ADI Finite Difference Discretization of the Heston–Hull–White PDE

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Abstract. This paper concerns the efficient numerical solution of the time-dependent, three-dimensional Heston–Hull–White PDE for the fair prices of European call options. The numerical solution method described in this paper consists of a finite difference discretization on non-uniform spatial grids followed by an Alternating Direction Implicit scheme for the time discretization and extends the method recently proved effective by In 't Hout & Foulon (2010) for the simpler, two-dimensional Heston PDE.

Keywords: financial option pricing, Heston model, Hull–White model, finite difference methods, ADI schemes.
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INTRODUCTION: OPTION PRICE MODEL

We consider the numerical valuation of European call options in a general asset price model given by the system of stochastic differential equations (SDEs)

$$\begin{cases} dS_\tau = R_\tau S_\tau \, d\tau + \sqrt{V_\tau} S_\tau \, dW_\tau^1, \\ dV_\tau = \kappa(\eta - V_\tau) \, d\tau + \sigma_1 \sqrt{V_\tau} \, dW_\tau^2, \\ dR_\tau = a(b(\tau) - R_\tau) \, d\tau + \sigma_2 \, dW_\tau^3 \end{cases} \tag{1}$$

for $0 < \tau \leq T$ with $T > 0$ the given maturity time of the option. Here S_τ, V_τ, R_τ denote random variables that represent the asset price, its variance, and the interest rate, respectively, at time τ. The asset price model (1) constitutes a natural extension of the well-known Black–Scholes model where the volatility $\sqrt{V_\tau}$ and the interest rate both evolve randomly over time. The stochastic models for V_τ and R_τ were proposed by Heston and Hull & White, respectively, cf. [5, 6, 7]. The quantities κ, η, σ_1 and a, σ_2 are given positive real constants; b is a given deterministic, positive function of time; $W_\tau^1, W_\tau^2, W_\tau^3$ are Brownian motions, with (arbitrary) given correlation factors $\rho_{12}, \rho_{13}, \rho_{23} \in [-1, 1]$.

Let $u(s, v, r, t)$ denote the fair price of a European call option if at time $\tau = T - t$ the asset price equals s, its variance equals v and the interest rate equals r. If $K > 0$ is the given strike price of the option, then the payoff defines

$$u(s, v, r, 0) = \max(0, s - K). \tag{2}$$

Financial option pricing theory yields that, under the model (1), the price for $0 < t \leq T$ is given by the expected value

$$u(s, v, r, t) = \mathbb{E}\left[e^{-\int_{T-t}^{T} R_\tau \, d\tau} \max\left(0, S_T - K\right) \,\big|\, S_{T-t} = s, V_{T-t} = v, R_{T-t} = r \right]$$

and satisfies the following partial differential equation (PDE),

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \tfrac{1}{2} \sigma_1^2 v \frac{\partial^2 u}{\partial v^2} + \tfrac{1}{2} \sigma_2^2 \frac{\partial^2 u}{\partial r^2} + \rho_{12} \sigma_1 s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{13} \sigma_2 s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r}$$

$$+ \rho_{23} \sigma_1 \sigma_2 \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} + rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T-t) - r) \frac{\partial u}{\partial r} - ru \tag{3}$$

for $s > 0$, $v > 0$, $-\infty < r < \infty$. We refer to (3) as the *Heston–Hull–White (HHW) PDE*. It can be viewed as a time-dependent convection-diffusion-reaction equation on an unbounded, three-dimensional spatial domain.

The aim of this paper is to describe an efficient numerical solution method for the HHW PDE. We follow the method-of-lines approach. For the space discretization, we use finite difference (FD) schemes on non-uniform Cartesian grids

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in the (s,v,r)-domain. For the time discretization, we apply a specific Alternating Direction Implicit (ADI) scheme that is tailored to semi-discrete systems stemming from multi-dimensional PDEs with mixed spatial-derivative terms. The numerical solution method formulated in this paper for the HHW PDE extends the method that has recently been proved effective [2] for the simpler, two-dimensional Heston PDE.

