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Classification of Strength-Three Multi-Level Orthogonal Arrays

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Abstract

Generalized Aberration (GA) is one of the most frequently used criteria to quantify the suitability of an orthogonal array (OA) to be used as an experimental design. We demonstrate that this criterion is less suitable than other criteria to classify multi-level OAs of strength 3. For this purpose, we classified complete series of three-level strength-3 OAs with 54 and 81 runs using the GA criterion, the estimation capacity, the projection estimation capacity, the rank of the second-order model matrix, the D-efficiency for a model with all the main effects and all the two-factor interactions, and a new model-robustness criterion. For all of the series, we provide a list of admissible designs according to the criteria of interest.

KEY WORDS: experimental design; resolution IV; non-regular design; regular design

1 Introduction

Scientific experiments are frequently conducted according to an orthogonal array (OA). An OA is an $N \times n$ matrix of symbols, whose rows correspond to the N runs of the experiment and whose columns correspond to the n factors studied in the experiment. The number of different symbols in a given column of an OA corresponds to the number of levels of the corresponding factor. Every OA has a certain strength t, which means that, for any t columns, every t-tuple of levels appears equally often in the array, while there is at least one set of $t+1$ columns for which the $(t+1)$-tuples of levels do not occur equally often (Rao, 1947). If all columns of an OA have the same number of levels, say s, then the array is called a symmetrical or pure-level array and denoted by $\mathrm{OA}(N; s^n; t)$. Arrays that have different numbers of levels, say $s_1, \ldots, s_n$, in their n columns are referred to as mixed-level arrays. We refer to Hedayat et al. (1999) for a comprehensive account on properties and explicit constructions of orthogonal arrays.

The numbers t, N, n and $s_1, \ldots, s_n$ are the parameters of an array. For any specific set of parameters, there may be many arrays which can be partitioned in so-called isomorphism classes. All arrays within one isomorphism class can be obtained from each other by a sequence of row permutations, column permutations and/or level permutations. When all the factors are considered as categorical variables, all arrays in the same isomorphism class are mathematically and statistically equivalent. Therefore, for statistical purposes, it is enough to study only one instance of every isomorphism class. This explains why much research has been done to determine minimum complete sets of OAs for given values of the parameters t, N, n and $s_1, \ldots, s_n$. Such sets have a single representative for each isomorphism class.

Minimum complete sets of regular two-level designs of up to 64 runs are given by Chen et al. (1993), whereas minimum complete sets of regular three-level designs of up to 729 runs are given in Xu (2005). Regular designs form a subset of the complete set of OAs. Designs from this subset can be constructed from a set of basic s-level factors, with s a prime number. The settings of additional factors are calculated by modulo-s addition of two or more basic factors specified in so-called generators (see, e.g., Mee, 2009; Wu and Hamada, 2000). However, only a small fraction of all OAs belong to the class of regular designs. Constructing minimum complete sets of OAs is therefore substantially more challenging than constructing minimum complete sets of regular designs. The generation of minimum complete sets which also contain non-regular designs is discussed by Sun et al. (2002) for pure two-level arrays of strength 2, Bulutoglu and Margot (2008) for pure-level OAs in general, and Schoen et al. (2010) for general (pure-level and mixed-level) OAs.

Once a minimum complete set of OAs has been obtained, it is important to rank its non-isomorphic OAs from best to worst. By far the most widely applied criterion to rank non-isomorphic OAs is the generalized aberration (GA) criterion proposed by Xu and Wu (2001) and Ma and Fang (2001). The GA-criterion reduces to the G_2-aberration criterion proposed by Tang and Deng (1999) for two-level OAs, which, in turn, reduces to the aberration criterion developed by Fries and Hunter (1980) for regular two-level designs .

The GA-criterion is based on the entries of the generalized word length vector $W = (A_3, A_4, A_5, \ldots, A_n)$. Roughly speaking, an entry A_p of the generalized word length pattern measures the extent to which effects involving q of the factors (with $q < p$) are confounded with effects involving $p - q$ of the factors. Small A_p values are generally desirable, especially for p as low as 3 and 4, as this implies that there is little confounding between main effects and two-factor interaction effects as well as among the two-factor interactions. An array R_1 is said to have less aberration and, hence, more preferable than an array R_2 if there is a p such that $A_p(R_1) < A_p(R_2)$ and $A_i(R_1) = A_i(R_2)$ for $i = 3, \ldots, p-1$. If there is no design with less aberration than R_1, then R_1 is a minimum generalized aberration (MGA) design.

The statistical justification for using the GA-criterion or the G_2-criterion is based on the expected bias resulting from the omission of higher-order terms in a statistical model containing the main effects. It turns out that, for pure-level designs, the MGA design sequentially minimizes the bias of the main effects' estimates due to interactions, starting with the bias due to two-factor interactions and ending with the bias due to $(n-1)$-factor interactions (Xu and Wu, 2001).

In this paper, we address the issue of classifying OAs of strength 3. While we believe that the GA-criterion is very useful for ranking strength-2 OAs, we find it less suitable for classifying strength-3 arrays, due to the following argument. Interactions involving three or more factors are usually negligible, so that the only practically relevant components of the generalized word length pattern W are A_3 and A_4. Now, strength-3 arrays have the property that $A_3 = 0$, which means that the factor-effect estimates for a model containing only the main effects are not biased by two-factor interactions. This leaves A_4 as the sole practically relevant component of W. Its value measures the total squared correlation among all two-factor interactions (Evangelaras et al., 2005). For regular two-level designs, it is directly related to the number of estimable models (Cheng et al., 1999). However, this is no longer the case when the factors have more than two levels, because two-factor interaction effects involving such factors use at least two degrees of freedom. Therefore, the relationship between the squared correlation and models that are estimable is unclear.

The goal of this paper is, first, to show empirically that the GA criterion is indeed less suitable for the classification of multilevel strength-3 arrays, and, second, to propose more suitable criteria to classify these arrays for statistical purposes. We study in particular how the GA-criterion relates to estimation capacity (Cheng et al., 1999), projection estimation capacity (Loeppky et al., 2007), the rank of the second-order model matrix (Cheng et al., 2008), the D-efficiency for a model with all the main effects and all the two-factor interactions, and a new model-robustness criterion. The usefulness of the different criteria is demonstrated through the classification of complete series of OAs of the type $\mathrm{OA}(N; 3^n; 3)$, with $N = 54$ and $N = 81$. The arrays with $N = 54$ and $n = 5$ were first enumerated by Hedayat et al. (1997), whereas those with $N = 54$ and $n = 4$ were catalogued by Schoen et al. (2010). To the best of our knowledge, the $\mathrm{OA}(81; 3^n; 3)$ family of arrays that we study here is new.

The rest of the paper is structured as follows. In Section 2, we define the classification criteria that we use. In Section 3, we discuss the classification of the three-level OAs of the types $\mathrm{OA}(54; 3^n; 3)$ and $\mathrm{OA}(81; 3^n; 3)$. Finally, in Section 4, we discuss the relationships between the various classification criteria and provide general recommendations for the classification of strength-3 arrays.

