Copulas and the distribution of cash flows with mixed signs

Marc Goovaerts[1,2], Ann De Schepper[3], Yong Hua[1]

Abstract

In a paper of 2000, Kaas, Dhaene and Goovaerts investigate the present value of a rather general cash flow. Making use of comonotonic risks, they derive upper and lower bounds for the distribution of the present value. These bounds are very close to the real distribution in case all payments have the same sign ; however, if there are both positive and negative payments, the upper bounds perform rather badly. In the present contribution we show what happens when solving this problem by means of copulas. The idea consists of splitting up the total present value in the difference of two present values with positive payments. Making use of a copula as an approximation for the joint distribution of the two sums, an approximation for the distribution of the original present value can be derived.

1 Description of the problem

As in Kaas, Dhaene and Goovaerts (2000), we consider a series of deterministic payments α_1, α_2, ..., α_n due at times 1, 2, ...n, which can be positive as well as negative. The present value then equals

$$S = \sum_{i=1}^{n} \alpha_i e^{-Y_1 - Y_2 - \dots - Y_i}, \tag{1}$$

where Y_i represents the stochastic continuous compounded rate of return over the period $[i-1, i]$.

[1] University of Leuven, Belgium
[2] University of Amsterdam, the Netherlands
[3] University of Antwerp, Belgium

In the classical assumption, prices are log-normally distributed, and the variables Y_i are independent and normally distributed with mean μ_i and variance σ_i^2.

For notational reasons, we will write this present value as

$$S = \sum_{i=1}^{n} \alpha_i X_i, \tag{2}$$

where for $i = 1, 2, ..., n$

$$X_i = e^{-Y(i)} \qquad \text{and} \qquad Y(i) = Y_1 + Y_2 + ... + Y_i. \tag{3}$$

Furthermore, we use the notations

$$\mu_{(i)} = E[Y(i)] \ , \quad \sigma_{(i)}^2 = Var[Y(i)] \quad \text{and} \quad \theta_{(i,j)} = \text{Cov}[Y(i), Y(j)] \tag{4}$$

for the parameters of the remaining variables.

Due to the fact that the variables X_i are correlated, it is impossible to derive the exact distribution of the sum S. Therefore in the same paper of Kaas, Dhaene and Goovaerts, among others, the authors present bounds in convexity order, that make use of the concept of comonotonic risks. This means that we replace the original sum S by a sum of components with the same marginal distributions as the variables X_i, but with the most or least "dangerous" dependence structure (see section 2 for details about these concepts).

It is shown in Kaas, Dhaene and Goovaerts (2000) that this upperbound equals

$$S_U = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} + \text{sign}(\alpha_i)\sigma_{(i)}\Phi^{-1}(U)}, \tag{5}$$

where U is a uniform (0,1) random variable, and where Φ is the standard normal cumulative distribution function.

If necessary, this upperbound can be improved, by conditioning on a random variable Z that is defined as

$$Z = \sum_{i=1}^{n} \beta_i Y_i, \tag{6}$$

in which case we need to calculate the correlation with $Y(i)$,

$$\rho_i = \text{Corr}[Y(i), Z]. \tag{7}$$

Kaas, Dhaene and Goovaerts show that by conditioning on this Z, the upperbound (5) can be improved to the closer bound

$$S_U^* = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} - \rho_i \sigma_{(i)} \Phi^{-1}(U) + \text{sign}(\alpha_i)\sqrt{1-\rho_i^2}\, \sigma_{(i)} \Phi^{-1}(V)}, \quad (8)$$

where U and V are mutually independent uniform (0,1) random variables, independent of the variable Z.

Since successive variables $Y(i)$ represent sums that only differ in one term, they are rather strongly (and positively) dependent, explaining the good performance of both of these bounds, which make use of the “strongest possible” dependence between the discount factors.
However, this strong affinity between exact and approximate distributions only holds in case all payments α_i have the same sign. When both positive and negative payments occur, the performance is much less adequate. This is due to the “negative” dependence structure between terms with different signs.

The conditioning on a random variable Z as in (6) can also be used (again, see Kaas, Dhaene and Goovaerts) to construct a lower bound for the original present value :

$$S_L = \sum_{i=1}^{n} \alpha_i e^{-\mu_{(i)} - \rho_i \sigma_{(i)} \Phi^{-1}(U) + \frac{1}{2}(1-\rho_i^2)\, \sigma_{(i)}^2}, \quad (9)$$

with U a uniform (0,1) distributed random variable.
In contrast with the upperbounds, this lowerbound seems to perform much better. However, due to the mixed signs of the payments α_i, the lowerbound is not a sum of comonotonic risks. As a consequence, its distribution is more difficult to obtain.

