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The Wijsman and Attouch–Wets topologies on hyperspaces revisited

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Abstract

In this paper we will prove that, for an arbitrary metric space X and a fairly arbitrary collection Σ of subsets of X, it is possible to endow the hyperspace $\mathrm{CL}(X)$ of all nonempty closed subsets of X (to be identified with their distance functionals) with a canonical distance function having the topology of uniform convergence on members of Σ as topological coreflection and the Hausdorff metric as metric coreflection. For particular choices of Σ, we obtain canonical distance functions overlying the Wijsman and Attouch–Wets topologies. Consequently we apply the general theory of spaces endowed with a distance function and compare the results with those obtained for the classical hyperspace topologies. In all cases we are able to prove results which are both stronger and more general than the classical ones.

Keywords: Wijsman topology; Attouch–Wets topology; Hausdorff metric; Approach space; Distance; Coreflection; Initial structures; Measure of compactness; Completeness

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1. Introduction

In the literature a large variety of topologies, uniformities and metrics have been considered on hyperspaces of topological, uniform or metric spaces. Especially in the case of metric spaces an impressive amount of work has been done in recent years by Attouch, Azé, Beer, Borwein, Cornet, Costantini, DiConcilio, DiMaio, Dolecki, Greco, Lechicki, Levi, Lucchetti, Naimpally, Pai, Wets and Zieminska, among others, in, e.g., [2–4,6–21,23]. In the case of metric spaces, in particular the so-called Wijsman

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topology and Attouch–Wets topology on the hyperspace of all nonempty closed subsets have been extensively studied. A remarkable fact about the Wijsman and the Attouch–Wets topology (and several other hyperspace topologies which we shall consider in future work) is that they are metric-dependent. Note that the Wijsman and the Attouch–Wets topology are closely related since, identifying sets with their distance-functionals, the Wijsman topology corresponds to the topology of pointwise convergence, and the Attouch–Wets topology corresponds to the topology of uniform convergence on bounded subsets. Another remarkable fact, which although evident may not be so apparent, is that both the Wijsman topology and the Attouch–Wets topology are "only" topologies. By this we mean that, as far as the structures are concerned, the transition from space to hyperspace goes parallel with a downward step in the hierarchy of structures (metric $\rightarrow$ uniformity $\rightarrow$ topology), and thus involves considerable loss of information. Even when these topologies are metrizable, compatible hyperspace metrics used are often defined in an ad-hoc way. This is noteworthy since in hyperspace structures such as the Vietoris topology and the Hausdorff metric this downward step does not occur. See, e.g., the work by Michael [30]. The reason for this phenomenon is that metric spaces are extremely unstable under constructions which need to preserve the underlying topologies. There is, in general, no product metric compatible with the product topology, no quotient metric compatible with the quotient topology and no sum metric compatible with the sum topology. Consequently many constructions, not only in hyperspace theory, but also in other areas of mathematics, although starting with metrics, out of necessity lead to topologies rather than to metrics.

A general solution to this problem was presented by E. Lowen and R. Lowen in [24,25] and by R. Lowen in [27]), where the concept of distances (or equivalently of approach structures) was introduced. Distances represent precisely what can be preserved from metrics if one performs the above mentioned constructions while maintaining compatibility with the underlying topologies.

In this paper we propose the use of such distances on hyperspaces of metric spaces. First we put both the Wijsman and the Attouch–Wets topologies in a common topological framework. This framework already shows that there are many other natural hyperspace topologies for metric spaces. Next we demonstrate that those topologies are canonically "distancizable".

Once we have fixed the framework we first prove a number of fundamental structural results showing the soundness of the set-up, and then we study completeness and compactness in our framework. We are able to prove general theorems which are richer and require less conditions than classical theorems. Moreover many classical results can be obtained as corollaries.

2. Preliminaries

In this section we recall some of the basic concepts of approach spaces which we shall be using throughout this paper. For more detailed information we refer the reader to E. Lowen and R. Lowen [24,25], R. Lowen [26,27] and R. Lowen and Robeys [28,29].

Given a set X we use the notations 2^X (respectively 2_0^X, $2^{(X)}$, $2_0^{(X)}$) to denote the set of all subsets (respectively nonempty subsets, finite subsets, nonempty finite subsets) of X. We shall require three different types of structures which are linked to each other: a distance between points and sets, systems of local distances, and a limit operator.

Let X be an arbitrary set, and consider a function $\delta : X \times 2^X \to [0,\infty]$. Given a subset $A \subset X$ and a positive real number $\varepsilon \in \mathbb{R}^+$, we denote $A^{(\varepsilon)} \doteqdot \{x \in X \mid \delta(x,A) \leqslant \varepsilon\}$.

The function

$$\delta : X \times 2^X \to [0,\infty]$$

is called a *distance* (on X) if it satisfies the following properties:

(D1) $\forall x \in X$: $\delta(x,\{x\}) = 0$,

(D2) $\forall x \in X$: $\delta(x,\emptyset) = \infty$,

(D3) $\forall x \in X$, $\forall A, B \in 2^X$: $\delta(x, A \cup B) = \delta(x,A) \wedge \delta(x,B)$,

(D4) $\forall x \in X$, $\forall A \in 2^X, \forall \varepsilon \in \mathbb{R}^+$: $\delta(x,A) \leqslant \delta(x, A^{(\varepsilon)}) + \varepsilon$.

A collection of order theoretic ideals (or dual filters) $\mathcal{A} \doteqdot (\mathcal{A}(x))_{x\in X}$ in $[0,\infty]^X$ is called an *approach system* (on X) if for all $x \in X$ the following properties are satisfied:

(A1) $\forall \varphi \in \mathcal{A}(x)$: $\varphi(x) = 0$,

(A2) $\forall \varphi \in [0,\infty]^X$:

$$\forall N \in \mathbb{R}^+,\ \forall \varepsilon \in \mathbb{R}_0^+\colon\ \exists \varphi_\varepsilon^N \in \mathcal{A}(x)\colon\ \varphi \wedge N \leqslant \varphi_\varepsilon^N + \varepsilon \Rightarrow \varphi \in \mathcal{A}(x),$$

(A3) $\forall \varphi \in \mathcal{A}(x)$, $\forall N \in \mathbb{R}^+$: $\exists (\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that

$$\forall z, y \in X\colon\ \varphi(y) \wedge N \leqslant \varphi_x(z) + \varphi_z(y).$$

The elements of $\mathcal{A}(x)$ are called *local distances*. Given $\varphi \in \mathcal{A}(x)$, the value $\varphi(y)$ in a point $y \in X$, is to be interpreted as "the distance from x to y according to φ". Often we are confronted not with an approach system but with a *basis for an approach system*, which means that (A2) need not be fulfilled. The approach system derived is then obtained by adding all functions which fulfil the condition stated in (A2).

Distances and approach systems are equivalent concepts. If one has a distance, then associated with it there is a unique approach system, and vice-versa. A detailed proof of these facts is given by R. Lowen [27]. We shall here restrict ourselves to giving the formulas for going from one system to the other.