2 Classification criteria

In this section, we define in detail the generalized aberration (GA) criterion (Xu and Wu, 2001; Ma and Fang, 2001), the estimation capacity (Cheng et al., 1999), the projection estimation capacity (Loeppky et al., 2007), the rank of the second-order model matrix (Cheng et al., 2008), a criterion based on D-efficiency and a model-robustness criterion that we use to classify strength-3 arrays. All but the last two ranking criteria focus on estimability of effects. The criterion based on D-efficiency focuses on the precision of the effect estimation. The robustness criterion measures the quality of a design if all the runs taken at some level of one of the factors produce bad results and an analysis is attempted without the results from these runs. Finally, to deal with multiple criteria, we use the concept of an admissible design.

2.1 Generalized aberration

The GA-criterion is based on the entries of the generalized word length pattern vector $W = (A_3, A_4, A_5, \ldots, A_n)$, where the entries A_p are obtained using the following procedure. First, replace every s-level column with $s-1$ orthogonal columns. Next, normalize all the columns so that each of them has a squared norm of N. The resulting columns are called main-effect columns. Then, define $\mathbf{x}_k^{(p)}$ as a column vector obtained by multiplying a particular set of p main-effect columns in an element-wise fashion, and $x_{ik}^{(p)}$ as its ith element. The A_p value is then given by

$$A_p = \frac{\sum_{k=1}^{n_p} \left| \sum_{i=1}^{N} x_{ik}^{(p)} \right|^2}{N^2},$$

where n_p is the total number of sets with p main-effect columns. When computing the A_p values, one should make sure that the p main-effect columns used for calculating each of the $\mathbf{x}_k^{(p)}$ vectors correspond to different factors. As discussed in Section 1, a design which sequentially minimizes the entry of vector W is preferred.

It is important to note that the A_p values do not depend on the orthogonalization used for the main-effect columns, provided the columns for one and the same factor are orthogonal with squared norm N (Xu and Wu, 2001). For a proof based on the average squared correlation between p-factor interaction components of multilevel factors, see Evangelaras et al. (2005). Finally, for regular s-level designs, the A_p values in W are $s-1$ times as large as the entries of the ordinary word length pattern (Xu and Wu, 2001).

2.2 Estimation capacity

Cheng et al. (1999) proposed classifying designs according to their capacity to estimate different models. They define the estimation capacity EC_k of an array as the percentage of estimable models with all the main effects and k different two-factor interactions among all possible such models. The authors further suggest sequentially maximizing $\mathrm{EC}_1, \mathrm{EC}_2, \ldots, \mathrm{EC}_c$, where $c = n(n-1)/2$ is the total number of two-factor interactions. Hence, the array with maximum estimation capacity is the one that sequentially maximizes $\mathrm{EC} = (\mathrm{EC}_1, \mathrm{EC}_2, \ldots, \mathrm{EC}_c)$. Since an OA of strength 3 has all its main effects clear from all two-factor interactions, such an array has $\mathrm{EC}_1 = 1$. Note that calculation of the estimation capacity requires extensive computation. For example, for an array with $n = 8$ factors, the evaluation of $\binom{c}{10} = \binom{28}{10} = 12,302,916$ models is required to calculate EC_{10}.

There are two ways of ranking OAs in terms of estimation capacity. One way uses a forward sorting of EC_i to sequentially maximize the $\mathrm{EC}_1, \mathrm{EC}_2, \ldots, \mathrm{EC}_c$ values, while the second way uses a backward sorting that sequentially maximizes the $\mathrm{EC}_c, \mathrm{EC}_{c-1}, \ldots, \mathrm{EC}_1$ values. The former approach guarantees maximum estimation capacity of the simplest possible models involving all main effects and few interaction effects, while the latter yields a maximum estimation capacity of the most complex models.

In this paper we consider arrays with three-level factors. Therefore, two-factor interactions involve more than one component. We consider a model containing k two-factor interactions estimable only if all components of all the k two-factor interactions are estimable.

2.3 Projection estimation capacity

Working with two-level designs, Loeppky et al. (2007) proposed a criterion based on the projections of an s-level array with n factors into $k = 2, 3, \ldots, q$ factors, where q is the largest integer such that $1 + q(s-1) + q(q-1)(s-1)^2/2 \leq N$. They define the projection estimation capacity PEC_k of an array as the percentage of estimable models among all possible models with all main effects and all two-factor interactions involving k of the factors. They classify designs based on the vector $\mathrm{PEC} = (\mathrm{PEC}_1, \mathrm{PEC}_2, \ldots, \mathrm{PEC}_q)$, and suggest sequentially maximizing this vector either by forward sorting from left to right, or by backward sorting from right to left. The forward sorting emphasizes the low-dimensional projections and focuses on the estimability of the simplest possible models. The backward sorting emphasizes high-dimensional projections and concentrates on the estimability of the most complex models.

Since a strength-3 OA contains all level combinations of each set of three factors equally often, any projection onto two or three factors involves a full factorial design and allows the independent estimation of all main effects and two-factor interaction effects. Hence, for strength-3 OAs, $\mathrm{PEC}_2 = \mathrm{PEC}_3 = 1$.

When computing the projection estimation capacity, we consider a model containing two-factor interactions estimable only if all components of these interaction effects are estimable.

2.4 Rank

Cheng et al. (2008) studied two-level OAs of strength 3 using the rank of the $N \times n(n-1)/2$ two-factor interaction matrix. For multi-level OAs such as the three-level arrays we study here, we propose using the rank r of the matrix $\mathbf{X}$ obtained by collecting all possible vectors $\mathbf{x}_k^{(2)}$, obtained by multiplying each pair of main-effect columns from different factors in an element-wise fashion. In this paper, we consider three-level designs only, so that there are $2n(n-1)$ possible vectors $\mathbf{x}_k^{(2)}$. Generally, a higher rank r makes an array more attractive to use as an experimental design. Note, however, that a higher rank does not necessarily imply a higher estimation capacity. This is because the estimability of a two-factor interaction requires specific two-factor interaction components to be estimable and the rank criterion does not prioritize estimability of any specific groups of two-factor interaction components. Note that the rank criterion is computationally less expensive than the criteria based on estimation capacity.

2.5 D-efficiency

For n-factor OAs with $\mathrm{PEC}_n = 1$, we also calculate the D-efficiency, $|\mathbf{X}'\mathbf{X}|^{1/p}/N$, of the matrix $\mathbf{X}$ involving all possible vectors $\mathbf{x}_k^{(2)}$, with $p = 2n(n-1)$. The D-efficiency is a measure for the precision with which the $n(n-1)/2$ two-factor interaction effects can be estimated. For n-factor OAs with $\mathrm{PEC}_n < 1$, $|\mathbf{X}'\mathbf{X}| = 0$. For such arrays, we suggest using the average D-efficiency for all models involving fewer than n factors as a secondary criterion for OAs that have a good estimation capacity, or a good projection estimation capacity; see Li and Nachtsheim (2000) and Loeppky et al. (2007) for similar suggestions.

