Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF PURE AND APPLIED ALGEBRA
ELSEVIER
Journal of Pure and Applied Algebra 207 (2006) 675–686
www.elsevier.com/locate/jpaa

Quantified functional analysis and seminormed spaces: A dual adjunction

M. Sioen[a], S. Verwulgen[b,*]

[a] *Free University of Brussels, Pleinlaan 2, B-1020 Brussels, Belgium*
[b] *University of Antwerp, Middelheimlaan 1, B-2020 Antwerp, Belgium*

Received 14 June 2004; received in revised form 5 October 2005; accepted 30 October 2005
Available online 19 January 2006
Communicated by J. Adámek

Abstract

In this work we investigate the natural algebraic structure that arises on dual spaces in the context of quantified functional analysis. We show that the category of absolutely convex modules is obtained as the category of Eilenberg–Moore algebras induced by the dualization functor $[-, \mathbb{R}]$ on locally convex approach spaces. We also establish a dual adjunction between the latter category and the category of seminormed spaces.

MSC: 41A65; 52A01; 46B10; 46M15

1. Introduction

Approach theory as started in [2] draws heavily upon the motivation of providing a remedy for the lack of stability under products of the concept of metrizability in topology: an arbitrary uncountable product of metrizable topological spaces is hardly ever metrizable,

* Corresponding address: Universiteit Antwerpen, Department Wiskunde en Informatica, Middelheimann 1, 2020 Anwerpen, Belgium.

E-mail addresses: msioen@vub.ac.be (M. Sioen), stijn.verwulgen@ua.ac.be (S. Verwulgen).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.10.021

and for countably infinite products, there is no preferred, 'canonical' metric on the product which does the job. However, suitably axiomatized point-set distances which cannot be retrieved from point–point distances (as in the metric case) seem to capture exactly that part of the metric information that can be retained in concordance with topological products. The topological construct **Ap** of approach spaces and contractions yields a category containing both the category **Top** of topological spaces and continuous maps, and the category **Met** of metric spaces and non-expansive maps as (full !) subcategories. We refer the reader to [3] for a more detailed account on this.

An interesting phenomenon occurs when one tries to 'merge' these approach structures with algebraic operations like for example vector space operations (in analogy to topological vector spaces): simply internalizing the vector space operations in **Ap** yields a totally unsatisfactory category, since the object $\mathbb{R}$ of the reals with its canonical Euclidean norm does not belong to it, for the addition is a 2-Lipschitz map and the multiplication not even Lipschitz. The category **ApVec**, introduced in [20], solves the problem with the products mentioned above for vector pseudometrics in the sense of [18]. This constitutes an acceptable candidate deserving the name of a 'quantification' of the category **TopVec** of topological vector spaces, since it contains both **TopVec** and the category of all vector-pseudometric spaces as full subcategories in a neat way. The situation becomes even nicer: by looking at the epireflective hull of $\mathbf{sNorm}_1$ (the category of seminormed spaces and linear non-expansive maps) in **ApVec**, the category **lcApVec** of locally convex approach spaces is obtained [20]. The introduction of **lcApVec** is solving again the productivity problem from above, now for seminorms in relation to locally convex topological spaces. Throughout the paper we denote the category of locally convex topological spaces by **lcTopVec**. So we propose **ApVec** and **lcApVec** as a framework for 'quantified functional analysis'. Classically for (non-metrizable/non-semi-normable) topological vector spaces, only an 'isomorphic' theory exists, whereas now, when we start e.g. from a well-chosen defining set of vector pseudometrics/seminorms, working on the approach level allows an 'isometric' theory, where canonical numerical concepts exist. For example, one can express 'how far is a vector away from being a limit point of a given net', rather than just knowing whether or not the filter or net converges to the vector. Examples and more results on these can be found e.g. in [4].

In this paper, we will study some aspects of duality theory by looking at the hom-functor $[-,\mathbb{R}] : \mathbf{lcApVec}^{\mathbf{op}} \to \mathbf{Set}$. These hom-sets bear the structure of an absolutely convex subset of a vector space and thus a major goal in this work is to demonstrate the existence of an intricate relationship between **AC**, the category of absolutely convex modules [12, 13], and locally convex approach spaces. We thereby use the theory of monads and their Eilenberg–Moore algebras, see e.g. [6,10,1], as a concrete connection. This is done in the third section. Notice that similar techniques were already successfully applied in the realm of functional analysis. We point in particular to the work of Negrepontis [8], where the investigation of the dual space of continuous functions from a compact Hausdorff space to the complex unit disc leads to the category of C^* algebras as Eilenberg–Moore algebras, with the Gelfand transformation as the comparison functor.

Finally we obtain, in the last section, a dual adjunction between $\mathbf{sNorm}_1$ and **lcApVec** which moreover implies that **lcApVec** not only contains $\mathbf{sNorm}_1$ but also a copy of its dual as a subcategory. We also use the introduced functors to yield an explicit description, by bi-dualization, of the adjoint to the comparison functor between $\mathbf{sNorm}_1$ and **AC**, alternative to the one given in [12].

