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Journal of Algebra 280 (2004) 577–589

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Quiver Hopf algebras ☆

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Received 2 October 2003

Available online 18 August 2004

Communicated by S. Montgomery

Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

Abstract

In this paper we study subHopfalgebras of graded Hopf algebra structures on a path coalgebra kQ^c. We prove that a Hopf structure on some subHopfquivers can be lifted to a Hopf structure on the whole Hopf quiver. If Q is a Schurian Hopf quiver, then we classified all simple-pointed subHopfalgebras of a graded Hopf structure on kQ^c. We also prove a dual Gabriel theorem for pointed Hopf algebras.

Introduction

Given a quiver Q and field k, Chin and Montgomery [4] have dualized the construction of the path algebra kQ^a to obtain the path coalgebra kQ^c; while Cibils and Rosso [7] have introduced the notion of the Hopf quiver $Q = Q(G, \chi)$ of a group G with a ramification χ, and then constructed all the graded Hopf algebras with length grading on kQ^c. It turns out that kQ^c admits a graded Hopf structure with length grading if and only if Q is a Hopf quiver. For other works on constructing Hopf algebras via quivers see [2,3,6,9].

☆ Supported by the Natural Science Foundation of China (No. 10271113 and No. 10301033), and the EC AsiaLink project "Algebras and Representations in China and Europe" (ASI/B7-301/98/679-11).
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.06.008

Note that in the algebra case, in general we consider quotient algebras of kQ^a. For this quiver techniques we refer Auslander, Reiten, Smalϕ [1], and Ringel [17]. It is then natural to study subcoalgebras of path coalgebra kQ^c, and the Hopf algebra structures on subcoalgebras of kQ^c. This is the consideration of this paper.

If Q is a Schurian Hopf quiver and A is a graded Hopf structure on kQ^c, with $\operatorname{char} k = 0$, then we classified the all simple-pointed subHopfalgebras of A (Theorem 2.7). If $\Gamma = Q(H, r)$ is a subHopfquiver of the Hopf quiver $Q = Q(G, \chi)$ with $G = N \rtimes H$, then we prove that any Hopf structure on $k\Gamma^c$ can be lifted to a Hopf structure on kQ^c (Theorem 3.2); we also give a dual version of the Gabriel theorem for pointed Hopf algebras (Theorem 4.5).

1. The construction of Cibils and Rosso

1.1. Path coalgebras

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 and Q_1 are the sets of vertices and of arrows, respectively, s and t are maps from Q_1 to Q_0, with $s(\alpha)$ and $t(\alpha)$ being the starting and the ending vertex of α, respectively. Assume that Q_0 and Q_1 are countable sets. A path p of length l in Q is a sequence $p = \alpha_l \cdots \alpha_1$ of arrows α_i with $t(\alpha_i) = s(\alpha_{i+1})$, $1 \leqslant i \leqslant l - 1$. Vertices are regarded as paths of length 0.

Denote by kQ the k-space with basis the set of all paths in Q. Then $kQ = \bigoplus_{n \geqslant 0} kQ_n$, where kQ_n is the k-space with basis the set of all paths of length n. By definition (see [4]), the path coalgebra kQ^c is the coalgebra with underlying space kQ, with comultiplication given by

$$\Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha,$$

and counit by $\varepsilon(p) = 0$ if path p is of length $\geqslant 1$ and $\varepsilon(p) = 1$ if p is of length 0.

For a coalgebra C, the set of group-like elements is $G(C) := \{0 \neq c \in C \mid \Delta(c) = c \otimes c\}$. A coalgebra C is said to be pointed if each simple subcoalgebra of C is of dimension one. For $x, y \in G(C)$, denote by $P_{x,y}(C) := \{c \in C \mid \Delta(c) = c \otimes x + y \otimes c\}$, the set of x, y-primitive elements in C. An x, y-primitive element c is said to be non-trivial if $c \notin k(x - y)$. The proof of the following useful fact is straightforward.

Lemma. *Let C be a subcoalgebra of kQ^c. Then*

(i) *C is pointed with $G(C) = Q_0 \cap C$.*
(ii) *For $x, y \in G(C)$ we have $P_{x,y}(C) = k(x - y) \oplus (C \cap {}^y kQ_1^x)$, where ${}^y kQ_1^x$ is the k-space spanned by all arrows from x to y.*

1.2. Cotensor coalgebras

A path coalgebra is in fact a cotensor coalgebra, and hence enjoys the universal property. Recall these from [14] and [15] (see also [13]).

Let M be a C-bicomodule, with structure maps δ_R and δ_L. Set $M^{\square 0} := C$, $M^{\square 1} := M$. For $n \geqslant 2$, define the nth cotensor product $M^{\square n}$ to be the kernel of the k-map:

$$M^{\otimes n} \xrightarrow{f} (M \otimes C \otimes M \otimes \cdots \otimes M) \oplus (M \otimes M \otimes C \otimes \cdots \otimes M) \oplus \cdots \\ \oplus (M \otimes M \otimes \cdots \otimes C \otimes M)$$

with $f = (\delta_R \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} - \mathrm{id} \otimes \delta_L \otimes \cdots \otimes \mathrm{id},\ \mathrm{id} \otimes \delta_R \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes \delta_L \otimes \cdots \otimes \mathrm{id},\ \ldots,\ \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \delta_L)$.