If δ is a distance then the associated approach system is given by

$$\mathcal{A}(x) \doteqdot \Big\{\varphi \in [0,\infty]^X \;\Big|\; \forall A \subset X\colon \inf_{y\in A} \varphi(y) \leqslant \delta(x,A)\Big\}, \quad x \in X.$$

Conversely, if $(\mathcal{A}(x))_{x\in X}$ is an approach system or a basis for an approach system, then the associated distance is given by

$$\delta(x,A) \doteqdot \sup_{\varphi\in\mathcal{A}(x)} \inf_{y\in A} \varphi(y), \quad x \in X,\ A \subset X.$$

In order not to overload the notation, unless there is possibility of confusion, we shall neither make reference to the distance giving rise to an approach system nor to the approach system giving rise to a distance. A set equipped with an approach system (or equivalently with a distance) is called an *approach space*.

As is the case for topological spaces, it is possible to define a notion of convergence in approach spaces. This can be done in a canonical way starting both from a distance and from an approach system.

Let (X,δ) be an approach space, and let $\mathcal{F}$ be a filter on X. Starting with primitive notion a distance, we define the function

$$\lambda\mathcal{F}: X \to [0,\infty]: \quad x \mapsto \sup_{U\in \sec \mathcal{F}} \delta(x,U),$$

where $\sec \mathcal{F}$ denotes the union of all ultrafilters finer than $\mathcal{F}$.

An alternative formula for $\lambda\mathcal{F}$, starting with primitive notion an approach system (or a basis for an approach system), is given as follows:

$$\lambda\mathcal{F}(x) = \sup_{\varphi\in\mathcal{A}(x)} \inf_{F\in\mathcal{F}} \sup_{y\in F} \varphi(y), \quad \forall x\in X.$$

Again we refer to, e.g., (E. Lowen and R. Lowen [24,26] and R. Lowen [27]) for proofs of the equivalence of these formulas and for more details. The interpretation is as follows. For each filter $\mathcal{F}$ and each point $x \in X$, the value $\lambda\mathcal{F}(x)$ indicates how far the point x is away from being a limit point of the filter $\mathcal{F}$.

If (X,δ) and (X',δ') are approach spaces then a function $f: X \to X'$ is called a *contraction* if

$$\forall x\in X,\ \forall A\in 2^X: \quad \delta'\big(f(x), f(A)\big) \leqslant \delta(x,A)$$

or equivalently in terms of the approach systems if

$$\forall x\in X,\ \forall \varphi'\in\mathcal{A}'(f(x)): \quad \varphi'\circ f\in\mathcal{A}(x).$$

Approach spaces and contractions form the objects and the morphisms of a topological category which we denote **AP**. For basic results on topological categories we refer to (Adamek, Herrlich and Strecker [1]). For a proof of the fact that **AP** is indeed topological we refer to (R. Lowen [27]). The most basic property of topological categories is the existence of initial structures. We recall how initial structures are constructed in **AP**.

Theorem 2.1. *Let X be a set and let J be a class. For each $j\in J$ let $(X_j,\mathcal{A}_j)$ be an approach space with approach system $\mathcal{A}_j \doteqdot (\mathcal{A}_j(x))_{x\in X}$, and consider the source*

$$\big(f_j: X \to (X_j,\mathcal{A}_j)\big)_{j\in J}.$$

Then for each $x\in X$ the set

$$\mathcal{B}(x) \doteqdot \Big\{ \sup_{j\in K} \xi_j\circ f_j \ \Big|\ K\subset J \text{ finite},\ \forall j\in K\colon\ \xi_j\in\mathcal{A}_j\big(f_j(x)\big)\Big\}$$

is a basis for the initial approach system on X. More precisely the initial distance is given by

$$\delta(x,A) = \sup_{K\in 2^{(J)}} \ \sup_{(\xi_j)_j\in\prod_{j\in K}\mathcal{A}_j(f_j(x))} \ \inf_{a\in A} \sup_{j\in K} \xi_j(f_j(a)). \qquad \square$$

We now briefly recall how both topological spaces and metric spaces can be viewed as approach spaces.

Given a topological space $(X,\mathcal{T})$, we can associate with it a unique approach space in the following way. We define the function

$$\delta_{\mathcal{T}}: X \times 2^X \to [0,\infty]: \quad (x,A) \mapsto \begin{cases} 0 & x \in \mathrm{cl}_{\mathcal{T}}(A), \\ \infty & x \notin \mathrm{cl}_{\mathcal{T}}(A). \end{cases}$$

Then $\delta_{\mathcal{T}}$ is a distance. Moreover if $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ are topological spaces then a function $f: X \to X'$ is continuous, when considered as a map between the topological spaces, if and only if it is a contraction, when considered as a map between the associated approach spaces. This implies that the category **TOP** of topological spaces is embedded as a full subcategory of the category of approach spaces.

The approach system associated with $\delta_{\mathcal{T}}$ is given by

$$\mathcal{A}_{\mathcal{T}}(x) \doteqdot \{\varphi \in [0,\infty]^X \mid \varphi(x) = 0,\ \varphi \text{ u.s.c. in } x\}, \quad \forall x \in X,$$

and a basis for this system is given by

$$\mathcal{B}_{\mathcal{T}}(x) \doteqdot \{\theta_V \mid V \in \mathcal{N}_{\mathcal{T}}(x)\}, \quad \forall x \in X,$$

where $\mathcal{N}_{\mathcal{T}}(x)$ stands for the neighborhood filter of x in the topological space $(X,\mathcal{T})$.

Here, for any subset A of X, θ_A stands for the indicator of A, i.e., the function attaining the value 0 in the points of A and the value ∞ outside of A.

The limit operator of $(X,\delta_{\mathcal{T}})$ is given by

$$\lambda_{\mathcal{T}}\mathcal{F}(x) = \begin{cases} 0, & \mathcal{F} \xrightarrow{\mathcal{T}} x, \\ \infty, & \mathcal{F} \underset{\mathcal{T}}{\not\to} x, \end{cases} \qquad \mathcal{F} \in \mathbf{F}(X),\ x \in X$$

where $\mathbf{F}(X)$ denotes the set of all filters on X.

Similarly, given an ∞p-metric space (X,d) (where ∞ means the value ∞ is allowed, and p, standing for pseudo, means the underlying topology need not be Hausdorff), we can associate with it a unique approach space in the following way. We define the function

$$\delta_d: X \times 2^X \to [0,\infty]: \quad (x,A) \mapsto \inf_{a \in A} d(x,a).$$

Then again δ_d is a distance on X. Again too if (X,d) and (X',d') are extended pseudometric spaces, then a function $f: X \to X'$ is nonexpansive, when considered as a map between the extended pseudometric spaces if and only if it is a contraction, when considered as a map between the associated approach spaces. This implies that the category $p\mathbf{MET}^{\infty}$ of extended pseudometric spaces is embedded as a full subcategory of the category **AP** of approach spaces.

