Graded Calabi Yau algebras of dimension 3

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Abstract

In this paper, we prove that Graded Calabi Yau algebras of dimension 3 are isomorphic to path algebras of quivers with relations derived from a superpotential. We show that for a given quiver $Q$ and a degree $d$, the set of good superpotentials of degree $d$, i.e. those that give rise to Calabi Yau algebras, is either empty or almost everything (in the measure theoretic sense). We also give some constraints on the structure of quivers that allow good superpotentials, and for the simplest quivers we give a complete list of the degrees for which good superpotentials exist.

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1. Introduction and motivation

If one studies boundary conditions of the $B$-model in superstring theory over an $n$-dimensional Calabi Yau manifold $X$, one obtains naturally the derived category of coherent sheaves $\mathcal{D}^{b}\text{Coh} X$ [11]. This category is called a Calabi Yau category of dimension three, i.e. the third shift in the derived category is a Serre Functor:

$$\forall A, B \in \mathcal{D}^{b}\text{Coh} X : \text{Hom}_{\mathcal{D}^{b}\text{Coh} X}(A, B) \cong \text{Hom}_{\mathcal{D}^{b}\text{Coh} X}(B, A[3])^{*},$$

where the isomorphisms are natural in $A$ and $B$. In general, this category is too big to study its structure directly and therefore it is interesting to look at full triangulated subcategories of $\mathcal{D}^{b}\text{Coh} X$ that can be modeled using derived categories of module categories of noncommutative algebras. In string theoretical papers, this is often done using path algebras of quivers with relations coming from a superpotential: if $Q$ is a quiver and $\mathbb{C}Q$ the corresponding path algebra, then a superpotential is an element of the vector space $\mathbb{C}Q/\mathbb{C}Q$, $\mathbb{C}Q$. On this space, we can define for every arrow $a$ a ‘derivation’ $\partial_a$ that cuts out $a$ (for a precise definition see Section 2.1). Given a superpotential $W$, one can construct the vacualgebra [3]

$$A_W := \mathbb{C}Q/(\partial_a W : a \in Q_1).$$

In the exemplary cases worked out by physicists, the derived category of finite dimensional modules of the vacualgebra is indeed a Calabi Yau category, and hence these algebras are called Calabi Yau algebras.
In this note, we will show that in the case of graded algebras, every graded path algebra with relations that is Calabi Yau of dimension 3 must be isomorphic to a vacualgebra of some superpotential. The converse is not true, but we will show that being a Calabi Yau algebra of dimension 3 corresponds to the exactness of a certain bimodule complex. Therefore, for a given quiver $Q$ and a given degree $d$, the subset of superpotentials of degree $d$ that give rise to Calabi Yau vacualgebras is either empty or almost everything. Furthermore, we will use Groebner basis techniques to show how one can determine the possible degrees of good superpotentials for some simple quivers.

The results in this paper build further on ideas introduced by Van den Bergh in [16]. Similar results on Calabi Yau algebras in different settings have been obtained by Reiten and Iyama [10], Rouquier and Chuang [15], and Ginzburg [9].

2. Preliminaries

2.1. Path algebras with relations

As usual, a quiver $Q$ is an oriented graph. We denote the set of vertices by $Q_0$, the set of arrows by $Q_1$ and the maps $h, t$ assign to each arrow its head and tail. A nontrivial path $p$ is a sequence of arrows $a_1 \cdots a_k$ such that $t(a_i) = h(a_{i+1})$, whereas a trivial path is just a vertex. We will denote the length of a path by $|p| := k$ and the head and tail by $h(p) = h(a_1), t(p) = t(a_k)$. A path is called a cycle if $h(p) = t(p)$. A quiver is called connected if it is not the disjoint union of two subquivers and it is strongly connected if there is a cycle through each pair of vertices.

The path algebra $\mathbb{C}Q$ is the complex vector space with, as basis, the paths in $Q$ and the multiplication of two paths $p, q$ is their concatenation $pq$ if $t(p) = h(q)$, or else 0. We can put a gradation on $\mathbb{C}Q$ using the length of the paths. The space spanned by all paths of nonzero length is a graded ideal of $\mathbb{C}Q$, and we will denote it by $\mathcal{J}$.

The vector space $\mathbb{C}Q/\mathcal{J}$ has as basis the set of cycles up to cyclic permutation of the arrows. We can embed this space into $\mathbb{C}Q$ by mapping a cycle onto the sum of all its possible cyclic permutations:

$$\circ : \mathbb{C}Q/\mathcal{J} \to \mathbb{C}Q : a_1 \cdots a_n \mapsto \sum_i a_i \cdots a_{n-1} a_1 a_{i-1}.$$

Another convention we will use is the inverse of arrows: if $p := a_1 \cdots a_n$ is a path and $b$ an arrow, then $pb^{-1} = a_1 \cdots a_{n-1}$ if $b = a_n$ and zero otherwise. Similarly one can define $b^{-1} p$. These new defined maps can be combined to obtain a ‘derivation’

$$\partial_b : \mathbb{C}Q/\mathcal{J} \to \mathbb{C}Q : p \mapsto \circ (p) a^{-1} = a^{-1} \circ (p).$$

From now on, $A$ will denote the quotient algebra $\mathbb{C}Q/\mathcal{I}$ by a finitely generated graded ideal $\mathcal{I} \subset \mathcal{J}^2$. The set $\mathcal{R} \subset \mathcal{I}$ will be a minimal set of homogeneous generators each sitting inside some $i\mathbb{C}Q, i, j \in Q_0$.

We denote the semi-simple (left) $A$-module $A/A_{\geq 1} \cong \mathbb{C}Q/\mathcal{J}$ by $S$. $S$ is the direct sum of $\#Q_0$ simple one-dimensional $A$-modules $S_i$, each corresponding to a vertex $i \in Q_0$ To each vertex, we can also assign a projective module $P_i$, which is the left ideal $Ai$ and $S_i = P_i/(P_i)_{\geq 1}$. Although it is a little sloppy, we will also use $S$ to denote the subring $A_0 \cong \mathbb{C}Q_0$, generated by the vertices.

2.2. Calabi Yau categories

Let $\mathcal{C}$ be an Abelian $\mathbb{C}$-linear category and $D^b\mathcal{C}$ its bounded derived category. Using the shift, we can define a graded functor $(s, \eta^s)$ in the sense of Proposition A.5.2 where $s$ is the shift functor and the $\eta^s$ gives natural isomorphisms

$$\eta^s_A : s(A[1]) \to (sA)[1] : x \mapsto -x.$$

As explained in the Appendix, these maps are uniquely determined by the demand of compatibility with the triangulated structure of $D^b\mathcal{C}$.

Definition 2.1. The category $D^b\mathcal{C}$ is called Calabi Yau of dimension $n$ if there are natural isomorphisms

$$v_{A, B} : \text{Hom}_{D^b\mathcal{C}}(A, B) \to \text{Hom}_{D^b\mathcal{C}}(B, s^n A)^*, \quad (* \text{ is the complex dual})$$

or, in other words, the $n$th shift is a Serre Functor.
Starting with a graded path algebra with relations $A$, we can construct the category of finite dimensional left $A$-modules: $\text{Rep} A$. This is an Abelian category, so we can construct its bounded derived category $D^b \text{Rep} A$. We will call $A$ a graded Calabi Yau Algebra of dimension $n$ if $D^b \text{Rep} A$ is a Calabi Yau category of dimension $n$.

Although the definition is asymmetric in the sense that one only uses left modules, it is easy to see that if $A$ is Calabi Yau, the derived category of finite dimensional right modules $D^b \text{Rep} A$ is also a Calabi Yau Category. This can be proved using the complex dual as an anti-equivalence between $D^b \text{Rep} A$ and $D^b \text{Rep} A$: let $M$, $N$ be complexes of right modules, and define

$$v_{M,N}^{\text{Rep} A} : \text{Hom}_{D^b \text{Rep} A}(M, N) \to \text{Hom}_{D^b \text{Rep} A}(N, s^n M)^*$$

by the equality

$$v_{M,N}^{\text{Rep} A}(f)(g) = v_{N^*,M^*}^{\text{Rep} A}(s^n(f^T))(s^n(g^T)).$$

The Calabi Yau property of the derived category can be tracked back to the original category to give us properties that we will often use

**Property 2.2.** If $A$ is Calabi Yau of dimension $n$, then

C1. The global dimension of $A$ is also $n$.

C2. If $X, Y \in \text{Rep} A$ then

$$\text{Ext}_{A}^k(X, Y) \cong \text{Ext}_{A}^{n-k}(Y, X)^*.$$  

C3. The identifications above gives us a pairings $\langle \cdot, \cdot \rangle_{XY} : \text{Ext}_{A}^k(X, Y) \times \text{Ext}_{A}^{n-k}(Y, X) \to \mathbb{C}$ which satisfy

$$\langle f, g \rangle_{XY} = \langle 1_X, g * f \rangle_{XY} = (-1)^{k(n-k)} \langle 1_Y, f * g \rangle_{YY},$$

where * denotes the standard composition of extensions.