In the sequel we show what happens when constructing an approximation by the introduction of copulas. The concepts of convex ordering and of copulas are explained in section 2. In section 3 we derive the approximation, while in section 4 some numerical illustrations are presented.

2 More about the concepts

2.1 Convex order and comonotonic risks

Many financial and actuarial applications are faced with the difficulty or even impossibility of finding an analytic expression for the distribution of the (stochastic) quantity under investigation. Also in the present case (see equation (1) and (2)), the stochastic variables $Y(i)$ or X_i are dependent, since they are constructed as successive sequences of the same independent variables.

The method of convex upper bounds is extremely helpful to deal with this kind of problems. In fact we replace the incalculable exact distribution by a simpler approximate distribution which is known to be associated with a quantity that is more dangerous than the original one.

In order to illustrate the fact that convex order nicely suites the notion of dangerousness, we mention three equivalent characterizations of this concept.

We say that W is an upper bound for V in convexity order, $V \leq_{cx} W$, if

a) $E[u(V)] \leq E[u(W)]$ for each convex function u ;
since convex functions take on their largest values in the tails, the variable W is more likely to take on extreme values than the variable V and thus more dangerous.

b) $E[u(-V)] \geq E[u(-W)]$ for each concave function u ;
each risk averse decision maker prefers a loss V over a loss W, and thus the variable W is more dangerous.

c) $E[V] = E[W]$ and $E[(V-k)_+] \leq E[(W-k)_+]$ for each value of k ;
the financial loss of realizations exceeding a number k (the so-called stop-loss premium) is always larger for W than for V and thus the variable W is more dangerous.

As a consequence, replacing a variable V with unknown distribution by a variable W (satisfying one of the previous properties) with known distribution, can be seen as a prudent strategy.

If in addition to this variable W a lower bound can be constructed for V, we also have a measure with respect to the reliability of the upperbound.

Returning to the problem of approximating distributions, e.g. of cash flows, we mention the following theorem which summarizes the most important result regarding this idea. A proof can be found in [5].

Proposition 2.1 *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + ... + \phi_n(X_n), \tag{10}$$

where the functions $\phi_t : \Re \to \Re : x \mapsto \phi_t(x)$ *are all increasing or all decreasing. The variable*

$$W = \phi_1(F_{X_1}^{-1}(U)) + \phi_2(F_{X_2}^{-1}(U)) + ... + \phi_n(F_{X_n}^{-1}(U)) \tag{11}$$

with U *an arbitrary random variable that is uniformly distributed on* $[0, 1]$ *then defines an upper bound in convexity order, or*

$$V \leq_{cx} W. \tag{12}$$

In the previous result, the notation $F_{X_j}(x)$ is used for the distribution function of X_j,

$$F_{X_j}(x) = Prob(X_j \leq x) \; ; \tag{13}$$

the inverse function is defined as ($p \in [0, 1]$)

$$F_{X_j}^{-1}(p) = \inf\{x \in \Re : F_{X_j}(x) \geq p\} \; . \tag{14}$$

Due to the construction of the variable W, the distribution of the bound can be determined rather easily as

$$F_W(s) = p_s \tag{15}$$

with p_s defined implicitely by

$$\sum_{i=1}^{n} \phi_i \left(F_{X_i}^{-1}(p_s) \right) = s. \tag{16}$$

2.2 Copulas

A copula C is a function that maps in a unique way the marginals F_1 and F_2 of a bivariate distribution F to the joint distribution :

$$C(u, v) : [0, 1] \times [0, 1] \to [0, 1] : (u, v) \mapsto F(F_1^{-1}(u), F_2^{-1}(v)) \tag{17}$$

such that

$$C(F_1(x), F_2(y)) = F(x, y). \tag{18}$$

Note that a copula is always symmetric in both arguments, and that the marginals of a copula are uniform distributions.

An Archimedian copula can be expressed in the form

$$C(u,v) = \psi^{-1}[\psi(u) + \psi(v)] \tag{19}$$

where the function

$$\psi : [0,1] \to [0,+\infty] \tag{20}$$

is a continuous, strictly decreasing and convex function.