The approach system associated with (X,δ_d) is given by

$$\mathcal{A}_{\delta_d}(x) \doteqdot \{\varphi \in [0,\infty]^X \mid \varphi \leqslant d(x,\cdot)\}, \quad \forall x \in X,$$

and a basis for this system is given by

$$\mathcal{B}_{\delta_d}(x) \doteqdot \{d(x,\cdot)\}, \quad \forall x \in X.$$

The limit operator of (X,δ_d) is given by

$$\lambda\mathcal{F}(x) = \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x,y), \quad \mathcal{F} \in \mathbf{F}(X),\ x \in X.$$

The embeddings of the categories **TOP** and p**MET**$^\infty$ in **AP** are quite well-behaved as described by our next result.

Theorem 2.2. **TOP** *and* p**MET**$^\infty$ *are embedded as coreflective subcategories of* **AP**. *Precisely if* (X,δ) *is an approach space then its topological coreflection is* $(X,\mathcal{T}_\delta)$ *where the closure operator of* $\mathcal{T}_\delta$ *is given by*

$$\mathrm{cl}_\delta(A)=\{x\in X \mid \delta(x,A)=0\},\quad A\subset X.$$

If $\mathcal{A}$ *is the approach system (or a basis for the approach system) of* (X,δ) *then the neighborhood system (or a basis for the neighborhood system) of* $\mathcal{T}_\delta$ *is given by*

$$\mathcal{N}_{\mathcal{T}_\delta}(x)\doteq\{\{\varphi<\varepsilon\}\mid \varphi\in\mathcal{A}(x),\ \varepsilon>0\},\quad x\in X.$$

The ∞p*-metric coreflection is* (X,d_δ) *where*

$$d_\delta(x,y)=\delta(x,\{y\})\vee\delta(y,\{x\}),\quad x,y\in X.$$

This theorem means that for each approach space (X,δ) there exist both a topological space $(X,\mathcal{T}_\delta)$ and an ∞p-metric space (X,d_δ) such that (1) $\mathcal{T}_\delta$ is the coarsest topology on X finer than δ and (2) d_δ is the coarsest ∞p-metric on X finer than δ. Loosely speaking $\mathcal{T}_\delta$ is the topology underlying δ, and d_δ is the ∞p-metric underlying δ. "Underlying" should be interpreted in the same way that we say a topology is underlying a metric. This is confirmed, e.g., by the fact that the topological coreflection of a metric approach space is indeed the topological space underlying the metric space.

The interpretation to give to the pair consisting of an approach space (X,δ) and its topological coreflection $(X,\mathcal{T}_\delta)$ is that the approach space is endowed with a distance which gives "metric-like" structure to X, and that $\mathcal{T}_\delta$ simply is the topology generated by that metric-like structure in the same way that a metric generates a topology.

3. The "distance of Σ-uniform convergence"

Let (X,d) be a metric space. $\mathrm{CL}(X)$ stands for the set of all nonempty closed subsets of X, called the hyperspace of X.

We begin by recalling the definitions of the Hausdorff metric and of the Wijsman, respectively the Attouch–Wets topology.

The Hausdorff metric on $\mathrm{CL}(X)$ can be defined in several equivalent ways. We will mention two characterizations, especially the first one of which is important for us. We denote this metric by h_d. It is given by the following formulas (see, e.g., Beer [9,11] for a proof). For any $A,B\in\mathrm{CL}(X)$

$$h_d(A,B)=\sup_{x\in X}|d(x,A)-d(x,B)|=\max\{e_d(A,B),e_d(B,A)\},$$

where

$$e_d(A,B)\doteq\sup_{x\in A}d(x,B)$$

is called the "Hausdorff excess of A over B".

The Wijsman topology, which will be denoted by $\mathcal{T}_{W_d}$ too can be introduced in several equivalent ways. Two of these are important for our considerations (see, e.g., Beer [10]).

First, it is the initial topology on $\mathrm{CL}(X)$ for the source

$$\big(\mathrm{CL}(X) \to \mathbb{R}^+\colon \quad A \mapsto d(x, A)\big)_{x \in X},$$

where $\mathbb{R}^+$ is equipped with the usual Euclidean topology.

Second it can also be characterized as being the initial topology for the source

$$\mathrm{CL}(X) \to (\mathbb{R}^+)^X\colon \quad A \mapsto d(\,\cdot\,, A),$$

where $\mathbb{R}^+$ is again equipped with the usual Euclidean topology and $(\mathbb{R}^+)^X$ is equipped with the product topology. In both cases the usual topology on the real line plays a crucial role and the Wijsman topology thanks its existence to the fact that we are able to construct initial topologies. In the first method we are simply constructing the Wijsman topology as an initial topology of the usual topology on $\mathbb{R}^+$ for a collection of maps, and in the second method, we are actually identifying $\mathrm{CL}(X)$ with the set of all distance functionals $\{d(\,\cdot\,, A) \mid A \in \mathrm{CL}(X)\}$ and we construct the Wijsman topology as a subspace topology of a product (or pointwise) topology.

If however we start, not with the usual topology on $\mathbb{R}^+$ but with the usual metric, then classically, we are unable to perform either of these constructions.

The same problems arise in an even more general setting and it is in this setting that we shall solve them. Let Σ be a set of subsets of X which satisfies the following natural conditions:

(T0) *$\Sigma \neq \emptyset$ and $\emptyset \notin \Sigma$.*

(T1) *Σ covers X.*

(T2) *Σ is closed under the formation of finite unions.*

Such a collection Σ of subsets will be called a "tiling of X". Then if we put

$$\mathcal{C}_{\Sigma,d}(A) \doteqdot \Big\{\Big\{B \in \mathrm{CL}(X) \mid \sup_{x \in F} |d(x, A) - d(x, B)| < \varepsilon\Big\} \mid F \in \Sigma,\ \varepsilon > 0\Big\},$$
$$A \in \mathrm{CL}(X)$$

$(\mathcal{C}_{\Sigma,d}(A))_{A \in \mathrm{CL}(X)}$ is a base for a topology on $\mathrm{CL}(X)$, which we will denote by $\mathcal{T}_{\Sigma,d}$ and which will be called the "topology of Σ-uniform convergence (with respect to the metric d)". It is easily seen that $\mathcal{T}_{\Sigma,d}$ corresponds to the topology of uniform convergence on members of Σ, always under the identification of $\mathrm{CL}(X)$ with

$$\big\{d(\,\cdot\,, A) \mid A \in \mathrm{CL}(X)\big\}.$$

If $\Sigma \doteqdot \{X\}$ or 2_0^X we have $\mathcal{T}_{\Sigma,d} = \mathcal{T}_{h_d}$, if $\Sigma \doteqdot 2_0^{(X)}$ we obtain $\mathcal{T}_{\Sigma,d} = \mathcal{T}_{W_d}$ and if $\Sigma \doteqdot \{B \in 2_0^X \mid B \text{ bounded}\}$, $\mathcal{T}_{\Sigma,d}$ coincides with the Attouch–Wets topology denoted by $\mathcal{T}_{AW_d}$.

