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The Numerical Solution of a Birth-Death Process Arising in Multimedia Synchronization

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Abstract—One of the most important features of multimedia applications is the integration of multiple media streams that have to be presented in a synchronized fashion. In this paper, we consider a distributed multimedia system where the communication between two nodes involve two media. Arrivals consist of two types of media packets, and the packets are processed for pairs of one packet from each media. We view this model as a two-dimensional finite birth-death process by considering the arrivals of the packets, following Poisson distribution, as births and the departures of the impatient packets, after waiting in the network for an exponential period, as deaths. We analyze the time-dependent behaviour of our model numerically. We study the various system characteristics like, time-dependent probabilities of the number of packets in each media, their averages, variances and the busy period. They are illustrated through tables and graphs.

Keywords—Two-dimensional birth-death process, Time-dependent analysis, Eigenvalues, Quasi-behaviour, Busy period.

1. INTRODUCTION

Multimedia technology is becoming an enabling and pervasive technology for many current and future application areas [1]. In multimedia applications, such as tele-conferencing and broadcast-quality presentations, multiple media such as continuous media (e.g., audio and video) and discrete media (e.g., text, data, and image) are integrated [2]. Multimedia applications are predominantly distributed in nature and require predictable quality of service (QoS). A distributed

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multimedia system is an integrated computing, communication, and information system; it enables the processing, management, delivery, and presentation of multimedia information with QoS guarantees.

In these multimedia communications and applications, synchronization problem arises when sounds, videos, images, and other media are brought together and integrated into a computer system [3]. Synchronization in multimedia systems refers to maintaining temporal relationship between packets of different media streams [1]. A distributed multimedia system is defined as an interactive multimedia environment which allows efficient real-time search and retrieval of multimedia information over the communication network. In such applications, it is necessary to have a synchronized presentation using the available media units of multimedia streams. But due to the dynamic delays by the network, the media units cannot reach the receiver within the time which leads to have a buffer at the receiver side [4,5]. The buffer content has to be studied for a better understanding of the system in such applications.

In this paper, we consider a distributed multimedia system where the communication between two nodes involve two media. Arrivals consist of two types of media packets, and the packets are processed for pairs of one packet from each media. We view this model as a two-dimensional finite birth-death process by considering the arrivals of the packets, following Poisson distribution, as births and the departures of the impatient packets, after exponential waiting time in the network, as deaths. By impatient packets we mean, arriving packets in one media cannot wait forever for the corresponding packets in the other media as they block the new arrivals in the network, and hence, they drop out from the network after a finite time. We reduce this two-dimensional process to a one-dimensional process by suitably labeling the states of the two-dimensional process.

Our main objective is to study the time-dependent behaviour of this multimedia system. Studying this behaviour analytically is difficult and often an impossible task. Even in the simple process where birth and death rates are constant, analytical solution involves an infinite series of Bessel function and their integrals (see, for example, [6,7]). But in real world problems the birth and death rates are the functions of the states of the birth-death models. So, the difficulty is compounded in the time-dependent analysis of such models. And hence, in the study of birth-death processes the emphasis had been on obtaining steady state solution. But in many potential applications steady state measures of system performance simply do not make sense when the practitioner needs to know how the system will operate up to some specified time [8]. Time-dependent analysis helps us to understand the behaviour of a system when the parameters involved are perturbed. There has been resurgence of interest in the time-dependent analysis of birth-death processes (see, for example, [9,10]). Recently, for a paired queueing system arising in multimedia synchronization the transient system size probabilities are obtained analytically [11]. Because of its importance we will study the time-dependent behaviour of our birth-death process, under consideration, numerically, where analytical solution is difficult and may be impossible.

Numerical methods provide an alternate approach to the analysis of time-dependent behaviour of birth-death processes. They are particularly useful in cases where it is not possible to obtain an analytic solution for the processes under study. The applicability of these numerical methods is limited by memory and CPU constraints. Computer memory requirements become severe when the state space of the process under study becomes very large. In this paper, we use an effective approach due to Rosenlund [12] to analyze the time-dependent behaviour of our model numerically. The advantage of this method is that the eigenvectors of the underlying matrix are defined recursively. We will study the various system characteristics, like, time-dependent probabilities of the number of packets in each media, their averages, variances, and busy periods. They are illustrated through tables and graphs. This approach involves polynomials which are defined recursively by three-term recurrence relations. A particular normalization was used to evaluate these recursively defined polynomials. Wilkinson [13] gave a detailed and illuminating discussion on the numerical instability, which arises because of this particular normalization, when the number of polynomials becomes larger, without giving any solution to overcome. Recently, Fernando [14] has given an efficient and simple way to overcome this problem. We use this Fernando's technique in this Rosenlund's approach to avoid numerical instability to certain extent. We will discuss in more detail about this tool in Section 3. In the next section, we describe the model under consideration and present a system of differential-difference equations governing the model.

2. MODEL DESCRIPTION

As we said in the previous section, we consider a distributed multimedia system where the communication between two nodes involve two media. Assume that each of these media uses separate real-time channels for transmission of packets. Such a system requires synchronization between media streams at the receiver. In the presence of network heterogeneity, the media packets arrive at the receiver at different time points. A packet from one media has to wait for its pair from other media, and hence, joins the queue at the receiver. We refer to these two media as Media-I and Media-II, respectively. The buffer in which the packets are stored has only finite capacity. We assume that the maximum number of packets which can be stored at the buffer is N_1 for Media-I and N_2 for Media-II. Also, the packets cannot wait forever as they will block the arrival of the other packets. We assume that each of the packets from Media-I and Media-II drops out of the system after an exponentially distributed time with means $1/\eta$ and $1/\theta$, respectively. We assume that the packets arrive at the system according to Poisson process with state-dependent rates λ_n and μ_n, respectively, when there are n packets in Media-I and Media-II, respectively in the system. This system can be modeled as a two-dimensional finite birth-death process $\mathcal{X} \equiv \{(U(t), V(t)),\ t \geq 0\}$, say, with state space given by

$$\mathcal{N} \equiv \{(0, N_2), \dots, (0,1), (0,0), (1,0), \dots, (N_1, 0)\}.$$

That is, if there are n_1 packets in Media-I and n_2 packets in Media-II then we say that the process is in state (n_1, n_2). We note that either n_1 and/or n_2 is always zero.