**Proof.** (1): if $i > n$ then $\text{Ext}_{A}^i(M, N) = \text{Ext}_{A}^{n-i}(M, N) = 0$, so $\text{gldim} A \leq n$ and $\text{Ext}_{A}^n(A/A_+, A/A_+) = \text{Hom}_{A}(A/A_+, A/A_+) = A/A_+ \neq 0$ so $\text{gldim} A \geq n$. For (2–3) see the Appendix. □

3. Graded Calabi Yau algebras of dimension $n \leq 3$

In this section, we will give descriptions of the types of quivers and relations that appear in graded Calabi Yau algebras of dimension 3.

From now on, we will also assume that the quiver $Q$ is connected. This is not a severe restriction, because $A$ is the direct sum of subalgebras defined over its connected components. Many properties like the Calabi Yau property transfer from the algebra to its direct summands: $A_1 \oplus A_2$ is Calabi Yau of dimension $n$ if both $A_1$ and $A_2$ are Calabi Yau of dimension $n$. This follows from the fact that the representation category (and hence the derived category) of $A$ decomposes as the direct sum of $\text{Rep} A_1$ and $\text{Rep} A_2$.

**Theorem 3.1.** If $A$ is Calabi Yau of dimension 3, then

1. there is a homogeneous superpotential $W \in \mathbb{C} Q / [\mathbb{C} Q, \mathbb{C} Q]$ such that

$$A \cong \mathbb{C} Q / (\partial a W : a \in Q_1),$$

2. every arrow in $Q$ is contained in a cycle of $\circ W$,

3. every vertex in $Q$ is the source of at least two arrows and the target of at least two arrows.

**Proof.** As the global dimension of $A$ must be 3, there is a minimal projective graded resolution

$$\bigoplus_{j \in Q_0} P_{j} \xrightarrow{(f_j)} \bigoplus_{i \in Q_1} P_{b(i)} \xrightarrow{(b^{-1})} \bigoplus_{i \in Q_1} P_{h(i)} \xrightarrow{(b)} P_i \xrightarrow{1} S_i.$$

In the diagram above, the $r$’s are elements of the minimal set of relations $R$ and the $b$’s are arrows, while the $f_r$ are maps that are not further specified. Using the Calabi Yau property and comparing dimensions, we can conclude that

1. $m_{ij} = \text{Dim} \text{Ext}^3(S_i, S_j) = \text{Dim} \text{Hom}(S_j, S_i) = \delta_{ij}$,

2. $\# \{ r \in R : h(r) = j, t(r) = i \} = \text{Dim} \text{Ext}^2(S_i, S_j) = \text{Dim} \text{Ext}^1(S_i, S_j) = \# \{ a \in Q_1 : i \xleftarrow{a} j \}.$
Because of (1), we can identify each \( f_r \) with an element in \( iAh(r) \). Consider the finite dimensional quotient algebra

\[
M = A/(f_r : r \in \mathcal{R}, A_n : n \geq N) \quad \text{where } \forall r : N > \deg f_r.
\]

The Calabi Yau property allows us to calculate the dimension of \( iM_j \):

\[
\dim iM_j = \dim \Hom(P_i, M_j) = \dim \Ext^3(S_i, M_j)^{\mathbb{C}} \cong \dim \Hom(M_j, S_i) = \delta_{ij},
\]

and conclude that \( M \) must be isomorphic to the degree zero part of \( A \). As (2) implies there are only as many \( f_r \) as there are arrows, we can conclude that the \( f_r \) must all have degree 1 and form a basis for \( A_1 \). Hence, by linearly combining our original relations, we can assume that the \( f_r \) can be identified with the arrows. Let \( r_a \) be the (nonzero) relation for which \( f_{r_a} = a \). This relation occurs only in the resolution of \( S_{t(r_a)} = S_{h(a)} \), and therefore \( h(a) = t(r_a) \) and \( t(a) = h(r_a) \).

Every arrow \( a \) is contained in a cycle: \( ar_a \), so if there is a path between two vertices, there is also a path in the opposite direction. This means that because \( Q \) is assumed to be connected, \( Q \) is also strongly connected. We will now prove that all the \( r_a \) have the same degree.

Let \( a \) be the arrow for which \( r_a \) has minimal degree. First of all, note that if two arrows \( a, b \) share their heads then \( \deg r_a = \deg r_b \), because they occur in the same resolution. Denote by \( r_{ab} := r_ab^{-1} \) the terms that appear in the middle map of the resolution. These terms are only nonzero if \( t(b) = h(a) \). The fact that the maps in the resolutions form a complex implies that \( \sum_{h(a)=i} ar_{ab} \) is zero in \( A \). If \( \deg r_a = \deg ar_{ab} \) is minimal, then there exist scalars \( (g_{bc}) \) such that

\[
\sum_{h(a)=i} ar_{ab} = \sum_{h(c)=h(b), t(c)=t(a)} g_{bc} r_c = \sum_{h(c)=h(b), t(c)=t(a)} g_{bc} \sum_{t(d)=h(b)} r_{cd} \quad \text{evaluated in } CQ.
\]

The \( \deg r_c \) (which is the same for all \( c \) with \( h(c) = h(b) \) including \( b \) itself) must also be minimal. All arrows following an arrow of minimal \( r_a \)-degree are also minimal, so by induction all arrows in \( Q \) have the same degree.

We will now prove that \( (g_{ab}) \) can be seen as a diagonal matrix. First note that

\[
\Ext^1(S_i, S_j) = \Hom \left( \bigoplus_{t(a)=i} P_{h(a)}, S_j \right) \cong \mathbb{C}^{[i \to j]}
\]

and on the other hand

\[
\Ext^2(S_j, S_i) = \Hom \left( \bigoplus_{t(r_a)=j} P_{h(r_a)}, S_i \right) = \Hom \left( \bigoplus_{h(a)=j} P_{t(a)}, S_i \right) \cong \mathbb{C}^{[i \to j]}.
\]

We can compose the spaces in two different ways:

\[
\Ext^1(S_i, S_j) \times \Ext^2(S_j, S_i) \to \Ext^3(S_j, S_j) \cong \mathbb{C} : (\eta_a) \ast (\eta_b) = \sum_a \xi_a \eta_a
\]

and

\[
\Ext^2(S_j, S_i) \times \Ext^1(S_j, S_j) \to \Ext^3(S_i, S_i) \cong \mathbb{C} : (\eta_b) \ast (\xi_a) = \sum_{a,b} g_{ab} \xi_a \eta_b.
\]

We only work out the last composition since the other one is similar. We extend the sequence \( (\eta_b) \) to a sequence running over all arrows by adding zeros. We push out (dotted lines) the map \( \eta \) forward along the resolution to obtain an exact sequence \( S_i \to \cdots \to S_j \):

\[
\begin{array}{ccccccc}
P_j & \overset{r_{cd}}{\longrightarrow} & \bigoplus_{h(c)=j} P_{t(c)} & \overset{r_{cd}}{\longrightarrow} & \bigoplus_{t(d)=j} P_{h(d)} & \overset{d}{\longrightarrow} & P_j & \longrightarrow & S_j \\
\downarrow (\eta_d) & & \downarrow S_i & \overset{r_{cd}}{\longrightarrow} & \bigoplus_{t(d)=j} P_{h(d)} & \overset{d}{\longrightarrow} & P_j & \longrightarrow & S_j. \\
S_i & \overset{(-\eta_c, r_{cd}, h(c)=j)}{\longrightarrow} & S_i & \overset{\bigoplus_{t(d)=j} P_{h(d)} \oplus P_j \quad \text{together}}{\longrightarrow} & \bigoplus_{t(d)=j} P_{h(d)} & \overset{(-\eta_d, r_{cd}, \gamma(d)=j)}{\longrightarrow} & S_j.
\end{array}
\]
We use this sequence to pull back (dotted arrows) the map \((\xi_b)\)

\[
\begin{array}{ccc}
P_i & \xrightarrow{a} & \bigoplus_{h(a)=i} P_{t(a)} \\
\; & & \; \xrightarrow{r_{ab}} \bigoplus_{t(a)=i} P_{h(b)} \xrightarrow{r_{ab}} P_i \\
\sum_{b} \xi_b \eta_c & \xrightarrow{m} & S_i \\
S_j & \xrightarrow{\bigoplus_{t(d)=j} P_{h(d)}} & S_j \\
\end{array}
\]

where

\[
m = \left(0, \sum_{ab} ar_{ab} \xi_b d^{-1}\right) = \left(0, \sum_{b} g_{bc} r_{ce} e \xi_b d^{-1}\right) = \left(0, \sum_{bc} g_{bc} r_{cd} \xi_b\right) = \left(\sum_{bc} g_{bc} \eta_c \xi_b, 0\right).
\]

Because of the Calabi Yau property, there exist traces \(\text{Tr}_{S_j} : \text{Ext}^3(S_j, S_j) \rightarrow \mathbb{C}\). As these Ext-spaces are one-dimensional, we can represent these traces by scalars \(\alpha_j\). Proposition A.5.2 can be rewritten as

\[
\text{Tr}_{S_j}((\xi_a) \ast (\eta_b)) = \text{Tr}_{S_j}((\eta_b) \ast (\xi_a))
\]

\[
\alpha_j \sum_a \xi_a \eta_a = \alpha_i \sum_{a,b} g_{ab} \xi_a \eta_b.
\]

As this holds for arbitrary \((\xi_a)\) and \((\eta_b)\) we can conclude that

\[g_{ab} = \frac{\alpha_{h(a)}}{\alpha_{t(a)}} \delta_{ab}.
\]

Now we construct the element

\[
\sum_{a,b \in Q_1} \alpha_{h(a)} a r_{ab} b,
\]

which is a sum of cycles. It is also a homogeneous element that is invariant under cyclic permutation:

\[
\sum_{a,b} \alpha_{h(a)} a r_{ab} b = \sum_{a,b} \alpha_{t(b)} r_{ab} a b = \sum_{a,b} \alpha_{t(b)} a r_{ba} a \frac{\alpha_{h(b)}}{\alpha_{t(b)}} b r_{ba} a = \sum_{a,b} \alpha_{h(b)} b r_{ba} a.
\]

This implies that we can identify it with \(\cap (W)\), where \(W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]\) such that \(r_a\) is a scalar multiple of \(\partial_a W\).