We add that all the dual adjunctions appearing in this work are based on the use of $\mathbb{R}$ as a schizophrenic object, i.e. they fit in the categorical framework described by Porst and Tholen in [9] and thus they are easy to understand from a structural point of view. For all other facts of a categorical nature used in this paper one may consult [1].

In order to start we have to recall some basic facts about the theory of absolutely convex modules and the theory of locally convex approach spaces.

2. Preliminaries

For an arbitrary set S, let $l_1^{\text{fin}} S$ denote the free vector space $l^{\text{fin}} S := \{a : S \to \mathbb{R} \mid \{a \neq 0\} \text{ is finite}\}$ equipped with the sum–norm $\|a\|_1 := \sum_{s \in S} |a(s)|$. The assignment $S \longmapsto l_1^{\text{fin}} S$ gives rise to a functor

$$l_1^{\text{fin}} : \mathbf{Set} \to \mathbf{sNorm}_1 : (S_1 \xrightarrow{f} S_2) \longmapsto (l_1^{\text{fin}} S_1 \xrightarrow{l_1^{\text{fin}} f} l_1^{\text{fin}} S_2),$$

with $l_1^{\text{fin}} f(a) := \sum_{s \in S_1} a(s)\delta_{f(s)}$, and where $\mathbf{sNorm}_1$ denotes the category of seminormed spaces and linear non-expansive maps. It is well known (see e.g. [12]) that this is a left adjoint to the functor

$$O : \mathbf{sNorm}_1 \to \mathbf{Set} : X \xrightarrow{f} Y \longmapsto OX \xrightarrow{f|_{OX}} OY,$$

where OX is the closed unit ball of a seminormed space X. In the following we put η' and ϵ' to denote respectively the unit and counit of the adjunction $l_1^{\text{fin}} \dashv O$. We write $\mathbf{T}' = (T', \eta', \mu')$ for the monad defined by this adjunction, i.e. $T' = O l_1^{\text{fin}}$ and $\mu' = O \epsilon' l_1^{\text{fin}}$. An absolutely convex module is the abstraction of the algebraic structure of the closed unit ball of a normed space. To be precise, the Eilenberg–Moore algebras induced by $\mathbf{T}'$ constitute the category **AC** of absolutely convex modules and absolutely affine maps. An absolutely convex module was characterized in [12] as a set X on which, for each finite sequence of scalars $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, with $\sum_{i=1}^n |\alpha_i| \leq 1$, an operation $\hat{\alpha} : X^n \to X$ is given, such that, with $\sum_{i=1}^n \alpha_i x_i := \hat{\alpha}(x_1, \ldots, x_n)$, the following identities are satisfied:

(AC1) $\sum_{i=1}^n \delta_i(k) x_i = x_k$,
(AC2) $\sum_{j=1}^m \beta_j (\sum_{i=1}^n \alpha_{ij} x_i) = \sum_{i=1}^n (\sum_{j=1}^m \beta_j \alpha_{ij}) x_i$.

The notation $\hat{O}X$ is used to denote OX endowed with the pointwise absolutely convex structure. It is in this way that the comparison functor $\hat{O} : \mathbf{sNorm}_1 \to \mathbf{AC}$ of the adjunction $l_1^{\text{fin}} \dashv O$ is obtained. We should add that the above was originally formulated in the setting of normed spaces, but it is easily seen to hold in the setting, needed for our development, of seminormed spaces. An absolutely convex module is called closed [21] if it is isomorphic to the closed unit ball of some seminormed space. Semadeni was the first to stress the importance of absolutely convex modules [19]. Then they were clearly defined and intensively studied by Pumplün and Röhrl [12–16,11].

Approach vector spaces were introduced in [5], as a result of describing the precise conditions an approach structure has to satisfy in order to fit well with the algebraic structure of a vector space. It was subsequently shown [20] that those approach vector spaces for which the local approach system has a basis of convex functions (they are called locally convex approach spaces) have a stunningly simple and useful characterization. A locally convex approach space is a pair $X = (X, \mathcal{M})$ consisting of a vector space X and a Minkowski system $\mathcal{M}$. The latter is a saturated ideal in the lattice of seminorms on X, i.e. an order theoretic ideal satisfying the saturation condition (S) below.

(S) If η is any seminorm on X such that for any $\epsilon > 0$, there exists a $\mu \in \mathcal{M}$ such that $\eta \leq (1+\epsilon)\mu$, then η is also in $\mathcal{M}$.