Now let Σ be a tiling of X, for each $F \in \Sigma$ define

$$d_F : \mathrm{CL}(X) \times \mathrm{CL}(X) \to [0, \infty]$$

by

$$d_F(A, B) \doteqdot \sup_{x \in F} \big|d(x, A) - d(x, B)\big|,$$

and consider the collection

$$\mathcal{D}_{\Sigma,d} \doteqdot \{d_F \mid F \in \Sigma\}.$$

Then clearly $\mathcal{D}_{\Sigma,d}$ is a collection of ∞p-metrics on $\mathrm{CL}(X)$ which is closed under the formation of finite suprema. As proved by R. Lowen and Robeys [28], such a collection of ∞p-metrics, closed under the formation of finite suprema generates a so-called "uniform approach structure". The following result describes this structure.

Proposition 3.1. *Let (X,d) be a metric space and let Σ be a tiling of X. Then the family $\mathcal{D}_{\Sigma,d}$ generates an approach structure on $\mathrm{CL}(X)$ with basis for the approach system*

$$\mathcal{B}_{\Sigma,d}(A) \doteqdot \{d_F(A,\cdot\,) \mid F \in \Sigma\}, \quad A \in \mathrm{CL}(X),$$

and distance

$$\delta_{\Sigma,d} : \mathrm{CL}(X) \times 2^{\mathrm{CL}(X)} \to [0,\infty]: \quad (A,\mathcal{A}) \mapsto \sup_{d' \in \mathcal{D}_{\Sigma,d}} \inf_{B \in \mathcal{A}} d'(A,B).$$

We shall refer to this distance as the *distance of Σ-uniform convergence*. If $\Sigma \doteqdot \{X\}$ or 2_0^X, $\delta_{\Sigma,d}$ coincides with the distance δ_{h_d} on $\mathrm{CL}(X)$ derived from the ∞p-metric h_d. If $\Sigma \doteqdot 2_0^{(X)}$ (respectively $\Sigma \doteqdot \{B \in 2_0^X \mid B \text{ bounded}\}$) then $\delta_{\Sigma,d}$ will be called the "Wijsman"- (respectively "Attouch–Wets"-)distance and will be denoted by δ_{W_d} (respectively δ_{AW_d}). Our first task of course is to justify this terminology.

Theorem 3.2. *Let (X,d) be a metric space and let Σ be a tiling of X. Then the topological coreflection of $(\mathrm{CL}(X), \delta_{\Sigma,d})$ is $\mathrm{CL}(X)$ equipped with the topology $\mathcal{T}_{\Sigma,d}$.*

Proof. It follows from Theorem 2.2 and Proposition 3.1 that a basis for the neighborhoods of $A \in \mathrm{CL}(X)$ in the underlying topology of $(\mathrm{CL}(X), \delta_{\Sigma,d})$ is given by the collection

$$\{B \in \mathrm{CL}(X) \mid d_F(A,B) < \varepsilon\}, \quad F \in \Sigma,\ \varepsilon > 0.$$

This however is precisely a basis for the neighborhoods of A in the topology $\mathcal{T}_{\Sigma,d}$. □

Corollary 3.3. *Let (X,d) be a metric space. Then the topological coreflection of $(\mathrm{CL}(X), \delta_{W_d})$ (respectively $(\mathrm{CL}(X), \delta_{AW_d})$) is $\mathrm{CL}(X)$ equipped with the Wijsman topology $\mathcal{T}_{W_d}$ (respectively the Attouch–Wets topology $\mathcal{T}_{AW_d}$).*

This corollary together with the construction of $\delta_{\Sigma,d}$ shows that δ_{W_d} and δ_{AW_d} canonically "distancize" the Wijsman and Attouch–Wets topologies.

Theorem 3.4. *Let (X,d) be a metric space and let Σ be a tiling of X. Then the ∞p-metric coreflection of $(\mathrm{CL}(X), \delta_{\Sigma,d})$ is $\mathrm{CL}(X)$ equipped with the Hausdorff metric h_d.*

Proof. It follows from Theorem 2.2 and Proposition 3.1 that for all $A, B \in \mathrm{CL}(X)$ we have

$$\begin{aligned} d_{\delta_{\Sigma,d}}(A,B) &= \delta_{\Sigma,d}\big(A,\{B\}\big) \vee \delta_{\Sigma,d}\big(B,\{A\}\big) \\ &= \sup_{F\in\Sigma} d_F(A,B) \\ &= \sup_{x\in\bigcup\Sigma} \big|d(x,A) - d(x,B)\big| \\ &= h_d(A,B), \end{aligned}$$

where the last step is valid since Σ covers X. $\square$

A natural saturation condition for tilings will allow us to compare the different hyperspace distances introduced. At the same time we will prove that for any tiling of X, there exists a canonically associated largest tiling of X which generates the same hyperspace distance with respect to d.

Definition 3.5. Let (X,d) be a metric space and let Σ be a tiling of X. Then we call Σ (d-)saturated iff it fulfils the following condition:

$$\forall F \in 2_0^X\colon\ \big(\forall \varepsilon > 0\ \exists G \in \Sigma\colon\ F \subset G^{(\varepsilon)}\big) \Rightarrow F \in \Sigma. \tag{Ts}$$

Proposition 3.6. *If (X,d) is a metric space and Σ is a d-saturated tiling of X, then Σ is closed under the formation of closures and nonempty subsets of its members.* $\square$

Proposition 3.7. *Let (X,d) be a metric space and let Σ be a tiling of X. If we define*

$$\Sigma_d^* \doteqdot \big\{F \in 2_0^X \mid \forall \varepsilon > 0\ \exists G \in \Sigma\colon\ F \subset G^{(\varepsilon)}\big\},$$

then Σ_d^* is a d-saturated tiling of X such that $\Sigma_d^* \supset \Sigma$ and $\delta_{\Sigma_d^*,d} = \delta_{\Sigma,d}$.

Proof. It is obvious that $\Sigma_d^* \supset \Sigma$ and because Σ fulfils (T0) and (T1) it automatically follows that Σ_d^* does too. To prove that Σ_d^* fulfils (T2), take $F, G \in \Sigma_d^*$ and fix $\varepsilon > 0$. Then there exist $H, K \in \Sigma$ such that $F \subset H^{(\varepsilon)}$ and $G \subset K^{(\varepsilon)}$, whence

$$F \cup G \subset H^{(\varepsilon)} \cup K^{(\varepsilon)} \subset (H \cup K)^{(\varepsilon)}.$$

Because Σ satisfies (T2) we know that $H \cup K \in \Sigma$ which by arbitrariness of ε shows that $F \cup G \in \Sigma_d^*$. From the definition of Σ_d^* it is clear that Σ_d^* is d-saturated, so only the equality $\delta_{\Sigma_d^*,d} = \delta_{\Sigma,d}$ remains to be proved. On the one hand, $\delta_{\Sigma_d^*,d} \geqslant \delta_{\Sigma,d}$ follows from the fact that $\Sigma_d^* \supset \Sigma$. On the other hand, $\delta_{\Sigma_d^*,d} \leqslant \delta_{\Sigma,d}$ follows from the fact that for any $\varepsilon > 0$ and $H \in \Sigma_d^*$, taking $G \in \Sigma$ such that $H \subset G^{(\varepsilon)}$, we have

$$d_H \leqslant d_G + 3\varepsilon. \qquad \square$$

Theorem 3.8. *Let (X,d) be a metric space and let Σ_1, Σ_2 be tilings of X. Then the following statements are equivalent:*

(1) $\delta_{\Sigma_1,d} \leqslant \delta_{\Sigma_2,d}$,

(2) $\mathcal{T}_{\Sigma_1,d} \subset \mathcal{T}_{\Sigma_2,d}$,
(3) $\Sigma_1 \subset (\Sigma_2)^*_d$,
(4) $(\Sigma_1)^*_d \subset (\Sigma_2)^*_d$.