Three special copulas are very illustrative :

- $C_1(u,v) = u \cdot v$
  represents the case of independent underlying variables ;
- $C_2(u,v) = \min(u,v)$
  is an upper bound, representing the case of most relating pair of variables with given marginals ;
- $C_3(u,v) = \max(0, u+v-1)$
  is a lower bound, representing the case of most antithetic pair of variables.

If (X,Y) has the bivariate distribution function F, with marginals F_1 and F_2, and if C is a copula as in equation (18), then Spearman's rho is given by

$$\rho_S(X,Y) = 12 \cdot \iint_{(0,1)^2} C(u,v)\,du\,dv - 3. \tag{21}$$

Note that the relation between Spearman's rho and Pearson correlation is given by

$$\rho_S(X,Y) = \rho(F_1(X), F_2(Y)). \tag{22}$$

See e.g. [3].

There are different manners in order to generate copulas that correspond to couples of variables with any correlation ρ_S between -1 and $+1$.

A first possibility consists of taking a convex combination of the three special copulas mentioned above,

$$C(u,v;\rho_S) = p_1 C_1(u,v) + p_2 C_2(u,v) + p_2 C_2(u,v) \tag{23}$$

with

$$\begin{cases} p_1, p_2, p_3 \geq 0 \\ p_1 + p_2 + p_3 = 1 \\ p_2 - p_3 = \rho_S. \end{cases} \tag{24}$$

The advantage of this approach can be found in the fact that we work with a combination of three extreme situations.

A second approach generates an Archimedian copula and makes use of a Gumbel function :

$$C(u, v; \rho_S) = exp\left\{ - \left((-\log u)^{1/\beta} + (-\log v)^{1/\beta} \right)^{\beta} \right\} \tag{25}$$

with $\beta \geq 0$ the unique solution of the equation

$$\int_0^1 \int_0^1 e^{-\left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta}} \, du \, dv = \frac{1}{12}(\rho_S + 3). \tag{26}$$

Note that a positive correlation corresponds to a value of β smaller than one, a negative correlation to a value larger than one (see figure 1).

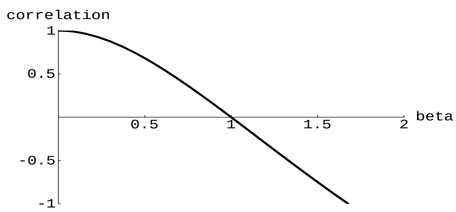

Figure 1: *Evolution of the correlation ρ_S with β*

Gumbel copulas as described in (25) have upper tail dependence, which means that

$$\lim_{u \nearrow 1} \text{Prob}\left[Y > F_2^{-1}(u) | X > F_1^{-1}(u) \right] > 0. \tag{27}$$

This is compatible with the present value under investigation, since we work with two components of the same cash flow, with the same interest rate structure.

3 Construction of the distribution function

3.1 Basic idea

Since the approximation described in section 1 performs excellent in case all payments have the same sign, it seems very reasonable to split the total present value

$$S = \sum_{i=1}^{n} \alpha_i X_i \tag{28}$$

into two separate parts, representing the positive and negative payments respectively.

This means that we write the sum S as

$$S = S^+ - S^- \tag{29}$$

where the terms are defined as

$$S^+ = \sum_{i=1}^{n} (\alpha_i)_+ X_i \tag{30}$$

and

$$S^- = \sum_{i=1}^{n} (-\alpha_i)_+ X_i. \tag{31}$$

The expression $(x)_+$ is used as a short-hand notation for $\max(x, 0)$.

The results of [7] can be used to find an adequate approximation for the distribution function of both S^+ and S^-, since each of them refers to a situation with equally signed cash flow payments.
The idea then is –starting from these two approximate distribution functions– to construct an adequate approximation for the joint distribution function of S^+ and S^-

$$H(s^+, s^-) = \text{Prob}[S^+ \leq s^+, S^- \leq s^-]. \tag{32}$$

Aftwerwards, the correct integration leads to an (approximate) distribution for the difference of both sums.

3.2 Upperbounds for S^+ and S^-

As explained before, very close upper bounds for S^+ and S^- can be found when applying the method of [7].