Table 1: Numbers of isomorphism classes for OAs of the type OA($N;3^n;3$). Numbers between brackets bear on regular OAs.

n	$N=27$	$N=54$	$N=81$
4	1(1)	7 (1)	32 (2)
5	-	4 (0)	17056 (2)
6	-	-	1099 (2)
7	-	-	486 (2)
8	-	-	235 (3)
9	-	-	1 (1)
10	-	-	1 (1)

2.6 Robustness

One might use an OA of strength 3 because of its robustness to missing data. The robustness criterion we use is based on the assumption that one level of one of the n factors leads to useless results, in which case all the rows involving that factor level are removed from the array. If the factor level that is removed corresponds to an s-level factor, then the remaining $(s-1)/N$ fraction of the original array is an OA of strength 2 (Hedayat et al., 1999). In general, the total number of possible sub-designs obtained by removing all the rows involving one level of a factor of an OA is the sum of the n numbers of factor levels, $\sum_{i=1}^{n} s_i$. For the three-level designs considered in this paper, the total number of sub-designs is $3n$. Each of the sub-designs has its own word length vector W, and its own A_3 value.

As a measure M for the lack of robustness of an OA, we use the maximum A_3 value over all $3n$ sub-designs that can be obtained by dropping one level of one of the n factors. Arrays with a small M value are desirable, as these guarantee that the design resulting from dropping the runs corresponding to one factor's level lead to the smallest possible extent of confounding.

2.7 Admissible arrays

When using more than one classification criterion, it is likely that some OAs are better than others according to one criterion but worse according to another criterion. In such cases, it is common to report admissible OAs. If, for a particular array OA_1, there exist an array OA_2 that is strictly better in terms of at least one of the criteria and that is at least as good in terms of the remaining criteria, then OA_1 is inadmissible. Otherwise, OA_1 is admissible. The concept of admissibility was introduced by Sun et al. (1997) in the context of blocking two-level designs. It is related to the concept of Pareto-optimality frequently used in multi-objective optimization and originating from economics: a solution is called Pareto-optimal if there exist no other solutions that perform strictly better with respect to one objective and at least as well with respect to all other objectives. Hence, admissible OAs are Pareto-optimal.

3 Classification of OA($N;3^n;3$) with $N \leq 81$

In this section, we present the classification of all arrays of the type OA($N;3^n;3$) with $N \leq 81$, i.e. of all strength-3 three-levels OAs with up to 81 runs. Table 1 shows the numbers of isomorphism classes for $N=27$, 54, and 81. To the best of our knowledge, the numbers for the 81-run arrays are new to the literature. The numbers in brackets are the numbers of non-isomorphic regular arrays among the total numbers of non-isomorphic OAs. The numbers of regular arrays were given earlier in Xu (2005).

There is just one isomorphism class for arrays of the type OA($27;3^4;3$). As a result, all 27-run strength-3 OAs with four factors are isomorphic. As a matter of fact, they are all isomorphic to the well-known regular 3_{IV}^{4-1} design with generator $D=ABC$. The A_4 value for that array equals 2, whereas the EC_2 and EC_3 values are 0.80 and 0.40, respectively. The M value is 0.50. The fact that there is only

Table 2: Classification of all arrays of the type OA(54; 3^4; 3).

ID	r	PEC_4	A_4	EC					M	D
				2	3	4	5	6		
54.4.1	18	0	2.00	.80	.40	0	0	0	0.500	0
54.4.2	24	1	1.00	1	1	1	1	1	0.500	0.882
54.4.3	23	0	1.00	.93	.80	.60	.33	0	0.250	0
54.4.4	24	1	0.78	1	1	1	1	1	0.250	0.916
54.4.5	24	1	0.67	1	1	1	1	1	0.250	0.931
54.4.6	24	1	0.61	1	1	1	1	1	0.167	0.939
54.4.7	24	1	0.50	1	1	1	1	1	0.125	0.949

Table 3: Classification of all arrays of the type OA(54; 3^5; 3).

ID	r	*GWLP*		PEC_4	EC								M
		A_4	A_5		2	3	4	5	6	7	8	9	
54.5.1	35	3.00	0.50	0.8	.98	.93	.87	.78	.65	.40	0	0	0.250
54.5.2	39	3.00	0.50	1	1	1	.99	.98	.93	.83	.67	.40	0.250
54.5.3	31	3.06	0.78	1	1	1	1	1	.88	0	0	0	0.167
54.5.4	36	3.06	0.61	1	1	1	1	1	.88	0	0	0	0.167

a single isomorphism class for arrays of type OA(27; 3^4; 3) and that all of its properties are well known implies that, for the purpose of this paper, we can concentrate on the 54-run and 81-run series. For these series, we denote the arrays by $N.n.q$, where N is the number of runs, n is the number of factors, and q is the lexicographic ranking of the lexicographically minimal representatives of each OA isomorphism class, for given N and n.

3.1 54 runs

3.1.1 Four factors

There are seven non-isomorphic arrays of the type OA(54; 3^4; 3). Their classification is presented in Table 2. For each of the seven isomorphism classes, this table shows the rank r of the two-factor interactions model matrix $\mathbf{X}$, the PEC_4 and A_4 value, the EC_k values for k ranging from 2 to 6, the M criterion value measuring the robustness, and the D-efficiency $|\mathbf{X}'\mathbf{X}|^{1/p}/N$.

One of the OA(54; 3^4; 3), namely the array labeled 54.4.1, can be viewed as regular, because it is a duplicated regular 3^{4-1}_{IV} fractional factorial design. For this reason, the rank r of the two-factor interactions matrix $\mathbf{X}$, and the A_4, EC_2, EC_3 and M values of the 54-run array are the same as for the 27-run array.

Five of the arrays of the type OA(54; 3^4; 3) permit estimation of a model with all the main effects and all the two-factor interaction effects. Therefore, these arrays have a PEC_4 value of one. Also, the rank of the two-factor interactions matrix $\mathbf{X}$ is $2n(n-1) = 24$ for these arrays, and their EC_i values are one for all i.

Array 54.4.7 is the only admissible array in the OA(54; 3^4; 3) series. It has the smallest A_4 and M values among all arrays that permit estimation of the full two-factor interactions model, and the largest D-efficiency. Array 54.4.7 is given in Table 9 of the Appendix.

3.1.2 Five factors

There are four non-isomorphic arrays of the type OA(54; 3^5; 3). Their classification is presented in Table 3. For each of the four isomorphism classes, the table shows the rank r of the two-factor interactions model matrix $\mathbf{X}$, the A_4 and A_5 values, the PEC_4 value, the EC_k values for k ranging from 2 to 9, and the M

criterion value.

To estimate all $5(5-1)/2 = 10$ two-factor interaction effects in a model involving five three-level factors, 40 degrees of freedom are required. However, Table 3 shows that the highest rank r of $\mathbf{X}$ among the four non-isomorphic arrays of the type OA$(54; 3^5; 3)$ is 39. As a result, none of the four arrays allows a model involving all two-factor interactions to be estimated. Hence, $\text{PEC}_5 = \text{EC}_{10} = 0$ for each of the arrays, and the D-efficiency of a full interaction model is zero as well.

Table 3 also shows that arrays 54.5.1 and 54.5.2 perform equally well in terms of the GA-criterion. However, array 54.5.1 is outperformed by array 54.5.2 in terms of the rank r, the PEC_4 value and all EC_i values. Hence, array 54.5.1 is not admissible. This demonstrates the usefulness of classification criteria other than the GA-criterion. Comparing arrays 54.5.3 and 54.5.4 learns that the two designs perform equally well on all criteria except two: array 54.5.4 has a smaller A_5 value and a higher value of r. For this reason, array 54.5.3, also, is inadmissible.