Proof. It is obvious that (1) implies (2). To prove that (2) implies (3), first note that for $i \in \{1,2\}$ basic neighborhoods of X in $\mathcal{T}_{\Sigma_i,d}$ are given by

$$\Gamma_i(F,\varepsilon) \doteq \left\{B \in \mathrm{CL}(X) \mid d_F(X,B) = \sup_{x\in F} d(x,B) < \varepsilon\right\}, \quad F \in \Sigma_i,\ \varepsilon > 0.$$

Therefore it follows from (2) that for any given $F \in \Sigma_1$ and $\varepsilon > 0$ there exist $G \in \Sigma_2$ and $\gamma > 0$ such that $\Gamma_2(G,\gamma) \subset \Gamma_1(F,\varepsilon)$. Consequently $\overline{G} \in \Gamma_1(F,\varepsilon)$ which implies that

$$\sup_{x\in F} d(x,G) < \varepsilon,$$

and thus $F \subset G^{(\varepsilon)}$. This proves that $\Sigma_1 \subset (\Sigma_2)^*_d$. The fact that (3) implies (4) is also clear since $(\Sigma_2)^*_d$ is d-saturated. To show that (4) implies (1), note that it follows from Proposition 3.7 that

$$\delta_{\Sigma_1,d} = \delta_{(\Sigma_1)^*_d,d} \leqslant \delta_{(\Sigma_2)^*_d,d} = \delta_{\Sigma_2,d}. \qquad \square$$

Corollary 3.9. *Let (X,d) be a metric space and let Σ_1, Σ_2 be tilings of X. Then the following statements are equivalent:*
(1) $\delta_{\Sigma_1,d} = \delta_{\Sigma_2,d}$,
(2) $\mathcal{T}_{\Sigma_1,d} = \mathcal{T}_{\Sigma_2,d}$,
(3) $(\Sigma_1)^*_d = (\Sigma_2)^*_d$.

Corollary 3.10. *Let (X,d) be a metric space and let Σ be a tiling of X, then $(\Sigma)^*_d$ is the largest tiling Ω of X such that $\delta_{\Omega,d} = \delta_{\Sigma,d}$.*

Corollary 3.11. *Let (X,d) be a metric space and let Σ be a tiling of X, then the following inequalities always hold:*
(1) $\delta_{\Sigma,d} \leqslant \delta_{h_d}$,
(2) $\delta_{W_d} \leqslant \delta_{\Sigma,d}$.

Corollary 3.12. *For any metric space (X,d) the following statements are equivalent:*
(1) $\delta_{W_d} = \delta_{h_d}$,
(2) *(X,d) is totally bounded.*

Proof. From Corollaries 3.10 and 3.11 it is clear that (1) is equivalent to the inequality $\delta_{h_d} \leqslant \delta_{W_d}$ which, since $\delta_{h_d} = \delta_{\{X\},d}$ and $\delta_{W_d} = \delta_{2_0^{(X)},d}$, is equivalent to following condition:

$$\forall \varepsilon > 0\ \exists F \in 2_0^{(X)}: \quad X \subset F^{(\varepsilon)}.$$

This in turn precisely means that (X,d) is totally bounded. $\square$

Corollary 3.13. *For any metric space (X,d) the following statements are equivalent:*
(1) $\delta_{W_d} = \delta_{AW_d}$,
(2) *every bounded subset of (X,d) is totally bounded.*

Proof. Again, it is clear from Corollaries 3.10 and 3.11 that (1) is equivalent to the following condition:

$$\forall B \in 2_0^X,\ B \text{ bounded},\ \forall \varepsilon > 0\ \exists F \in 2_0^{(X)}:\quad B \subset F^{(\varepsilon)}.$$

This precisely means that every bounded subset is totally bounded. □

Corollary 3.14. *For any metric space (X,d) the following statements are equivalent:*
(1) $\delta_{AW_d} = \delta_{h_d}$,
(2) *(X,d) is bounded.*

Proof. Again, it is clear from Corollaries 3.10 and 3.11 that (1) is equivalent to the following condition:

$$\forall \varepsilon > 0\ \exists B \in 2_0^X,\ B \text{ bounded}:\quad X = B^{(\varepsilon)}.$$

This obviously is equivalent to the boundedness of X. □

Together with Corollary 3.9, Corollary 3.12 implies the well-known topological analogue of Corollary 3.12 which can be found in Beer, Lechicky, Levi and Naimpally [14] or in Beer [11]. Analogously, combining Corollary 3.9 with Corollaries 3.13 and 3.14 allows us to deduce topological analogues concerning the Attouch–Wets topologies, which can be added to the extensive list of similar results in [14], which can also be found in [11].

By Lechicky and Levi [23] it was shown that the Wijsman topology on $\mathrm{CL}(X)$ is metrizable if and only if $(X,\mathcal{T}_d)$ is separable and it is also a well-known fact that the Attouch–Wets topology on $\mathrm{CL}(X)$ always is metrizable (see, e.g., Beer and Di-Concilio [13]). Notice that conditions in our case are not required. $(\mathrm{CL}(X),\mathcal{T}_{W_d})$ (respectively $(\mathrm{CL}(X),\mathcal{T}_{AW_d})$) is always canonically "distancizable" by δ_{W_d} (respectively δ_{AW_d}).

4. Relations between X and $\mathrm{CL}(X)$: completeness and compactness

Before considering properties of X and $\mathrm{CL}(X)$ we first look at some structural relations.

A first relation which we would like to highlight concerns the admissibility (see Michael [30]) of the distances $\delta_{\Sigma,d}$. A topological (or uniform structure) on a hyperspace is called admissible if the map $x \mapsto \{x\}$ is well-defined and an embedding in its respective category. For topological structures this means it has to be a homeomorphism onto the image and for uniform structures this means it has to be a uniform isomorphism onto the image. Analogously we shall say that a distance δ on $\mathrm{CL}(X)$ is *admissible* if

$$(X,\delta_d) \to (\mathrm{CL}(X),\delta),\quad x \mapsto \{x\}$$

is an embedding in **AP**.