The transition probabilities $\{P_{n_1,n_2}(t)\}$ are defined by

$$P_{n_1,n_2}(t) \equiv \Pr(U(t) = n_1, V(t) = n_2 \mid U(0) = m_1, V(0) = m_2), \qquad (n_1, n_2), (m_1, m_2) \in \mathcal{N}.$$

Then these probabilities satisfy the Kolmogorov forward differential-difference equations

$$\begin{aligned}
P'_{00}(t) &= (\mu_0 + \eta)P_{10}(t) + (\lambda_0 + \theta)P_{01}(t) - (\lambda_0 + \mu_0)P_{00}(t), \\
P'_{n_1,0}(t) &= \lambda_{n_1-1}P_{n_1-1,0}(t) + (\mu_0 + (n_1+1)\eta)P_{n_1+1,0}(t) \\
&\quad - (\mu_0 + \lambda_{n_1} + n_1\eta)P_{n_1,0}(t), \qquad 1 \leq n_1 < N_1, \\
P'_{N_1,0}(t) &= \lambda_{N_1-1}P_{N_1-1,0}(t) - (\mu_0 + N_1\eta)P_{N_1,0}(t), \\
P'_{0,n_2}(t) &= \mu_{n_2-1}P_{0,n_2-1}(t) + (\lambda_0 + (n_2+1)\theta)P_{0,n_2+1}(t) \\
&\quad - (\lambda_0 + \mu_{n_2} + n_2\theta)P_{0,n_2}(t), \qquad 1 \leq n_2 < N_2, \\
P'_{0,N_2}(t) &= \mu_{N_2-1}P_{0,N_2-1}(t) - (\lambda_0 + N_2\theta)P_{0,N_2}(t).
\end{aligned} \tag{1}$$

By equating the right-hand side of (1) to zero, we get after some simple calculation the stationary probabilities $p_{n_1,0}$, p_{0,n_2}, and $p_{0,0}$,

$$p_{n_1,0} = \left(\prod_{j=0}^{n_1-1} \frac{\lambda_j}{\mu_0 + (j+1)\eta} \right) p_{0,0}, \qquad n_1 = 1, 2, \dots, N_1, \tag{2}$$

$$p_{0,n_2} = \left(\prod_{j=0}^{n_2-1} \frac{\mu_j}{\lambda_0 + (j+1)\theta} \right) p_{0,0}, \qquad n_2 = 1, 2, \dots, N_2, \tag{3}$$

$$p_{0,0} = \left(1 + \sum_{n_1=1}^{N_1} \prod_{j=0}^{n_1-1} \frac{\lambda_j}{\mu_0 + (j+1)\eta} + \sum_{n_2=1}^{N_2} \prod_{j=0}^{n_2-1} \frac{\mu_j}{\lambda_0 + (j+1)\theta} \right)^{-1}. \tag{4}$$

Since one the components of each state in $\mathcal{N}$ is zero, it is possible to view this two-dimensional birth-death process $\mathcal{X}$ as a one-dimensional birth-death process $\tilde{\mathcal{X}} \equiv \{X(t),\ t \geq 0\}$, say, with state space $\tilde{\mathcal{N}} \equiv \{0, 1, \ldots, N_1 + N_2 (\equiv N)\}$. That is, we relabel the states of $\mathcal{N}$ as follows:

$$\begin{aligned}
(0, N_2) &\rightarrow 0 \\
(0, N_2 - 1) &\rightarrow 1 \\
\vdots\ &\ \ \vdots \\
(0, 1) &\rightarrow N_2 - 1 \\
(0, 0) &\rightarrow N_2 \\
(1, 0) &\rightarrow N_2 + 1 \\
\vdots\ &\ \ \vdots \\
(N_1, 0) &\rightarrow N_1 + N_2 \equiv N.
\end{aligned}$$

With this relabeling system (1) takes the following form:

$$\begin{aligned}
P_0'(t) &= -(\lambda_0 + N_2\theta)P_0(t) + \mu_{N_2-1}P_1(t), \\
P_n'(t) &= (\lambda_0 + (N_2 - n + 1)\theta)P_{n-1}(t) - (\lambda_0 + \mu_{N_2-n} + (N_2 - n)\theta)P_n(t) \\
&\quad + \mu_{N_2-n-1}P_{n+1}(t), \qquad n = 1, 2, \ldots, N_2 - 1, \\
P_{N_2}'(t) &= (\lambda_0 + \theta)P_{N_2-1}(t) - (\lambda_0 + \mu_0)P_{N_2}(t) + (\mu_0 + \eta)P_{N_2+1}(t), \\
P_n'(t) &= \lambda_{n-N_2-1}P_{n-1}(t) - (\mu_0 + \lambda_{n-N_2} + (n - N_2)\eta)P_n(t) \\
&\quad + (\mu_0 + (n - N_2 + 1)\eta)P_{n+1}(t), \qquad n = N_2 + 1, \ldots, N_1 + N_2 - 1, \\
P_N'(t) &= \lambda_{N_1-1}P_{N-1}(t) - (\mu_0 + N_1\eta)P_N(t).
\end{aligned} \tag{5}$$

This reduction of dimension of the original process $\mathcal{X}$ gives more clarity in the time-dependent analysis of the model under observation. Further if

$$\tilde{\lambda}_n \equiv \begin{cases} \lambda_0 + (N_2 - n)\theta, & n = 0, 1, \ldots, N_2, \\ \lambda_{n-N_2}, & n = N_2 + 1, N_2 + 2, \ldots, N - 1, \end{cases} \tag{6}$$

and

$$\tilde{\mu}_n \equiv \begin{cases} \mu_{N_2-n}, & n = 1, 2, \ldots, N_2, \\ \mu_0 + (n - N_2)\eta, & n = N_2 + 1, \ldots, N, \end{cases} \tag{7}$$

then the above system can be written as

$$\begin{aligned}
P_0'(t) &= -\tilde{\lambda}_0 P_0(t) + \tilde{\mu}_1 P_1(t), \\
P_n'(t) &= \tilde{\lambda}_{n-1}P_{n-1}(t) - (\tilde{\lambda}_n + \tilde{\mu}_n)P_n(t) + \tilde{\mu}_{n+1}P_{n+1}(t), \\
&\qquad n = 1, 2, \ldots, N_1 + N_2 - 1, \\
P_N'(t) &= \tilde{\lambda}_{N-1}P_{N-1}(t) - \tilde{\mu}_N P_N(t).
\end{aligned} \tag{8}$$

In matrix notation (8) may be written as

$$\mathbf{P}'(t) = RP(t), \tag{9}$$

where

$$R \equiv \begin{bmatrix} -\tilde{\lambda}_0 & \tilde{\mu}_1 & & & & \\ \tilde{\lambda}_0 & -\tilde{\lambda}_1 - \tilde{\mu}_1 & \tilde{\mu}_2 & & & \\ & \tilde{\lambda}_1 & -\tilde{\lambda}_2 - \tilde{\mu}_2 & \tilde{\mu}_3 & & \\ & & & & \ddots & \\ & & & & \tilde{\lambda}_{N-1} & -\tilde{\mu}_N \end{bmatrix}_{(N+1)\times(N+1)}. \tag{10}$$

System (8) is nothing but the forward Kolmogorov equations. We observe that $\tilde{\lambda}_N = \tilde{\mu}_0 = 0$. Hence, $\tilde{\mathcal{X}}$ is a finite birth-death process taking values in $\tilde{\mathcal{N}}$ with birth rate being $\tilde{\lambda}_n$ and death rate being $\tilde{\mu}_n$ when the process is in state $n \in \tilde{\mathcal{N}}$. In the next section, we give the solution of system (8).

3. TIME-DEPENDENT ANALYSIS

The basic numerical method to solve a system of ODEs is finding eigenvalues and eigenvectors of the coefficient matrix and express the solution as a linear combination of these quantities and the involved constants are found using the initial conditions. In this section, we briefly discuss Rosenlund's approach [12] to calculate the time-dependent system size probabilities. The advantage of using Rosenlund's technique is that these eigenvectors can be recursively computed using three-term recurrence relations, and hence, more stable numerically in comparison with the direct computation of eigenvectors of R.