Interestingly, backward and forward sorting of the EC vector in Table 3 results in different designs being ranked first. Forward sorting leads to arrays 54.5.3 and 54.5.4 being ranked first, while array 54.5.2 is ranked first if backward sorting is used. There is a single set of four out of ten two-factor interaction effects that are not jointly estimable with array 54.5.2, which explains why the EC_9 is 0.40. This OA is recommended if many interactions are expected, while 54.5.3 and 54.5.4 do better if few interactions are expected.

As a conclusion, the arrays 54.5.2 and 54.5.4 are the two admissible designs in the OA$(54; 3^5; 3)$ series. The former is better in terms of the GA-criterion, the rank r and EC_i values for large i. Array 54.5.4 is better in terms of the EC_i values for low i and in terms of the robustness criterion M. Both arrays have $PEC_4 = 1$. We have listed the two admissible designs in Table 9 of the Appendix.

3.2 81 runs

3.2.1 Four factors

As indicated in Table 1, there are 32 non-isomorphic arrays of the type OA$(81; 3^4; 3)$. Two arrays of this type do not have full rank r for the two-factor interactions matrix $\mathbf{X}$. For the first one, this is due to the fact that it is a triplicated regular 3^{4-1}_{IV} design. As a result, the classification criterion values for this array are identical to those of the 27-run array discussed at the start of this section. For the second array that does not have full-rank $\mathbf{X}$, the low rank is due to the fact that the array consists of one replicate of array 54.4.3 and one replicate of the regular 3^{4-1}_{IV} design. Each of these building blocks has a low rank for $\mathbf{X}$ and a relatively low (projection) estimation capacity.

The full factorial 3^4 design is the only admissible design of the series. This design allows estimation of a fourth-order saturated model, and, as a result has a zero A_4 value, EC_i values of 100% and a D-efficiency of 1. In addition, dropping 27 runs from the design corresponding to one level of any one of the factors results in a full factorial $3^3 2^1$ design. For this reason, $M = 0$ for this design.

Finally, note that array 54.4.7 permits estimation of all the two-factor interactions with a high D-efficiency at lower cost than the full factorial 3^4 design.

3.2.2 Five factors

In the five-factor series labeled OA$(81; 3^5; 3)$, 40 out of the 17056 non-isomorphic arrays do not permit all two-factor interaction effects to be estimated. The only admissible design is the regular 3^{5-1}_{V} fractional factorial design with generator $E = ABCD$. Using this array, all two-factor interaction effects can be estimated with 100% D-efficiency. Hence, the array has a zero A_4 value and EC_i values of 100%. Leaving out any 27 runs corresponding to a particular level of one of the factors results in a $3^4 2^1$ array of strength 3. Therefore, $M = 0$ for this design too.

Table 4: The four admissible arrays (upper panel) and the two regular arrays (lower panel) of the type $\mathrm{OA}(81; 3^6; 3)$

ID	(A_4, A_5)	r	$(\mathrm{EC}_2, \ldots, \mathrm{EC}_{15})$	$(\mathrm{PEC}_4, \ldots, \mathrm{PEC}_6)$	M	D
81.6.778	(4, 4)	60	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1)	.759	.732
81.6.831	(4, 4)	60	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1)	.778	.745
81.6.1051	(4.2, 3.6)	59	(1, 1, 1, 1, 1, 1, 1, .99, .97, .93, .83, .67, .40, 0)	(1, 1, 0)	.741	0
81.6.1099	(4, 4)	56	(1, 1, 1, 1, 1, .89, .76, .63, .48, .33, .18, .67, 0, 0)	(1, 1, 0)	.667	0
81.6.1	(6, 0)	44	(1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0.8, 0, 0)	1	0
81.6.4	(4, 4)	48	(1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0.87, 0.33, 0)	1	0

Note that the 54-run array 54.5.2 only lacks one degree of freedom to estimate all the two-factor interaction effects. That array is therefore a good alternative if budgetary constraints render the use of the 81-run 3_V^{5-1} fractional factorial design impossible.

3.2.3 Six factors

Table 4 presents the classification of all admissible and all regular arrays of the type $\mathrm{OA}(81; 3^6; 3)$. The admissible designs are listed in Table 10 in the Appendix.

The arrays 81.6.1 and 81.6.4 are regular designs. The former array has generators $E = ABC$ and $F = ABD$, with generalized word length pattern $W = (6, 0, 2)$, whereas the latter array has generators $E = ABC^2$ and $F = ABCD$ and generalized word length pattern $W = (4, 4, 0)$. Array 81.6.4 is one of 94 designs with minimum generalized aberration (MGA), which have A_4 and A_5 values of 4. However, like 90 other MGA-designs, array 81.6.4 is inadmissible because it performs worse on all of the other classification than the admissible MGA-arrays 81.6.778, 81.6.831 and 81.6.1099. The most striking weakness of the regular array 81.6.4 is that the rank r of its two-factor interaction matrix $\mathbf{X}$ is only 48, while arrays 81.6.778 and 81.6.831 have a full rank two-factor interactions matrix ($r = 4\binom{6}{2} = 60$). Array 81.6.1099 does not have a full-rank two-factor interactions matrix. It is nevertheless admissible, because it has the best value for the robustness criterion M. The remaining admissible design, 81.6.1051, has a slightly better M value than the full-rank arrays 81.6.778 and 81.6.831 and a better rank r than array 81.6.1099.

3.2.4 Seven factors

Table 5 contains the classification of all admissible arrays and the two regular arrays of the type $\mathrm{OA}(81; 3^7; 3)$. The admissible designs are listed in Table 11 in the Appendix. The two regular OAs are arrays 81.7.1, which has generators $E = ABCD$, $F = AB^2C$ and $G = AB^2D$, and array 81.7.4, which has generators $E = ABCD$, $F = AB^2C$ and $G = AC^2D$. Both of the regular arrays are inadmissible, even though the latter is a MGA-design. Both regular designs perform poorly with respect to estimation capacity, projection estimation capacity and the robustness criterion, and they have a low rank for the two-factor interactions matrix.

Among the OAs of the type $\mathrm{OA}(81; 3^7; 3)$, there are seven MGA-designs. Their generalized word length pattern is $W = (10, 12, 2, 2)$. MGA-array 81.7.247 is the only admissible MGA-design because, among all MGA-arrays, it performs best in terms of the classification criteria other than the GA-criterion. It even has the overall best value for the robustness criterion M, and it allows 60 two-factor interaction components to be estimated, because the rank r of the two-factor interactions matrix is 60. There are, however, many non-MGA arrays which allow substantially more two-factor interaction components to be estimated. The best array in this respect is array 81.7.461, whose 80 available degrees of freedom can be used to estimate all seven main effects (this requires 14 degrees of freedom) and 66 two-factor interaction components.