Proposition 4.1. *For any metric space (X,d) and any tiling Σ of X, the distance of Σ-uniform convergence on $\mathrm{CL}(X)$ is admissible.*

Proof. Since $(X,\mathcal{T}_d)$ is Hausdorff the function

$$\psi:(X,\delta_d)\to\big(\mathrm{CL}(X),\delta_{\Sigma,d}\big):\quad x\mapsto\{x\}$$

is well-defined. On the other hand ψ clearly is injective, so only initiality remains to be verified. Now if $x\in X$ and $A\subset X$ then by (S1) there exists $F_0\in\Sigma$ with $x\in F_0$, whence on the one hand we have

$$\begin{aligned}\delta_{\Sigma,d}\big(\psi(x),\psi(A)\big)&=\sup_{F\in\Sigma}\inf_{a\in A}d_F\big(\psi(x),\psi(a)\big)\\&=\sup_{F\in\Sigma}\inf_{a\in A}\sup_{y\in F}\big|d(y,x)-d(y,a)\big|\\&\geqslant\inf_{a\in A}\sup_{y\in F_0}\big|d(y,x)-d(y,a)\big|\\&\geqslant\inf_{a\in A}\big|d(x,x)-d(x,a)\big|\\&=\delta_d(x,A),\end{aligned}$$

whereas on the other hand we have

$$\begin{aligned}\delta_{\Sigma,d}\big(\psi(x),\psi(A)\big)&=\sup_{F\in\Sigma}\inf_{a\in A}d_F\big(\psi(x),\psi(a)\big)\\&=\sup_{F\in\Sigma}\inf_{a\in A}\sup_{y\in F}\big|d(y,x)-d(y,a)\big|\\&\leqslant\sup_{F\in\Sigma}\inf_{a\in A}\sup_{y\in F}d(x,a)\\&=\delta_d(x,A).\end{aligned}$$

Consequently ψ is an embedding. □

Corollary 4.2. *For any metric space (X,d) and any tiling Σ of X, the topology of Σ-uniform convergence on $\mathrm{CL}(X)$ is admissible.* □

Upon identifying a set with the set of its singletons and a metric space with its model in **AP**, loosely speaking 4.1 says that (X,d) is a metric subspace of $(\mathrm{CL}(X),\delta_{\Sigma,d})$, or that $\delta_{\Sigma,d}|_{X\times 2^X}=\delta_d$, where Σ is an arbitrary tiling of X. Notice that this implies that the link between the metric on X and the distance of Σ-uniform convergence is much stronger than between the metric on X and the topology of Σ-uniform convergence, where under certain conditions, different metrics on X may generate the same topology of Σ-uniform convergence, as was shown for the Wijsman topology by Costantini, Levi and Zieminska [19] and for the Attouch–Wets topology by Beer and DiConcilio [13]. In the case of distances this is not possible, and the foregoing result actually shows that, for each tiling Σ of X, $d\mapsto\delta_{\Sigma,d}$ is an injection. We also have following results as easy corollaries.

Corollary 4.3. *Let X be a nonempty set, d and ϱ two equivalent metrics on X and Σ_1, Σ_2 two tilings of X. Then the following statements are equivalent:*

(1) $\delta_{\Sigma_1,d} = \delta_{\Sigma_2,\varrho}$,
(2) $d = \varrho$ *and* $(\Sigma_1)^*_d = (\Sigma_2)^*_d$.

Corollary 4.4. *Let X be a nonempty set, d and ϱ two equivalent metrics on X and Σ a tiling of X. Then following statements are equivalent:*
(1) $\delta_{\Sigma,d} = \delta_{\Sigma,\varrho}$,
(2) $d = \varrho$.

Corollary 4.5. *Let X be a nonempty set, d and ϱ two equivalent metrics on X and Σ a tiling of X. Then $\delta_{\Sigma,d} \leqslant \delta_{\Sigma,\varrho}$ implies that $d \leqslant \varrho$.*

Counterexample 4.6. We now provide a counterexample to the converse implication in foregoing theorem. Let $X \doteqdot \mathbb{N}$,

$$d: X \times X \to [0, \infty], \quad (x, y) \mapsto \begin{cases} 0, & x = y, \\ 1, & x \neq y,\ x, y \in \mathbb{N}_0, \\ 2, & \text{otherwise} \end{cases}$$

and

$$\varrho: X \times X \to [0, \infty], \quad (x, y) \mapsto \begin{cases} 0, & x = y, \\ 2, & x \neq y. \end{cases}$$

Then d and ϱ are equivalent metrics on X, both generating the discrete topology on X, whence $\mathrm{CL}(X) = 2^X_0$. Obviously $d \leqslant \varrho$, but taking $A \doteqdot \{0\}$ and

$$\mathcal{A} \doteqdot \{B \in \mathrm{CL}(X) \mid B \supsetneq \{0\}\},$$

we obtain that

$$\delta_{W_d}(A, \mathcal{A}) \geqslant \inf_{B \in \mathcal{A}} d_{\{1\}}(A, B) \geqslant 1 > 0 = \delta_{W_\varrho}(A, \mathcal{A}). \qquad \square$$

We now focus our attention on completeness and compactness. By Beer [9,11] it was shown that if (X, d) is complete and separable then $(\mathrm{CL}(X), \mathcal{T}_{W_d})$ is Polish, i.e., it is separable and completely metrizable, and this result was generalized by Costantini, who proved in [18] that the Wijsman topology for any compatible metric for a Polish space is Polish. This means that among all metrics which metrize $(\mathrm{CL}(X), \mathcal{T}_{W_d})$ a complete one can be found. Concerning the Attouch–Wets topology an analogous result was shown by Attouch, Lucetti and Wets [2]: if (X, d) is complete, then $(\mathrm{CL}(X), \mathcal{T}_{AW_d})$ is completely metrizable. We shall prove that, without any separability conditions and for any tiling Σ of X, if (X, d) is complete, then $(\mathrm{CL}(X), \delta_{\Sigma,d})$ itself is already complete.

In the setting of approach spaces a notion of completeness was introduced by R. Lowen and Robeys [28] which coincides with usual completeness for metric spaces and which is such that every topological space is complete. Recall that in an approach space (X, δ) we have at our disposal a limit operator λ which for any filter $\mathcal{F}$ on X and any point

$x \in X$ tells us how far x is away from being a limit point of $\mathcal{F}$, namely the value $\lambda\mathcal{F}(x)$. The filter $\mathcal{F}$ "really" converges to x if $\lambda\mathcal{F}(x) = 0$, which also means that $\mathcal{F}$ converges to x in the underlying topology. Therefore it is logical to define $\mathcal{F}$ to be a δ-Cauchy filter if $\inf_{x \in X} \lambda\mathcal{F}(x) = 0$. This definition captures the spirit of a Cauchy filter as being a filter with all the properties of a convergent filter, except that the limit point may be missing, i.e., the infimum might not be a minimum. Now a space is said to be complete if for no Cauchy filter this can happen, i.e., for every Cauchy filter there exists a point to which it "really" converges, i.e., if the infimum is always a minimum.