Set $\mathrm{CoT}_C(M) := C \oplus M \oplus M^{\square 2} \oplus \cdots \oplus M^{\square n} \oplus \cdots$. Then $\mathrm{CoT}_C(M)$ has a coalgebra structure, which is called the cotensor coalgebra of bicomodule M over C: the counit ε is given by $\varepsilon|_{M^{\square i}} = 0$ for $i \geqslant 1$, and $\varepsilon|_C = \varepsilon_C$; the comultiplication is given by $\Delta|_C := \Delta_C$, $\Delta|_M := \delta_L \oplus \delta_R$, and for $\sum m_1 \otimes \cdots \otimes m_n \in M^{\square n}$, $n \geqslant 2$,

$$\begin{aligned}
&\Delta\Big(\sum m_1 \otimes \cdots \otimes m_n\Big) \\
&\quad := \sum \delta_L(m_1) \otimes m_2 \otimes \cdots \otimes m_n + \sum m_1 \otimes (m_2 \otimes \cdots \otimes m_n) + \cdots \\
&\qquad + \sum (m_1 \otimes m_2 \otimes \cdots \otimes m_{n-1}) \otimes m_n + \sum m_1 \otimes \cdots \otimes m_{n-1} \otimes \delta_R(m_n) \\
&\quad \in \big(C \otimes M^{\square n}\big) \oplus \big(M \otimes M^{\square (n-1)}\big) \oplus \cdots \oplus \big(M^{\square (n-1)} \otimes M\big) \oplus \big(M^{\square n} \otimes C\big) \\
&\quad \subseteq \mathrm{CoT}_C M \otimes \mathrm{CoT}_C(M).
\end{aligned}$$

Note that for any quiver Q, kQ_n is a kQ_0-bicomodule for $n \geqslant 0$ via $\delta_L(p) := t(p) \otimes p$ and $\delta_R(p) := p \otimes s(p)$; and that $kQ^c \simeq \mathrm{CoT}_{kQ_0}(kQ_1)$ as coalgebras.

We need the following universal property (see, e.g., [7], also [15]).

Lemma. *Let $X \xrightarrow{\psi} \mathrm{CoT}_C(V)$ be a coalgebra map. Set $\psi_n := p_n \psi : X \to V^{\square n}$ to be the projection. Then $\psi_0 : X \to C$ is a coalgebra map, and $\psi_1 : X \to V$ is a C-bicomodule map, where X has the induced C-bicomodule structure via ψ_0; for $n \geqslant 2$, ψ_n is exactly the C-bicomodule map given by the composition*

$$\psi_n : X \xrightarrow{\Delta} X \otimes X \xrightarrow{\Delta \otimes \mathrm{id}} X \otimes X \otimes X \to \cdots \to X \otimes X \otimes \cdots \otimes X \xrightarrow{\psi_1^{\otimes n}} V^{\otimes n}.$$

That is, ψ is uniquely determined by ψ_0 and ψ_1.

Conversely, let $\psi_0 : X \to C$ be a coalgebra map, and $\psi_1 : X \to V$ a C-bicomodule map. Let $\psi_n : X \to V^{\otimes n}$ be the composition given as above. Then ψ_n is a C-bicomodule map with $\mathrm{Im}(\psi_n) \subseteq V^{\square n}$. If for each $x \in X$ there are only finite i such that $\psi_i(x) \neq 0$, then $\psi := \sum_{i \geqslant 0} \psi_i : X \to \mathrm{CoT}_C(V)$ is a coalgebra map.

1.3. Hopf bimodules over group algebras

Let H be a Hopf algebra. An H-Hopf bimodule M is an H-bimodule and simultaneously an H-bicomodule such that the structure maps $\delta_L : M \to H \otimes M$ and $\delta_R : M \to M \otimes H$ are both H-bimodule maps, where the H-module structures of $H \otimes M$ and $M \otimes H$ are diagonal.

Let C be a coalgebra, and $x, y \in C$. For a C-bimodule M, denote

$$ {}^{y}M^{x} := \{ m \in M \mid \delta_L(m) = y \otimes m,\ \delta_R(m) = m \otimes x \}. $$

The following theorem gives the description of the category $\mathcal{B}(kG)$ of kG-Hopf bimodules by the module categories of subgroups of G.

Theorem (Cibils–Rosso [6]). *For each conjugacy class C of G, choose an element $u(C) \in C$, and let Z_C be the centralizer of $u(C)$ in G. Then $\mathcal{B}(kG) \simeq \prod_{C \in \mathcal{C}} \operatorname{mod} kZ_C$.*

More precisely, let $M \in \mathcal{B}(kG)$. Then ${}^{u(C)}M^1 \in \operatorname{mod} kZ_C$ via $m \cdot g := g^{-1}mg$, $\forall g \in Z_C$, $m \in {}^{u(C)}M^1$. Conversely, let $M = (M_C)_{C \in \mathcal{C}} \in \prod_{C \in \mathcal{C}} \operatorname{mod} kZ_C$. Then $\mathcal{V}(M) := \bigoplus_{C \in \mathcal{C}} kG \otimes M_C \otimes_{kZ_C} kG$ has a kG-Hopf bimodule structure with ${}^{y}\mathcal{V}(M)^{x} = x \otimes M_C \otimes g_i$, and a kZ_C-module isomorphism ${}^{u(C)}\mathcal{V}(M)^1 \simeq M_C$, where g_i is an element in G such that $g_i^{-1}u(C)g_i = x^{-1}y$.

(Note that the left kG-module $\mathcal{V}(M)$ is trivial, the right kG-module $\mathcal{V}(M)$ is diagonal, and the kG-bicomodule structure of $\mathcal{V}(M)$ is defined as $\delta_L(g \otimes m_C \otimes g_i) = gg_i^{-1}u(C)g_i \otimes (g \otimes m_C \otimes g_i)$ and $\delta_R(g \otimes m_C \otimes g_i) = (g \otimes m_C \otimes g_i) \otimes g$.)

1.4. Hopf quivers

Let G be a group, with a ramification χ (i.e., a function $\chi : \mathcal{C} \to \mathbb{N}_0$), denoted by $\chi = \sum_{C \in \mathcal{C}} \chi_C C$, where $\mathcal{C}$ is the set of conjugacy classes of G. By definition [7], the corresponding Hopf quiver $Q = Q(G, \chi)$ has the set of vertices $Q_0 = G$, and for each $x \in Q_0$, $c \in C \in \mathcal{C}$, one has χ_C arrows from x to cx.

The following theorem answers the question that on which quivers one can construct graded Hopf algebras with length grading.