To prove the last condition on the structure of the quiver, assume first that \(v\) is the tail of a unique arrow \(a\) and let the \(b_i\) be the vertices whose head is \(t(a)\). As \(r_a = \sum_i b_i r_{ba} \) and \(r_a \neq 0\) in \(\mathbb{C}Q\), there must be at least one \(r_{ba} \neq 0\) in \(\mathbb{C}Q\), and because of its degree it is also nonzero in \(A\). Now \(r_{ba} W\) sits inside the kernel of \(P_{h(a)} \xrightarrow{a} P_{t(a)}\) because \(\partial_a W = r_{ba} a\). This would imply that the resolution for \(S_{h(a)}\) is not exact. Using right modules instead of left ones proves that every vertex is also the tail of at least two arrows. \(\square\)

For reasons of completeness, we also include the descriptions of Calabi Yau algebras of smaller dimension, because the techniques for doing this are similar.

The zero-dimensional case is trivial and consists of the semi-simple algebras, i.e. quivers without arrows. The one-dimensional case consists of direct sums of \(\mathbb{C}[X]\) (disjunct unions of one-vertex-one-loop quivers). This is a consequence of property \(C2 : \#(i \rightarrow j) = \dim \text{Ext}^1(S_i, S_j) = \dim \text{Hom}(S_j, S_i) = \delta_{ij}\).

**Theorem 3.2.** If \(A\) is Calabi Yau of dimension 2, then \(A\) is the preprojective algebra of a non-Dynkin quiver (for a definition of a preprojective algebra see [7]).

**Proof.** As the global dimension of \(A = \mathbb{C}Q/\mathcal{I}\) is now 2, the projective graded resolutions look like

\[
\bigoplus_{t(r)=i} P_{h(r)} \xrightarrow{r_{a}^{-1}} \bigoplus_{t(a)=i} P_{h(a)} \xrightarrow{a} P_i \rightarrow S_i.
\]

From the Calabi Yau property \(C2\), we deduce that

\[
\#(r \in \mathcal{R} | h(r) = i, t(r) = j) = \dim \text{Ext}^2(S_i, S_j) = \dim \text{Hom}(S_j, S_i) = \delta_{ij},
\]

i.e. for every vertex, there is a unique relation and vice versa.
Now, similarly to the three dimensional case, we consider the finite dimensional quotient algebra $M = A/(ra^{-1} : r \in R, a \in Q_1, A_n : n \geq N)$ where $\forall r : N > \deg r$. The Calabi Yau property allows us to calculate the dimension of $iMj$:

$$\dim iMj = \dim \text{Hom}(P_i, Mj) = \dim \text{Ext}^2(S_i, Mj) \cong \dim \text{Hom}(Mj, S_i) = \delta_{ij}$$

and conclude that $M$ must be isomorphic to the degree zero part of $A$. This implies that the $ra^{-1}$ are all linear and span $A_1$. For every $a$, there is also at most one $r$ such that $ra^{-1}$ is nonzero: the unique $r$ with $t(r) = t(a)$. If we group the relations together in $R = \sum_{r \in R} r$, then there exists an invertible complex matrix $g_{ab}$ such that

$$Ra^{-1} = \sum_{a,b} g_{ab}b.$$ 

We can use this $g$ to explicitly calculate the pairing (the calculation is analogous to the thee-dimensional case).

$$\text{Ext}^1(S_j, S_i) \times \text{Ext}^1(S_j, S_i) \to \text{Ext}^2(S_i, S_i)(\xi_a) \ast (\eta_b) = \sum_{a,b} g_{ab}\xi_a\eta_b.$$ 

Property C3 now implies that there are scalars $\alpha_i, i \in Q_0$ (coming from the traces) such that $\alpha_i g_{ab}$ is antisymmetric and non-degenerate, so using a base transformation on the arrows, we can put $\alpha_i g_{ab}$ in its standard symplectic form.

The fact that $g_{ab} \neq 0 \implies \theta(a) = \theta(b)$ indicates that this base transformation only mixes arrows with identical head and tail. In this new basis, the arrows can be partitioned in couples $(a, a^*)$ with $\alpha_i g_{aa^*} = 1$ and $\alpha_i g_{ab} = 0$ if $b \neq a^*$. The relation $R' = \sum_{r \in R} \alpha_i(r) r = \sum_{a,b} \alpha_i g_{ab} ba$ (which is equivalent with $R$) assumes the form of the standard preprojective relations:

$$\sum_a a a^* - (a^*) a$$

where $a$ runs over the unstarred half of the arrows. Also, $Q$ cannot be the double of a Dynkin quiver, because $A$ must have global dimension 2; see [7].

4. Selfdual resolutions

In this section, we use the notion of selfdual resolutions to give a criterion to check whether a vacualgebra $A_w$ is indeed Calabi Yau.

4.1. Projective $A$-modules

Let $A$ be a finitely generated graded algebra that is the quotient of a path algebra $\mathbb{C}Q$ and let $S = A_0$.

For every finite dimensional $S$-bimodule $T$, we can define a projective $A$-bimodule

$$F_T := A \otimes_S T \otimes_S A.$$ 

We denote the full subcategory of $\text{Mod} A - A$ containing these projective modules as $\mathcal{P}$. The basic objects of this category are of the form $F_{ij} := F_{Si \otimes jS} = Ai \otimes jA$ with $i, j \in Q_0$.

The bimodule homomorphisms between $F_T \in \mathcal{P}$ and a bimodule $M \in \text{Mod} A - A$ can be identified with

$$\text{Hom}_{A - A}(F_T, M) \cong T^* \otimes_{S-S} M.$$ 

The tensor product in this formula tensors over both the left and the right $S$ action. The identification can be expressed explicitly as

$$\theta \otimes m : b_1 \otimes_S t \otimes_S b_2 \mapsto \sum_{i,j \in Q_0} \theta(itj)b_1imbjb_2.$$ 

A special role is played by $F_{S \otimes S} \cong A \otimes A$. We will denote this space by $F$. On this vector space, we can define two commuting $A$-bimodule structures

$$F_{\text{Outer}} : (a_1(b_1 \otimes b_2)a_2) = a_1b_1 \otimes b_2a_2,$$

$$F_{\text{Inner}} : (a_1(b_1 \otimes b_2)a_2) = b_1a_2 \otimes a_1b_2.$$
If we use no subscript, we automatically assume the outer structure. These structures are both isomorphic as bimodules to the free bimodule of rank one, and the isomorphism between them is given by the twist

$$\tau : F_{\text{outer}} \to F_{\text{inner}} : (b_1 \otimes b_2) \mapsto (b_2 \otimes b_1).$$

The existence of these two commuting structures implies that for any $A$-bimodule $M$, the object $\text{Hom}_{A-A}(M, F_{\text{outer}})$ is again an $A$-bimodule using the inner structure. This bimodule will be denoted by $M^\vee$. Maps can also be dualized in the standard way to turn $-^\vee$ into a functor:

$$\forall f \in \text{Hom}_{A-A}(M, N) : \forall m \in M : \forall v \in N^\vee : f^\vee(v)(m) := v(f(m)).$$

For the standard projective bimodules, we have the following natural identities

- $F_T^\vee = \text{Hom}_{A-A}(F_T, F) \cong \left( T^* \otimes_{S-S} F^\vee \right)_{\text{inner}} \cong A \otimes_S T^* \otimes_S A = F_T^*,$
- $\text{Hom}_{A-A}(F_T, M) \cong T^* \otimes_{A-A} M \cong F_T^\vee \otimes_{A-A} M \cong F_T^\vee \otimes M.$

We can also write out the duality for the morphisms:

$$\left( \theta \otimes a_1 \otimes_S t \otimes_S a_2 \right)^\vee = t \otimes a_2 \otimes_S \theta \otimes_S a_1.$$