Theorem 4.7. *For any metric space (X, d) and any tiling Σ of X, the following are equivalent:*

(1) *$(\mathrm{CL}(X), \delta_{\Sigma,d})$ is complete.*
(2) *$(\mathrm{CL}(X), h_d)$ is complete.*
(3) *(X, d) is complete.*

Proof. The equivalence of (2) and (3) is well known and can be found, e.g., in (Kuratowski [22]).

To prove that (2) implies (1), let $\mathfrak{F}$ be a $\delta_{\Sigma,d}$-Cauchy filter. This means that

$$\inf_{A \in \mathrm{CL}(X)} \lambda_{\Sigma,d}(\mathfrak{F})(A) = 0,$$

where

$$\lambda_{\Sigma,d}(\mathfrak{F})(A) = \sup_{F \in \Sigma} \inf_{\mathcal{F} \in \mathfrak{F}} \sup_{B \in \mathcal{F}} d_F(A, B).$$

This is equivalent to

$$\forall \varepsilon > 0\ \exists A \in \mathrm{CL}(X)\ \forall F \in \Sigma\colon\ B_{d_F}(A, \varepsilon) \in \mathfrak{F}.$$

From 3.7, 3.8 and 3.11 we may assume without loss of generality that $\{\{x\} \mid x \in X\} \subset \Sigma$, and consequently this in turn implies that

$$\forall x \in X\ \forall \varepsilon > 0\ \exists A \in \mathrm{CL}(X)\colon\ B_{d_{\{x\}}}(A, \varepsilon) \in \mathfrak{F}.$$

Now, for all $x \in X$, we consider the filterbase $\mathfrak{F}(x)$ defined as

$$\mathfrak{F}(x) \doteqdot \{\mathcal{F}(x) \mid \mathcal{F} \in \mathfrak{F}\},$$

where for any $\mathcal{F} \in \mathfrak{F}$, $\mathcal{F}(x) \doteqdot \{\delta_d(x, A) \mid A \in \mathcal{F}\}$, then this implies that for all $x \in X$, $\mathfrak{F}(x)$ is a Cauchy filter base in $\mathbb{R}^+$. Consequently there exists $f \in (\mathbb{R}^+)^X$ such that for all $x \in X$

$$\mathfrak{F}(x) \longrightarrow f(x) \quad \text{in } \mathbb{R}^+.$$

This means that

$$\forall n \in \mathbb{N}_0\ \forall x \in X\ \exists \mathcal{F}_0 \in \mathfrak{F}\colon\quad \mathcal{F}_0(x) \subset B\left(f(x), \frac{1}{2n}\right).$$

Since $\mathfrak{F}$ is moreover a $\delta_{\Sigma,d}$-Cauchy filter we also have that

$$\forall n \in \mathbb{N}_0\ \exists A_n \in \mathrm{CL}(X)\ \forall F \in \Sigma\colon\quad B_{d_F}\left(A_n, \frac{1}{2n}\right) \in \mathfrak{F}.$$

Combining this we deduce that for all $x \in X$ and $n \in \mathbb{N}_0$

$$\left|d(x, A_n) - f(x)\right| < \frac{1}{n},$$

and consequently

$$\left(d(\,\cdot\,, A_n)\right)_n \longrightarrow f \quad \text{uniformly.}$$

Now since $(\mathrm{CL}(X), h_d)$ is isometric with the set of all distance functionals equipped with the supremum metric and since $(\mathrm{CL}(X), h_d)$ is complete this means that there exists $A \in \mathrm{CL}(X)$ such that $f = d(\,\cdot\,, A)$. Since uniform convergence implies $\mathcal{T}_{\Sigma,d}$-convergence we moreover have that $\lambda_{\Sigma,d}(\mathfrak{F})(A) = 0$ and thus it follows that $(\mathrm{CL}(X), \delta_{\Sigma,d})$ is complete.

To prove that (1) implies (2) let $(A_n)_n$ be a Cauchy sequence in $(\mathrm{CL}(X), h_d)$. Then we have that

$$\forall \varepsilon > 0\ \exists n_0\ \forall n \geqslant n_0\colon \quad \sup_{x \in X} \left|d(x, A_n) - d(x, A_{n_0})\right| \leqslant \varepsilon.$$

Now let $\mathfrak{F}$ stand for the filter generated by the sequence $(A_n)_n$, then it follows that

$$\lambda_{\Sigma,d}(\mathfrak{F})(A_{n_0}) = \sup_{F \in \Sigma} \inf_{m} \sup_{k \geqslant m} d_F(A_{n_0}, A_k) \leqslant \varepsilon,$$

and thus $\mathfrak{F}$ is a $\delta_{\Sigma,d}$-Cauchy filter. Since $(\mathrm{CL}(X), \delta_{\Sigma,d})$ is complete there exists $A \in \mathrm{CL}(X)$ such that $\lambda_{\Sigma,d}(\mathfrak{F})(A) = 0$. This implies that for all $x \in X$

$$\limsup_{n \to \infty} \left|d(x, A_n) - d(x, A)\right| = 0,$$

i.e., that for all $x \in X$

$$\left(d(x, A_n)\right)_n \longrightarrow d(x, A).$$

However since $(A_n)_n$ is a Cauchy sequence in $(\mathrm{CL}(X), h_d)$, the corresponding sequence of distance functionals is a uniform Cauchy sequence, which implies that

$$\left(d(\,\cdot\,, A_n)\right)_n \longrightarrow d(\,\cdot\,, A) \quad \text{uniformly.}$$

This proves that $(\mathrm{CL}(X), h_d)$ is complete. □

Given an approach space (X, δ), the following *measure of compactness* of (X, δ) was defined by R. Lowen [26]:

$$\mu_c(X, \delta) \doteqdot \sup_{\mathcal{U} \in \mathbb{U}(X)} \inf_{x \in X} \lambda\mathcal{U}(x).$$

It can be proved that, with $\mathcal{A} \doteqdot (\mathcal{A}(x))_{x \in X}$ the associated approach system and $\mathcal{B} \doteqdot (\mathcal{B}(x))_{x \in X}$ an arbitrary base for $\mathcal{A}$, the measure of compactness can be reformulated in the following way:

$$\begin{aligned}
\mu_c(X, \delta) &= \sup_{\varphi \in \prod_{x \in X} \mathcal{A}(x)} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} \varphi(x)(z) \\
&= \sup_{\psi \in \prod_{x \in X} \mathcal{B}(x)} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} \psi(x)(z).
\end{aligned}$$