Of particular interest are seven arrays with $(\mathrm{PEC}_4, \mathrm{PEC}_5, \mathrm{PEC}_6) = (1, 1, 0.43)$. All projections onto five factors of each of these arrays permit the estimation of all main effects and all two-factor interactions

Table 5: The 21 admissible arrays (upper panel) and the two regular arrays (lower panel) of the type $\mathrm{OA}(81; 3^7; 3)$.

ID	(A_4, A_5)	r	$(\mathrm{EC}_2, \ldots, \mathrm{EC}_7)$	$(\mathrm{EC}_{14}, \mathrm{EC}_{15}, \mathrm{EC}_{16})$	$(\mathrm{PEC}_4, \ldots, \mathrm{PEC}_6)$	M
81.7.31	(10.44, 10.67)	60	(0.99, 0.96, 0.91, 0.86, 0.79, 0.7)	(0.03, 0, 0)	(0.97, 0.86, 0)	1.611
81.7.80	(10.44, 10.67)	60	(0.99, 0.96, 0.92, 0.86, 0.79, 0.71)	(0.03, 0, 0)	(0.97, 0.86, 0)	1.611
81.7.87	(10.44, 10.67)	64	(0.98, 0.94, 0.89, 0.82, 0.74, 0.65)	(0.05, 0.02, 0)	(0.94, 0.71, 0.29)	1.611
81.7.180	(10.67, 10)	64	(0.98, 0.94, 0.89, 0.82, 0.74, 0.65)	(0.07, 0.03, 0.01)	(0.94, 0.71, 0.29)	1.667
81.7.246	(10.54, 10.37)	62	(0.99, 0.96, 0.92, 0.86, 0.79, 0.72)	(0.07, 0.01, 0)	(0.97, 0.86, 0.29)	1.611
81.7.247	(10, 12)	60	(0.99, 0.96, 0.91, 0.85, 0.77, 0.68)	(0.01, 0, 0)	(0.97, 0.86, 0)	1.5
81.7.250	(10.69, 9.93)	62	(0.99, 0.96, 0.92, 0.86, 0.79, 0.72)	(0.09, 0.02, 0)	(0.97, 0.86, 0.57)	1.611
81.7.294	(10.84, 9.48)	62	(0.99, 0.96, 0.92, 0.86, 0.79, 0.72)	(0.09, 0.03, 0)	(0.97, 0.86, 0.57)	1.63
81.7.297	(10.74, 9.78)	62	(0.99, 0.96, 0.91, 0.86, 0.79, 0.72)	(0.09, 0.03, 0)	(0.97, 0.86, 0.29)	1.63
81.7.438	(10.59, 10.22)	62	(0.99, 0.96, 0.91, 0.86, 0.78, 0.69)	(0.03, 0, 0)	(0.97, 0.67, 0)	1.574
81.7.461	(10.44, 10.67)	66	(0.99, 0.96, 0.91, 0.86, 0.79, 0.71)	(0.05, 0.01, 0)	(0.97, 0.76, 0.29)	1.611
81.7.476	(10.91, 9.7)	64	(1, 1, 1, 1, 0.99, 0.98)	(0, 0, 0)	(1, 0.86, 0)	1.63
81.7.477	(10.67, 10)	64	(0.99, 0.96, 0.91, 0.86, 0.78, 0.69)	(0, 0, 0)	(0.91, 0.43, 0)	1.556
81.7.478	(11.11, 8.67)	63	(1, 1, 1, 1, 1, 1)	(0.44, 0.18, 0)	(1, 1, 0.43)	1.685
81.7.479	(11.11, 8.89)	63	(1, 1, 1, 1, 1, 1)	(0.47, 0.23, 0)	(1, 1, 0.43)	1.685
81.7.480	(11.06, 8.96)	63	(1, 1, 1, 1, 1, 1)	(0.37, 0.12, 0)	(1, 1, 0.43)	1.667
81.7.481	(11.06, 8.81)	63	(1, 1, 1, 1, 1, 1)	(0.32, 0.09, 0)	(1, 1, 0.43)	1.667
81.7.482	(10.89, 9.33)	60	(1, 1, 0.99, 0.96, 0.92, 0.84)	(0.01, 0, 0)	(1, 0.71, 0)	1.611
81.7.484	(11.36, 8.52)	63	(1, 1, 1, 1, 1, 1)	(0.95, 0.76, 0.01)	(1, 1, 0.43)	1.667
81.7.485	(11.36, 8.59)	63	(1, 1, 1, 1, 1, 1)	(0.96, 0.77, 0.01)	(1, 1, 0.43)	1.667
81.7.486	(11.21, 8.81)	63	(1, 1, 1, 1, 1, 1)	(0.79, 0.39, 0)	(1, 1, 0.43)	1.648
81.7.1	(12, 6)	56	(0.91, 0.75, 0.54, 0.33, 0.17, 0.06)	(0, 0, 0)	(0.83, 0.14, 0)	2
81.7.4	(10, 12)	56	(0.93, 0.79, 0.61, 0.42, 0.25, 0.12)	(0, 0, 0)	(0.86, 0.29, 0)	1.5

between the five factors. The best arrays of this subset are the arrays 81.7.484 and 81.7.485, because they have $\mathrm{EC}_{14} \geq 0.95$. The arrays 81.7.478, 81.7.479, 81.7.480, 81.7.481 and 81.7.486 have a substantially poorer estimation capacity when a lot of interactions are present in the model (as reflected by the EC_{14} and EC_{15} values, for example). These arrays are admissible only because they have a slightly better generalized word length pattern than arrays 81.7.484 and 81.7.485, and/or a better M value.

3.2.5 Eight factors

Table 6 presents the classification of all admissible and all regular arrays of the type $\mathrm{OA}(81; 3^8; 3)$. Again, the admissible designs are listed in the Appendix; see Table 12. The $\mathrm{OA}(81; 3^8; 3)$ series consists of 235 arrays, including three regular 3^{8-4}_{IV} designs. One of the three regular designs, array 81.8.7, has generators $E = ABCD$, $F = AB^2C$, $G = AC^2D$, and $H = AB^2D^2$. It is admissible because it is the only MGA-design. Its generalized word length pattern is $W = (20, 32, 8, 16, 4)$. While this array is desirable in terms of the GA-criterion, it has the smallest number of estimable two-factor interaction components of all admissible designs. As a matter of fact, the rank of its two-factor interactions matrix $\mathbf{X}$ is only 60, whereas all other admissible designs have rank $r = 64$, which is the maximum possible rank for eight factors and strength 3.

Perhaps the most interesting arrays in the $\mathrm{OA}(81; 3^8; 3)$ series are the arrays 81.8.234 and 81.8.235. These arrays have $\mathrm{PEC}_4 = \mathrm{PEC}_5 = 1$ and $\mathrm{PEC}_6 = 0.43$ as well as excellent estimation capacities. Thus, all projections onto five factors of these two arrays permit the estimation of all main effects and all two-factor interactions between the five factors. The two arrays have a slightly different generalized word length pattern.

Table 6: The 18 admissible arrays, including the regular array 81.8.7 (upper panel), and the two inadmissible regular arrays (lower panel) of the type OA$(81; 3^8; 3)$.