Taking Laplace transformation of (8) gives

$$s\bar{P}_n(s) - \delta_{m,n} = -\left(\tilde{\lambda}_n + \tilde{\mu}_n\right)\bar{P}_n(s) + \tilde{\lambda}_{n-1}\bar{P}_{n-1}(s) + \tilde{\mu}_{n+1}\bar{P}_{n+1}(s), \qquad n \in \tilde{\mathcal{N}}, \tag{11}$$

where $\delta_{m,j}$ is the Kronecker delta and $m \in \tilde{\mathcal{N}}$ is the initial state corresponding to $(m_1, m_2) \in \mathcal{N}$.

Equation (11) can be written in matrix form as follows:

$$A\left[\bar{P}_0(s), \bar{P}_1(s), \dots, \bar{P}_N(s)\right]^{\top} = \left[\delta_{m,0}, \delta_{m,1}, \dots, \delta_{m,N}\right]^{\top},$$

where A is a square matrix of order $N+1$ with element $a_{i,j}$ in row i and column j ($i, j \in \tilde{\mathcal{N}}$) defined by

$$a_{i,j} \equiv \begin{cases} -\tilde{\lambda}_{i-1}, & j = i-1, \\ s + \tilde{\lambda}_i + \tilde{\mu}_i, & j = i, \\ -\tilde{\mu}_{i+1}, & j = i+1. \end{cases}$$

Using Cramer's rule we have,

$$\bar{P}_n(s) = \frac{Q_{m,n}(s)}{B_{N+1}(s)}, \qquad n \in \tilde{\mathcal{N}}. \tag{12}$$

where the cofactor $Q_{m,n}(s)$ of the element $a_{m,n}(s)$ is given by

$$Q_{m,j}(s) = c_{mn} A_{N-\max(m,n)}(s) B_{\min(m,n)}(s). \tag{13}$$

Here

$$\begin{aligned} B_1(s) &= \left(s + \tilde{\lambda}_0\right) B_0(s), \\ B_n(s) &= \left(s + \tilde{\lambda}_{n-1} + \tilde{\mu}_{n-1}\right) B_{n-1}(s) - \tilde{\lambda}_{n-2}\tilde{\mu}_{n-1} B_{n-2}(s), \qquad n = 2, 3, \dots, N, \end{aligned} \tag{14}$$

$$\begin{aligned} B_{N+1}(s) &= \left(s + \tilde{\mu}_N\right) B_N(s) - \tilde{\lambda}_{N-1}\tilde{\mu}_N B_{N-1}(s), A_1(s) = \left(s + \tilde{\mu}_N\right) A_0(s), \\ A_n(s) &= \left(s + \tilde{\lambda}_{N+1-n} + \tilde{\mu}_{N+1-n}\right) A_{n-1}(s) - \tilde{\lambda}_{N+1-n}\tilde{\mu}_{N+2-n} A_{n-2}(s), \\ & \qquad n = 2, 3, \dots, N, \end{aligned} \tag{15}$$

$$A_{N+1}(s) = \left(s + \tilde{\lambda}_0\right) A_N(s) - \tilde{\lambda}_0\tilde{\mu}_1 A_{N-1}(s),$$

$$c_{mj} = \begin{cases} \tilde{\lambda}_m \tilde{\lambda}_{m+1} \dots \tilde{\lambda}_{j-1}, & m < j, \\ 1, & m = j, \\ \tilde{\mu}_{j+1}\tilde{\mu}_{j+2} \dots \tilde{\mu}_m, & m > j. \end{cases} \tag{16}$$

We observe that $B_n(s)$ is the determinant obtained by considering the first n rows and columns of $\det(A)$ and $A_n(s)$ is the determinant obtained by considering the last n rows and columns of $\det(A)$. Hence, $A_{N+1}(s) = B_{N+1}(s) = \det(A)$.

An alternate expression for $Q_{m,n}(s)$, given by (13), in terms of $B_n(s)$ alone is given by

$$Q_{m,n}(s) = \frac{c_{mN} c_{Nn} B_m(s) B_n(s)}{B_N(s)}, \qquad m, n \in \tilde{\mathcal{N}}. \tag{17}$$

Equation (17) is not straight-forward to derive and for details one can refer to [15]. In our analysis we strict to (17).

We note that $B_{N+1}(s)$, with $B_0(s) = 1$, is the characteristic polynomial of R. In [10], the authors show that the eigenvalues of R are all real, negative, distinct, and one of them is zero. Suppose, $\xi_0, \xi_1, \dots, \xi_N$ are the roots of $B_{N+1}(s) = 0$ such that $0 \geq \xi_0 > \xi_1 > \cdots > \xi_N$ then (12) can be written as

$$\bar{P}_n(s) = \frac{Q_{m,n}(s)}{\prod_{k=0}^{N} (s - \xi_k)}, \qquad m, n \in \tilde{\mathcal{N}}.$$

Using partial fractions technique we have,

$$\bar{P}_n(s) = \sum_{j=0}^{N} \frac{Q_{m,n}(\xi_j)}{\prod_{i=0, i \neq l}^{N} (\xi_j - \xi_i)(s - \xi_j)}, \qquad n \in \tilde{\mathcal{N}}. \tag{18}$$

On inverting (18) we get,

$$P_n(t) = \sum_{j=0}^{N} \frac{Q_{m,n}(\xi_j)}{\prod_{i=0, i \neq l}^{N} (\xi_j - \xi_i)} \exp(\xi_j t), \qquad n \in \tilde{\mathcal{N}}, \tag{19}$$

where

$$Q_{m,n}(\xi_j) = \frac{c_{mN} c_{Nn} B_m(\xi_j) B_n(\xi_j)}{B_N(\xi_j)}, \qquad m, n \in \tilde{\mathcal{N}}. \tag{20}$$

Indeed, the coefficients of the exponential functions in (19) are the eigenvectors of the corresponding eigenvalues ξ_j's.

Hence, for the two-dimensional process $\mathcal{X}$ the probabilities $P_{n_1,0}(t)$, $P_{0,n_2(t)}$, and $P_{0,0}(t)$ are given by

$$P_{0,n_2}(t) = \sum_{j=0}^{N} \frac{Q_{m,N_2-n_2}(\xi_j)}{\prod_{i=0, i \neq l}^{N} (\xi_j - \xi_i)} \exp(\xi_j t), \qquad n_2 = 1, 2, \dots, N_2, \tag{21}$$

$$P_{0,0}(t) = \sum_{j=0}^{N} \frac{Q_{m,N_2}(\xi_j)}{\prod_{i=0, i \neq l}^{N} (\xi_j - \xi_i)} \exp(\xi_j t), \tag{22}$$

$$P_{n_1,0}(t) = \sum_{j=0}^{N} \frac{Q_{m,N_2+n_1}(\xi_j)}{\prod_{i=0, i \neq l}^{N} (\xi_j - \xi_i)} \exp(\xi_j t), \qquad n_1 = 1, 2, \dots, N_1. \tag{23}$$