Theorem (Cibils–Rosso [7]). *Let Q be a quiver. Then the following are equivalent*:

(i) *Q is the Hopf quiver of some (G, χ);*
(ii) *Q_0 is a group and kQ_1 is a kQ_0-Hopf bimodule with $\delta_L(p) := t(p) \otimes p$ and $\delta_R(p) := p \otimes s(p)$ for each arrow p;*
(iii) *kQ^c admits a graded Hopf structure with length grading.*

1.5. Concrete construction

Given a Hopf quiver $Q = Q(G, \chi)$ with $\chi = \sum_{C \in \mathcal{C}} \chi_C C$, in order to construct a graded Hopf structure on kQ^c with length grading, firstly, we construct a kG-Hopf bimodule $\mathcal{V}(M)$: for each conjugacy class $C \in \mathcal{C}$ and a fixed element $u(C)$ in C, choose a right kZ_C-module M_C of dimension χ_C, where Z_C is the centralizer of $u(C)$ in G, and set $M := (M_C)_{C\in\mathcal{C}} \in \prod_{C\in\mathcal{C}} \text{mod } kZ_C$. This gives a KG-Hopf bimodule $\mathcal{V}(M) := \bigoplus_{C\in\mathcal{C}} kG \otimes M_C \otimes_{kZ_C} kG$. Secondly, make kQ_1 into a kQ_0-Hopf bimodule: since $kQ_1 = \bigoplus_{x,y\in G} {}^y kQ_1^x$ and $\mathcal{V}(M) = \bigoplus_{x,y\in G} {}^y\mathcal{V}(M)^x$, with $\dim_k {}^y kQ_1^x = \dim_k {}^y\mathcal{V}(M)^x = \chi_{[x^{-1}y]}$, we can identify kQ_1 with $\mathcal{V}(M)$ by identifying ${}^y kQ_1^x$ with ${}^y\mathcal{V}(M)^x$. Finally, construct the corresponding graded, associative multiplication on kQ^c as follows: let ψ_0 be the composition

$$X \xrightarrow{p_0\otimes p_0} kQ_0 \otimes kQ_0 \to kQ_0,$$

where $X = kQ^c \otimes kQ^c$, and ψ_1 be the composition

$$X \xrightarrow{p_0\otimes p_1 \oplus p_1\otimes p_0} (kQ_0 \otimes kQ_1) \oplus (kQ_1 \otimes kQ_0) \to kQ_1.$$

Then ψ_0 is a coalgebra map and ψ_1 is a kQ_0-bicomodule map. Then by Lemma 1.2 we have a coalgebra map

$$\psi : kQ^c \otimes kQ^c \to kQ^c \simeq \mathrm{CoT}_{kQ_0}(kQ_1).$$

In this way one gets a graded associative multiplication ψ on kQ^c. Since a pointed bialgebra whose set of group-like elements is a group is a Hopf algebra (see Takeuchi [20]), it follows that (kQ, ψ, Δ) is a graded Hopf algebra.

By construction we have in particular

$$\beta \cdot \alpha = \big[t(\beta).\alpha\big]\big[\beta.s(\alpha)\big] + \big[\beta.t(\alpha)\big]\big[s(\beta).\alpha\big] \tag{$*$}$$

for any arrows α and β in Q, where $[t(\beta).\alpha]$ denotes the action of $t(\beta)$ on α.

Note that any graded Hopf structure on kQ^c is given by the construction of Cibils–Rosso described above. For details we refer to [7].

2. Simple-pointed Hopf algebras of Schurian Hopf quivers

2.1. Radford introduced the concept of simple-pointed Hopf algebra, and classified finite-dimensional, simple-pointed Hopf algebras over an algebraically closed field. Recall that a Hopf algebra H is simple-pointed if H is pointed, not cocommutative, and if L is a proper subHopfalgebra of H then $L \subseteq H_0$. See [16].

Given a graded Hopf structure A on kQ^c, we are interested in determining all possible (not necessarily finite-dimensional) simple-pointed subHopfalgebras H of A. The aim of this section is to do this for Q a Schurian Hopf quiver.

By definition, a Hopf quiver $Q = Q(G, \chi)$ is said to be *Schurian* if $\chi_C \leqslant 1$ for each conjugacy class C of G. That is, for each pair (x, y) of vertices of Q, there are at most one arrow from x to y.

2.2. Let $q \in k$ be an nth root of unity. For non-negative integers l and m, the Gaussian binomial coefficient is defined to be $\binom{l+m}{l}_q := (l+m)!_q / l!_q m!_q$, where $l!_q := 1_q \cdots l_q$, $0!_q := 1$, $l_q := 1 + q + \cdots + q^{l-1}$.

2.3. Denote by Z_n the basic cycle of length n, i.e., Z_n is the quiver with set of vertices the cyclic group $\{1, g, \ldots, g^{n-1}\}$, and a unique arrow α_i from g^i to g^{i+1} for each $0 \leqslant i \leqslant n-1$. Then Z_n is a Schurian Hopf quiver of the cyclic group $\langle g \rangle$ with ramification $\chi_C = 0$ for $C \neq [g]$, and 1 for $C = [g]$. Let p_i^l denote the path in Z_n of length l starting at g^i (thus, $p_i^0 = g^i$, $p_i^1 = \alpha_i$). For each nth root $q \in k$ of unity, Cibils and Rosso [7] have defined a graded Hopf algebra structure $kZ_n(q)$ (with length grading) on the path coalgebra kZ_n^c by

$$p_i^l \cdot p_j^m = q^{jl} \binom{l+m}{l}_q p_{i+j}^{l+m} \qquad (**)$$

with antipode S mapping p_i^l to $(-1)^l q^{-l(l+1)/2 - il} p_{n-l-i}^l$.

Note that if $q = 1$ and $\operatorname{char} k = 0$, then by $(**)$ we see that g and α_0 are generators of $kZ_n(1)$, since $\binom{l+m}{l} \neq 0$ for any l and m.