These formulas imply that there is a natural equivalence between

$$\left( -^\vee \otimes_{A-A} -^\vee \right) = \mathcal{P} \times \text{Rep} A - A \to \text{Mod} \mathbb{C}$$

and between $-^\vee |_{\mathcal{P}}$ and $\mathcal{P} \leftrightarrow \text{Mod} A - A$. These functors and identities can be transferred to complexes if we assume that

$$(M^*)^* = (M_{-i}^* - (d_{-i}^M)^*)^* \quad \text{and} \quad (P^*)^\vee = (P_{-i}^\vee - (d_{-i}^P)^\vee).$$

Keeping all this in mind, we can propose the following definition:

**Definition 4.1.** A projective resolution $P^*$ of left $A$-bimodules is selfdual with shift $n$ if and only if there exists a commutative diagram

$$
\begin{array}{ccccccccc}
P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 \\
\downarrow{\alpha_n} & & \downarrow{\alpha_{n-1}} & & \cdots & & \downarrow{\alpha_1} & & \downarrow{\alpha_0} \\
P_0^\vee & \xleftarrow{-d_n^\vee} & P_1^\vee & \xleftarrow{-d_{n-1}^\vee} & \cdots & \xleftarrow{-d_1^\vee} & P_1 & \xleftarrow{-d_0^\vee} & P_0^\vee
\end{array}
$$

for which the $\alpha_i$ are isomorphisms of $A$-bimodules. In shorthand, we can write $P^* \cong (P^*)^\vee[n]$.

**Theorem 4.2.** If an algebra $A$ has a selfdual resolution of length $n$ with entries in $\mathcal{P}$, then $A$ is Calabi Yau of dimension $n$.

**Proof.** Let $M^*$ and $N^*$ be two complexes in $\text{Rep} A$. Standard homological algebra allows us to identify naturally

$$\text{Hom}_{\mathcal{D}^b \text{Rep} A}(M^*, N^*) \cong \text{Hom}_{\mathcal{D}^b \text{Mod} A - A}(A, N^* \otimes (M^*)^*)$$

$$\cong \text{Hom}_{\mathcal{D}^b \text{Mod} A - A}(P^*, N^* \otimes (M^*)^*)$$

$$\cong H^0 \text{RHom}_{\mathcal{D}^b \text{Mod} A - A}(P^*, N^* \otimes (M^*)^*)$$

$$\cong H^0 \text{Hom}^*_{\text{Mod} A - A}(P^*, N^* \otimes (M^*)^*).$$

So if we can prove that there is a natural equivalence between

$$\text{Hom}^*_{\text{Mod} A - A}(P^*, V^*) \quad \text{and} \quad \text{Hom}^*_{\text{Mod} A - A}(P^*, (V^*)^*[n])$$

where $V^*$ is a bounded complex of finite dimensional $A$-bimodules, we are done (just take $V^* = N^* \otimes (M^*)^*$).
Now using the fact that the resolution is composed of projectives in \( T \), we can make the following identifications

\[
(\text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, V^{*})^*) \cong \left( (P^{*})^{\vee} \otimes_{A \rightarrow A} V^{*} \right)^*
\]

\[
\cong (P^{*})^{\vee \vee} \otimes_{A \rightarrow A} (V^{*})^*
\]

\[
\cong (P^{*})^{\vee} \otimes_{A \rightarrow A} (V^{*})^*[n]
\]

\[
\cong \text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, (V^{*})^*[n])
\]

which are natural in \((V^{*})^*\). □

For an explicit write-out of the corresponding pairing between \( \text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, N^{*} \otimes (M^{*})^*) \) and \( \text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, M^{*} \otimes (N^{*})^*[n]) \), we first need some notation: for simplicity, we will work with elements that are pure tensors:

\[
f \in \text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, N^{*} \otimes (M^{*})^*): f^{ij} = \phi^{ij}_{A \rightarrow A} \mu^{ij} \in P^{i \vee}_{A \rightarrow A} N^{i+j} \otimes M^{j^*}
\]

\[
g \in \text{Hom}_{\text{KMod}}^{*} A \rightarrow A (P^{*}, M^{*} \otimes (N^{*})^*[n]): g^{ij} = \gamma^{ij}_{A \rightarrow A} m^{ij} \in P^{i \vee}_{A \rightarrow A} M^{j} \otimes (N^{n-i+j})^*.
\]

With these expressions for \( f \) and \( g \), we can track back the pairing in the previous identifications:

\[
(f, g)_{M^{*} N^{*}} = \sum_{ij} T_{ij} \mu^{ij} \circ \phi^{ij}_{A \rightarrow A} \phi^{-1}_{n-i} (\gamma^{n-i,j}) m^{n-i,j}.
\]

### 4.2. Superpotentials and selfduality

In the case of a graded algebra \( A := \mathbb{C} Q / I, I \subset J^{2} \), one can construct its minimal resolution using standard presentations of \( T^{n} / T^{n+1} \). These objects, introduced in [5], consist of quintuples \((U, V, r, l, \Delta)_{n}\), where

1. \( U, V \subset T^{n} \) are \( S \)-bimodule complements such that

   \[
   T^{n} = U \oplus J T^{n} + T^{n} J \quad \text{and} \quad J T^{n} \cap T^{n} J = V \oplus J T^{n} J.
   \]

2. \( r, l : T^{n} \rightarrow A \otimes_{S} U \otimes_{S} A \) are a \( Q - S \) and a \( S - \mathbb{C} Q \)-bimodule section of the \( \mathbb{C} Q \)-bimodule morphism

   \[
e : A \otimes_{S} U \otimes_{S} A \rightarrow T^{n} : 1 \otimes_{S} u \otimes_{S} 1 \mapsto u
   \]

   and use these to define a map

   \[
d : A \otimes_{S} V \otimes_{S} A \rightarrow A \otimes_{S} U \otimes_{S} A : 1 \otimes_{S} v \otimes_{S} 1 \mapsto l(v) - r(v)
   \]

3. \( \Delta : T^{n} \rightarrow A \otimes_{S} V \otimes_{S} A \) is a \( \mathbb{C} Q \)-bimodule derivation (i.e. a \( S \)-bimodule morphism satisfying \( \Delta(azb) = \Delta(az)b + a \Delta(zb) - a \Delta(z)b \)), such that \( d \Delta = l - r \) and \( \forall v \in V : \Delta(v) = 1 \otimes_{S} v \otimes_{S} 1 \).

Although the map \( d \) is a morphism as \( \mathbb{C} Q \)-bimodules, it can also be considered as a morphism of \( A \)-modules \( d_{A} : F_{V} \rightarrow F_{V} \) because the \( \mathbb{C} Q \)-action factors over \( A \). The same can be done with \( e \) provided we factor out \( T^{n+1} \) in the target: \( e_{A} : F_{U} \rightarrow T^{n} / T^{n+1} \). To turn \( \Delta \) into a \( A \)-bimodule morphism, we have to do two things: look at the subspace \( T^{n+1} \subset T^{n} \) (this turns the derivation law into a morphism law) and mod out \( T^{n+2} \) (this turns the domain into a \( A \)-bimodule):

\[
c_{A} : \frac{T^{n+1}}{T^{n+2}} \rightarrow F_{V} : x + T^{n+2} \mapsto \Delta(x).
\]

These maps can be packed together in sequences of \( A - A \) bimodules

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{T^{n+1}}{T^{n+2}} & \xrightarrow{c_{A}} & F_{\frac{T^{n+1}}{T^{n+2}}} & \xrightarrow{d_{A}} & F_{\frac{T^{n+1}}{T^{n+2}}} & \xrightarrow{e_{A}} & \frac{T^{n}}{T^{n+1}} & \rightarrow & 0.
\end{array}
\]
In [5], it is proved that these sequences are exact and they can be spliced together to get a projective bimodule resolution of \( I^0/I^1 = A \). This resolution is not minimal, but it can be made minimal if one cuts out the excess summands that occur at the splicing boundaries. These terms are of the form

\[
A \otimes_S \frac{I^{n+1}}{I^{n+1} \cap J I J} \otimes_S A \cong A \otimes_S \frac{I^{n+1} + J I^n J}{J I^n J} \otimes_S A \subset F_{\frac{I^n I J J}{J I^n J}}.
\]

We will now apply this to the case of Calabi Yau algebras of dimension 3. As we already know from Theorem 3.1, the ideal is generated by

\[ \partial_y W, \ a \in Q_1 \]

where \( W \in \mathbb{C}Q/\mathbb{C}Q \) is a superpotential, and as the global dimension is 3, we only need to look at the standard presentations for \( n = 0, 1 \).

The case \( n = 0 \) has the same form for every algebra

- \( U_0 = S, V_0 = \mathbb{C}Q_1 \)
- \( l_0 : a \mapsto a \otimes_S t(a) \otimes_S 1 \)
- \( r_0 : a \mapsto 1 \otimes_S h(a) \otimes_S a \),
- \( \Delta : a_1 \cdots a_k \mapsto \sum_{1 \leq j \leq k} a_1 \cdots a_{j-1} \otimes_S a_j \otimes_S a_{j+1} \cdots a_k \).