The smallest set of seminorms containing a given collection of seminorms and satisfying (S) is called the saturation of that set. This saturation is a Minkowski system, if the original set is an ideal basis. A morphism or linear contraction between locally convex approach spaces $(X_1, \mathcal{M}_1)$ and $(X_2, \mathcal{M}_2)$ is a linear map $f : X \to Y$ such that, for all η in $\mathcal{M}_2$, $\eta \circ f$ is in $\mathcal{M}_1$. The category of locally convex approach spaces has many good properties, among which is the fact that the forgetful functor **lcApVec** $\longrightarrow$ **Vec** is topological. In terms of Minkowski systems, this yields that for any source

$$(X \xrightarrow{f_i} (X_i, \mathcal{M}_i))_{i \in I},$$

where X is a vector space, $(X_i, \mathcal{M}_i)$ are locally convex approach spaces and f_i are linear maps, there exists a smallest Minkowski system $\mathcal{M}$ on X such that all the f_i's are contractions. This $\mathcal{M}$ is given by the saturation of the set

$$\left\{ \sup_{j=1}^{n} \eta_{i_j} \circ f_{i_j} \mid n \in \mathbb{N}_0, \forall j \in \{1, \dots, n\} : i_j \in I,\ \eta_{i_j} \in \mathcal{M}_{i_j} \right\}. \tag{1}$$

The set of continuous seminorms of a given locally convex topology is a Minkowski system, and the identification of the topology with it (see e.g. [17,18]) yields a full concrete embedding **lcTopVec** $\hookrightarrow$ **lcApVec**. Moreover we have a right adjoint

$$\mathbf{lcApVec} \underset{\hookleftarrow}{\longrightarrow} \mathbf{lcTopVec}.$$

The coreflection arrow is concrete, given by $\mathrm{id} : (X, \mathbb{R}^+\mathcal{M}) \longrightarrow (X, \mathcal{M})$, with $\mathbb{R}^+\mathcal{M} := \{k\eta \mid k > 0, \eta \in \mathcal{M}\}$. Note that the corresponding topology is precisely the topology induced by $\mathcal{M}$. The identification $(X, \eta) \leftrightarrow (X, \mathcal{M}_\eta)$, with $\mathcal{M}_\eta := \{\mu \text{ seminorm on } X \mid \mu \leq \eta\}$ yields a full embedding of $\mathbf{sNorm}_1$ into **lcApVec**. Moreover, we have a right adjoint $N : \mathbf{lcApVec} \longrightarrow \mathbf{sNorm}_1$ to this embedding. The coreflection arrow, for a locally convex approach space $(X, \mathcal{M})$, is given by the embedding $N(X, \mathcal{M}) \longrightarrow (X, \mathcal{M})$, where $N(X, \mathcal{M})$ is the space $\{x \in X \mid \sup_{\mu \in \mathcal{M}} \mu(x) < \infty\}$, equipped with the seminorm $x \longmapsto \sup_{\mu \in \mathcal{M}} \mu(x)$.

We will only consider vector spaces over the field of real numbers throughout this work. It is under the above identification that the scalar field $\mathbb{R} = (\mathbb{R}, |\ |)$ is considered to be a locally convex approach space.

Note that the following lemma, which we recall from [21], gives us a linkage between the product locally convex approach space $\mathbb{R}^S$ and the free **AC** algebra on a set S. This will be of paramount importance in the establishment of the main theorem in the following section.

Lemma 2.1. *Let S be a non-empty set. Then for any $\phi \in [\mathbb{R}^S, \mathbb{R}]$ there exists a non-empty finite set $S' \subset S$ and a collection of scalars $(b_s)_{s\in S'}$ with $\sum_{s\in S'} |b_s| \leq 1$ such that $\phi(-) = \sum_{s\in S'} b_s \mathrm{ev}(-, s)$.*

Here we consider $\mathbb{R}^S$ with the **lcApVec**-product structure, that is the Minkowski system given by the saturation of the basis $\{\sup_{s\in S'} |\mathrm{ev}(-, s)| : S' \subset S \text{ finite}\}$.

In the following we use the notation $\mathcal{M}_{(X,X^{\odot})}$ to denote the initial Minkowski system of the source $(X \xrightarrow{\phi} \mathbb{R})_{\phi\in X^{\odot}}$, where $X^{\odot}$ is a subset of the algebraic dual of a vector space X.

3. The algebra on dual spaces

In the following we put KX to denote the hom-set $[X, \mathbb{R}]$, i.e. the set of all linear contractive functionals on a locally convex approach space X.

Definition 3.1. It was shown in [21] that KX is the closed unit ball of a seminorm on the set of all linear functionals that are continuous with respect to the topological coreflection of X; the resulting seminormed space is denoted by $L_{\hat{K}} X$.

This definition is crucial in the development of the following section.