2.4. Denote by $C_d(n)$, $d \geqslant 2$, the subcoalgebra of kZ_n^c with basis the set of all paths of length strictly less than d.

If q is an nth root of unity, with multiplicative order $d \geqslant 2$, then $\binom{l+m}{l}_q = 0$ for m, $l \leqslant d-1$, $l+m \geqslant d$. It follows from $(**)$ that $C_d(n)$ is a subHopfalgebra of $kZ_n(q)$. Denote this Hopf algebra by $C_d(n,q)$. Note that

$$\binom{l+m}{l}_q \neq 0 \quad \text{for } l+m \leqslant d-1,$$

and hence by $(**)$ we deduce that $C_d(n,q)$ is generated by g and α_0 as an algebra.

Let q is an nth root of unity, with multiplicative order $d \geqslant 2$. Recall that by definition $A_{n,d}(q)$ is an associative algebra generated by elements g and x, with relations

$$g^n = 1, \qquad x^d = 0, \qquad xg = qgx.$$

Then $A_{n,d}(q)$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = x \otimes 1 + g \otimes x, \qquad \epsilon(g) = 1, \qquad \epsilon(x) = 0,$$
$$S(g) = g^{-1} = g^{n-1}, \qquad S(x) = -xg^{-1} = -q^{-1} g^{n-1} x.$$

In particular, if q is an nth primitive root of unity, then $A_{n,d}(q)$ is the n^2-dimensional Hopf algebra introduced by Taft [19]. For this reason $A_{n,d}(q)$ is called a generalized Taft algebra in [10]. Mapping g to g and x to α_0, we get a Hopf algebra isomorphism $A_{n,d}(q) \simeq C_d(n,q)$.

2.5. Denote by A_∞^∞ the quiver with set of vertices the infinite cyclic group $\langle g \rangle$, and a unique arrow α_i from g^i to g^{i+1} for each $i \in Z$. Then A_∞^∞ is a Schurian Hopf quiver of the cyclic group $\langle g\rangle$ with ramification $\chi_C = 0$ for $C \neq [g]$, and 1 for $C = [g]$. Again let p_i^l denote the path in A_∞^∞ of length l starting at g^i. For each $0 \neq q \in K$, Cibils and Rosso [7] have defined a graded Hopf algebra structure $kA_\infty^\infty(q)$ (with length grading) on the path coalgebra $kA_\infty^{\infty c}$, with multiplication given again by $(**)$.

If q is not a root of unity, then by $(**)$ we see that g and α_0 are generators of $kA_\infty^\infty(q)$, since in this case $\binom{l+m}{l}_q \neq 0$ for any l and m.

If q is a root of unity of order $d \geqslant 2$, then by $(**)$ we see that the subalgebra of $kA_\infty^\infty(q)$ generated by g and α_0 is an infinite-dimensional Hopf algebra spanned by all paths of lengths strictly less than d, which is denoted by $kA_\infty^\infty(d,q)$. This is because that

$$\binom{l+m}{l}_q = 0 \quad \text{for } m,\ l \leqslant d-1,\ l+m \geqslant d, \quad \text{and}$$
$$\binom{l+m}{l}_q \neq 0 \quad \text{for } l+m \leqslant d-1.$$

If $q = 1$ and $\operatorname{char} k = 0$, then by $(**)$ we see that g and α_0 are generators of $kA_\infty^\infty(1)$.

Lemma 2.6. *Let A be a graded Hopf structure on a Schurian Hopf quiver $Q = Q(G,\chi)$, and $\alpha : 1 \to g$ be an arrow of Q, where 1 is the identity element of group G.*

(i) *If the order $o(g) = n$, then A has a subHopfalgebra isomorphic to $kZ_n(q)$.*
(ii) *If $o(g) = \infty$, then A has a subHopfalgebra isomorphic to $kA_\infty^\infty(q)$.*

Proof. By the definition of the Schurian Hopf quiver we see that $\chi_{[g]} = 1$, where $[g]$ is the conjugacy class. If $o(g) = n$, then Q has a subquiver Z_n with arrows $\alpha_i : g^i \to g^{i+1}$ for each $0 \leqslant i \leqslant n-1$, where $\alpha = \alpha_0$. We claim that the product of p_i^l and p_j^m in A is again given by the formula $(**)$, with q an nth root of unity in k, and hence A has a subHopfalgebra isomorphic to $kZ_n(q)$.

In order to prove this, note that the algebra structure of A is determined by a kG-Hopf bimodule $\mathcal{V}(M)$, where $M = (M_C)_{C\in\mathcal{C}} \in \prod_{C\in\mathcal{C}} \operatorname{mod} kZ_C$, with $\dim_k M_C = \chi_C$ (see 1.5), and $\mathcal{V}$ is the functor defined in Theorem 1.3. Choose $u(C)$ such that $u([g]) = g$ (see Theorem 1.3). Since $\chi_{[g]} = 1$, it follows that $\dim_k M_{[g]} = 1$. Set $M_{[g]} = kv$ with $vg = qv$ for an nth root q of unity. Identify kQ_1 with $\mathcal{V}(M)$. In this identification we have ${}^{g^{i+1}}(kQ_1)^{g^i} = g^{i+1}(kQ_1)g^i = k(g^i \otimes v \otimes_{kZ_{[g]}} 1)$, and hence we may identify α_i with $g^i \otimes v \otimes_{kZ_{[g]}} 1$. It follows that

$$g\alpha_i = g\big(g^i \otimes v \otimes_{kZ_{[g]}} 1\big) = g^{i+1} \otimes v \otimes_{kZ_{[g]}} 1 = \alpha_{i+1}$$

and

$$\begin{aligned}\alpha_i g &= \big(g^i \otimes v \otimes_{kZ_{[g]}} 1\big)g = g^{i+1} \otimes v \otimes_{kZ_{[g]}} g = g^{i+1} \otimes vg \otimes_{kZ_{[g]}} 1\\ &= qg^{i+1} \otimes v \otimes_{kZ_{[g]}} 1 = q\alpha_{i+1}.\end{aligned}$$