It is of interest to observe the system size probabilities when the packets are coming only in one media and other media is empty, that is one media is keep building up while the other remains empty. This type of analysis is useful in finding the mean and variance of the number of packets in the building up media known as quasimean (quasiaverage) and quasivariance. And the conditional probabilities are referred as quasiprobabilities. We denote these quasiprobabilities by $P^q_{n_1,0}(t)$ and $P^q_{0,n_2(t)}$. Then clearly,

$$P^q_{n_1,0}(t) = \frac{P_{n_1,0}(t)}{1 - \sum_{n_2=0}^{N_2} P_{0,n_2}(t)}, \qquad n_1 = 1, 2, \ldots, N_1, \tag{24}$$

$$P^q_{0,n_2}(t) = \frac{P_{0,n_2}(t)}{1 - \sum_{n_1=0}^{N_1} P_{n_1,0}(t)}, \qquad n_2 = 1, 2, \ldots, N_2. \tag{25}$$

In the next section, we briefly discuss about the numerical instability, which arises when N becomes large, in calculating $B_n(s)$ using (14) and the solution given by Fernando [14] to overcome this instability.

4. NUMERICAL INSTABILITY IN (14)

As we see from the expression of $Q_{m,n}(s)$, the polynomials $B_n(s)$ play a major role. Since $B_{N+1}(\xi_j) = 0$, for $j = 0, 1, \ldots, N$, the last equation in (14) will have only two terms, namely, B_{N-1} and B_N. Also, the first equation has only two terms, namely, B_0 and B_1. Hence, these are underdetermined systems of equations, at least one equation in the system is redundant. If the k^{th} equation is redundant, then one may assume that $B_k(s) = 1$ and solve the rest of the equations. It has been the normal practice to assume that the superfluous equation is the first, that is, $B_0(s) = 1$. Wilkinson [13] gives a detailed and illuminating discussion on the numerical instability, which arises because of this particular normalization, and it leads to disastrous results in computing other $B_n(s)$ recursively for larger values of N. Recently, Fernando [14] has provided a solution to overcome this instability. The idea is to find out the equation, among $N + 1$ equations, to be treated as a redundant one by setting the corresponding element to unity. This is achieved by computing the diagonal entries of the matrix M, which is obtained by elementwise reciprocation of the inverse of the matrix $(sI - R)^\top$ (see [13,14] for more details). This is based on LDU and UDL factorizations of the shifted tridiagonal matrix $(sI - R)$ (see [16] for more details) and we briefly discuss below.

We consider the LDU factorization of the unreduced tridiagonal matrix A, where, L is lower bidiagonal, U is upper bidiagonal, and D is diagonal. The diagonal elements $d_i(s)$ of D are given recursively as follows:

$$\begin{aligned} d_0(s) &= a_{0,0}, \\ d_i(s) &= a_{i,i} - \frac{a_{i-1,i}\, a_{i,i-1}}{d_{i-1}(s)}, \qquad i = 1, 3, \ldots, N, \end{aligned} \tag{26}$$

where s is an eigenvalue of the matrix R.

Now, we consider the UDL factorization of the unreduced tridiagonal matrix A. The diagonal elements $\delta_i(s)$ of D are given recursively as follows:

$$\begin{aligned} \delta_N(s) &= a_{N,N}, \\ \delta_i(s) &= a_{i,i} - \frac{a_{i+1,i}\, a_{i,i+1}}{d_{i-1}(s)}, \qquad i = N-1, N-2, \ldots, 0. \end{aligned} \tag{27}$$

Then the diagonal elements $\eta_k(s)$ of the matrix M are given by

$$\begin{aligned} \eta_0(s) &= \delta_0(s), \\ \eta_i(s) &= \delta_i - \frac{a_{i-1,i}\, a_{i,i-1}}{d_{i-1}(s)}, \qquad i = 1, 2, 3, \ldots, N. \end{aligned} \tag{28}$$

The following algorithm may be used for computing $B_n(s)$.

ALGORITHM.

1. Compute $\eta_k = \min_{0 \le i \le N}\{\eta_i\}$.
2. Set $B_k(s) = 1$.
3. Compute other $B_n(s)$ using

$$B_n(s) = -\frac{a_{n,n+1}}{d_n(s)} B_{n+1}(s), \qquad n = k-1, k-2, \dots, 0,$$
$$B_n(s) = -\frac{a_{i,i-1}}{\delta_n(s)} B_{n-1}(s), \qquad n = k+1, k+2, \dots, N.$$

So, to avoid numerical instability to a greater extent we use the above algorithm to compute $B_n(s)$ instead of (14). In the next section, we consider some particular models and discuss the numerical results.

In the next section, we will carry out the busy period analysis for both Media-I and Media-II by suitably modifying the analysis discussed in Section 3.

5. BUSY PERIOD ANALYSIS

Busy period analysis also forms an integral part of the study of any queueing system and has been carried out for every queueing system discussed in the literature [7,17]. It plays a vital role in understanding various operations taking place in any queueing system.

For the busy period analysis of our original process $\mathcal{X}$, we assume that the state $(0,0)$ is an absorbing state, that is, $\lambda_0 = 0 = \mu_0$. Equivalently, in the one-dimensional process $\tilde{\mathcal{X}}$ we have the following changes: we divide busy periods for our model into (1) Media-I periods and (2) Media-II periods, which have obvious interpretations. For Media-I periods, we set $N_2 = 0$ and consider an initial state in $\{1, 2, \dots, N_1\}$, a reflecting barrier at N_1, and an absorbing barrier at 0. For Media-II periods, we set $N_1 = 0$ and consider an initial state in $\{0, 1, \dots, N_2 - 1\}$, a reflecting barrier at 0 and an absorbing barrier at N_2. Hence, for Media-I periods $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ are given by

$$\begin{aligned} \tilde{\lambda}_n &= \lambda_n, \qquad n = 1, 2, \dots, N_1 - 1, \\ \tilde{\mu}_n &= n\eta, \qquad n = 1, 2, \dots, N_1, \end{aligned} \tag{29}$$

and for Media-II periods they are given by

$$\begin{aligned} \tilde{\lambda}_n &= (N_2 - n)\theta, \qquad n = 0, 1, \dots, N_2 - 1, \\ \tilde{\mu}_n &= \mu_{N_2 - n}, \qquad n = 1, 2, \dots, N_2. \end{aligned} \tag{30}$$