SPACE DISCRETIZATION OF THE HESTON–HULL–WHITE PDE

As a preliminary step, the spatial domain is restricted to a bounded set $[0,S_{\max}] \times [0,V_{\max}] \times [-R_{\max},R_{\max}]$ with fixed values $S_{\max}, V_{\max}, R_{\max}$ chosen sufficiently large. The following boundary conditions are imposed,

$$u(0,v,r,t) = 0, \qquad \qquad \frac{\partial u}{\partial r}(s,v,-R_{\max},t) = 0,$$

$$\frac{\partial u}{\partial s}(S_{\max},v,r,t) = 1, \qquad u(s,V_{\max},r,t) = s, \qquad \frac{\partial u}{\partial r}(s,v,R_{\max},t) = 0. \qquad (4)$$

Clearly, these conditions are of Dirichlet and Neumann type. We assume throughout that the Feller condition $2\kappa\eta > \sigma_1^2$ holds, which implies that $v = 0$ forms an outflow boundary and no information concerning u is required there.

For the space discretization of the initial-boundary value problem (2)–(4) for the HHW PDE we consider an extension of the FD discretization on Cartesian grids described in [2] for the two-dimensional Heston PDE. The spatial grid is such that in the s- and v-directions non-uniform meshes $0 = s_0 < s_1 < \ldots < s_{m_1} = S_{\max}$ and $0 = v_0 < v_1 < \ldots < v_{m_2} = V_{\max}$ are defined with relatively many points in the neighborhood of $s = K$ and $v = 0$, respectively. This choice is natural for numerical reasons, since the initial function (2) is non-smooth at $s = K$ and for $v \approx 0$ the HHW PDE is convection-dominated in the v-direction. Moreover, values (s,v) that lie near $(K,0)$ are of main practical interest. In the r-direction, we define a non-uniform mesh as well, where many points are placed near the so-called reversion level $b(t)$ at each time t. For ease of presentation, we suppose here that b is constant, $b(t) = b$. Let integer $m_3 \geq 1$ and parameter $c > 0$ and let equidistant $\xi_0 < \xi_1 < \ldots < \xi_{m_3}$ be given by

$$\xi_k = \sinh^{-1}((-R_{\max}-b)/c) + k \cdot \Delta\xi \quad (0 \leq k \leq m_3) \quad \text{with} \quad \Delta\xi = \frac{1}{m_3}[\sinh^{-1}((R_{\max}-b)/c) - \sinh^{-1}((-R_{\max}-b)/c)].$$

Then a non-uniform mesh $-R_{\max} = r_0 < r_1 < \ldots < r_{m_3} = R_{\max}$ is defined through the transformation

$$r_k = b + c \sinh(\xi_k) \quad (0 \leq k \leq m_3).$$

The parameter c controls the relative number of mesh points r_k that lie near the reversion level b. The above mesh is smooth in the sense that there exist real constants $C_0, C_1, C_2 > 0$ such that the mesh widths $\Delta r_k = r_k - r_{k-1}$ satisfy

$$C_0 \Delta\xi \leq \Delta r_k \leq C_1 \Delta\xi \quad \text{and} \quad |\Delta r_{k+1} - \Delta r_k| \leq C_2 (\Delta\xi)^2 \quad \text{(uniformly in } k, m_3\text{)}.$$

The type of mesh for r is the same as employed in [2] for s and v. It has previously been considered e.g. in [7]. In our application, we have chosen $c = K/10$, $c = V_{\max}/500$, $c = R_{\max}/400$ for the s-, v-, r-directions, respectively.

To formulate the FD schemes under consideration, let $f : \mathbb{R} \to \mathbb{R}$ be any given function, let $\{x_i\}_{i \in \mathbb{Z}}$ be any given increasing sequence of mesh points, and $\Delta x_i = x_i - x_{i-1}$ for all i. We use the following FD approximations of the derivatives $f'(x_i)$ and $f''(x_i)$:

$$f'(x_i) \approx \frac{\Delta x_i}{\Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)} f(x_{i-2}) + \frac{-\Delta x_{i-1} - \Delta x_i}{\Delta x_{i-1}\Delta x_i} f(x_{i-1}) + \frac{\Delta x_{i-1} + 2\Delta x_i}{\Delta x_i(\Delta x_{i-1} + \Delta x_i)} f(x_i), \qquad (5a)$$

$$f'(x_i) \approx \frac{-\Delta x_{i+1}}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} f(x_{i-1}) + \frac{\Delta x_{i+1} - \Delta x_i}{\Delta x_i \Delta x_{i+1}} f(x_i) + \frac{\Delta x_i}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})} f(x_{i+1}), \qquad (5b)$$