Note that the product of p_i^l and p_j^m in A is uniquely determined by the actions of g^j on α_i for $i, j = 0, 1, \ldots, n-1$ (for details we refer to Theorem 3.8 in [7]), and hence the product of p_i^l and p_j^m in A is again given by the formula $(**)$ (for details we refer to the proof of Proposition 3.17 in [7]). For example, by $(*)$ in 1.5 we have

$$\begin{aligned}\alpha_i\alpha_j &= \big[t(\alpha_i).\alpha_j\big]\big[\alpha_i.s(\alpha_j)\big] + \big[\alpha_i.t(\alpha_j)\big]\big[s(\alpha_i).\alpha_j\big]\\ &= \big[g^{i+1}.\alpha_j\big]\big[\alpha_i.g^j\big] + \big[\alpha_i.g^{j+1}\big]\big[g^i.\alpha_i\big]\\ &= q^j\alpha_{i+j+1}\alpha_{i+j} + q^{j+1}\alpha_{i+j+1}\alpha_{i+j} = q^j\binom{2}{1}_q\alpha_{i+j+1}\alpha_{i+j}.\end{aligned}$$

This proves (i), and (ii) is similarly proved. □

The main result of this section is as follows.

Theorem 2.7. *Let Q be a Schurian Hopf quiver and H be a subHopfalgebra of a graded Hopf algebra structure on kQ^c with $\operatorname{char} k = 0$. Then H is simple-pointed if and only if H is isomorphic as a Hopf algebra to one of the following*:

(i) $C_d(n,q)$, *for an nth root q of unity with multiplicative order $d \geqslant 2$.*
(ii) $kZ_n(1)$ *for some $n \geqslant 1$.*
(iii) $kA_\infty^\infty(q)$, *where $0 \neq q \in k$ is not a root of unity.*
(iv) $kA_\infty^\infty(d,q)$, *where q is a primitive dth root of unity, $d \geqslant 2$.*
(v) $kA_\infty^\infty(1)$ *for some $n \geqslant 1$.*

Proof. First, we claim that any Hopf algebra H given in (i)–(v) is simple-pointed. Assume that L is a subHopfalgebra of H which is not contained in kG, where $G = G(H) = \langle g\rangle$. Denote by $\{L_n\}$ the coradical filtration of L. Since $L \neq L_0$, it follows that $L_1 \neq L_0$, and hence by the Taft–Wilson theorem (see, e.g., [12, p. 68]) that there is a non-trivial x, y-primitive element $\alpha \in L$ for some $x, y \in G(L)$, and hence $x^{-1}\alpha$ is a non-trivial $1, x^{-1}y$-primitive element in L. While by the quiver structure of A_∞^∞ (or of Z_n) we have $x^{-1}y = g$ and $P_{1,g}(H) = k(1-g) \oplus k\alpha_0$. It follows that $\alpha_0 \in L$. Since in all cases of (i)–(v), g and α_0 are generators of H, it follows that $L = H$. This proves that H is simple-pointed.

Conversely, let H be a simple-pointed subHopfalgebra of A, where A is a graded Hopf structure on Schurian Hopf quiver $Q = Q(G,\chi)$. Since $H \neq H_0$, it follows that $H_1 \neq H_0$, and hence by the Taft–Wilson theorem that there is a non-trivial x, y-primitive element $\alpha \in H$ for some $x, y \in G(H)$, and hence $x^{-1}\alpha$ is a non-trivial $1, x^{-1}y$-primitive element in H. This means that in quiver Q there is an arrow $\alpha_0 : 1 \to g := x^{-1}y$. Since Q is Schurian and $x^{-1}\alpha \in H$, it follows that $\alpha_0 \in H$.

If $o(g) = n$, then by Lemma 2.6 A has a subHopfalgebra $L \simeq kZ_n(q)$. If $o(q) = d \geqslant 2$, then the subalgebra of L generated by g and α_0 is isomorphic to $C_d(n,q)$; and if $d = 1$, then the subalgebra of L generated by g and α_0 is isomorphic to $kZ_n(1)$. In both cases, this subalgebra is a subHopfalgebra and contained in H. While H is simple-pointed, it follows that $H \simeq C_d(n,q)$ with $d \geqslant 2$, or $H \simeq kZ_n(1)$.

If $o(g) = \infty$, the similar argument proves that H is isomorphic to one of the Hopf algebras given in (iii)–(v). □

3. SubHopfquivers

Definition 3.1.

(i) Let G be a group with ramification χ, and H be a group with ramification r. We say $(H, r) \leqslant (G, \chi)$ provided that H is a subgroup of G, and $r_{[h]_H} \leqslant \chi_{[h]_G}$, $\forall h \in H$, where $[h]_H$ and $[h]_G$ denote the conjugacy class of h in H and in G, respectively.
(ii) If $(H, r) \leqslant (G, \chi)$, then the Hopf quiver $\Gamma = Q(H, r)$ of (H, r) is said to be a subHopfquiver of the Hopf quiver $Q = Q(G, \chi)$ of (G, χ).

Given a Hopf quiver Q and a subHopfquiver Γ of Q, it is natural to ask that when a Hopf structure B on $k\Gamma^c$ can be lifted to a Hopf structure on kQ^c. For this we have

Theorem 3.2. *Let $Q = Q(G, \chi)$ be the Hopf quiver of (G, χ), and $\Gamma = Q(H, r)$ a subHopfquiver of Q, with $G = N \rtimes H$ for some normal subgroup N of G. Then for each graded Hopf structure B (with length grading) on $k\Gamma^c$, there is a graded Hopf structure A (with length grading) on kQ^c, such that B is a subHopfalgebra of A.*

In order to prove the theorem we need some preparations. By the construction of a cotensor coalgebra, it is straightforward to verify the following fact.