We observe that from (29), $\tilde{\lambda}_0 = 0 = \tilde{\mu}_0$. Therefore, busy periods of Media-I are nothing but the first passage times to state 0 in one-dimensional process $\tilde{\mathcal{X}}$ with $N_2 = 0$. Whereas, in (30), $\tilde{\lambda}_{N_2} = 0 = \tilde{\mu}_{N_2}$. Therefore, busy periods of Media-II are the first passage times to state N_2 in one-dimensional process $\tilde{\mathcal{X}}$ with $N_1 = 0$. Now, we modify the analysis discussed in Section 3 accordingly for the busy period analysis of Media-I by treating state 0 as an absorbing barrier and $N_2 = 0$. We denote by $\tilde{P}_n(t) (\equiv P_{n,0}(t))$ $1 \le n \le N_1$ the probability that there are n packets in Media-I given 0 is an absorbing barrier and $N_2 = 0$. Then these probabilities satisfy the following system of equations with $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ given by (29):

$$\begin{aligned} \tilde{P}_0'(t) &= \tilde{\mu}_1 \tilde{P}_1(t), \\ \tilde{P}_1'(t) &= -\left(\tilde{\lambda}_1 + \tilde{\mu}_1\right) \tilde{P}_n(t) + \tilde{\mu}_{n+1} \tilde{P}_{n+1}(t), \\ \tilde{P}_n'(t) &= \tilde{\lambda}_{n-1} \tilde{P}_{n-1}(t) - \left(\tilde{\lambda}_n + \tilde{\mu}_n\right) \tilde{P}_n(t) + \tilde{\mu}_{n+1} \tilde{P}_{n+1}(t), \\ & \qquad n = 1, 2, \dots, N_1 - 1, \\ \tilde{P}_{N_1}'(t) &= \tilde{\lambda}_{N_1 - 1} \tilde{P}_{N_1 - 1}(t) - \tilde{\mu}_{N_1} \tilde{P}_{N_1}(t). \end{aligned} \tag{31}$$

We note that $\tilde{P}_0'(t)$ gives the probability density function of the busy period of Media-I. In (31) we ignore the first equation for time being and consider the rest. By closely following the analysis similar to the one given in Section 3, to solve (8), we get the following expression for $\tilde{P}_n(t)$:

$$\tilde{P}_n(t) = c_{mN}c_{Nn}\sum_{j=1}^{N_1} \frac{B_m(s_j)B_n(s_j)}{B_{N-1}(s_j)B_N'(s_j)} \exp(\xi_j t), \qquad m,n = 1,2,\ldots,N_1, \tag{32}$$

where m is the initial state, ξ_j, $j = 1,2,\ldots,N_1$, are the roots of $B_{N_1}(s)$ defined recursively by

$$\begin{aligned} B_1(s) &= \left(s + \tilde{\lambda}_1 + \tilde{\mu}_1\right) B_0(s), \\ B_n(s) &= \left(s + \tilde{\lambda}_n + \tilde{\mu}_n\right) B_{n-1}(s) - \tilde{\lambda}_{n-1}\tilde{\mu}_n B_{n-2}(s), \qquad n = 2,4,\ldots,N_1 - 1, \\ B_{N_1}(s) &= (s + \tilde{\mu}_{N_1}) B_{N_1-1}(s) - \tilde{\lambda}_{N_1-1}\tilde{\mu}_{N_1} B_{N-2}(s), \end{aligned} \tag{33}$$

and c_{mj}, $m,j = 1,2,\ldots,N_1$ are given by (16). We note that in (16) m and j do take value 0 but here they do not.

From the first equation of (31) the busy period density function of Media-I is given by $\tilde{P}_0'(t) = \tilde{\mu}_1\tilde{P}_1(t)$. Hence, from (32), we have

$$\tilde{P}_0'(t) = \tilde{\mu}_1 c_{mN}c_{Nn}\sum_{j=1}^{N_1} \frac{B_m(\xi_j)B_1(\xi_j)}{B_{N-1}(\xi_j)B_N'(\xi_j)} \exp(\xi_j t), \qquad m = 1,2,\ldots,N_1. \tag{34}$$

Integrating (34) with respect to t gives the busy period distribution function $\tilde{P}_0(t)$ which is

$$\tilde{P}_0(t) = \tilde{\mu}_1 c_{mN}c_{Nn}\sum_{j=1}^{N_1} \frac{B_m(\xi_j)B_1(\xi_j)}{s_j B_{N-1}(\xi_j)B_N'(\xi_j)} \left[\exp(\xi_j t) - 1\right]. \tag{35}$$

The mean busy period M_1 of Media-I is given by $\int t\tilde{P}_0'(t)$, and hence,

$$M_1 = \tilde{\mu}_1 c_{mN}c_{Nn}\sum_{j=1}^{N_1} \frac{B_m(\xi_j)B_1(\xi_j)}{(\xi_j)^2 B_{N-1}(\xi_j)B_N'(\xi_j)}. \tag{36}$$

We use Fernando's technique, discussed in Section 4, by suitably modifying it to calculate $B_n(s)$ to avoid numerical instability which will arise if we use (33).

Similarly, one can find the expressions for the busy period density function and its mean for Media-II.

In the next section, we give some numerical results by assuming the arrival rates λ_n and μ_n as linear and quadratic functions of n.

6. NUMERICAL RESULTS

For the numerical illustrations we assume the following forms for λ_n and μ_n.

Linear Rates:

$$\lambda_n = (N_1 - n)\lambda, \qquad \lambda > 0, \quad n = 0,1,\ldots,N_1, \tag{37}$$

$$\mu_n = (N_2 - n)\mu, \qquad \mu > 0, \quad n = 0,1,\ldots,N_2. \tag{38}$$

Quadratic Rates:

$$\lambda_n = (N_1 - n)(\lambda - (n-1)a), \qquad \lambda > 0, \frac{-\lambda}{N_1 + 2} < a < 0, \quad n = 0, 1, \dots, N_1, \tag{39}$$

$$\mu_n = (N_2 - n)(\mu - (n-1)b), \qquad \mu > 0, \frac{-\mu}{N_2 + 2} < b < 0, \quad n = 0, 1, \dots, N_2. \tag{40}$$

Conditions on the parameters in the above rates are to ensure the positivity of the rates λ_n and μ_n.

In Figure 1, the time-dependent system size probabilities are drawn for quadratic rates (39) and (40) with the assumption that the system starts with ten packets in Media-II at time $t = 0$. The parameters values are $N_1 = 25$, $N_2 = 25$, $\lambda = 0.2$, $\mu = 0.3$, $a = -0.001$, $b = -0.002$, $\eta = 0.1$, and $\theta = 0.2$. For the sake of clarity only few probability curves are depicted in the figure. We observe that probability curves corresponding to the states $(0,7), \dots, (0,15)$ increase initially and decrease to the stationary values whereas other curves increase gradually to the stationary values except the curve for $P_{0,10}$ corresponding to the initial condition $(0,10)$, which decrease from one to the stationary value. These stationary values are prevailing around 8.5 time units.