This results in (see equation (5))

$$S_U^+ = \sum_{i=1}^{n} (\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)}\Phi^{-1}(U_1)\right) \tag{33}$$

and

$$S_U^- = \sum_{i=1}^{n} (-\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)}\Phi^{-1}(U_2)\right) \tag{34}$$

with U_1 and U_2 independent uniform (0,1) random variables.
Both sums an be written alternatively as

$$S_U^+ = \sum_{i=1}^{n} (\alpha_i)_+ exp\left(-\sum_{j=1}^{i} Z_j^1\right) \tag{35}$$

and

$$S_U^- = \sum_{i=1}^{n} (-\alpha_i)_+ exp\left(-\sum_{j=1}^{i} Z_j^2\right) \tag{36}$$

with Z_j^1 and Z_j^2 independent normal (μ_j, σ_j) random variables $(j = 1, ..., n)$.

Due to the construction of these sums (see section 2) the distribution of both of them can be derived rather easily. Indeed, we have

$$F_{S_U^+}(s) = \text{Prob}[S_U^+ \leq s] = \Phi(v_s) \tag{37}$$

and

$$F_{S_U^-}(s) = \text{Prob}[S_U^- \leq s] = \Phi(w_s) \tag{38}$$

with v_s and w_s defined implicitly by

$$\sum_{i=1}^{n} (\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)} v_s\right) = s \tag{39}$$

and

$$\sum_{i=1}^{n} (-\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)} w_s\right) = s. \tag{40}$$

3.3 Joint distribution of S^+ and S^-

The second step now consists of mapping the two approximate distributions for the sums into a approximation for their joint distribution. As the distributions of both S_U^+ and S_U^- are very close to the distributions of S^+ and S^- and as –in contrast to the exact ones– they are very well calculable, this seems a reasonable approach.
The idea is to realize this mapping by means of a copula, making use of the approximate distributions in combination with the correlation of S_U^+ and S_U^-. We then get a bivariate distribution for which the marginals are equal to the approximate distributions of the two terms in the difference, and for which the underlying variables have the correct correlation.

In other words, we will construct a copula $C(u, v; \rho_S^U)$, where ρ_S^U is the correlation between S_U^+ and S_U^-.
Following (32), we then have

$$H(s^+, s^-) \approx C\left(F_{S_U^+}(s^+), F_{S_U^-}(s^-); \rho_S^U\right). \tag{41}$$

The construction of the copula can be performed e.g. by means of a suitable combination of the special copulas as mentioned in equations (23) and (24) or by means of a Gumbel copula as mentioned in equation (25). In the latter case, the value of β can be determined as the unique solution of equation (26).

In both cases, we still need a value for the correlation ρ_S^U. Therefore, we note that

$$\rho_S^U = \rho_S\left(S_U^+, S_U^-\right) = \rho\left(F_{S_U^+}(S_U^+), F_{S_U^-}(S_U^-)\right); \tag{42}$$

since both distributions are known, a suitable value for this correlation can be found by simulation.

Therefore, we generate N values from a standard normal distribution, denoted by ϵ_k $(k = 1, ..., N)$. With these values, we construct the values

$$z_{j,k} = \mu_j + \sigma_j\ \epsilon_k \qquad j = 1, ..., n\ ;\ \ k = 1, ..., N \tag{43}$$

and

$$\begin{cases} s_k^+ &= \sum_{i=1}^n (\alpha_i)_+\ exp\left(-\sum_{j=1}^i z_{j,k}\right) \\ s_k^- &= \sum_{i=1}^n (-\alpha_i)_+\ exp\left(-\sum_{j=1}^i z_{j,k}\right) \end{cases} \qquad k = 1, ..., N. \tag{44}$$

Relying on equations (37)-(40), we calculate

$$F_{S_U^+}(s_k^+) = \Phi(v_k^+) \tag{45}$$

where v_k^+ follows from

$$\sum_{i=1}^{n} (\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)} v_k^+\right) = s_k^+ \tag{46}$$

and

$$F_{S_U^-}(s_k^-) = \Phi(w_k^-) \tag{47}$$

where w_k^- follows from

$$\sum_{i=1}^{n} (-\alpha_i)_+ exp\left(-\mu_{(i)} + \sigma_{(i)} w_k^-\right) = s_k^-. \tag{48}$$