$$f'(x_i) \approx \frac{-2\Delta x_{i+1} - \Delta x_{i+2}}{\Delta x_{i+1}(\Delta x_{i+1} + \Delta x_{i+2})} f(x_i) + \frac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+1}\Delta x_{i+2}} f(x_{i+1}) + \frac{-\Delta x_{i+1}}{\Delta x_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})} f(x_{i+2}), \qquad (5c)$$

$$f''(x_i) \approx \frac{2}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} f(x_{i-1}) + \frac{-2}{\Delta x_i \Delta x_{i+1}} f(x_i) + \frac{2}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})} f(x_{i+1}). \qquad (5d)$$

Note that (5b), (5d) are central schemes and (5a), (5c) are upwind schemes. If $f : \mathbb{R}^2 \to \mathbb{R}$ is any given function of two variables (x,y), then we approximate the mixed derivative $f_{xy}(x_i,y_j)$ at any point (x_i,y_j) by successive application

of (5b) in the *x*- and *y*-directions. This is equivalent to a FD approximation based on a 9-point stencil centered about (x_i, y_j). For smooth non-uniform meshes, all FD schemes above possess a second-order truncation error.

The actual FD discretization of (2)–(4) is performed as follows. In view of the Dirichlet boundary conditions in (4), the relevant set of grid points is $\mathcal{G} = \{(s_i, v_j, r_k) : 1 \leq i \leq m_1, 0 \leq j \leq m_2 - 1, 0 \leq k \leq m_3\}$. At this grid, each spatial derivative appearing in (3) is replaced by its corresponding central FD approximation, except:

- At the boundary $s = S_{\max}$ the derivatives in the *s*-direction need to be considered. The Neumann condition clearly yields $\partial u / \partial s$ and it implies that the mixed derivatives $\partial^2 u / \partial s \partial v$, $\partial^2 u / \partial s \partial r$ vanish there. Next, $\partial^2 u / \partial s^2$ is approximated at $s = S_{\max}$ by (5d) using a virtual point $s_{m_1+1} > S_{\max}$ where the value at this point is defined by extrapolation.
- At the outflow boundary $v = 0$ the derivative $\partial u / \partial v$ is approximated by the upwind scheme (5c). All other derivatives in the *v*-direction have a factor v or $\sqrt{v}$ in front in (3), and thus can conveniently be set equal to zero.
- In the region $v > 1$, we apply the upwind scheme (5a) for $\partial u / \partial v$ whenever the flow in the *v*-direction is towards $v = V_{\max}$. This is done so as to avoid spurious oscillations in the FD solution when σ_1 is small.
- At the boundaries $r = \pm R_{\max}$ the Neumann conditions are incorporated similarly as for s at the boundary $s = S_{\max}$.

The FD discretization of the initial-boundary value problem for the HHW PDE gives rise to an initial value problem for a large system of stiff ordinary differential equations (ODEs),

$$U'(t) = A(t)U(t) + g(t) \quad (0 \leq t \leq T), \quad U(0) = U_0. \tag{6}$$

Here $A(t)$, for any $t \geq 0$, is a given real $m \times m$–matrix and $g(t)$, U_0 are given real m–vectors, with $m = m_1 m_2 (m_3 + 1)$. The vector function g depends on the boundary conditions (4) and the vector U_0 is directly obtained from the initial condition (2). For each $t > 0$, the entries of the solution vector $U(t)$ to (6) form approximations to the exact option values $u(s, v, r, t)$ at the grid points $(s, v, r) \in \mathcal{G}$, ordered in a convenient way.