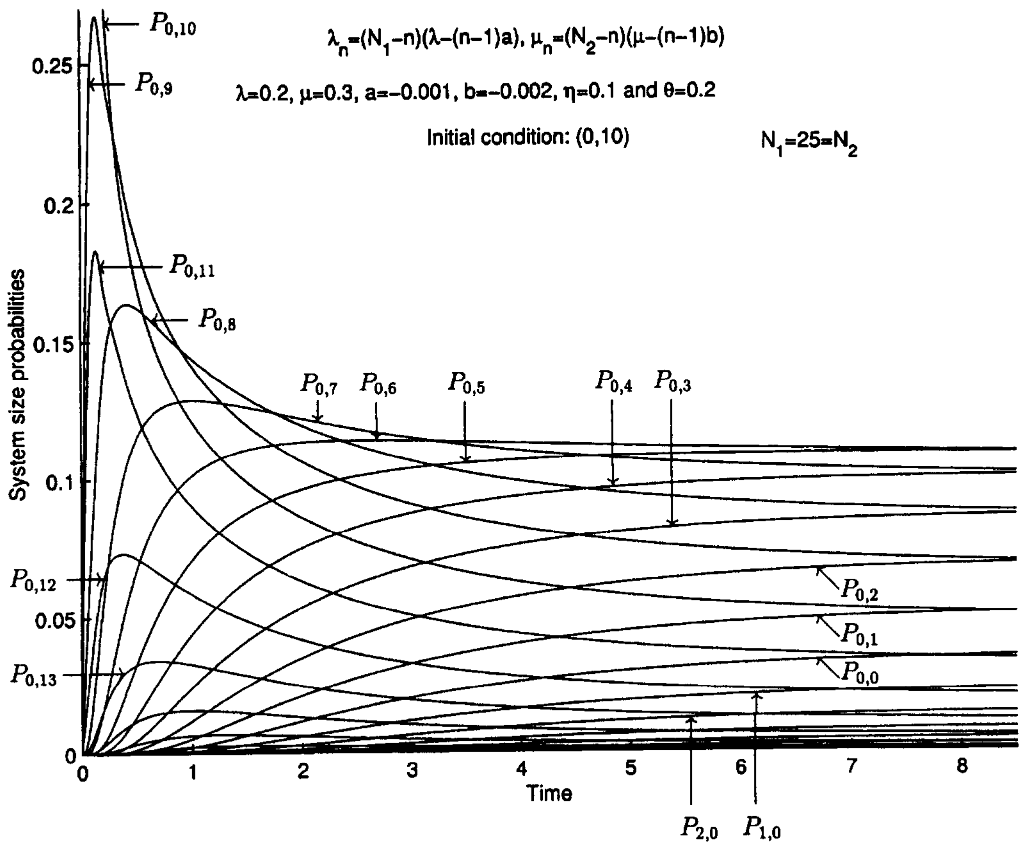

Figure 1. Probabilities of the number of Media-I and Media-II packets.

The quasiaverage of the number of packets in Media-II are plotted in Figure 2 for linear rates (37) and (38) and for different values of N_1 and N_2. The parameter values are $\lambda = 0.2$, $\mu = 0.3$, $\eta = 0.1$, and $\theta = 0.2$ and the initial condition is $(0,10)$. These quantities are the means of the quasiprobabilities of Media-II given by (25). We note that for $N_1 < N_2$ the curves decrease from ten gradually to the stationary values whereas for $N_1 \geq N_2$ they increase from ten gradually to the stationary values. These stationary values prevail around five time units.

The quasiaverage of the number of packets in Media-I are plotted in Figure 3 for linear rates (37) and (38) and for different initial conditions. The parameter values are $N_1 = 50$, $N_2 = 50$, $\lambda = 0.2$,

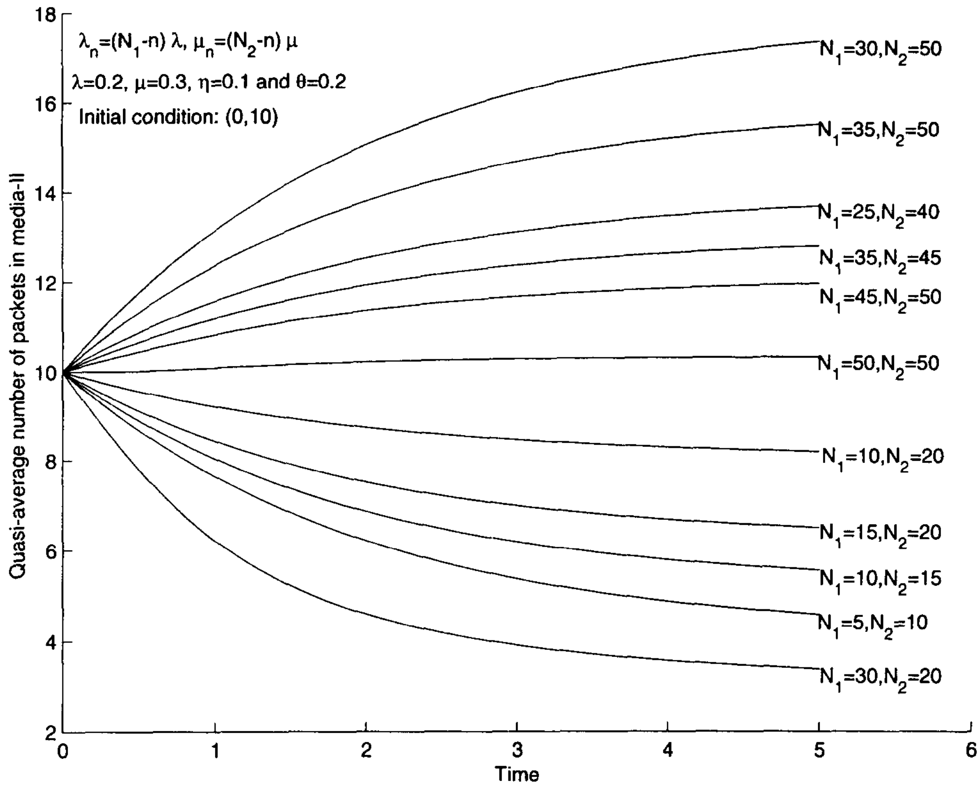

Figure 2. Quasi-average number of Media-II packets for different values of N_1 and N_2.

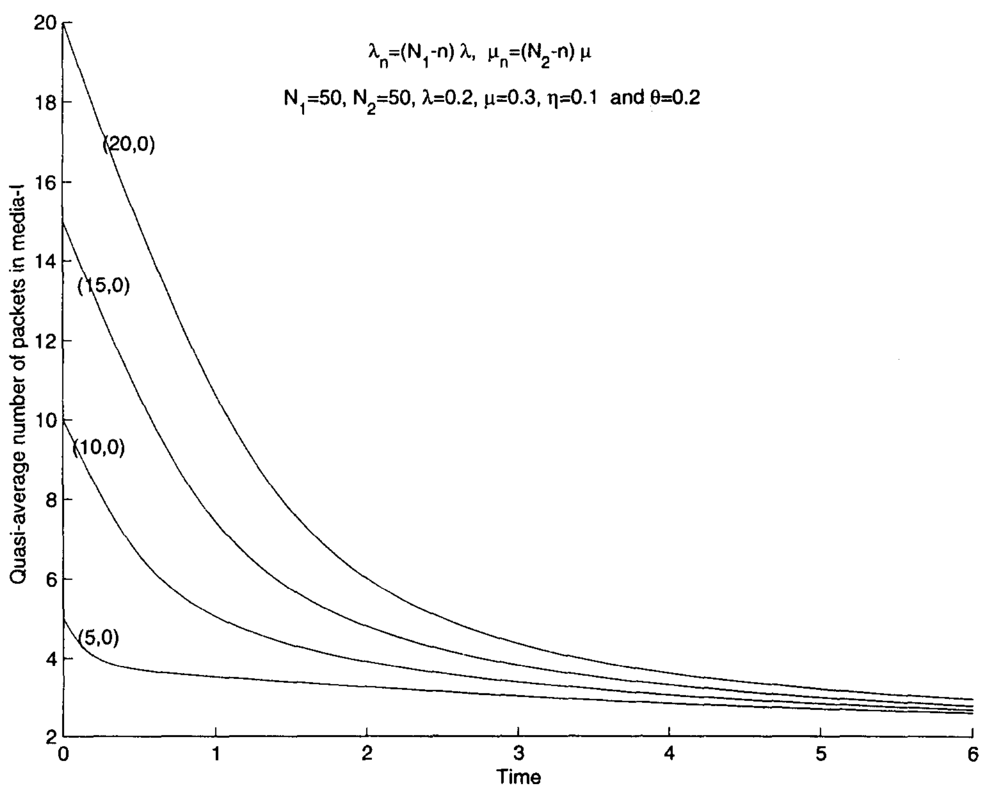

Figure 3. Quasiaverage number of Media-I packets for different initial conditions.

