essentiële ingrediënt hierbij is lemma 2.3.4.

Doordat een gelijkaardig lemma geldt voor Dubrovin-valuatieringen met een niet-idempotent Jacobson radicaal — een klasse van ringen met gelijkaardige eigenschappen als klassieke valuatieringen — hebben we (in hoofdstuk drie) ook voor deze ringen een aritmetische pseudo-valuatie kunnen construeren met de correcte ring van positieven. Bovendien neemt deze aritmetische pseudo-valuatie, voor noetherse Dubrovin-valuatieringen, waarden aan in $\mathbb{Z}$. Gebruikmakend van het reeds bekende feit dat lokalisaties van begrensde Krull orders noetherse Dubrovin-valuatieringen zijn, hebben we voor dergelijke ringen een divisorentheorie kunnen invoeren (cfr. hoofdstuk vier). Dit opent nieuwe mogelijkheden, zoals bijvoorbeeld een Riemann-Roch stelling voor begrensde Krull orders.

Een heel andere insteek wordt gevolgd in hoofdstuk vijf. Zoals reeds vermeld zijn simpel artinse ringen in zekere zin de correcte niet-commutatieve veralgemening van lichamen. Dergelijke ringen zijn, dankzij een oude stelling van Wedderburn, noodzakelijk isomorf met matrixringen over scheve lichamen. Aangezien matrixringen groepoïde-gegradeerd zijn, is het logisch om de theorie van gegradeerde valuaties naar deze context te veralgemenen en verder te ontwikkelen. Hiertoe geven we in hoofdstuk vijf een aanzet. Niet alleen worden daar de nodige concepten ingevoerd en een aantal veralgemeningen van klassieke bewijzen gegeven, maar er wordt ook een link gelegd met andere veralgemeende valuaties — met name Dubrovin-valuatieringen.

In hoofdstuk zes wordt een andere interessante invalshoek belicht. Aan elke valuatie kan een (gesepareerde, exhaustieve) filtratie worden geassocieerd (die bovendien voldoet aan $F_\gamma R \subsetneq F_\delta R$ voor $\gamma < \delta$), dus het is logisch om deze filtraties te beschouwen als veralgemeende valuaties. Aan dergelijke filtraties kan een canonieke pseudo-valuaties worden geassocieerd met $F_0 R$ als ring van positieve elementen. Helaas lijkt het karakteriseren van die deelringen die pseudo-valuatieringen zijn in het algemeen een tamelijk wild probleem. Voor reguliere pseudo-valuaties is de kans op een volledige classificatie groter — zeker als alleen deelringen van simpele artinse ringen worden beschouwd — maar hier wordt op het moment nog aan gewerkt.

Zoals bij elk wiskundig onderzoek zijn ook bij deze thesis een aantal pistes op niets uitgedraaid. Ik zou nog verder willen gaan: het meeste werk dat gedaan is, heeft uiteindelijk niets opgeleverd. Een aantal denkpistes die wel interessante resultaten gaven maar die wat verder van de rest van de thesis afstaan, zijn verzameld in de appendices. In appendix A wordt een vroeg en enigszins naïef vermoeden van mij, namelijk dat totale deelringen misschien in $1-1$-verband zouden kunnen staan met rechts valuaties, de kop ingedrukt. Appendix B is het resultaat van een nevenproject over een onderwerp dat in se weinig te maken heeft met (niet-commutatieve) valuatietheorie. De bekomen resultaten vallen volledig binnen het domein van de analyse in plaats van de algebra, maar tonen de kracht van ringtheorie. Met name de localisatietheorie van Ore is nodig om operatoren uit $S_\alpha^-$ te inverteren, waarop de constructie van de Appell-sequences berust.

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So I have come to end of this thesis while I feel like I am only now getting started. I have spent nearly six years in this familiar (and slightly creaky) swivelling chair with a view over the same familiar oak (although the pigeon sitting in it might be a different one) and, behind it, the same familiar cemetery (with, presumably, the same occupants). I have learned a lot in those six years, mainly thanks to other people.

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