The approximate correlation ρ_S^U then equals

$$\rho_S^U = \frac{\frac{1}{N}\sum_{k=1}^{N} \Phi(v_k^+)\Phi(w_k^-) - \frac{1}{N}\sum_{k=1}^{N} \Phi(v_k^+) \cdot \frac{1}{N}\sum_{\ell=1}^{N} \Phi(w_\ell^-)}{\sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\Phi(v_k^+) - \frac{1}{N}\sum_{\ell=1}^{N}\Phi(v_\ell^+)\right)^2}\sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\Phi(w_k^-) - \frac{1}{N}\sum_{\ell=1}^{N}\Phi(w_\ell^-)\right)^2}}. \tag{49}$$

3.4 Distribution of the present value

Suppose that the exact bivariate distribution of S^+ and S^- is expressed by means of a copula as

$$\begin{aligned} H(s^+, s^-) &= \text{Prob}[S^+ \leq s^+, S^- \leq s^-] \\ &= C\left(F_{S^+}(s^+), F_{S^-}(s^-); \rho_S\right). \end{aligned} \tag{50}$$

Starting from this joint distribution, the cumulative distribution function of the difference $S = S^+ - S^-$ can be written as

$$\begin{aligned} F_S(s) &= \text{Prob}[S^+ - S^- \leq s] \\ &= \int_0^\infty \int_0^\infty \frac{\partial^2}{\partial s^+ \partial s^-} H(s^+, s^-)\, \mathcal{I}_{(-s^+ + s^- + s \geq 0)}\, ds^+ ds^- \\ &= \int_0^1 \int_0^1 \frac{\partial^2}{\partial u \partial v} C(u, v; \rho_S)\, \mathcal{I}_{(-F_{S^+}^{-1}(u) + F_{S^-}^{-1}(v) + s \geq 0)}\, du\, dv \\ &= \int_0^1 du \int_{F_{S^-}\left(F_{S^+}^{-1}(u) - s\right)}^1 \frac{\partial^2}{\partial u \partial v} C(u, v; \rho_S)\, dv. \end{aligned} \tag{51}$$

Splitting up the integration over u, and carrying through the integration over v, this results in

$$
\begin{aligned}
F_S(s) &= F_{S^+}(s) + \int_{F_{S^+}(s)}^{1} du \int_{F_{S^-}\left(F_{S^+}^{-1}(u)-s\right)}^{1} \frac{\partial^2}{\partial u \partial v} C(u,v;\rho_S)\, dv \\
&= F_{S^+}(s) \\
&\quad + \int_{F_{S^+}(s)}^{1} du \left[\frac{\partial}{\partial u} C\left(u,1;\rho_S\right) - \frac{\partial}{\partial u} C\left(u, F_{S^-}\left(F_{S^+}^{-1}(u) - s\right); \rho_S\right)\right] \\
&= 1 - \int_s^{\infty} dF_{S^+}(k) \frac{\partial}{\partial u} C\left(F_{S^+}(k), F_{S^-}(k-s); \rho_S\right). \qquad (52)
\end{aligned}
$$

The previous result is exact. Making use of the approximation for the joint distribution as explained in the previous section, together with (52) we get an appropriate approximation for the distribution of the present value (1), or

$$
F_S \approx F_{S_{cop}} \qquad (53)
$$

with

$$
F_{S_{cop}}(s) = 1 - \int_s^{\infty} dF_{S_U^+}(k) \frac{\partial}{\partial u} C\left(F_{S_U^+}(k), F_{S_U^-}(k-s); \rho_S^U\right). \qquad (54)
$$

Note that in case we use a copula as mentioned in equation (23), we have

$$
\frac{\partial}{\partial u} C(u,v;\rho) = p_1 \cdot u + p_2 \cdot \mathcal{U}(v-u) + p_3 \cdot \mathcal{U}(u+v-1), \qquad (55)
$$

where $\mathcal{U}(x)$ denotes the Heaviside function, or

$$
\mathcal{U}(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}. \qquad (56)
$$

If we use a Gumbel copula as in suggested in equation (25), we have

$$
\begin{aligned}
\frac{\partial}{\partial u} C(u,v;\rho) &= \frac{1}{u} \cdot (-\log u)^{1/\beta - 1} \left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta - 1} \\
&\quad \cdot exp\left\{ -\left((-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right)^{\beta}\right\}. \qquad (57)
\end{aligned}
$$

4 Numerical illustration and conclusion

In this last section, we want to examine the accuracy of our approximation compared to the exact distribution of the present value and compared to the comonotonic bounds in [7]. In order to do so, we investigate the cumulative distribution function of the present value in two examples with a different cash flow structure :