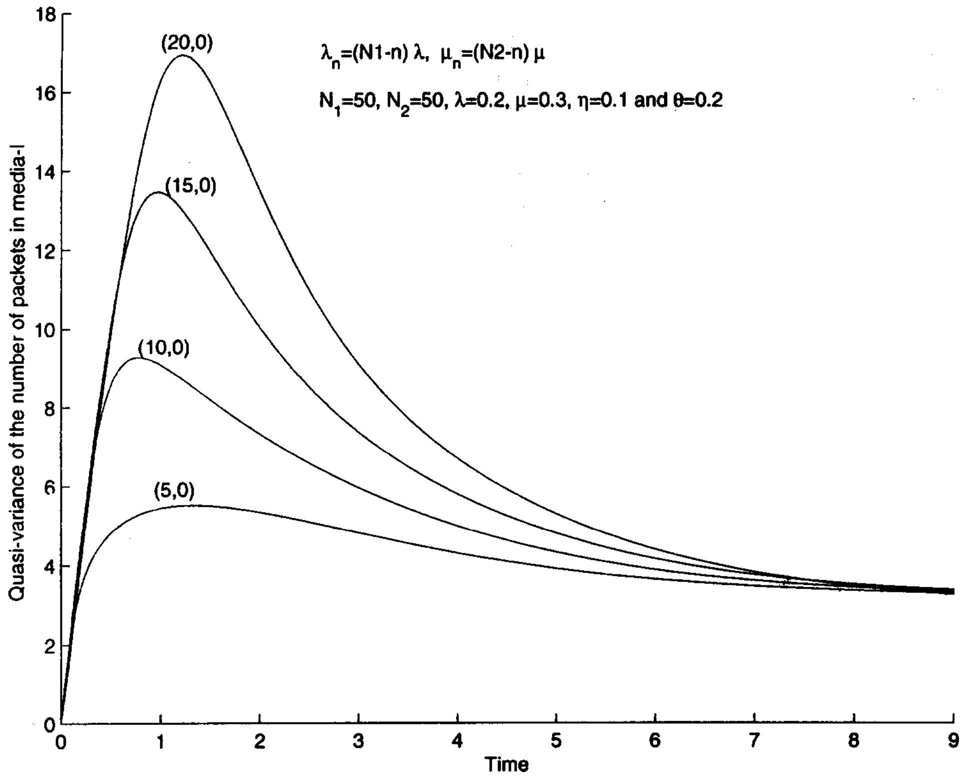

Figure 4. Quasivariance of the Media-I packets for different initial conditions.

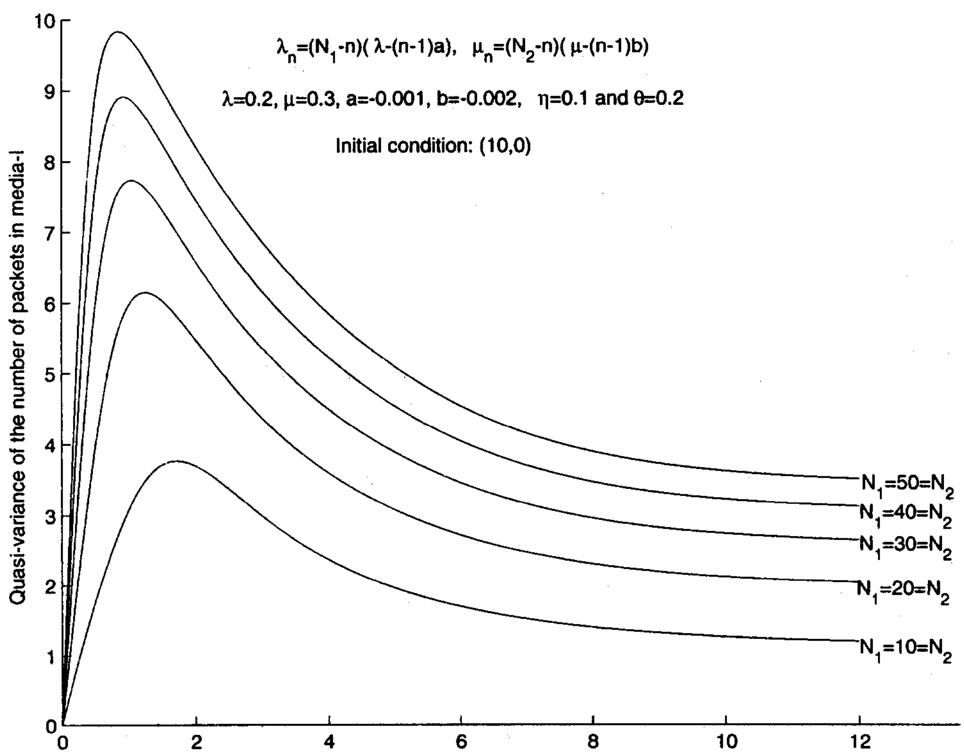

Figure 5. Quasivariance of the Media-I packets for different values of N_1 and N_2.