## TIME DISCRETIZATION: THE MODIFIED CRAIG–SNEYD SCHEME

The selection of a suitable time discretization scheme for the semi-discrete HHW system (6) is crucial to acquiring an effective full numerical solution method for the HHW initial-boundary value problem. Standard implicit schemes such as Crank–Nicolson are in general computationally too demanding. For the efficient time discretization of (6), we propose in this paper a splitting scheme of the ADI type. We decompose the matrix $A(t)$ into four simpler matrices,

$$A(t) = A_0 + A_1 + A_2 + A_3(t). \tag{7}$$

Here A_0 represents the part of $A(t)$ that stems from the FD discretization of all mixed derivative terms in the HHW PDE. Note that A_0 is nonzero whenever one of the correlation factors ρ_{12}, ρ_{13}, ρ_{23} is nonzero. In line with the traditional ADI idea, the matrices A_1, A_2, $A_3(t)$ represent the parts of $A(t)$ that stem from the FD discretization of all spatial derivatives in the *s*-, *v*- and *r*-directions, respectively, and the *ru* term from (3) is distributed evenly over A_1, A_2, $A_3(t)$. We decompose $g(t) = g_0 + g_1 + g_2 + g_3(t)$ analogous to that of $A(t)$.

Let $\theta > 0$ be a given real parameter, let $\Delta t = T/N$ with integer $N \geq 1$, and $t_n = n \cdot \Delta t$. The following scheme generates, in a one-step manner, approximations U_n to the exact solution values $U(t_n)$ of (6) for $n = 1, 2, \ldots, N$:

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t \left(A(t_{n-1}) U_{n-1} + g(t_{n-1}) \right), \\
Y_j = Y_{j-1} + \theta \Delta t A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2), \\
Y_3 = Y_2 + \theta \Delta t \left(A_3(t_n) Y_3 - A_3(t_{n-1}) U_{n-1} + g_3(t_n) - g_3(t_{n-1}) \right), \\
\widehat{Y}_0 = Y_0 + \theta \Delta t A_0 \left(Y_3 - U_{n-1} \right), \\
\widetilde{Y}_0 = \widehat{Y}_0 + \left(\tfrac{1}{2} - \theta \right) \Delta t \left(A(t_n) Y_3 - A(t_{n-1}) U_{n-1} + g(t_n) - g(t_{n-1}) \right), \\
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta \Delta t A_j \left(\widetilde{Y}_j - U_{n-1} \right) \quad (j = 1, 2), \\
\widetilde{Y}_3 = \widetilde{Y}_2 + \theta \Delta t \left(A_3(t_n) \widetilde{Y}_3 - A_3(t_{n-1}) U_{n-1} + g_3(t_n) - g_3(t_{n-1}) \right), \\
U_n = \widetilde{Y}_3.
\end{cases}
\tag{8}
$$
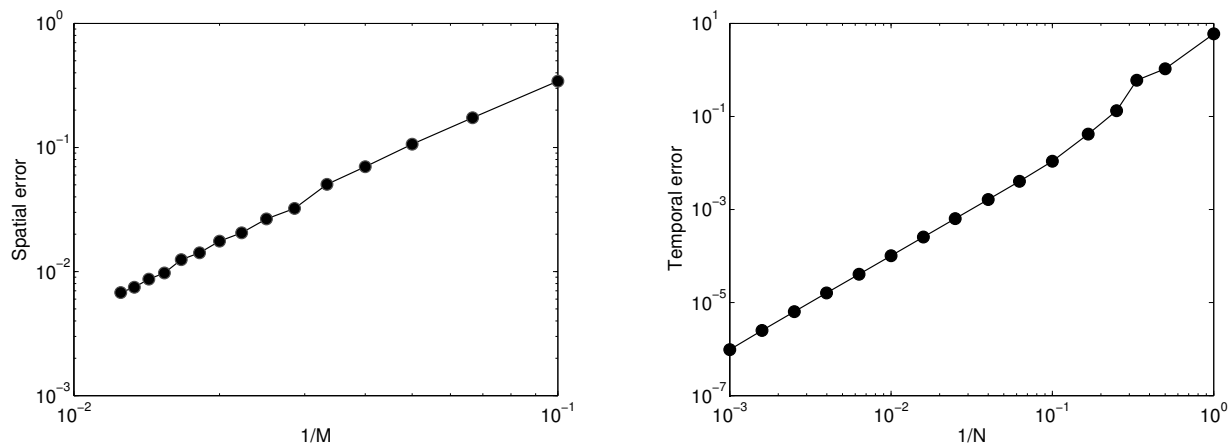
The scheme (8) is called the *Modified Craig–Sneyd (MCS) scheme*. It has recently been introduced, in a slightly more general form, by In 't Hout & Welfert [4]. Taylor expansion yields that the MCS scheme has a classical order of consistency equal to two for any given θ. A perusal of (8) shows that the A_0 part, representing all mixed derivative terms, is always treated in an *explicit* fashion. The $A_1, A_2, A_3(t)$ parts are successively treated in an *implicit* fashion.