We put K for the covariant hom-functor $[-, \mathbb{R}] : \mathbf{lcApVec}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and, for a set S, we denote by FS the **lcApVec**-product $\mathbb{R}^S$. It is straightforward to verify that the map

$$\eta_S : S \longrightarrow KFS : s \longmapsto \mathrm{ev}(-, s) \tag{2}$$

is universal for K. Hence the assignment $(S_1 \xrightarrow{f} S_2) \longmapsto (FS_2 \xrightarrow{Ff} FS_1 : a \longmapsto a \circ f)$ defines a left adjoint $F : \mathbf{Set} \to \mathbf{lcApVec}^{\mathrm{op}}$ to K. The counit ϵ of the adjunction $F \dashv K$ is given, on a locally convex approach space, by

$$\epsilon_X : X \to FKX : x \longmapsto \mathrm{ev}(-, x). \tag{3}$$

Let $\mathbf{T} = (T, \eta, \mu)$ denote the monad of this adjunction.

For each s in a fixed set S, the map $\mathrm{ev}(-, s) : FS \to \mathbb{R}$ is in $KFS = TS$. Since TS is closed under the formation of pointwise absolutely convex combinations, we obtain a map

$$\tau_S : Ol_1^{\mathrm{fin}} = T'S \longrightarrow TS : b \longmapsto \sum_{s\in S} b(s)\mathrm{ev}(-, s), \tag{4}$$

which is easily seen to be an injection. Then Lemma 2.1 asserts that τ_S is bijective. Before stating our main result we mention that, whenever $f : X \to Y$ is a morphism between locally convex approach spaces, $Kf : \hat{K}Y \to \hat{K}X$ is a morphism between totally convex modules, the convention being to denote KX endowed with the pointwise absolutely convex module structure by $\hat{K}X$. We thus actually have a functor

$$\hat{K} : \mathbf{lcApVec}^{\mathrm{op}} \to \mathbf{AC}. \tag{5}$$

Theorem 3.2. *The collection of mappings $\tau := (\tau_S)_{S\in|\mathbf{Set}|}$ defines a monad isomorphism $\tau : \mathbf{T}' \to \mathbf{T}$, so τ induces an isomorphism between $\mathbf{Set}^{\mathbf{T}}$, the category of Eilenberg–Moore algebras of the monad induced by the adjunction $F \dashv K$, and the category* **AC**. *Moreover the functor $\hat{K}$ is the comparison functor with respect to this representation.*

Proof. If we show that τ is a monad morphism then, since each τ_S is a bijection, it follows that τ is an isomorphism between $\mathbf{T}'$ and $\mathbf{T}$. The remainder of the assertion follows from categorical routine work. Here are the needed verifications.

1. $\tau : T' \to T$ is a natural transformation. Take a map $f : S_1 \longrightarrow S_2$ between sets. We have to verify the commutation of

$$\begin{tikzcd} T'S_1 \arrow[r, "T'f"] \arrow[d, "\tau_{S_1}"'] & T'S_2 \arrow[d, "\tau_{S_2}"] \\ TS_1 \arrow[r, "Tf"'] & TS_2. \end{tikzcd}$$

Let $b = \sum_{s\in S_1} b(s)\delta_s \in T'S_1$. Then on the one hand we have that

$$\begin{aligned}(\tau_{S_2} \circ T'f)(b) &= \tau_{S_2}\left(\sum_{s\in S_1} b(s)\delta_{f(s)}\right)\\ &= \sum_{s\in S_1} b(s)ev(-, f(s))\end{aligned}$$

and on the other hand we have

$$\begin{aligned}(Tf \circ \tau_{S_1})(b) &= KFf\left(\sum_{s\in S_1} b(s)ev(-, s)\right)\\ &= \sum_{s\in S_1} b(s)(ev(-, s) \circ Ff)\\ &= \sum_{s\in S_1} b(s)ev(-, f(s)).\end{aligned}$$

2. τ rewrites the unit, i.e. we have to check that for an arbitrary set S the diagram

$$\begin{tikzcd} S \arrow[r, "\eta'_S"] \arrow[dr, "\eta_S"'] & T'S \arrow[d, "\tau_S"] \\ & TS \end{tikzcd}$$

is commutative. But this is trivial: $(\tau_S \circ \eta'_S)(s) = \tau_S(\delta_s) = \mathrm{ev}(-, s) = \eta_S(s)$ for all $s \in S$.

3. Last we have to verify that τ rewrites the multiplication, that is, for any set S we have the commutation of the diagram

$$\begin{tikzcd} T'^2S \arrow[r, "\mu'_S"] \arrow[d, "T'\tau_S"] \arrow[dd, bend right=60, "\tau_S^2"'] & T'S \arrow[dd, "\tau_S"] \\ T'TS \arrow[d, "\tau_{TS}"] & \\ T^2S \arrow[r, "\mu_S"'] & TS. \end{tikzcd}$$

In order to show this, let $B \in T'^2S$, where we write $B = \sum_{b\in T'S} B(b)\delta_b$. Then $\mu'_S(B) = \sum_{b\in T'S} B(b)b$, so

$$(\tau_S \circ \mu'_S)(B) = \sum_{s\in S}\sum_{b\in T'S} B(b)b(s)\mathrm{ev}(-, s).$$