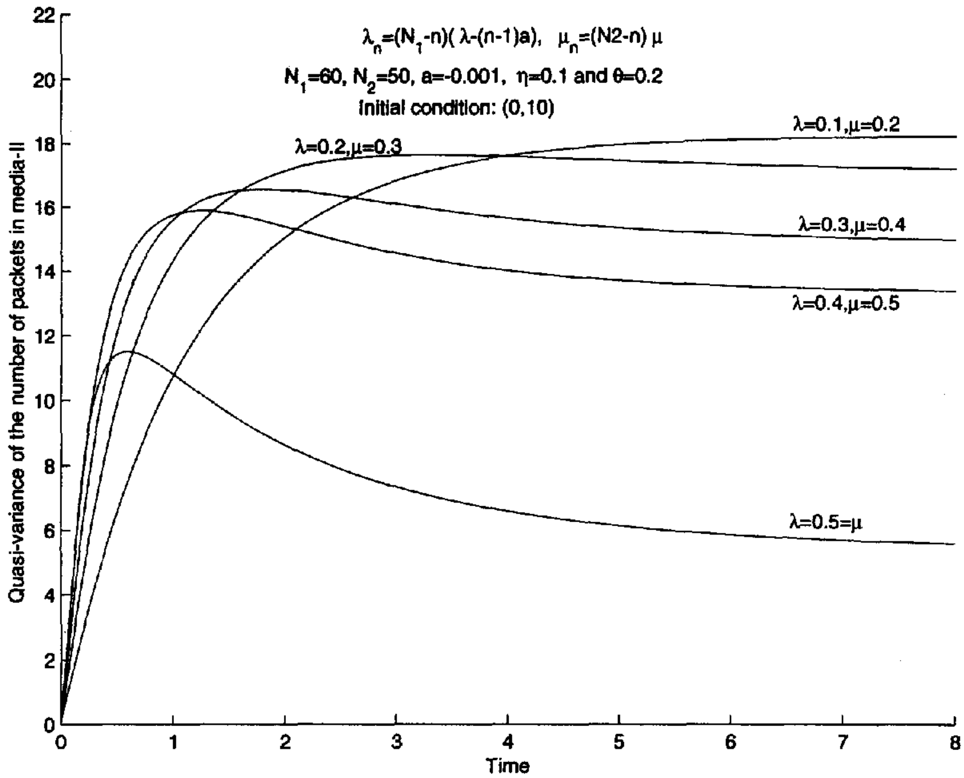

Figure 6. Quasivariance of the Media-II packets for different values of λ and μ.

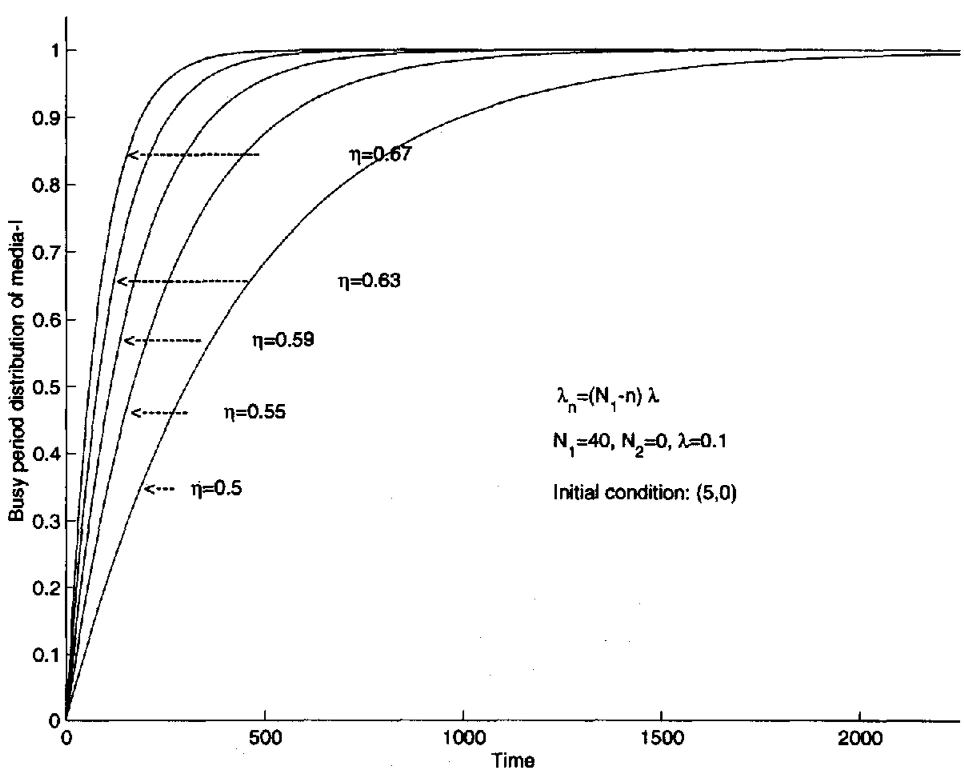

Figure 7. Busy period distribution of the Media-I packets for different values of η.

$\mu = 0.3$, $\eta = 0.1$, and $\theta = 0.2$. We observe that immaterial of the different initial conditions all curves gradually reduce to the stationary value 3.5 (approx.) around six time units. For the same set of parameters the quasivariance of the Media-II packets are drawn in Figure 4. We note that these curves increase initially and decrease to the stationary value four (approx.) around nine time units immaterial of the different initial conditions.

In Figure 5, the quasivariance of the Media-I packets are plotted for different values of N_1 and N_2 for the quadratic rates (39) and (40) with the assumption that the system starts with ten packets in Media-I at time $t = 0$. We observe that the curves increase to maximum values and decrease gradually to the stationary values around 12 time units. We also note that the maximum values reached by the curves increase with the values of N_1 and N_2. Also, these curves become closer as N_1 and N_2 increase which clearly indicates that there will not be any effect in the system after certain values of N_1 and N_2.

The quasivariance of the Media-II packets are drawn in Figure 6 with the assumption that the arrival rate of Media-I packets is quadratic (39) and for Media-II it is linear (38). These curves are plotted for different values of λ and μ. The parameters values are $N_1 = 60$, $N_2 = 50$, $a = -0.001$, $\eta = 0.1$, and $\theta = 0.2$ and the initial condition is $(0, 10)$. We observe that all curves increase initially and decrease to the stationary values around eight time units except for the curve corresponding to the case $\lambda_0.1$, $\mu = 0.2$ which increase gradually to the stationary value.

In Figure 7, the busy period distribution of Media-I packets are plotted for the linear rate (37) and for different values of η. The parameters values are $N_1 = 40$, $\lambda = 0.1$ and the initial condition is $(5, 0)$. We observe that for $\eta = 0.5$ the busy period comes to an around 2000 time units, for $\eta = 0.55$ it comes to an end around 1500 units, for $\eta = 0.59$ it comes to an end around 900 time units and for $\eta = 0.63, 0.67$ it comes to an end around 600 time units. It clearly indicates that as η increases the busy period duration decreases.

In Table 1, the mean busy period of Media-II packets are tabulated with μ_n given by (40) and for different initial conditions. The parameters values are $N_2 = 80, \mu = 0.3, b = -0.002, \theta = 0.2$. We observe that the mean busy period decreases as the initial number of packets decreases and for the initial conditions 55, 50, 45, 35, 30, 25, and 20 the first seven digits coincide.

Table 1. Mean busy period of Media-II with μ_n given by (40) and for different initial conditions. The parameters values are $N_2 = 80, \mu = 0.3, b = -0.002, \theta = 0.2$.

Initial Condition (Media-II)	Mean Busy Period (Media-II)
75	8.5611721032698400(14)
70	8.5396064906496738(14)
65	8.5395280644784200(14)
60	8.5395372853163612(14)
55	8.5395383347741450(14)
50	8.5395384466356038(14)
45	8.5395384564163650(14)
40	8.5395384569664638(14)
35	8.5395384576113600(14)
30	8.5395384641071350(14)
25	8.5395384874284588(14)
20	8.5395380333892250(14)

Where (k) denotes 10^k.

REMARK. Computationally, the methods presented in this paper give a reasonably good approximation for larger values of N, though it takes a longer execution time. For some values of the parameters, the diagonal elements of the matrix become large and thus, causing overflow problem while calculating the eigenvalues.

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