Lemma 3.3. *Let C and D be coalgebras, U and V be a C-bicomodule and a D-bicomodule, respectively. If $f_0 : D \to C$ is a coalgebra map, and $f_1 : V \to U$ is a C-bicomodule map, where V has the induced C-bimodule structure via f_0, then there exists a unique coalgebra, graded map $f : \mathrm{CoT}_D(V) \to \mathrm{CoT}_C(U)$ with $f_n : V^{\square n} \to U^{\square n}$ being exactly the restriction of $f_1^{\otimes n} : V^{\otimes n} \to U^{\otimes n}$ to $V^{\square n}$, $n \geqslant 2$.*

Moreover, f is injective if and only if f_0 and f_1 are injective.

Note that the last assertion follows from a theorem duo to Heyneman and Radford (see [11], or [12, Theorem 5.3.1]).

Lemma 3.4. *Let H be a subgroup of G, V and U be a kH and a kG-Hopf bimodules, respectively. Denote by B and A the corresponding induced graded Hopf structures on $\mathrm{CoT}_{kH}(V)$ and on $\mathrm{CoT}_{kG}(U)$, respectively. If $f_1 : V \to U$ is a kH-bimodule map, and simultaneously a kG-bicomodule map, where V has the induced kG-bicomodule structure via the coalgebra embedding $kH \to kG$, then there exists a unique graded Hopf algebra map $f : B \to A$ with $f_n = f_1^{\otimes n}|_{V^{\square n}}$, $n \geqslant 2$.*

Moreover, f is injective if and only if f_1 is injective.

Proof. By Lemma 3.3 it remains to prove that f is an algebra map, i.e., $fm_B = m_A(f \otimes f)$. Set $\psi = fm_B$ and $\psi' = m_A(f \otimes f)$. Since both maps are coalgebra maps, it follows from Lemma 1.2 that it suffices to verify $\psi_0 = \psi'_0 : B \otimes B \to A \twoheadrightarrow kG$, and $\psi_1 = \psi'_1 : B \otimes B \to A \twoheadrightarrow U$. This follows from the assumption that f_1 is a kH-bimodule map. □

3.5. Let $G = N \rtimes H$ for some normal subgroup N of G. Denote by $\mathcal{C}_G$ and $\mathcal{C}_H$ the set of conjugacy classes of G and H, respectively. Set $\mathcal{C}_1 := \{C \in \mathcal{C}_G \mid C \cap H \neq \emptyset\}$ and $\mathcal{C}_2 := \{C \in \mathcal{C}_G \mid C \cap H = \emptyset\}$.

Since $G = N \rtimes H$, it follows that for each conjugacy class C of G in $\mathcal{C}_1$, there is only one conjugacy class of H contained in C. It follows that, for each $C \in \mathcal{C}_G$, we can always fix an element $u(C)$ in C, such that if $C \in \mathcal{C}_1$, then $u(C) \in H$.

(In fact, if $h_1 \sim h_2$ in G, then $h_2 = nhh_1h^{-1}n^{-1}$ for some $n \in N$, $h \in H$. Set $h' := hh_1h^{-1} \in H$. Then $h'^{-1}(nh'n^{-1}) = h'^{-1}h_2 \in H \cap N = \{1\}$. Thus, $h_2 = h' = hh_1h^{-1}$, i.e., $h_1 \sim h_2$ in H.)

Denote by $Z_G(u(C))$ and $Z_H(u(C))$ the centralizer of $u(C)$ in G and in H, respectively. Then for each $C \in \mathcal{C}_1$ we have $Z_G(u(C)) = Z_N(u(C)) \rtimes Z_H(u(C))$.

(In order to see this, note that $Z_G(u(C)) = Z_N(u(C))Z_H(u(C))$ with $Z_N(u(C)) = N \cap Z_G(u(C))$ being normal in $Z_G(u(C))$. In fact, if $nh \in Z_G(u(C))$ with $n \in N$, $h \in H$, then

$$hu(C)h^{-1}\big(u(C)\big)^{-1} = n^{-1}u(C)n\big(u(C)\big)^{-1} \in N \cap H = \{1\},$$

which implies $n \in Z_N(u(C))$ and $h \in Z_H(u(C))$.)

Lemma 3.6. *Let $(H, r) \leqslant (G, \chi)$ with $G = N \rtimes H$. Then for each kH-Hopf bimodule V with $\dim_k({}^{u(c)}V^1) = r_C$, $\forall C \in \mathcal{C}_H$, there exists a kG-Hopf bimodule U with $\dim_k({}^{u(C)}U^1) = \chi_C$, $\forall C \in \mathcal{C}_G$, such that V is kH-subbimodule of U, and simultaneously a kG-subbicomodule of U, where V has the induced kG-bicomodule structure via embedding $kH \hookrightarrow kG$.*

Proof. By Theorem 1.3 we have $V = \mathcal{V}(N)$ for some

$$N = (N_C)_{C \in \mathcal{C}_H} \in \prod_{C \in \mathcal{C}_H} \operatorname{mod} kZ_H\big(u(C)\big)$$

with $\dim_k N_C = r_C$, where $\mathcal{V}(N)$ is defined as in Theorem 1.3. Now, construct

$$M = (M_C)_{C \in \mathcal{C}_G} \in \prod_{C \in \mathcal{C}_G} \operatorname{mod} kZ_G\big(u(C)\big)$$

with $\dim_k M_C = \chi_C$ as follows. If $C \in \mathcal{C}_2$, then just choose a right $kZ_G(u(C))$-module M_C of dimension χ_C, say the trivial module. If $C \in \mathcal{C}_1$, then define

$$M_C := S_C \oplus N_C, \quad \text{where } S_C \text{ is a right } kZ_N\big(u(C)\big)\text{-module of dimension } \chi_C - r_C,$$

with action of $Z_G(u(C)) = Z_N(u(C)) \rtimes Z_H(u(C))$ on M_C being diagonal.