On the other hand we should compute $(\mu_S \circ \tau_S^2)(B) = (\mu_S \circ \tau_{TS} \circ T'\tau_S)(B)$. First, note that

$$T'\tau_S(B) = \sum_{b\in T'S} B(b)\delta_{\tau_S(b)},$$

so

$$(\tau_{TS} \circ T'\tau_S)(B) = \sum_{b\in T'S} B(b)\mathrm{ev}(-, \tau_S(b))$$

and

$$(\mu_S \circ \tau_{TS} \circ T'\tau_S)(B) = \sum_{b\in T'S} B(b)\mathrm{ev}(-, \tau_S(b)) \circ \epsilon_{FS}.$$

For all $a \in FS$ we have that

$$\begin{aligned} \mathrm{ev}(-, \tau_S(b)) \circ \epsilon_{FS}(a) &= \mathrm{ev}(\mathrm{ev}(-, a), \tau_S(b)) \\ &= \mathrm{ev}(\tau_S(b), a) \\ &= \tau_S(b)(a) \\ &= \left(\sum_{s\in S} b(s)\mathrm{ev}(-, s)\right)(a), \end{aligned}$$

hence

$$\begin{aligned} (\mu_S \circ \tau_{TS} \circ T'\tau_S)(B) &= \sum_{b\in T'S} B(b)\left(\sum_{s\in S} b(s)\mathrm{ev}(-, s)\right) \\ &= \sum_{b\in T'S}\sum_{s\in S} B(b)b(s)\mathrm{ev}(-, s). \quad \square \end{aligned}$$

Example 3.11 in [21] teaches us that the adjunction $F \dashv K$ is not premonadic, so we can not use this fact to conclude that $\hat{K}$ is part of an adjunction (see [10]). Notice that the vector space structure puts the structure of an absolutely convex module on $\mathbb{R}$ and that WZ, the set of all absolutely affine mappings from an absolutely convex set Z to $\mathbb{R}$, is a vector space in a pointwise way. Now for all $z \in Z$, $\eta''_Z(z) := \mathrm{ev}(-, z)$ is a linear functional on WZ, so WZ can be endowed with the locally convex approach structure $\mathcal{M}_{(WZ,\eta''(Z))}$. Then it is elementary to verify that the absolutely affine map

$$\begin{aligned} \eta''_Z : \quad Z &\longrightarrow \hat{K}WZ \\ z &\longmapsto \mathrm{ev}(-, z) \end{aligned} \tag{6}$$

is universal for $\hat{K}$, i.e. the assignment

$$W : \mathbf{AC} \longrightarrow \mathbf{lcApVec}^{\mathrm{op}} : (Y \xrightarrow{f} Z) \longmapsto (WZ \xrightarrow{Wf} WY : a \longmapsto a \circ f)$$

defines a functor that is left adjoint. The counit of the adjunction $W \dashv \hat{K}$ is given on a locally convex approach space X by the map

$$\begin{aligned} \epsilon''_X : \quad X &\longrightarrow W\hat{K}X \\ x &\longmapsto \mathrm{ev}(-, x). \end{aligned} \tag{7}$$

It is fascinating to see that we can retrieve the weak locally convex approach structure $\mathcal{M}_{(X,KX)}$ [21] out of this adjunction. Indeed, all the diagrams in the collection

$$\begin{tikzcd} W\hat{K}X \arrow[r, "\eta_{\hat{K}X}(f)"] & \mathbb{R} \\ (X, \mathcal{M}_{(X,KX)}) \arrow[u, "\epsilon''_X"] \arrow[ur, "f"'] & \end{tikzcd} \qquad f \in KX$$

are commutative, and the horizontal source and the diagonal source are initial, so $\mathcal{M}_{(X,KX)}$ is initial for the linear map in (7).

4. The dual adjunction with seminormed spaces

We know from [12] that the adjunction $l_1^{\mathrm{fin}} \dashv O$ is premonadic. Since the KX is the closed unit ball of $L_{\hat{K}}X$ we thus deduce from the comparison functor $\hat{K}$ another dualization functor

$$L_{\hat{K}} : \mathbf{lcApVec}^{\mathrm{op}} \longrightarrow \mathbf{sNorm}_1 : (X \xrightarrow{f} Y) \longmapsto (L_{\hat{K}}Y \xrightarrow{L_{\hat{K}}f} L_{\hat{K}}X : a \longmapsto a \circ f).$$

Note that if $f : X \longrightarrow Y$ is a morphism between seminormed spaces then the map $L_{\hat{K}}f$ is the well-known adjoint of f (see e.g. [17]). If a vector space X is endowed with the trivial seminorm then $L_{\hat{K}}X = \{0\}$ and any linear map from X to X is a morphism. So it is seen that $L_{\hat{K}}$ is not faithful. The functor $L_{\hat{K}}$ is not full either since this would imply that every seminormed space is isomorphic to its bidual.