The MCS scheme retains the key advantage of traditional ADI schemes in the presence of mixed derivative terms: the arising linear systems are efficiently solved by *LU* factorization, as the pertinent matrices are essentially tridiagonal or pentadiagonal. For standard implicit methods, the matrices that arise have a (very) large bandwidth and solving by *LU* factorization is in general too demanding. Moreover, with the splitting (7), the time-dependency of A is only passed onto the simpler matrix A_3, which yields an additional advantage of (8) over standard implicit methods.

The MCS scheme with $\theta = \frac{1}{3}$ has recently been found effective [2] in the numerical solution of the Heston PDE. In [3, 4] positive stability results were proved for the MCS scheme when applied to two-dimensional convection-diffusion equations with mixed derivative terms. A stability analysis that is directly relevant to its application in the case of the three-dimensional HHW PDE is non-trivial and yet incomplete. Ample numerical evidence indicates, however, the interesting result that the MCS scheme with the choice $\theta = \frac{1}{3}$ is also stable and effective in this case. It is our aim of future research to rigorously investigate this. In the following section we provide a numerical illustration.

NUMERICAL EXPERIMENT

If the Brownian motion W_τ^3 is not correlated to W_τ^1 and W_τ^2 (i.e., if $\rho_{13} = \rho_{23} = 0$), then a semi closed-form analytic formula for the exact European call option price function u is known [1]. We employ this formula to gain insight into the *global spatial error*, defined as the maximal absolute error between $u(s_i, v_j, r_k, T)$ and the corresponding entry of the solution vector $U(T)$ to (6) over a region of interest, chosen here as $\frac{1}{2}K < s_i < \frac{3}{2}K$, $0 < v_j < 1$, $0 < r_k < \frac{1}{4}$. Together with this we study the *global temporal error*, defined as the maximal absolute error between corresponding entries of $U(T)$ and its approximation U_N (where $T = N \cdot \Delta t$) over the same region of interest. For the numerical illustration we consider the Heston and Hull–White parameter values

$$\kappa = 3, \quad \eta = 0.12, \quad \sigma_1 = 0.04, \quad a = 0.1, \quad b(t) \equiv 0.06, \quad \sigma_2 = 0.01, \quad \rho_{12} = 0.6, \quad \rho_{13} = 0, \quad \rho_{23} = 0.$$

The maturity time, strike price and spatial domain are taken as $T = 1$, $K = 100$ and $[0, 14K] \times [0, 5] \times [-4, 4]$.

The left part of the figure displays the estimated global spatial errors versus $1/M$ for $M = 10, 15, \ldots, 80$ where $m_1 = 2M$, $m_2 = M$ and $m_3 = M$. The observed order of convergence for the spatial error is approximately equal to two, which is as desired.

The right part of the figure concerns the MCS scheme with $\theta = \frac{1}{3}$. It displays the estimated global temporal errors versus $1/N$ for a range of step numbers $1 \le N \le 1000$ when $M = 50$. We note that this scheme was applied with $N = 5000$ to obtain a reference solution for $U(T)$. Clearly, the temporal errors are uniformly bounded from above by a moderate value and decay monotonically as N increases. Further experiments show that the temporal errors are only weakly affected by the chosen number of spatial grid points. These observations are compatible with an

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unconditionally stable behavior of the pertinent MCS scheme. Subsequently, a convergence behavior is observed with an order approximately equal to two, which agrees with its classical order of consistency.

We note that the implementation has been done in Matlab, version 7.8, where all matrices were defined as sparse. As an indication for the computing times, our implementation of the MCS scheme took about 0.20 cpu-sec per time step if $M = 50$ (i.e., $m \approx 2.5 \times 10^5$) on a Intel Core Duo T7250 2.00 GHz processor with 4 GB memory.

In addition to the stability analysis mentioned at the end of the previous section, we shall elaborate in future research more on the performance of the numerical solution method described in this paper. Among others, time-dependent reversion levels, correlation factors that are all nonzero, and exotic options will be considered as well as the combined spatial-temporal errors.

ACKNOWLEDGEMENTS

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