For \( n = 1 \), we do not need to bother about the \( \Delta_1 \), because it does not affect the minimal resolution:

- \( I = \mathbb{C}[\partial_a W, a \in Q_1] \oplus J I + I J, U_1 = \mathbb{C}[\partial_a W, a \in Q_1] \cong \mathbb{C}Q_1^{op} \).

For \( V_1 \), we choose a complement that contains the subspace \( \mathbb{C}[i \otimes W, i \in Q_0] \).

- \( l_1 : x \partial_y Wy \mapsto x \otimes_S \partial_y W \otimes_S y \) if \( y \notin I \),
- \( r_1 : x \partial_y Wy \mapsto x \otimes_S \partial_y W \otimes_S y \) if \( x \notin I \).

Because the \( i \otimes W \) are not contained in \( I^2 \), they are not cut out by restricting them to the minimal resolution. Moreover, because

\[
\text{Ext}^1_A(S, S) = \text{Hom}_S \left( \frac{V_1}{I^2 \cap V_1}, S \right)
\]

\[
\cong \text{Hom}_A(S, S)^* \cong \text{Hom}_S(\mathbb{C}[i, i \in Q_0], S)^* \cong \text{Hom}_S(\mathbb{C}[i \otimes W, i \in Q_0], S)^*.
\]

we have that the third term in the minimal resolution must be \( F \xrightarrow{\delta_1} F \xrightarrow{\delta_2} F \xrightarrow{\delta_3} F \xrightarrow{\delta_4} 0 \) with maps

\[
\delta_1 (1 \otimes_S a \otimes_S 1) = a \otimes_S t(a) \otimes_S 1 - 1 \otimes_S h(a) \otimes_S a
\]

\[
\delta_2 (1 \otimes_S \partial_a W \otimes_S 1) = \Delta(\partial_a W)
\]

\[
\delta_3 (1 \otimes_S W \otimes_S 1) = \sum_{a \in Q_1} a \otimes_S \partial_a W \otimes_S 1 - 1 \otimes_S \partial_a W \otimes_S a.
\]

A more explicit write-out of the complex \( C_W \), whose 0th homology is equal to \( A \), in terms of the basic projective \( F_{ij} \), looks like

\[
C_W : \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{(\tau_{\partial_a W})} \bigoplus_{a \in Q_1} F_{(a)h(a)} \xrightarrow{\partial_a^2 W} \bigoplus_{b \in Q_1} F_{h(b)r(b)} \xrightarrow{(db)} \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{m} A
\]
where the differential is \( da := a \otimes t(a) - h(a) \otimes a \), and the second derivatives are \( \partial_{ba}^2 W = \pi_{F_i(h(a))} \Delta h_i W \). More explicitly, if \( c \) is a cycle, then
\[
\partial_{ba}^2 c = \sum_{p_1, p_2 : \circlearrowright a p_1 b p_2 = \circlearrowright c} p_1 \otimes p_2.
\]
Note that because \( \circlearrowright W \) is invariant under cyclic permutation, \( \partial_{ba}^2 W = \tau \partial_{ba}^2 W \)
\[
\begin{align*}
\bigoplus_{i \in Q_0} F_{ii} & \xrightarrow{(\tau da_+)} \bigoplus_{a \in Q_1} F_{(a)h(a)} \xrightarrow{(\partial_{ba}^2 W -)} \bigoplus_{b \in Q_1} F_{h(b)h(b)} \xrightarrow{(db_+)} \bigoplus_{j \in Q_0} F_{jj} \\
= \bigoplus_{j \in Q_0} F_{jj} & \xrightarrow{(\tau db_+)} \bigoplus_{b \in Q_1} F_{h(b)h(b)} \xrightarrow{(\partial_{ba}^2 W -)} \bigoplus_{a \in Q_0} F_{(a)h(a)} \xrightarrow{(\tau da_+)} \bigoplus_{i \in Q_0} F_{ii} \\
= \bigoplus_{j \in Q_0} F_{jj} & \xrightarrow{(\tau db_+)} \bigoplus_{b \in Q_1} F_{h(b)h(b)} \xrightarrow{(\partial_{ba}^2 W -)} \bigoplus_{a \in Q_0} F_{(a)h(a)} \xrightarrow{(\tau da_+)} \bigoplus_{i \in Q_0} F_{ii}.
\end{align*}
\]
This complex is selfdual, and the isomorphism connecting the complex with its dual is composed of the standard identifications we used in the previous paragraph.

So the sufficient condition of selfduality is also necessary for Calabi Yau algebras of dimension 3.

**Theorem 4.3.** A vacualgebra \( A_W \) is Calabi Yau of dimension 3 if and only if the complex \( C_W \) is a projective resolution of \( A_W \) as an \( A_W \)-bimodule.

This fact has a nice interpretation for the classification of good superpotentials, i.e. superpotentials with a vacualgebra that is indeed Calabi Yau.

**Corollary 4.4.** For a given quiver \( Q \) and a given dimension \( d \), the subset of \( \text{Sup}_d Q \) of good superpotentials of degree \( d \) is either the empty set or almost everything (in the measure theoretic sense).

**Proof.** The condition we must check is that the standard complex is indeed a resolution. To do this, we first lift the complex \( C_W \) to a sequence of linear maps between \( \mathbb{C} Q \)-bimodules
\[
\tilde{C}_W : \bigoplus_{i \in Q_0} \mathbb{C} Q_i \otimes i \mathbb{C} Q \xrightarrow{d_2} \bigoplus_{a \in Q_1} \mathbb{C} Q^t(a) \times h(a) \mathbb{C} Q \xrightarrow{\cdots} .
\]
Note that this sequence is not a complex any more. To check that \( C_W \) is a resolution, we must check that \( \text{Im} d_i + \text{Ker} \pi_i = d_{i-1}^{-1}(\text{Ker} \pi_{i-1}) + \text{Ker} \pi_i \) where the \( d_i \) are the differentials in \( \tilde{C}_W \) and the \( \pi_i \) are the maps from \( \tilde{C}_W \) to \( C_W \).

Because the resolution is graded, we can check this separately for every degree. We know that the equalities that must be checked are already inclusions, so they impose Zariski open conditions. Hence, the subspace of good superpotentials is an intersection of a countable number of Zariski open sets. If one of these sets is empty we’re in the first case; otherwise the complement of this set is a countable union of hypersurfaces, which has measure zero for the standard measure on \( \mathbb{C}^n \). \qed

**Remark 4.5.** For global dimension two, we can do a similar thing. Recall that if \( A \) is Calabi Yau of dimension two, then the set of arrows partitions in pairs \((a, a^*)\) with opposite head and tail.

The selfdual resolution now looks like
\[
\bigoplus_{i \in Q_0} F_{ii} \xrightarrow{(\tau da^*_+)} \bigoplus_{(a,a^*)} F_{(a)h(a)} \oplus F_{(a^*)h(a^*)} \xrightarrow{(\tau da^*_+)} \bigoplus_{i \in Q_0} F_{ii} \xrightarrow{m} A.
\]
This is indeed the standard resolution for preprojective algebras of non-Dynkin quivers (for extended Dynkin quivers this was shown in [7]; for general quivers this can be deduced from the Hochschild cohomology calculated in [6]).
4.3. The matrix valued Hilbert polynomial

For a graded algebra $A = \mathbb{C}Q/(\mathcal{R})$, one can define the matrix valued Hilbert series

$$H_A(t) := h_0 + h_1 t + h_2 t^2 + \cdots$$

where the $h_k$ are matrices in $\text{Mat}_{\#Q_0 \times \#Q_0}(\mathbb{C})$ and

$$(h_k)_{ij} = \dim i A_k j \quad (A_k \text{ is the degree } k \text{ part of } A).$$

The matrix valued Hilbert series of a Calabi Yau algebra can be computed from its bimodule resolution:

**Theorem 4.6.** If a vacualgebra $A_W$ with $\deg W = d \geq 3$ is Calabi Yau then

$$H_{A_W}(t) = \frac{1}{1 - M_Q t + M_Q^t t^{d-1} - t^d}$$

where $M_Q$ is the incidence matrix of $Q$. This equality must be evaluated in the ring of formal power series $\text{Mat}_{\#Q_0 \times \#Q_0}(\mathbb{C}[t])$.

**Proof.** The Hilbert polynomial of $F_{kl}$ is equal to

$$H_{F_{kl}}(t) = H_A(t)e_{kl}H_A(t)$$

where $e_{kl}$ is the matrix with 1 on the entry $k, l$ and zero elsewhere. So from the exactness of the resolution $C_W$ and the fact that $H_0(\mathcal{P}^*) = A$, we get

$$H_A(t) = H_{F_S} - t(H_{F_{C_{Q_1}}} - t^{d-2}(H_{F_{C_{Q_1}^{op}}} - t H_{F_S}))$$

$$= H_A(t) H_A(t) - t H_A(t) M_Q H_A(t) + t^{d-1} H_A(t) M_Q^t H_A(t) - t^d H_A(t) H_A(t).$$

Note that $H_A(t)$ is invertible because $H_A(0) = 1$. Multiplying to the left and the right by $H_A(t)^{-1}$ and taking the inverse, we obtain the equality. \quad \square

The bimodule resolution gives us also resolutions of the left modules $S$. Writing out the dimensions of these resolutions gives the equation

$$1 = H_A(t) - t M_Q H_A(t) + t^{d-1} M_Q^t H_A(t) - t^d H_A(t).$$

This is nothing new, but as this equation corresponds to a real resolution, we can derive certain inequalities that must be met:

11. $H_A(t) \geq 0$
12. $(M_Q^t - t) H_A(t) \geq 0$
13. $(M_Q - M_Q^t t^{d-2} - t^{d-1}) H_A(t) \geq 0.$

Note that a matrix valued series $f(t)$ is positive if all its entries $(f_k)_{ij}$ are positive. These inequalities can be useful to check whether quivers have good superpotentials of a given degree.