- $\alpha_i = \begin{cases} -1 & i = 1, \ldots, 5 \\ +1 & i = 6, \ldots, 20, \end{cases}$
- $\alpha_i = \begin{cases} -1 & i = 1, 3, 5, \ldots, 19 \\ +1 & i = 2, 4, 6, \ldots, 20. \end{cases}$

The parameters of the log-normally distributed rates of return are chosen as in [7] :

- $\mu_i = \mu = 0.07$
- $\sigma_i = \sigma = 0.1.$

For the first cash flow, the estimated values for correlation and Gumbel parameter are

- $\rho_S = 0.6391160$
- $\beta = 0.5372741.$

Due to the specific structure of the second cash flow, the correlation there is much higher :

- $\rho_S = 0.992937688$
- $\beta = 0.075625.$

For both examples, we first computed the quantiles of the variables S_U^+ and S_U^-. Afterwards, a simulation as explained in equations (43)-(49), provided us with an estimator for the correlation ρ_S.
Both examples are elaborated by means of a Gumbel copula ; the value of β is calculated from the estimator for the correlation as mentioned in (26).

Figure 2 shows the quantiles for the present value for the cash flows in the first example. One can see that the convex upperbound S_U performs very badly. On the other hand, our new copula approximation S_{cop} seems to be very accurate in approximating the exact distribution, the convex lowerbound

S_L is even almost indistinguishable from the cdf of the exact present value, obtained by simulation.
The same observations can be made for the second example, for which a graph is shown in figure 3.

It should be remarked that, since the calculation of an estimator for the correlation ρ_S is rather complicated, it seems that the efficiency of the approximation by means of copulas is rather low compared to the very good result for the lowerbound as obtained in [7]. Although the lowerbound is not a sum of comonotonic risks, the calculation of its distribution function is still more easy than it is the case for the copula approximation.

Acknowledgement

The authors would like to express their thanks to Mr. Ronny Suetens. He helped in programming the computations needed for the last section, and he provided a part of the numerical results.

Corresponding author :

Ann De Schepper
University of Antwerp, Faculty of Applied Economics Ufsia-Ruca
Middelheimlaan 1
B-2020 Antwerp
Belgium
tel : +32-3-218.07.86
fax : +32-3-218.07.00
email : ann.deschepper@ua.ac.be

References

[1] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. & Vyncke D. (2002). "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics and Economics*, vol.31(1), p.3-34.

[2] Dhaene J., Denuit M., Goovaerts M.J., Kaas R. & Vyncke D. (2002). "The concept of comonotonicity in actuarial science and finance: applications", *Insurance: Mathematics and Economics*, vol.31(2), p.133-162.

[3] Embrechts P., Lindskog, F. & McNeil, A. (2001). "Modelling Dependence with Copulas and Applications to Risk Management", *to be published*.

[4] Genest C. & MacKay R.J. (1986). "Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données", *The Canadian Journal of Statistics*, vol.14(2), p.145-159.

[5] Goovaerts M., Dhaene J., & De Schepper A. (2000). "Stochastic upper bounds for present value functions", *Journal of Risk and Insurance*, vol.67(1), p.1-14.

[6] Goovaerts M., Kaas R. (2002). "Some problems in actuarial finance involving sums of dependent risks", *Statistica Neerlandica*, vol.56(3), p.253-269.

[7] Kaas R., Dhaene J., & Goovaerts M. (2000). "Upper and lower bounds for sums of variables", *Insurance: Mathematics and Economics*, vol.27(2), p.151-168.

[8] Nelsen R.B. (1999). *An Introduction to Copulas*, Springer-Verlag, New York, 216 p.

[9] Vyncke D., Goovaerts M. & Dhaene J. (2000). "Convex upper and lower bounds for present value functions", *Applied Stochastic Models in Business and Industry*, vol.17(2), p.149-164.