ID	(A_4, A_5)	r	$(\mathrm{EC}_2, \ldots, \mathrm{EC}_7)$	$(\mathrm{EC}_{14}, \mathrm{EC}_{15}, \mathrm{EC}_{16})$	$(\mathrm{PEC}_4, \ldots, \mathrm{PEC}_6)$	M
81.8.6	(20.89, 28.44)	64	(0.97, 0.9, 0.81, 0.69, 0.56, 0.42)	(0, 0, 0)	(0.94, 0.71, 0)	2.722
81.8.7	(20, 32)	60	(0.92, 0.77, 0.58, 0.39, 0.22, 0.11)	(0, 0, 0)	(0.86, 0.29, 0)	2.5
81.8.34	(21.6, 25.58)	64	(0.98, 0.95, 0.9, 0.84, 0.76, 0.67)	(0.01, 0, 0)	(0.97, 0.86, 0.14)	2.889
81.8.35	(21.38, 26.47)	64	(0.98, 0.95, 0.9, 0.84, 0.76, 0.66)	(0.01, 0, 0)	(0.97, 0.86, 0.18)	2.722
81.8.39	(21.33, 26.67)	64	(0.98, 0.94, 0.88, 0.8, 0.72, 0.62)	(0.01, 0, 0)	(0.94, 0.71, 0)	2.778
81.8.43	(21.6, 25.58)	64	(0.98, 0.95, 0.91, 0.84, 0.77, 0.69)	(0.03, 0, 0)	(0.97, 0.86, 0.07)	2.833
81.8.56	(21.68, 25.28)	64	(0.98, 0.95, 0.91, 0.84, 0.77, 0.69)	(0.05, 0.01, 0)	(0.97, 0.86, 0.39)	2.815
81.8.58	(21.63, 25.48)	64	(0.98, 0.95, 0.9, 0.84, 0.76, 0.67)	(0.03, 0, 0)	(0.97, 0.71, 0)	2.778
81.8.127	(21.33, 26.67)	64	(0.97, 0.92, 0.85, 0.76, 0.65, 0.53)	(0, 0, 0)	(0.91, 0.29, 0)	2.722
81.8.155	(20.89, 28.44)	64	(0.96, 0.89, 0.8, 0.69, 0.57, 0.46)	(0, 0, 0)	(0.86, 0.29, 0)	2.611
81.8.186	(21.63, 25.48)	64	(0.98, 0.95, 0.9, 0.84, 0.76, 0.67)	(0.02, 0, 0)	(0.97, 0.79, 0)	2.778
81.8.217	(21.68, 25.28)	64	(0.98, 0.95, 0.91, 0.84, 0.77, 0.69)	(0.03, 0, 0)	(0.97, 0.86, 0)	2.722
81.8.230	(21.83, 25.88)	64	(1, 1, 1, 1, 0.99, 0.97)	(0, 0, 0)	(1, 0.86, 0)	2.759
81.8.231	(21.33, 26.67)	64	(0.98, 0.95, 0.9, 0.84, 0.76, 0.65)	(0, 0, 0)	(0.91, 0.43, 0)	2.667
81.8.232	(22.22, 23.56)	64	(1, 1, 1, 1, 0.99, 0.99)	(0.30, 0.11, 0)	(1, 1, 0.43)	2.815
81.8.233	(22.12, 23.8)	64	(1, 1, 1, 1, 0.99, 0.99)	(0.23, 0.06, 0)	(1, 1, 0.43)	2.815
81.8.234	(22.72, 22.77)	64	(1, 1, 1, 1, 1, 1)	(0.79, 0.47, 0.03)	(1, 1, 0.43)	2.870
81.8.235	(22.42, 23.51)	64	(1, 1, 1, 1, 1, 0.99)	(0.53, 0.20, 0.01)	(1, 1, 0.43)	2.870
81.8.1	(22, 24)	64	(0.91, 0.75, 0.54, 0.34, 0.18, 0.08)	(0, 0, 0)	(0.84, 0.21, 0)	3
81.8.2	(24, 16)	64	(0.9, 0.73, 0.51, 0.3, 0.14, 0.05)	(0, 0, 0)	(0.83, 0.14, 0)	3

3.2.6 Nine and ten factors

To complete the classification of all arrays of the type OA$(81; 3^n; 3)$, there is a single array with nine factors and a single array with ten factors. Both of these arrays are regular. The array of type OA$(81; 3^9; 3)$ has generators $E = ABCD$, $F = AB^2C$, $G = AC^2D$, $H = AB^2D^2$, and $P = BCD^2$, an A_4 value of 36 and an A_5 value of 72. The array of type OA$(81; 3^{10}; 3)$ has generators $E = ABCD$, $F = AB^2C$, $G = AC^2D$, $H = AB^2D^2$, $P = BCD^2$, and $Q = ABC^2D^2$, an A_4 value of 60 and an A_5 value of 144. The rank r of the two-factor interactions matrix $\mathbf{X}$ is 60 in both cases. For the ten-factor array, 60 is the maximum rank possible for the two-factor interaction matrix given that 80 degrees of freedom are available and 20 parameters are required to model the ten main effects.

4 Discussion

The goal of this paper is to propose suitable criteria for the classification of multi-level orthogonal arrays of strength 3. For that purpose, we studied the generalized aberration criterion, the estimation capacity, the projection estimation capacity, a new robustness criterion and the D-efficiency of the complete set of all non-isomorphic strength-3 three-level arrays with 27, 54 and 81 runs.

In the introduction, we claim that the generalized aberration criterion is less suitable for orthogonal arrays of strength 3, especially when applied to multilevel arrays. This claim is strongly supported by the rankings of some admissible arrays of the type OA$(81; 3^n; 3)$ with n values ranging from 6 to 8 obtained using the different classification criteria. The rankings we obtained are given in Table 7. For each number of factors n, we sorted the admissible designs in increasing order of their generalized aberration and report the results for the best and the worst arrays according to that criterion.

The six-factor admissible MGA designs include one array, labeled 81.6.1099, that does not permit estimation of a model involving all main effects and two-factor interactions. That array is therefore only ranked 5th in terms of the rank criterion r, which indicates how many two-factor interaction components

Table 7: Rankings of some admissible arrays of the type $OA(81; 3^n; 3)$. The table shows results for $n = 6, 7$, and 8 in the top, the middle, and the bottom panel, respectively.

ID	GA	r	PEC		EC		M	D
			forward	backward	forward	backward		
81.6.778	1	1	1	1	1	1	3	4
81.6.831	1	1	1	1	1	1	4	1
81.6.1099	1	5	2	2	106	100	1	105
81.6.1051	2	2	2	2	15	11	2	105
81.7.247	1	6	8	12	270	302	1	
81.7.485	29	3	1	2	2	4	7	
81.8.7	1	2	59	48	186	186	1	
81.8.234	38	1	1	1	1	1	9	

are estimable, 106th in terms of its estimation capacity in case of forward sorting, 100th in terms of its estimation capacity in case of backward sorting, and 105th in terms of its D-efficiency (which equals zero because the model involving the two-factor interactions is not estimable). Clearly, achieving minimum generalized aberration does not, in itself, lead to the best results in terms of estimation capacity.