5. Applications

5.1. Groebner bases and superpotentials

To show that for a given quiver and a given degree there exist good superpotentials, one has to check whether one can find a superpotential $W$ such that $C_W$ is exact. To do this, we will use the technique of Groebner bases as outlined in [13], adapted to path algebras. Suppose that $Q$ is a quiver with $n$ arrows, and put an order on the arrows: $a_1 > \cdots > a_n$. One can extend this order to the set of paths with nonzero length using the *deglex ordering* method:

$$a_{i_1} \cdots a_{i_p} < a_{j_1} \cdots a_{j_q}$$
if and only if $p < q$ or $p = q$ and $\exists v \leq p : a_{i_\nu} < a_{j_\mu} \land \forall \mu < v : i_\mu = j_\mu$. We denote the leading monomial term (according to the deglex ordering) of $f \in \mathbb{C}Q$ by $\text{lt}(f)$. Recall that a (not necessarily finite) set of elements $G \subset \mathcal{I} \triangleleft \mathbb{C}Q$ is a Groebner basis if all $\text{lt}(g), g \in G$ are different and

$$\text{lt}(\mathcal{I}) := (\text{lt}(f) : f \in \mathcal{I}) = (\text{lt}(g) : g \in G)$$

where the equality is taken as ideals in $\mathbb{C}Q$. Groebner bases are very useful in determining the structure of an algebra. They can be used to determine the Hilbert polynomial because

$$H_{\mathbb{C}Q/\mathcal{I}} = H_{\mathbb{C}Q/\text{lt}\mathcal{I}}$$

and they can be used to check whether certain expressions in $\mathbb{C}Q$ are zero in $\mathbb{C}Q/\mathcal{I}$:

$$f \in \mathcal{I} \implies \text{lt}(f) \in \text{lt}(\mathcal{I}) = (\text{lt}(g) : g \in G).$$

To check whether a given set of relations is indeed a Groebner basis, one can use the method of Bergman diamonds [2]. For any $f$ in $\mathbb{C}Q$, an elementary reduction of $f$ by $g \in G$ is the new expression

$$\rho_g(f) := \begin{cases} f - \zeta agb & \text{if } a, b \text{ are paths and } \zeta \in \mathbb{C} \text{ s.t. } \text{lt}(f) = \zeta \text{alt}(g)b, \\ f & \text{otherwise}. \end{cases}$$

If $G$ is a set of relations, then a triple of monomial terms $(a, b, c)$ is called an ambiguity if $ab = \text{lt}(g_1), bc = \text{lt}(g_2)$ with $g_1, g_2 \in G$. An ambiguity is called resolvable if there is a sequence of elementary reductions such that

$$\rho_1 \cdots \rho_m(g_1 c - ag_2) = 0.$$ 

Now Bergman’s diamond lemma states that if all leading terms are different and all ambiguities are resolvable, then $G$ is a Groebner basis.

We will now give a useful criterion to find good superpotentials.

**Lemma 5.1.** Suppose every vertex in $Q$ is the source and the target of at least two arrows, and $W$ is a superpotential such that

- The leading terms of the relations $\partial_a W$ are all different and the ambiguities are in 1 to 1 correspondence to the vertices, and are of the form
  
  $$\text{amb}_v = (a, \text{lt}(\partial_a W)b^{-1}, b) = (a, a^{-1}\text{lt}(\partial_a W), b) \quad \text{with allt}(\partial_a W) = \text{lt}(\partial_a W)b = \text{lt}(vWv),$$

- For every vertex $v$ there is at least one arrow $a, t(a) = v$ such that $\forall b \in Q_0 : \text{lt}(\partial_a W)a^{-1} = 0$

then $A_W$ is Calabi Yau.

**Proof.** First note that the condition implies that $\{\partial_a W : a \in Q_1\}$ is a Groebner basis: an ambiguity of the form $\text{amb}_v(a, \text{lt}(\partial_a W)b^{-1}, b)$ is resolvable because

$$\sum_{h(c)=v} c\partial_c W = \sum_{t(c)=v} \partial_c Wc$$

and hence

$$a\partial_a W - \partial_b Wb = \sum_{t(c)=v, c \neq b} \partial_c Wc - \sum_{h(c)=v, c \neq a} c\partial_c W.$$

Note that the leading terms of the summands in the right hand side are all different, because the $\text{lt}(\partial_a W)$ are and there is only one ambiguity corresponding to $v$. We can remove each term using an elementary reduction, starting with the one with the highest leading term. Therefore, the ambiguity is resolvable.

To calculate the Hilbert series, one must calculate $(h_k)_{vw}$, which is equal to the number of words between $v$ and $w$ of a given length $k$ not containing $\text{lt}(\partial_a W)$’s. This can be done using recursion:

$$(h_k)_{vw} = \sum_u h_{vu}^{k-1} \# \{u \leftarrow w\} - \sum_u h_{vu}^{k-d+1} \# \{w \leftarrow u\} + h_{vw}^{k-d}.$$
There are no further terms needed: a word ending \( w \) can only be double counted once, because of the form and number of the ambiguities. The Hilbert series of \( A_W \) is thus

\[
H_{A_W}(t) = \frac{1}{1 - M_W t + M_W^2 t^{d-1} - t^d}.
\]

Using the exactness of the first 2 terms of \( C_W \), we can calculate the Hilbert series of the kernel of the third map

\[
H_{A_W}(t) - k t H_{A_W}(t)^2 + k t^{d-1} H_{A_W}(t)^2 = t^d H_{A_W}(t)^2.
\]

This is the same as the Hilbert series of the last term, so if we can prove that the last map is an injection we are done. There is indeed no element \( C_W \).

The deglex ordering on \( C \) can be transferred to an ordering on the monomials of \( CQ \):\[v_1 \otimes v_2 > w_1 \otimes w_2 \iff v_1 > w_1 \text{ or } v_1 = w_1 \text{ and } v_2 > w_2.\]This ordering is compatible with the multiplicative structure on \( CQ \). Let \( f_1 \otimes g_1 \) be the highest order term; then the highest order term of \( \sum_j f_j \otimes g_j - f_j \otimes bg_j \) is \( f_1 \otimes g_1 \). Therefore \( f_1 \not\in (\lt(a, W) : a \in Q_1) \) but \( f_1 b \in \lt(a, W) \) for every \( b \in Q_0 \). This would imply that for every \( b \) with \( h(b) = t(f_1) \), there is a \( c \in Q_1 \) such that \( f_1 b \) ends in \( \lt(a, W) \), contradicting the second condition on \( W \).

The conditions imposed on the superpotential are very strict, and there are far better superpotentials that do not meet these conditions. In general, the ideal generated by a good superpotential will not have a finite Groebner basis. However for many quivers and degrees, we will be able to find superpotentials that satisfy the demands of the lemma.

5.2. The one vertex situation

First note that if \( Q \) has only one vertex and one loop, then none of the vacualgebras can be Calabi Yau of dimension 3, because these algebras are finite dimensional and hence \( H_A(t) \) cannot be the inverse of the polynomial \( 1 - t + t^{d-1} - t^d \).

So, in this section, let \( Q \) be a quiver with one vertex and \( k \geq 2 \) loops, and let \( \text{Sup}_d \subset CQ/[[CQ, CQ]] \) be the subspace of all superpotentials of degree \( d \) with \( d \geq 3 \). We will show that the space of good superpotentials is non-empty if and only if \( (k, d) \neq (2, 3) \).

If \( (k, d) = (2, 3) \), then there are no good superpotentials because the inequality (12) does not hold:

\[
(2 - t) \frac{1}{1 - 2t + 2t^2 - t^3} = 2 + 3t + 2t^2 + t^3 + 2t^4 + \cdots \not\geq 0.
\]

For every other couple \((k, d)\), we can find at least one good superpotential.

**Lemma 5.2.** Take \( CQ \cong C(X_1, \ldots, X_n) \) and \( X_1 > X_2 > \cdots > X_n \), then the following superpotentials are good:

1. \( W = X_1X_2X_3 + X_1X_3X_2 + \sum_{j>3} X_1X_j^2 + [CQ, CQ] \).
2. \( W = \sum_{k\geq j>1} X_1^{d-2}X_jX_k + [CQ, CQ] \).