Set $U := \mathcal{V}(M)$, where $\mathcal{V}(M)$ is defined as in Theorem 1.3. Then U is a kG-Hopf bimodule. By the construction given in Theorem 1.3 the assertion follows. □

3.7. **Proof of Theorem 3.2.** By 1.5 the graded Hopf structure B on $k\Gamma^c$ corresponds a kH-Hopf bimodule V with $\dim_k({}^{u(C)}V^1) = r_C$, $C \in \mathcal{C}_H$. Then the assertion follows from Lemmas 3.6 and 3.4. □

4. Dual Gabriel theorem for pointed Hopf algebras

4.1. Let C be a coalgebra, D and E two subspaces of C. The wedge of D and E is defined to be $D \wedge E := \Delta^{-1}(D \otimes C + C \otimes E)$ (see, e.g., [18]). Montgomery [13] has introduced the quiver $\Gamma(C)$ of C as follows: the vertices of $\Gamma(C)$ are isoclasses of simple subcoalgebras of C; for two simple subcoalgebras S_1 and S_2, there are exactly $\dim_k((S_1 \wedge S_2)/(S_1 + S_2))$ arrows from S_1 to S_2.

The following fact was observed in [5], see also [13, p. 2345].

Lemma 4.2. *Let C be a coalgebra, $x, y \in G(C)$. Then we have*

$$\dim_k\big((ky \wedge kx)/(kx + ky)\big) = \dim_k P_{x,y}(C) - 1.$$

That is, the number of arrows from kx to ky in $\Gamma(C)$ is exactly $\dim_k P_{x,y}(C) - 1$.

4.3. Now, let C be a pointed Hopf algebra with $G(C) = G$. By Lemma 4.2 the quiver $\Gamma(C)$ in this case can be interpreted as: the set of vertices of $\Gamma(C)$ is G; for $x, y \in G$, the number of arrows from x to y is $\dim_k P_{x,y}(C) - 1$. For example, the quiver of path coalgebra kQ^c is exactly Q, i.e., $\Gamma(kQ^c) = Q$.

Proposition 4.4. *Let H be a pointed Hopf algebra. Then $\Gamma(H)$ is a Hopf quiver.*

Proof. For any $x, z, z' \in G(H)$ with z, z' conjugate in $G(H)$, we have $\dim_k P_{x,zx} = \dim_k P_{x,z'x}$. In fact, if $z' = gzg^{-1}$, then $c \mapsto gcx^{-1}g^{-1}x$ gives a one to one correspondence from $P_{x,zx}$ to $P_{x,gzg^{-1}x}$. □

Chin and Montgomery have proved that if C is a pointed coalgebra, then C can be embedded into $k(\Gamma(C))^c$ such that $C \supseteq k(\Gamma(C))_0 \oplus k(\Gamma(C))_1$ (see [4, Theorem 4.3]). This is the dual Gabriel theorem for pointed coalgebras.

Let H be a pointed Hopf algebra with coradical filtration $\{H_n\}$. Then by Lemma 5.2.8 in [12] that $\mathrm{gr}(H) := H_0 \oplus H_1/H_0 \oplus \cdots \oplus H_n/H_{n-1} \oplus \cdots$ has the induced graded Hopf structure. The following result can be regarded as a dual Gabriel theorem for pointed Hopf algebras.

Theorem 4.5. *Let H be a pointed Hopf algebra. Then there exists a Hopf structure A on path coalgebra $k(\Gamma(H))^c$ such that there is a graded Hopf algebra embedding* $\mathrm{gr}(H) \hookrightarrow A$ *such that* $\mathrm{gr}(H) \supseteq k(\Gamma(H))_0 \oplus k(\Gamma(H))_1$.

Proof. Set $Q := \Gamma(H)$ and $G := G(H)$. Then Q is a Hopf quiver by Proposition 4.4. It follows from Theorem 1.4 that kQ^c admits a graded Hopf structure. We choose such a Hopf structure A on kQ^c as follows. By the Taft–Wilson theorem we have $H_1/H_0 = \bigoplus_{x,y\in G} P'_{x,y}(H)$, where $P'_{x,y}(H)$ is a subspace of $P_{x,y}(H)$ such that $P_{x,y}(H) = k(x-y) \oplus P'_{x,y}(H)$. Since the number of arrows from x to y in Q is exactly $\dim_k P'_{x,y}(H)$, it follows that one can identify ${}^y kQ_1^x$ with $P'_{x,y}(H)$. Under this identification, for each arrow α starting at x and ending at y, and $g \in G$, the product $g\alpha$ and αg are both non-trivial gx, gy-primitive elements of H, it follows that

$$g\alpha = h + \beta$$

with $h \in k(gx - gy)$ and $\beta \in P'_{gx,gy}(H)$. This makes kQ_1 a left kG-Hopf module by defining $g\alpha = \beta$. Similarly, we have a kG-Hopf bimodule kQ_1, and then we obtain the corresponding graded Hopf structure A on kQ^c.

In order to get a coalgebra map $\psi : \mathrm{gr}(H) \to kQ^c = \mathrm{CoT}_{kQ_0}(kQ_1)$, by Lemma 1.2 it suffices to construct a coalgebra map $\psi_0 : \mathrm{gr}(H) \to kQ_0$, and a kQ_0-bicomodule map $\psi_1 : \mathrm{gr}(H) \to kQ_1$, such that for each $x \in \mathrm{gr}(H)$ there are only finite i with $\psi_i(x) \neq 0$, where ψ_i is defined as in Lemma 1.2. In fact, set ψ_0 to be the projection $\mathrm{gr}(H) \to (\mathrm{gr}(H))_0 = H_0 = kG$, and ψ_1 to be the projection $\mathrm{gr}(H) \to \bigoplus_{x,y\in G} P'_{x,y}(H) = kQ_1$. Since $\{\bigoplus_{i\leqslant n} H_i/H_{i-1}\}$ is the coradical filtration of $\mathrm{gr}(H)$, we deduce that $\psi_n(H_m/H_{m-1}) = 0$ for $n \neq m$. It follows that we have a coalgebra map $\psi : \mathrm{gr}(H) \to kQ^c$ which is graded. Since $\psi|_{\mathrm{gr}(H)_0 \oplus \mathrm{gr}(H)_1}$ is injective, it follows from a theorem due to Heyneman and Radford (see [11], or [12, Theorem 5.3.1]) that ψ is injective.