Similarly, the best seven-factor and eight-factor array according to the generalized aberration criterion performs rather poorly in terms of estimation capacity. For the eight-factor case, it is the admissible array with the worst performance in terms of generalized aberration that is ranked best according to the rank r and the various estimation capacity criteria. For the seven-factor case, the discrepancy between the ranking in terms of generalized aberration and in terms of estimation capacity is almost as extreme. As a matter of fact, the admissible array with the worst ranking according to the generalized aberration criterion, array 81.7.485, has the best projection estimation capacity in case of forward sorting and ranks very highly according to the other estimation-capacity criteria.

Table 7 suggests that the best arrays according to any of the four estimation-capacity criteria also rank high according to the other three criteria. This is confirmed by the rank correlation coefficients for these criteria for the arrays of type $\mathrm{OA}(81; 3^n; 3)$ with $5 \leq n \leq 8$.

Table 8 presents rank correlation matrices for the five- up to eight-factor arrays for the GA-criterion, the rank r of the two-factor interaction matrix, forward sorting of estimation capacity, the robustness criterion M and the D-efficiency. The numbers in brackets given in the column headings of the tables are the numbers of distinct rank orders obtained from the various criteria.

For the five- and six-factor series, there is a large correlation between the rank r of the two-factor interactions matrix $\mathbf{X}$ and the estimation capacity. This is probably due to the fact that there are arrays with a full-rank $\mathbf{X}$ matrix in these series, each of which automatically has the best estimation capacity. For the seven- and eight-factor arrays, where none of the $\mathbf{X}$ matrices is of full rank, this correlation is substantially smaller.

We also obtained large correlations between the ranking in terms of the GA criterion and in terms of D-efficiency for the 5-factor series of arrays, between the D-efficiency and the rank r of $\mathbf{X}$ and between the D-efficiency and the estimation capacity for the six-factor series, and between the GA criterion and the robustness criterion M in the seven-factor series. However, there are no large correlations between the rankings in terms of generalized aberration, on the one hand, and the rank r or the estimation capacity, on the other hand.

Our study of the various classification criteria leads us to the following conclusions. First, the best arrays in terms of the GA criterion do not perform best in terms of estimation capacity as measured by the rank r of the two-factor interactions matrix, the EC values and the PEC values. Therefore, it is best not to rely solely on the GA criterion when selecting a multi-level array, and use the rank r, the EC values and the PEC values as criteria of primary interest. Second, the model-based criteria (the rank r of the two-factor interactions matrix, the EC values and the PEC values) are closely related. As a result,

Table 8: Rank correlations for various classification criteria for arrays of the types OA$(81; 3^n; 3)$ for n=5, 6, 7 and 8

$n = 5$					
	GA (252)	r (4)	EC (7)	M (21)	D (148)
GA	–	0.04	0.04	0.74	0.99
r		–	1	0.03	0.06
EC			–	0.03	0.06
M				–	0.74
D					–

$n = 6$					
	GA (53)	r (32)	EC (166)	M (18)	D (105)
GA	–	0.62	0.57	0.49	0.57
r		–	0.95	0.45	0.79
EC			–	0.45	0.79
M				–	0.29
D					–

$n = 7$				
	GA (32)	r (8)	EC (474)	M (10)
GA	–	−0.01	0.19	0.77
r		–	0.44	−0.02
EC			–	0.04
M				–

$n = 8$				
	GA (40)	r (2)	EC (188)	M (11)
GA	–	−0.13	0.21	0.57
r		–	0.14	−0.16
EC			–	0.13
M				–

Table 9: Admissible arrays of the type $\mathrm{OA}(54, 3^n, 3)$ for $n = 4$ and $n = 5$.

54.4.7
125251512251512125512125251
54.5.2
048521521521152215521215152
125536671536084716671716325
54.5.4
055512521512125251521251118
057563736736615381563372615

Table 10: Admissible arrays of the type $\mathrm{OA}(81, 3^6, 3)$.

778	1051
0458458028014468038045045278064064564562	0478462178161462165165075178161075164078
5165165165165112585107248055231125372048	5125238412536548020148166732247535205505
5521635516635246242251637372642242516367	5235523707172172712237521723705642365507
831	1099
0458458028014468038045045278064064564562	1178178048035265135138138175162162462465
5165165165165115282117538055211155062348	3781781121125664364382382147076076546532
5646505651505242646237251372642242516367	5183642637516716635512167635642507653505

one could use just one of these criteria to select a suitable array. As determining the estimation capacity is computationally more intensive than determining the projection estimation capacity, we recommend using the projection estimation capacity. Finally, there is no consistent strong relation among the criteria, except for the four estimation capacity criteria. Therefore, we recommend calculating the estimation capacity criteria, along with the D-efficiency and the GA criterion and to choose one of the resulting admissible designs.

Appendix: Admissible non-regular three-level strength-3 arrays

The Tables 9–12 list all admissible non-regular arrays with 54 and 81 runs discussed in the text. The arrays are given in a condensed form. First, we omitted the first three columns of each array, because these columns simply form a duplicated 3^3 full factorial design for all arrays with 54 runs and a triplicated 3^3 full factorial design for all arrays with 81 runs. The rows of the two or three replicates of the full factorial design are sorted lexicographically, so that, for a given run of the full factorial design, the two or three replicates appear in pairs or triples. Also, we omitted the first row of the 81-run arrays, because that row is a zero row for each of the 81-run arrays. The next step was to transpose the arrays, so that we obtained an $(n-3) \times 54$ matrix for the 54-run arrays and an $(n-3) \times 80$ matrix for the 81-run arrays. The resulting matrices thus have either 27 or 40 pairs of digits. The final step was to replace each pair of successive digits, which can be viewed as a base-3 number, by its base-10 representation. The end result is an $(n-3) \times 27$ matrix for the 54-run arrays and an $(n-3) \times 40$ matrix for the 81-run arrays.

We illustrate this procedure by reconstructing the fourth column of array 81.6.778. In the first row of Table 10, we find

$$0458458028014468038045045278064064564562$$

as the condensed form of this fourth column (note that the first three columns were omitted because they correspond to the 3^3 factorial design). We then convert every digit of that column into the two digits of the corresponding base-three number. So we replace 0 with 00, 4 with 11, etc. Subsequently, we add a 0 as the first symbol of the column (because we dropped the first row of every 81-run array). The resulting column, in transposed form, then is

$$000111222111222000222000111111202200102200111200111202212200201100201112201112200 2.$$

Table 11: Admissible arrays of the type OA$(81, 3^7, 3)$.