Remark that the classical, topological, hom-functor $[-, \mathbb{R}]_t : \mathbf{lcTopVec}^{\mathrm{op}} \to \mathbf{Set}$ has a left adjoint. Moreover it can be shown using the technique from Section 3 that the bijection $l^{\mathrm{fin}}S \to L\mathbb{R}^S : b \longmapsto \sum_{s \in S} b(s)\mathrm{ev}(-, s)$ induces a representation of the corresponding Eilenberg–Moore algebras as the category of vector spaces, where the comparison functor

$$\widehat{[-, \mathbb{R}]_t} : \mathbf{lcTopVec}^{\mathrm{op}} \to \mathbf{Set}$$

is described by considering the pointwise vector space structure on dual spaces. This justifies, from a structural point of view, the common intuition that the conjugate of a locally convex topological space is in a natural way a vector space. Notice the commutativity of the diagram

$$\begin{tikzcd} \mathbf{lcApVec}^{\mathrm{op}} \arrow[r, "L_{\hat{K}}"] \arrow[d] & \mathbf{sNorm}_1 \arrow[d] \\ \mathbf{lcTopVec}^{\mathrm{op}} \arrow[r, "\widehat{[-,\mathbb{R}]}_t"'] & \mathbf{Vec}, \end{tikzcd}$$

where the vertical arrows are the usual forgetful functors.

Note that in the situation

$$\begin{tikzcd} \mathbf{lcApVec}^{\mathrm{op}} \arrow[rr, "L_{\hat{K}}"] \arrow[dr, shift left, "\hat{K}"] & & \mathbf{sNorm}_1 \arrow[dl, "\hat{O}"] \\ & \mathbf{AC}, \arrow[ul, shift left, "W"] & \end{tikzcd}$$

we have $\hat{O}L_{\hat{K}} = \hat{K}$.

Recall that in [4] a notion of weak*-distance was given on the topological dual of a normed space. Here we introduce the same notion on the algebraic dual of a seminormed space, i.e. for a seminormed space X we put $L_{\mathrm{pcu}}X$ to denote the algebraic dual of X, endowed with $\mathcal{M}_{(X^*,(OX)^{**})}$. We have $OX^{**} := \{\mathrm{ev}(-,x) : X^* \longrightarrow \mathbb{R} \mid x \in OX\}$, so the underlying topology is the topology of pointwise convergence on the unit ball.

Theorem 4.1. *Let X be a seminormed space. Then the map*

$$\begin{array}{rccc} \alpha_X : & X & \longrightarrow & L_{\hat{K}}L_{\mathrm{pcu}}X \\ & x & \longmapsto & \mathrm{ev}(-,x) \end{array} \tag{8}$$

is an isomorphism of seminormed spaces which is universal for $L_{\hat{K}}$.

Proof. Let us put α_{OX} to denote the restriction of α_X to OX. We know from the remark at the beginning of this section that for the first assertion it suffices to show that α_{OX} yields an isomorphism between the absolutely convex modules $\hat{O}X$ and $\hat{K}L_{\mathrm{pcu}}X$. By definition, $\alpha_{OX} : OX \to (OX)^{**}$ is a surjection. Take $x \neq y \in OX$. Then, with the axiom of choice, one obtains a linear basis $S \subset X$ containing $\{x, y\}$ and, with $f : X \to \mathbb{R}$ the linear extension of the assignment

$$S \to \mathbb{R} : s \longmapsto \begin{cases} 1 & s = x, \\ 0 & \text{otherwise}, \end{cases}$$

we have that $f(x) \neq f(y)$, from which $\alpha_{OX}(x) \neq \alpha_{OX}(y)$ follows. It is trivial to verify that, for any $x_1, \ldots, x_n \in OX$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^{n} |\alpha_i| \leq 1$, the bijection

$\alpha_{OX} : OX \to (OX)^{**}$ satisfies the identity

$$\alpha_{OX}\left(\sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i\alpha_{OX}(x_i). \tag{9}$$

So $(OX)^{**}$ is an absolutely convex subset of $(L_{\mathrm{pcu}}X)^*$ that is closed in the sense of [21]. So we can apply Theorem 2.10 [21] and we obtain that $KL_{\mathrm{pcu}}X = (OX)^{**}$.

Let Y be a locally convex approach space, $f : X \to L_{\hat{K}}Y$ be a linear non-expansive map and suppose that we have a linear contraction $\hat{f} : Y \to L_{\mathrm{pcu}}X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & L_{\hat{K}}L_{\mathrm{pcu}}X \arrow[dl, "L_{\hat{K}}\hat{f}"] \\ L_{\hat{K}}Y & \end{tikzcd}$$

commutative. Then, for all $x \in X$ and $y \in Y$, $f(x)(y) = L_{\hat{K}}\hat{f}(\alpha_X(x))(y) = L_{\hat{K}}\hat{f}(\mathrm{ev}(-,x))(y) = \mathrm{ev}(-,x)\circ\hat{f}(y) = \mathrm{ev}(-,x)(\hat{f}(y)) = \hat{f}(y)(x)$.