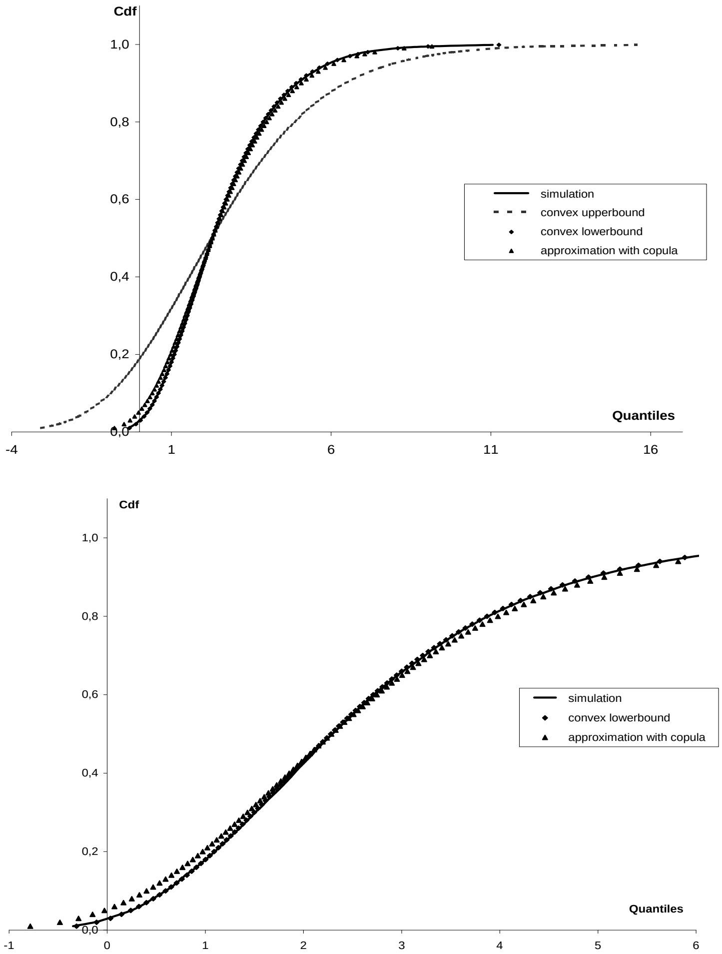

Figure 2: *CDF for the present value of the first cash flow, with and without the convex upperbound*

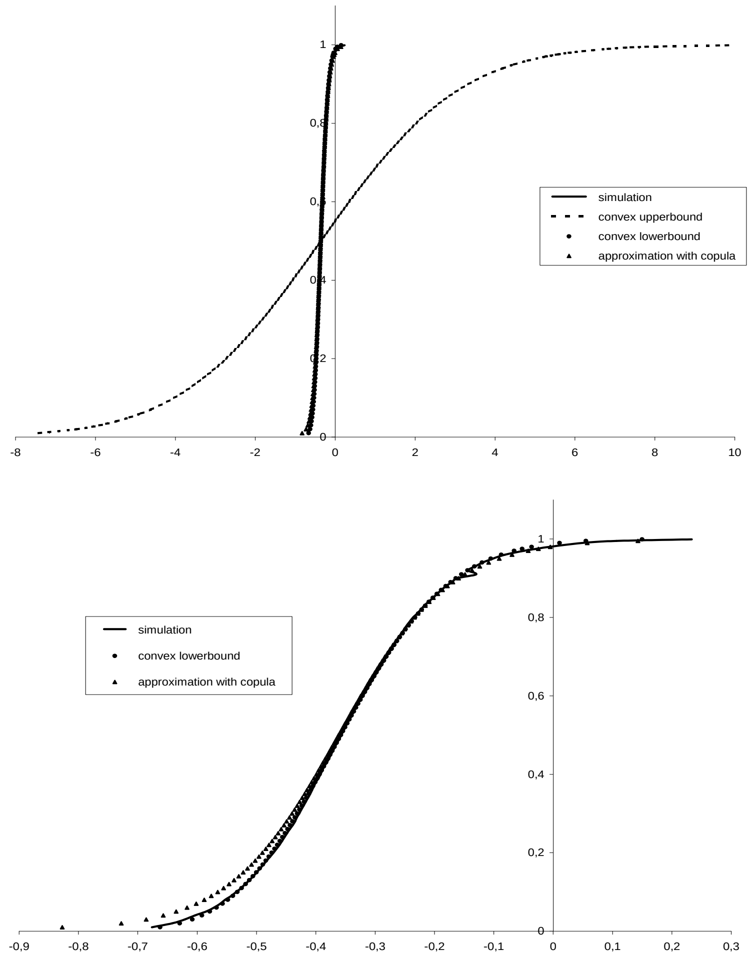

Figure 3: *CDF for the present value of the second cash flow, with and without the convex upperbound*