31 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516521651521635181505521516651165505 5521176651505635646646505176176181181646	80 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516646651521521165181521521646165176 5521505521635165651635516516521181521165
87 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516646651521521165646181521181505176 5521505181646651505521505176165651505521	180 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516646651651505165516651516651505165 5521635176521521521176635165635521505521
246 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516651521516651165521181646181505176 5521176253642635712635507507376523651165	247 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516651521516651165521181646181505176 5521176505651635181635516516646505651165
250 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516651521516651165521181646181505176 5521242176721707521635365505642516721167	294 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516651521521635635505521516181521165 5521242253712707716365381242242653707176
297 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5165516651521521635635505521516181521165 5521253176381365516635721251253646365176	438 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5167376721646646635235523383376521637165 5521646646165165516635521651646176181176
461 0458458028014458028014014888014014874860 5165165165165165165165165165165165165165 5176176651521521521176635165651521176505 5505646181646181635646505516521505651165	476 0458468048064468035075074278064074264265 5165115285211115278204235385211505161486 5365653716165251705376635376367505246651 5521646251381505723251381365705165521637
477 0458468048064468048064014878064014875261 5165115285211115285211165185211165163468 5167651516367653716165172172365705167653 5521505716246181365705521181381635521507	478 0458458028014468038045045278064064564562 5165165165165105572235567055501242631378 5176516181651521637352508335176722335422 5637712512253383381172172646242642512507
479 0458458028014468038045045278064064564562 5165165165165105572235567055501242631378 5176516181651521637352508335176722335422 5642253507712732706537172636737532523642	480 0458458028014468038045045278064064564562 5165165165165105572237565055501242361648 5183642637516381183172365646642246376507 5637653512505736162571704806553731665811
481 0458458028014468038045045278064064564562 5165165165165105572237565055501242361648 5183642637516381183172365646642246376507 5646176646181521507367635505165723376651 484 0458468048064468032175164278064164261565 5165105585501505246167642365512581031655 5183381176642112781172846057633512633525 5637736166553183366571531672642641812804	482 0458458028014468038045045278064064564562 5165165165165115275263263485211211771766 5521635516635176721246381181235376181523 5646505651505505721365251505705381646507 485 0458468048064468032175164278064164261565 5165105585501505246167642365512581031655 5183381176642112781172846057633512633525 5646521505165536702467523636537732523635
486 0458468048064468032175164278064164261565 5165115285211181177065215635505805334182 5367642646642317684076283427613617241652 5521181365705635183563703776521231842770	

Table 12: Admissible arrays of the type $\mathrm{OA}(81, 3^8, 3)$.

6 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165151652165116518165116517651 6 552117663552152118117663551663552118117 6 564650565150518150564650564618164618150 5	7 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165151652165116518165116517651 6 552117663552163552117663551663552117663 5 552163551663517665117652116552163551663 5
34 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165152163518150552151665116550 5 552117663738365324664250517217670718164 6 552152118324217271663763550752138318117 6	35 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165152163518150552151665116550 5 552117663738365324664250517217670718164 6 552163551270750737665316565363524218151 6
39 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165152163518150552151665116550 5 552117665150563564664650517617618118164 6 552163550564651617665116565163550518151 6	43 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551652165152163518150552151665116550 5 552118150738352324664263517218170718151 6 552150565324264271663716550750538318164 6
56 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551664665152152116518152152164616517 6 552117625138352364263550716717272152117 6 552163537670716752363551264265324652116 5	155 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551664665152152164616518152116518151 6 552150518164616552152152117665163550550 5 564617665117652116552164617663550565117 6
186 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516551665152151665116552118164618150517 6 552117618164663564663550550551652165116 5 552152164616517650563565152118151618117 6	217 045845802801445802801401488801401487486 0 516516516516516516516516516516516516516 5 516737665152352337625118118164670516737 6 552163550552118164663551617617663552163 5 564271652116750770538164617663523564271 6
230 045846804806446803507507427806407426426 5 516511528521111527820423538521150516148 6 536565371616525170537663537636750524665 1 552164625138150572325138136570516552163 7 572150770550538138636415167624213568650 4	231 045846804806446804806401487806401487526 1 516511528521111528521116518521116516346 8 516765151636765371616517217236570516765 3 552150571624618136570552118138163552150 7 552318118138364224664650716770536552318 3
233 045845802801446803804504527806406456456 2 516516516516510557223756505550124236164 8 518364263751638118317236564664224637650 7 563765351250573616257170480655373166581 1 564617664618152150736763550516572337665 1	234 045846804806446803217516427806416426156 5 516510558550150524616764236551258103165 5 518338117664211278117284605763351263352 5 563773616655318336657153167264264181280 4 564652150516553670246752363653773252363 5
235 045846804806446803217516427806416426156 5 516511528521118117706521563550580533418 2 536764264664231768407628342761361724165 2 552118136570563518356370377652123184277 0 572363717652324264637663524264218363564 2	

References

Bulutoglu, D. A. and Margot, F. (2008). Classification of orthogonal arrays by integer programming. *Journal of Statistical Planning and Inference*, 138:654–666.

Chen, J., Sun, D. X., and Wu, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *International Statistical Review*, 61:131–145.

Cheng, C. S., Mee, R. W., and Yee, O. (2008). Second order saturated orthogonal arrays of strength three. *Statistica Sinica*, 18:105–119.

Cheng, C. S., Steinberg, D. M., and Sun, D. X. (1999). Minimum aberration and model robustness for two-level factorial dessigns. *Journal of the Royal Statistical Society Series B*, 61:85–94.

Evangelaras, H., Koukouvinos, C., Dean, A. M., and Dingus, C. A. (2005). Projection properties of certain three level orthogonal arrays. *Metrika*, 62:241–257.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics*, 22:601–608.

Hedayat, A. S., Seiden, E., and Stufken, J. (1997). On the maximal number of factors and the enumeration of 3-symbol orthogonal arrays of strength 3 and index 2. *Journal of Statistical Planning and Inference*, 58:43–63.

Hedayat, A. S., Sloane, N. J. A., and Stufken, J. (1999). *Orthogonal arrays: Theory and applications*. Springer.

Li, W. and Nachtsheim, C. J. (2000). Model-robust factorial designs. *Technometrics*, 42:345–352.

Loeppky, J. L., Sitter, R. R., and Tang, B. (2007). Nonregular designs with desirable projection properties. *Technometrics*, 49:454–467.

Ma, C. X. and Fang, K. T. (2001). A note on generalized aberration in fractional designs. *Metrika*, 53:85–93.

Mee, R. W. (2009). *A comprehensive guide to factorial two-level experimentation*. Springer-Verlag, New York, NY, USA.

Rao, C. R. (1947). Factorial experiments derivable from combinatorial arrangements of arrays. *Journal of the Royal Statistical Society Supplement*, 9:128–139.

Schoen, E. D., Eendebak, P. T., and Nguyen, M. V. M. (2010). Complete enumeration of pure-level and mixed-level orthogonal arrays. *Journal of Combinatorial Designs*, 18:123–140.

Sun, D. X., Li, W., and Ye, K. Q. (2002). An algorithm for sequentially constructing nonisomorphic orthogonal designs and its applications. Technical report, Department of Applied Mathematics and Statistics, SUNY at Stony Brook.

Sun, D. X., Wu, C. F. J., and Chen, Y. (1997). Optimal blocking schemes for 2^n and 2^{n-p} designs. *Technometrics*, 39:298–307.

Tang, B. and Deng, L. Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Annals of Statistics*, 27:1914–1926.

Wu, C. J. F. and Hamada, M. (2000). *Experiments: Planning, Analysis and Parameter Design Optimization*. Wiley, New York, NY, USA.

Xu, H. (2005). A catalogue of three-level regular fractional factorial designs. *Metrika*, 62:259–281.

Xu, H. and Wu, C. F. J. (2001). Generalized minimum aberration for asymmetrical fractional factorial designs. *Annals of Statistics*, 29:1066–1077.