**Proof.** We calculate the leading terms of the relations

1. \( \lt(\partial X_1 W) = X_2X_3, \lt(\partial X_2 W) = X_3X_2, \lt(\partial X_3 W)X_1X_2, \lt(\partial X_4 W)X_1X_3, \ldots \)
2. \( \lt(\partial X_1 W) = X_1^{d-3}X_2^3, \lt(\partial X_2 W) = X_1^{d-1}X_1X_2, \ldots, \lt(\partial X_3 W) = X_1^{d-1}X_2. \)

The only ambiguity we can construct is

1. \( (X_1, X_2, X_3) \) between \( \partial X_1 W \) and \( \partial X_2 W \),
2. \( (X_1, X_4^{d-3}X_2, X_3) \) between \( \partial X_1 W \) and \( \partial X_2 W \).

We also see that none of the leading terms ends in \( X_1 \).

**Remark 5.3.** In the cases where \( (k, d) \) equals \( (2, 4) \) or \( (3, 3) \), one can obtain a complete classification of the good superpotentials, because then we are in the case of Artin–Shelter regular algebras [1].
5.3. Special quivers

The simplest quivers with more than one vertex that can have good potentials are

\[ Q_1 := \begin{array}{c}
\bullet \\
\downarrow a_1 a_2 \\
\bullet \\
\downarrow a_3 a_4 \\
\bullet
\end{array} \quad Q_2 := \begin{array}{c}
\bullet \\
\downarrow b_1 b_3 \\
\bullet \\
\downarrow b_4 b_2 \\
\bullet
\end{array} \]

**Theorem 5.4.**
- $\mathbb{C}Q_1/\mathbb{C}Q_1$ contains good superpotentials if and only if $d \geq 4$ and $d$ is even.
- $\mathbb{C}Q_2/\mathbb{C}Q_2$ contains good superpotentials if and only if $d \geq 4$.

**Proof.** For both quivers, $d$ must be bigger than or equal to 4, because otherwise the inequalities I1–I3 are not satisfied. For $Q_1, d$ must be even, because every cycle has even length.

Assume the orders $a_1 > a_2 > a_3 > a_4, b_1 > b_2 > b_3 > b_4$ and define the following superpotentials

\[
Q_1 : a_1 a_3 (a_2 a_4)^{d-1} + a_3 a_1 (a_4 a_2)^{d-1} + [\mathbb{C}Q, \mathbb{C}Q] \\
Q_2 : b_1^{d-2} b_3 b_4 + b_2^{d-2} b_4 b_3 + [\mathbb{C}Q, \mathbb{C}Q].
\]

The leading terms of the relations are

\[
Q_1 : a_3 a_1 a_4 (a_2 a_4)^{d-2}, a_1 a_2 a_4 (a_4 a_2)^{d-2}, a_1 a_3 a_2 (a_4 a_2)^{d-2}, a_1 a_3 a_2 (a_4 a_2)^{d-2}, \\
Q_2 : b_1^{d-3} b_3 b_4, b_2^{d-3} b_4 b_3, b_2^{d-2} b_4 b_3, b_1^{d-2} b_3.
\]

For each of the quivers, there are two ambiguities (one for each vertex)

\[
Q_1 : (a_1, a_3 a_2 a_4 (a_2 a_4)^{d-2}, a_4), (a_3, a_1 a_4 (a_2 a_4)^{d-2}, a_2) \\
Q_2 : (b_1, b_1^{d-3} b_3 b_4, b_2, b_2^{d-3} b_4 b_3).
\]

Finally, none of the relations end in $a_1, a_3$ and $b_1, b_2$. □

The method described above can be extended to lots of other quivers and degrees, especially quivers of the form

\[ Q \tilde{p} \]

The number of arrows between consecutive vertices can differ (but is ≥2).

**Theorem 5.5.** Let $Q$ be a quiver of the form above with $k \geq 2$ vertices, and let $p_i \geq 2$ be the number of arrows between the $i$th and the $i + 1$th vertex. If $d = \ell k$ with $\ell \geq 2$ then $Q$ has good superpotentials of dimension $d$.

**Proof.** For every vertex $v \in Q_0$, we will denote the consecutive vertex by $v + 1$, so $\forall a \in Q_1 : h(a) = t(a) + 1$. Fix an order on the arrows of $Q$, and let $a_i, b_i$ the highest and second highest arrow arriving in the vertex $i$.

Define the superpotential

\[
W := \sum_{i \in Q_0} a_i a_{i-1} b_{i-2} \cdots b_{i-\ell k + 1} \\
+ \sum_{c \neq a, b} c a_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1} (c b_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1})^{\ell-1} + [\mathbb{C}Q, \mathbb{C}Q].
\]

The leading terms of the relations now look like

\[
\text{lt}(\partial_{a_i} W) = a_{i-1} b_{i-2} \cdots b_{i-\ell k + 1} \\
\text{lt}(\partial_{b_i} W) = a_{i-1} a_{i-2} b_{i-3} \cdots b_{i-\ell k + 1} \\
\text{lt}(\partial_{c} W) = a_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1} (c b_{h(c)-1} b_{h(c)-2} \cdots b_{h(c)-k+1})^{\ell-1}.
\]
It is easy to check that all ambiguities are of the form $(a_i, a_{i-1}b_{i-2} \cdots b_{i-\ell k+2}, b_{i-\ell k+1})$, and that none of the relations ends in some $a_i$. □

**Remark 5.6.** If $\ell = 1$, the situation is more complicated, because the solutions of the inequalities I1–I3 are not easy to determine. It is not the case that if they are satisfied for $Q_{\vec{p}}$, then they are also satisfied for a quiver $Q_{\vec{p}'}$ with $(p'_1, \ldots, p'_k) \succeq (p_1, \ldots, p_k)$. E.g. a quiver with arrows $\vec{p} = (2, 2, 2, 2)$ has good superpotentials, but one with $\vec{p} = (6, 2, 2, 2)$ has not.

The method of finding these very special superpotentials does not always work. As a counterexample, consider the quiver

\[ \begin{array}{c}
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\leftarrow \\
\downarrow \\
\leftarrow \\
\downarrow \\
\rightarrow \\
\end{array} \]

One can check that there are no superpotentials of dimension 4 satisfying the conditions from Lemma 5.1, although the Groebner basis computations in GAP [8] (up to a certain degree because the full Groebner basis could be infinite) seem to indicate that a generic superpotential is indeed good.

The general picture that arises from computations is that as soon as conditions I1–I3 are met by the Hilbert series, good superpotentials do exist, but we have no proof for this statement.

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**Appendix. The signs of Serre duality by Michel Van den Bergh**

**A.1. Introduction**

In this self-contained Appendix, we determine the exact signs which occur in Serre duality (see for example Proposition A.5.2 for the Calabi-Yau case). Although the answer is the obvious, the verification turned out to be slightly more tricky than foreseen.

We thank Bernhard Keller for pointing out Example A.3.2 (see [12,17] for further information) and suggesting that, likewise, the correct signs in Serre duality should be determined by the requirement that the Serre functor be exact.

**A.2. Graded categories**

**Definition A.2.1.** A graded (pre-additive) category is a pair $(\mathcal{C}, S)$ where $\mathcal{C}$ is a pre-additive category and $S$ is an automorphism of $\mathcal{C}$.

**Remark A.2.2.** It is customary to only require $S$ to be an autoequivalence. The stronger condition that $S$ be an automorphism is usually satisfied in practice, and up to an appropriate notion of equivalence we may always reduce to this case.

In a graded category $(\mathcal{C}, S)$ we may define the graded Hom-sets between objects by

\[ \text{Hom}^i_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}}(A, S^i B) \]

and

\[ \text{Hom}^{gr}_{\mathcal{C}}(A, B) = \bigoplus_i \text{Hom}^i_{\mathcal{C}}(A, B). \]
There is an obvious graded composition

\[ - \ast - : \text{Hom}^j_C(B, C) \times \text{Hom}^i_C(A, B) \to \text{Hom}^{i+j}_C(A, C) : (g, f) \mapsto S^i(g) f. \]

We denote by \( C^{gr} \) the category \( C \) equipped with graded Hom-sets.

A graded functor between graded categories \((C, S), (D, T)\) is an additive functor \( U : C \to D \) together with a natural isomorphism \( \eta^U : U \circ S \to T \circ U \). By a slight abuse of notation, we will write the composition

\[ U \circ S^i \to T \circ U \circ S^{i-1} \to \cdots \to T^i \circ U \]

as \( (\eta^U)_i \).

Associated to \((U, \eta^U)\) there is a functor \( U^{gr} : C^{gr} \to D^{gr} \) given by

\[ U^{gr}(f) = (\eta^U)_B \circ U(f) \]  

for \( f_i \in \text{Hom}^i_C(A, B) \). It is clear that the formation of \((-)^{gr}\) is compatible with compositions.

### A.3. Triangulated categories

We will assume that triangulated categories have a strictly invertible shift functor. Up to equivalence we may always reduce to this case.

**Definition A.3.1.** An exact functor \( U : S \to T \) between triangulated categories is a graded functor \((U, \eta^U) : (S, [1]) \to (T, [1])\) such that for any distinguished triangle

\[ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \]

the following triangle

\[ UA \xrightarrow{Uf} UB \xrightarrow{ Ug } UC \xrightarrow{\eta^U_B \circ Uh} (UA)[1] \]

is distinguished.