It remains to prove that ψ is a Hopf algebra map, or, that ψ is an algebra map, i.e., $\psi m_{\mathrm{gr}(H)} = m_A(\psi \otimes \psi)$, where m_A and $m_{\mathrm{gr}(H)}$ denotes respectively the multiplication of A and $\mathrm{gr}(H)$. Set $\phi := \psi m_{\mathrm{gr}(H)}$ and $\phi' := m_A(\psi \otimes \psi)$. Then both ϕ and ϕ' are coalgebra maps from $\mathrm{gr}(H) \otimes \mathrm{gr}(H)$ to A. By the constructions of ψ and A we see that $\phi_0 = \phi'_0$ and $\phi_1 = \phi'_1$, where ϕ_i and ϕ'_i, $i = 0, 1$, are defined as in Lemma 1.2. It follows from Lemma 1.2 that $\phi = \phi'$. This completes the proof. □

4.6. Note that a Hopf quiver $Q(G, \chi)$ is connected if and only if the union $\bigcup_{\chi_C \neq 0} C$ generates G. We point out here, by a theorem of Montgomery, it is clear that in some sense we may restrict ourselves to connected Hopf quivers.

In fact, let Q be the Hopf quiver of (G, χ) with connected components $\{\Gamma_i\}$, and $1 = 1_G \in \Gamma_1$, N_i the set of vertices of Γ_i. Then it is straightforward to verify that N_1 is a normal subgroup of G and $G = N \cup N_2 \cup \cdots \cup N_t \cup \cdots$ is the coset decomposition of G respect to N; that Γ_1 is the Hopf quiver of (N_1, r), where $r_D = \chi_C$ if the conjugacy class D of N is contained in a conjugacy class C of G, and Γ_i is isomorphic to Γ_1 as quivers; and that $\chi_C = 0$ for each conjugacy class C of G with $C \cap N = \emptyset$.

In [13, Theorem 2.1 and Corollary 2.2], Montgomery proved that if C is a coalgebra with $\Gamma(C)$ having connected components $\{\Gamma_i\}$, then $C = \bigoplus_i C_i$, where C_i is indecomposable as a coalgebra with $\Gamma(C_i) = \Gamma_i$ for each i. In particular, C is indecomposable if and only if $\Gamma(C)$ is a connected quiver, see also [8]. In [13] Theorem 3.2 it is proved that if H is a pointed Hopf algebra with $G = G(H)$, then $H_{(1)}$ is a subHopfalgebra with

$N := G(H_{(1)})$ a normal subgroup of G, where $H_{(1)}$ is the coalgebra indecomposable component of H containing 1, and H is a crossed product of $H_{(1)} \#_\sigma k(G/N)$ with cocycle $\sigma : G/N \times G/N \to N$.

By Montgomery's result cited above we immediately have

Corollary 4.7. *Let Q be the Hopf quiver of (G, χ) and $\Gamma_{(1)}$ the connected component of Q containing* 1*. Then for each Hopf structure H on kQ^c, $H_{(1)} = k\,(\Gamma_{(1)})^c$ is a subHopfalgebra of H such that $H \simeq H_{(1)} \#_\sigma k(G/N)$ with cocycle $\sigma : G/N \times G/N \to N$, where N is the normal subgroup of G consisting of vertices of $\Gamma_{(1)}$.*

Acknowledgment

We thank the referee for the helpful comments and suggestions.

References

[1] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, in: Cambridge Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1995.
[2] C. Cibils, A quiver quantum group, Comm. Math. Phys. 157 (1993) 459–477.
[3] X.W. Chen, H.L. Huang, Y. Ye, P. Zhang, Monomial Hopf algebras, J. Algebra 275 (2004) 212–232.
[4] W. Chin, S. Montgomery, Basic coalgebras, in: AMS/IP Stud. Adv. Math., vol. 4, 1997, pp. 41–47.
[5] W. Chin, I. Musson, Hopf algebra duality, injective modules, and quantum groups, Comm. Algebra 22 (1994) 4661–4692.
[6] C. Cibils, M. Rosso, Algebres des chemins quantique, Adv. Math. 125 (1997) 171–199.
[7] C. Cibils, M. Rosso, Hopf quivers, J. Algebra 254 (2002) 241–251.
[8] J.A. Green, Locally finite representations, J. Algebra 41 (1976) 137–171.
[9] E.L. Green, Ø. Solberg, Basic Hopf algebras and quantum groups, Math. Z. 229 (1998) 45–76.
[10] H.L. Huang, H.X. Chen, P. Zhang, Generalized Taft Algebras, Algebra Colloq. 11 (3) (2004), in press.
[11] R.G. Heyneman, D.E. Radford, Reflexivity and coalgebras of finite type, J. Algebra 28 (2) (1974) 215–246.
[12] S. Montgomery, Hopf Algebras and Their Actions on Rings, in: CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
[13] S. Montgomery, Indecomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. Amer. Math. Soc. 123 (1995) 2343–2351.
[14] W.D. Nichols, Bialgebras of type one, Comm. Algebra 6 (15) (1978) 1521–1552.
[15] D.E. Radford, On the structure of pointed Hopf algebras, J. Algebra 77 (1982) 1–14.
[16] D.E. Radford, Finite-dimensional simple-pointed Hopf algebras, J. Algebra 211 (1999) 686–710.
[17] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, in: Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[18] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[19] E.J. Taft, The order of the antipode of finite dimensional Hopf algebras, Proc. Natl. Acad. Sci. USA 68 (1971) 2631–2633.
[20] M. Takeuchi, Free Hopf algebras generated by coalgebras, J. Math. Soc. Japan 23 (1971) 561–582.