To show the existence of such $\hat{f}$, note that for each $y \in Y$, the map $\hat{f}(y) : X \to \mathbb{R} : x \longmapsto f(x)(y)$ is linear, so we indeed have a map $\hat{f} : Y \to L_{\mathrm{pcu}}X$, and it is trivial to verify that it is a linear one. For any $x \in OX$ we have that $f(x)$ is in $OL_{\hat{K}}Y = KY$ and from the identity $\mathrm{ev}(-,x)\circ\hat{f} = f(x)$ we obtain by initiality that $\hat{f}$ is a morphism between locally convex approach spaces. It is trivial to verify that $f = \alpha_X \circ L_{\hat{K}}f$. □

By the previous result we obtain a dualization functor

$$L_{\mathrm{pcu}} : \mathbf{sNorm}_1 \longrightarrow \mathbf{lcApVec}^{\mathrm{op}} : (X \xrightarrow{f} Y) \longmapsto (L_{\mathrm{pcu}}Y \xrightarrow{L_{\mathrm{pcu}}f} L_{\mathrm{pcu}}X : a \longmapsto a\circ f)$$

that is left adjoint to $L_{\hat{K}}$. Note that L_{pcu} is equivalent to $W\hat{O}$. We conclude, from the fact that the unit

$$L_{\hat{K}}L_{\mathrm{pcu}} \xrightarrow{\alpha} \mathrm{id}_{\mathbf{sNorm}_1}$$

of this adjoint situation is a natural isomorphism, that the image of $L_{\hat{K}}$ is equivalent to the category $\mathbf{sNorm}_1$. As a consequence of the categorical framework concerning dual adjunctions and dualities developed in [9], one obtains that **lcApVec** not only contains $\mathbf{sNorm}_1$ but also, up to categorical equivalence, $\mathbf{sNorm}_1{}^{\mathrm{op}}$ as a full reflective subcategory. It would be interesting to obtain an internal description of $\mathbf{sNorm}_1^{\mathrm{op}}$ and compare this to the category of Waelbroeck spaces [7].

Theorem 4.1 also yields that the image of the comparison functor $\hat{K} : \mathbf{lcApVec}^{\mathrm{op}} \to \mathbf{AC}$ is isomorphism-dense in the class of closed unit balls. Moreover, note that the category $\mathbf{sNorm}_1$ is isomorphic to the full subcategory **clAcSet** of **AC**, formed by the class of closed unit balls of seminormed spaces. This isomorphism induces an isomorphism between the co-domain restriction $\hat{K} : \mathbf{lcApVec}^{\mathrm{op}} \longrightarrow \mathbf{clAcSet}$ and Ł$_{\hat{K}}$. So the proposition below, showing how a left adjoint of $\hat{O} : \mathbf{sNorm}_1 \longrightarrow \mathbf{AC}$ [12] can be obtained by bi-dualization, automatically holds if **AC** is replaced by **clAcSet**. However extending the left adjoint to the whole of **AC** does not follow for categorical reasons but requires a separate proof.

Proposition 4.2. *The functor $L_{\hat{K}}W : \mathbf{AC} \to \mathbf{sNorm}_1$ is left adjoint to $\hat{O} : \mathbf{sNorm}_1 \longrightarrow \mathbf{AC}$.*

Proof. Let Z be an absolutely convex module. Then $\hat{O}L_{\hat{K}}WZ = \hat{K}WZ$. We claim that the map $\eta''_Z : Z \to \hat{O}L_{\hat{K}}WZ$ from (6) is universal for $\hat{O}$. Take a seminormed space Y and an absolutely affine map $f : Z \longrightarrow \hat{O}Y$. It follows from Theorem 4.1 that there exists a locally convex approach space X together with an absolutely affine isomorphism $\alpha : \hat{O}Y \longrightarrow \hat{K}X$. By the $\hat{K}$-universality of η''_Z, we find a unique linear contraction $\overline{\alpha \circ f} : X \longrightarrow WZ$ such that $\alpha \circ f = \hat{K}(\overline{\alpha \circ f}) \circ \eta''_Z$. There exists a linear contractive lift $g : L_{\hat{K}}WZ \longrightarrow Y$ of $\alpha^{-1} \circ \hat{K}(\overline{\alpha \circ f}) : \hat{K}WZ \longrightarrow \hat{O}Y$, i.e. such that $\hat{O}g = \alpha^{-1} \circ \hat{K}(\overline{\alpha \circ f})$. So $\hat{O}g \circ \eta'' = f$. The uniqueness of this g is a straightforward consequence of the fact that $L_{\hat{K}}WZ$ is the linear hull of the set $\{\mathrm{ev}(-, z) : WZ \to \mathbb{R} \mid z \in Z\}$. Indeed: a linear map $a : WZ \to \mathbb{R}$ is continuous if and only if $a(-) \leq k \sup_{i=1}^{n} |\mathrm{ev}(-, z_i)|$ for a finite number $z_1, \ldots, z_n \in Z$ and for some $k > 0$, which implies (see for example [17]) that a is a linear combination of $(\mathrm{ev}(-, z_i))_{i=1}^{n}$. $\square$