**Example A.3.2.** Let \( s : A \to A \) be the functor which coincides with \([1]\) on objects and maps, but for which \( \eta^s_A : s(A[1]) \to (sA)[1] \) is given by \(-id_{A[2]}\). Then \((s, \eta^s)\) is an exact endofunctor on \( A \). Note in contrast that \([1]\) itself, while being a graded endofunctor, is not exact.

### A.4. Serre functors

Let \( k \) be a field and assume that \( C \) is a Hom-finite \( k \)-linear category.

**Definition A.4.1.** \( C \) satisfies *Serre duality* if there is an auto-equivalence \( F : C \to C \) together with isomorphisms

\[ \text{Hom}_C(A, B) \to \text{Hom}_C(B, FA)^* \]  

natural in \( A, B \). Such an \( F \) is called a *Serre functor* for \( C \).

Putting \( B = A \) in (2) yields a canonical element

\[ \text{Tr}_A : \text{Hom}_C(A, FA) \to k \]

corresponding to the identity in \( \text{Hom}_C(A, A) \). It is easy to see that \( \text{Tr}_A(\ast) \) defines a non-degenerate pairing

\[ \text{Hom}_C(B, FA) \times \text{Hom}_C(A, B) \to k \]

and that the map (2) is given by \( f \mapsto \text{Tr}_A(\ast f) \). In addition, we have the following fundamental identity [14]

\[ \text{Tr}_A(g \circ f) = \text{Tr}_B(Ff \circ g). \]  

Now assume that \((C, S)\) is graded and assume that \(C\) has a Serre functor \(F\). We may make \(F\) into a graded functor as follows: we have to give maps
\[
\eta_A^F : (F \circ S)(A) \to (S \circ F)(A)
\]
natural in \(A\). Using non-degeneracy of the trace pairing we define these maps via the requirement
\[
\text{Tr}_A(S^{-1}(\eta_A^F \circ f)) = -\text{Tr}_A(f)
\tag{4}
\]
for any \(f : SA \to (F \circ S)(A)\).

**Remark A.4.2.** The minus sign in this formula is an arbitrary choice in the graded context, but it is forced in the triangulated context. See the proof of **Theorem A.4.4**.

**Proposition A.4.3 (Graded Serre Duality).** For \(f_i \in \text{Hom}_C(A, B), g_{-i} \in \text{Hom}_{C^{-1}}(B, FA)\) we have
\[
\text{Tr}_A(g_{-i} \ast f_i) = (-1)^i \text{Tr}_B(F_{gr} f_i \ast g_{-i}).
\]

**Proof.** We have
\[
\text{Tr}_B(F_{gr} f_i \ast g_{-i}) = \text{Tr}_B(S^{-i}(F_{gr} f_i) \circ g_{-i}) \quad \text{(by Section A.2)}
= \text{Tr}_B(S^{-i}((\eta_A^F)^i_B \circ F(f_i) \circ S' g_{-i})) \quad \text{(by (1))}
= (-1)^i \text{Tr}_{S'B} (F(f_i) \circ S' g_{-i}) \quad \text{(by (4))}
= (-1)^i \text{Tr}_A(S' g_{-i} \circ f_i) \quad \text{(by (3))}
= (-1)^i \text{Tr}_A(g_{-i} \ast f_i) \quad \text{(by Section A.2).}
\]

Assume now that \(A\) is a Hom-finite \(k\)-linear triangulated category with a Serre functor \(F\).

**Theorem A.4.4 ([4]).** \(F\) is an exact functor when equipped with the graded structure obtained from (4) (with \(S = [1]\)).

**Proof.** This is proved by Bondal and Kapranov in [4]. We give a somewhat more direct proof which makes the signs evident.

We start with a distinguished triangle.
\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].
\]

We have to construct a map \(\delta\) such that the following diagram is a morphism of distinguished triangles
\[
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC & \xrightarrow{\eta_A^F \circ Fh} & (FA)[1] \\
\| & & \| & & \| & & \|
FA & \xrightarrow{Ff} & FB & \xrightarrow{a} & X & \xrightarrow{\beta} & (FA)[1]
\end{array}
\]

where \(X\) is the cone of \(Ff\).

In equations:
\[
\eta_A^F \circ Fh \circ \delta = \beta \quad \tag{5}
\]
\[
\delta \circ a = Fg. \quad \tag{6}
\]

For any \(x : A \to X[-1]\), we deduce from (5)
\[
(\eta_A^F \circ Fh \circ \delta)[-1] \circ x = \beta[-1] \circ x.
\]

Using (4), this is equivalent to
\[
\text{Tr}_{A[1]}(Fh \circ \delta \circ x[1]) = -\text{Tr}_A(\beta[-1] \circ x)
\]
which using (3) can be further rewritten as

\[
\text{Tr}_C(\delta \circ x[1] \circ h) = -\text{Tr}_A(\beta[1] \circ x).
\]  

(7)

Using the non-degeneracy of the trace pairing, we see that (5) is equivalent to the validity of (7) for all \( x : A \to X[1] \).

Similarly, (6) is equivalent to

\[
\text{Tr}_C(\delta \circ \alpha \circ y) = \text{Tr}_C(Fg \circ y) = \text{Tr}_B(y \circ g)
\]

for all \( y : C \to FB \).

Summarizing: we have to find \( \delta \) such that the following equations

\[
\text{Tr}_C(\delta \circ x[1] \circ h) = -\text{Tr}_A(\beta[1] \circ x)
\]

\[
\text{Tr}_C(\delta \circ \alpha \circ y) = \text{Tr}_B(y \circ g)
\]

(8)

hold for all \( x \in \text{Hom}_A(A, X[-1]) \) and \( y \in \text{Hom}_A(C, FB) \).

We may view the Eq. (8) as fixing the value of the function \( \text{Tr}_C(\delta \circ -) \) on two sub vector spaces of \( \text{Hom}_A(C, X) \). Since \( \text{Tr}_C \) is non-degenerate, such a system can be solved provided we give the same value on the intersection. Thus we have to show

\[
\alpha \circ y = x[1] \circ h \quad \text{then} \quad \text{Tr}_B(y \circ g) = -\text{Tr}_A(\beta[1] \circ x).
\]

To prove this, assume \( \alpha \circ y = x[1] \circ h \) and consider the following commutative diagram

\[
\begin{array}{ccc}
F A & \xrightarrow{Ff} & FB \\
\psi \downarrow & & \downarrow \alpha \\
B & \xrightarrow{g} & C \\
\end{array}
\quad
\begin{array}{ccc}
& X & \xrightarrow{\beta} & (FA)[1] \\
\psi[1] \downarrow & & \downarrow \beta[1] \\
\end{array}
\]  

(9)

where \( \psi \) exists because of the axioms of triangulated categories.

We compute

\[
\text{Tr}_B(y \circ g) = \text{Tr}_B(Ff \circ \psi) = \text{Tr}_A(\psi \circ f) = -\text{Tr}_A(\beta[1] \circ x).
\]

In the third line, we have used the commutativity of the rightmost square in (9). □

A.5. The Calabi-Yau case

Definition A.5.1. A triangulated category with Serre functor \( F \) is \textit{Calabi-Yau of dimension} \( n \) if \( F \cong s^n \) as graded functors, where \( s \) is as in Example A.3.2.

Proposition A.5.2. Assume that \( A \) is Calabi-Yau of dimension \( n \). Then for \( f_i \in \text{Hom}_A(A, B) \), \( g_{n-i} \in \text{Hom}^{n-i}_A(B, A) \) we have

\[
\text{Tr}_A(g_{n-i} \ast f_i) = (-1)^{i(n-i)}\text{Tr}_B(f_i \ast g_{n-i}).
\]  

(10)

Proof. We view \( g_{n-i} \) as an element of \( \text{Hom}_A^{-i}(A, FB) \) by using the naive identification on objects \((FB)[-i] = B[n][-i] = B[n-i]\). To avoid confusion, we put \( h_{-i} = g_{n-i} \).

The graded Serre duality now reads as

\[
\text{Tr}_A(h_{-i} \ast f_i) = (-1)^i\text{Tr}_B((s^{2g})^n(f_i) \ast h_{-i}).
\]

Writing out everything explicitly, we get

\[
\text{Tr}_A(h_{-i}[i] \circ f_i) = (-1)^i\text{Tr}_B(((\eta^s)^{n_i}_A \circ f_i[n])[-i] \circ h_{-i})
\]

\[
= (-1)^i\text{Tr}_B((\eta^s)^{n_i}_A[-i] \circ f_i[n-i] \circ h_{-i}).
\]
Now composing with \((\eta^s)^{ni}_A[-i]\) is just multiplying by \((-)^{ni}\). Thus, we obtain
\[
\text{Tr}_A(h_{-i}[i] \circ f_i) = (-1)^{i+ni}\text{Tr}_B(f_i[n-i] \circ h_{-i})
\]
which translates into (10). □

References