Then it was also shown by R. Lowen [26] that for any topological space $(X,\mathcal{T})$, we have that $\mu_c(X,\delta_{\mathcal{T}}) = 0$ iff $(X,\mathcal{T})$ is compact, whereas for any metric space (X,d), we have that $\mu_c(X,\delta_d) = 0$ iff (X,d) is totally bounded. These results indeed indicate that μ_c can be interpreted as a measure of compactness for approach spaces. We conclude this remark by mentioning that for an arbitrary metric space (X,d), $\mu_c(X,\delta_d)$ coincides with the well-known *Haussdorff measure of noncompactness* (see, e.g., Banaś and Goebel [5]), which is given by

$$m_h(X,d) \doteqdot \inf\left\{\varepsilon \in \mathbb{R}^+ \mid \exists x_1,\dots,x_n \in X\colon\ X = \bigcup_{i=1}^{n} B(x_i,\varepsilon)\right\}.$$

Theorem 4.8. *For any metric space (X,d) and any tiling Σ of X, we have following equality:*

$$\mu_c(X,\delta_d) = \mu_c\big(\mathrm{CL}(X),\delta_{\Sigma,d}\big).$$

Proof. First of all we note that

$$\mu_c(X,\delta_d) = \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(x,z)$$

and

$$\mu_c\big(\mathrm{CL}(X),\delta_{\Sigma,d}\big) = \sup_{(G_A)_A\in\Sigma^{\mathrm{CL}(X)}} \inf_{\mathcal{B}\in 2^{(\mathrm{CL}(X))}} \sup_{C\in\mathrm{CL}(X)} \inf_{A\in\mathcal{B}} d_{G_A}(A,C).$$

We start by showing that $\mu_c(X,\delta_d) \leqslant \mu_c(\mathrm{CL}(X),\delta_{\Sigma,d})$. For every $A \in \mathrm{CL}(X)$, fix $x_A \in A$. According to Proposition 3.7, Theorem 3.8 and Corollary 3.11 we can assume without loss of generality that $\Sigma \supset \{\{x\} \mid x\in X\}$. Therefore we define $G_A^0 \doteqdot \{x_A\}$ for every $A\in\mathrm{CL}(X)$. Then if $\mathcal{B}\in 2^{(\mathrm{CL}(X))}$ is arbitrary and we put $Y \doteqdot \{x_A \mid A\in\mathcal{B}\}$, we have

$$\begin{aligned}
\sup_{z\in X}\inf_{x\in Y} d(x,z) &= \sup_{z\in X}\inf_{A\in\mathcal{B}} d\big(x_A,\{z\}\big)\\
&= \sup_{z\in X}\inf_{A\in\mathcal{B}} \big|d(x_A,A) - d\big(x_A,\{z\}\big)\big|\\
&= \sup_{z\in X}\inf_{A\in\mathcal{B}} d_{G_A^0}\big(A,\{z\}\big)\\
&\leqslant \sup_{C\in\mathrm{CL}(X)}\inf_{A\in\mathcal{B}} d_{G_A^0}(A,C).
\end{aligned}$$

To prove that $\mu_c(X,\delta_d) \geqslant \mu_c(\mathrm{CL}(X),\delta_{\Sigma,d})$, first of all note that we have

$$\begin{aligned}
\mu_c\big(\mathrm{CL}(X),\delta_{\Sigma,d}\big) &\leqslant \inf_{\mathcal{B}\in 2^{(\mathrm{CL}(X))}} \sup_{C\in\mathrm{CL}(X)} \inf_{A\in\mathcal{B}} h_d(A,C)\\
&= \mu_c\big(\mathrm{CL}(X),\delta_{h_d}\big).
\end{aligned}$$

Thus it suffices to verify that $\mu_c(X,\delta_d) \geqslant \mu_c(\mathrm{CL}(X),\delta_{h_d})$. Suppose

$$X = \bigcup_{i=1}^{k} B(x_i,\varepsilon).$$

Let $\mathcal{B}$ stand for the set of all nonempty subsets of $\{x_1, \ldots, x_k\}$. Clearly $\mathcal{B} \in 2^{(\mathrm{CL}(X))}$. Now take $A \in \mathrm{CL}(X)$ arbitrary. Then it is obvious that

$$I_A \doteqdot \{i \in \{1, \ldots, k\} \mid A \cap B(x_i, \varepsilon) \neq \emptyset\} \neq \emptyset$$

and that $\bigcup_{i \in I_A} B(x_i, \varepsilon) \supset A$. If we define $B_A \doteqdot \{x_i \mid i \in I_A\}$, then $B_A \in \mathcal{B}$ and

$$h_d(A, B_A) = \Big(\sup_{a \in A} d(a, B_A)\Big) \vee \Big(\sup_{i \in I_A} d(x_i, A)\Big) \leqslant \varepsilon.$$

Consequently

$$\forall n \in \mathbb{N}_0\colon\ \mathrm{CL}(X) = \bigcup_{B \in \mathcal{B}} B_{h_d}\left(B, \varepsilon + \frac{1}{n}\right),$$

from which it immediately follows that

$$\forall n \in \mathbb{N}_0\colon\ \mu_c\big(\mathrm{CL}(X), \delta_{h_d}\big) \leqslant \varepsilon + \frac{1}{n}. \qquad \Box$$

We also obtain the following classical results as simple corollaries: the first one can be found in Kuratowski [22] and the second one is a generalization of analogous results concerning the Hausdorff metric topology (see Kuratowski [22]) and the Wijsman topology (see Lechicki and Levi [23]).

Corollary 4.9 (Kuratowski). *For any metric space (X, d) the following are equivalent:*
(1) *(X, d) is totally bounded.*
(2) *$(\mathrm{CL}(X), h_d)$ is totally bounded.*

Corollary 4.10 (Kuratowski, Lechicki and Levi). *For any metric space (X, d) and any tiling Σ of X the following are equivalent:*
(1) *(X, d) is compact.*
(2) *$(\mathrm{CL}(X), \mathcal{T}_{\Sigma,d})$ is compact.*

Example 4.11. For each $\alpha \in [0, \infty]$ there exists a metric space, the hyperspace of which, endowed with the distance of Σ-uniform convergence for an arbitrary tiling Σ of X, has precisely α as measure of compactness. From Theorem 4.8 it follows that this problem amounts to constructing, for each $\alpha \in [0, \infty]$ a metric space with measure of compactness $\alpha \in [0, \infty]$. For $\alpha = 0$, we can take $(X, d) \doteqdot ([0, 1], d_E)$, whereas for $\alpha = \infty$, $(\mathbb{R}, d_E)$ answers the question, so we only need to consider the case that $\alpha \in]0, \infty[$. If we let $X \doteqdot \mathbb{R}^{\mathbb{N}}$ and

$$d(x, y) \doteqdot \Big(\sup_{n \in \mathbb{N}} d_E(x_n, y_n)\Big) \wedge \alpha, \quad x, y \in \mathbb{R}^{\mathbb{N}},$$

then d is a metric on X with $d \leqslant \alpha$ from which it is clear that $\mu_c(X, \delta_d) \leqslant \alpha$. On the other hand, $\{x^{(n)} \doteqdot (\alpha.\delta_k^n)_{k \in \mathbb{N}} \mid n \in \mathbb{N}\}$ is a countably infinite α-discrete subset of X, which yields that $\mu_c(X, \delta_d) \geqslant \alpha$.

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