There is an interesting relation between the embedding $\mathbf{sNorm}_1 \hookrightarrow \mathbf{lcApVec}$ and its left adjoint (the seminorm coreflection functor $N : \mathbf{lcApVec} \longrightarrow \mathbf{sNorm}_1$) on one hand, and the adjunction induced by $L_{\hat{K}}$ on the other hand. According to the Hahn–Banach theorem, we have the commutation of the diagram

$$\begin{tikzcd}
\mathbf{sNorm}_1 \arrow[r, hook] \arrow[d, "L_{\mathrm{pcu}}"'] & \mathbf{lcApVec} \arrow[d, "L_{\hat{K}}^{\mathrm{op}}"] \\
\mathbf{lcApVec}^{\mathrm{op}} \arrow[r, "N^{\mathrm{op}}"'] & \mathbf{sNorm}_1^{\mathrm{op}}.
\end{tikzcd}$$

After interpreting the dual seminormed space, of a locally convex approach space X, as a locally convex approach space, one could take the dual again and thus obtain a seminormed space. It is not hard to assert that the map

$$NX \longrightarrow L_{\hat{K}}L_{\hat{K}}X : x \longmapsto \mathrm{ev}(-, x)$$

is initial in $\mathbf{sNorm}_1$ over $\mathbf{Vec}$.

Acknowledgement

We thank the referee for numerous suggestions, amongst which was pointing us to the categorical duality framework [9], leading e.g. to the fact that $\mathbf{sNorm}_1^{\mathrm{op}}$ is fully and reflectively embedded in $\mathbf{lcApVec}$.

References

[1] J. Adámek, H. Herrlich, G. Strecker, Abstract and Concrete Categories, J. Wiley and sons, 1990.

[2] R. Lowen, Approach spaces: A common supercategory of top and met, Mathematische Nachrichten 141 (1989) 183–226.

[3] R. Lowe, Approach spaces: The Missing Link in the Topology–Uniformity–Metric Triad, in: Oxford Mathematical Monographs, Oxford University Press, 1997.
[4] R. Lowen, M. Sioen, Approximations in functional analysis, Results in Mathematics 37 (2000) 345–372.
[5] R. Lowen, S. Verwulgen, Approach vector spaces, Houston Journal of Mathematics 30 (4) (2004) 1127–1142.
[6] S. Mac Lane, Categories for the Working Mathematician, Springer, 1998.
[7] P. Michor, Funktoren zwischen kategorien von banach-und waelbroeck-rumen, Sitzungsberichte sterreichische Akademie Wiss., Abt II 182 (1974) 43–65.
[8] J.W. Negrepontis, Duality in analysis from the point of view of triples, Journal of Algebra 19 (1971) 228–253.
[9] H.E. Porst, W. Tholen, Concrete dualities, in: Category Theory at Work, 1991, pp. 111–136.
[10] D. Pumplün, Eilenberg–Moore algebras revisited, Seminarberichte 29 (1988) 57–144.
[11] D. Pumplün, Absolutely convex modules and Saks spaces, Journal of Pure and Applied Algebra 155 (2001) 257–270.
[12] D. Pumplün, H. Röhrl, Banach spaces and totally convex spaces i, Communications in Algebra 12 (8) (1984) 953–1019.
[13] D. Pumplün, H. Röhrl, Banach spaces and totally convex spaces ii, Communications in Algebra 13 (5) (1985) 1047–1113.
[14] D. Pumplün, H. Röhrl, Separated totally convex spaces, Manuscripta Mathematica 50 (1985) 145–183.
[15] D. Pumplün, H. Röhrl, The coproduct of totally convex spaces, Beiträge zur Algebra und Geometrie 24 (1987) 249–278.
[16] D. Pumplün, H. Röhrl, Convexity theories V: Extensions of absolutely convex modules, Applied Categorical Structures 8 (2000) 527–543.
[17] W. Rudin, Functional Analysis, in: International Series in Pure and Applied Mathematics, McGraw-Hill, 1991.
[18] H.H. Schaefer, Topological Vector Spaces, in: Graduated Texts in Mathematics, Springer-Verlag, 1999.
[19] Z. Semadeni, Monads and their Eilenberg–Moore algebras in functional analysis, Queen's Papers in Pure and Applied Mathematics 33 (1973) 1–98.
[20] M. Sioen, S. Verwulgen, Locally convex approach spaces, Applied General Topology 4 (2) (2003) 263–279.
[21] M. Sioen, S. Verwulgen, Quantified functional analysis: Recapturing the dual unit ball, Results in Mathematics 45 (3–4) (2004) 359–369.