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On the sample mean after a group sequential trial[1]

Ben Berckmoes[2], Anna Ivanova, Geert Molenberghs

Abstract

A popular setting in medical statistics is a group sequential trial with independent and identically distributed normal outcomes, in which interim analyses of the sum of the outcomes are performed. Based on a prescribed stopping rule, one decides after each interim analysis whether the trial is stopped or continued. Consequently, the actual length of the study is a random variable. It is reported in the literature that the interim analyses may cause bias if one uses the ordinary sample mean to estimate the location parameter. For a generic stopping rule, which contains many classical stopping rules as a special case, explicit formulas for the expected length of the trial, the bias, and the mean squared error (MSE) are provided. It is deduced that, for a fixed number of interim analyses, the bias and the MSE converge to zero if the first interim analysis is performed not too early. In addition, optimal rates for this convergence are provided. Furthermore, under a regularity condition, asymptotic normality in total variation distance for the sample mean is established. A conclusion for naive confidence intervals based on the sample mean is derived. It is also shown how the developed theory naturally fits in the broader framework of likelihood theory in a group sequential trial setting. A simulation study underpins the theoretical findings.

Keywords: bias, confidence interval, group sequential trial, likelihood theory, mean squared error, sample mean

1. Introduction

Throughout the paper, $X_1, X_2, \ldots$ will be a fixed sequence of independent and identically distributed random variables with law $N(\mu, \sigma^2)$, and $\psi_1, \psi_2, \ldots$ a fixed sequence of Borel measurable maps of $\mathbb{R}$ into $[0, 1]$.

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[2] postal address: Universiteit Antwerpen, Departement Wiskunde-Informatica, Middelheimlaan 1, 2020 Antwerpen, Belgium; email: ben.berckmoes@uantwerpen.be.

For natural numbers $L \in \mathbb{N}_0$ and $0 < m_1 < m_2 < \ldots < m_L < n$, we consider a random sample size N with the following properties:

(a) N can take the values $m_1, m_2, \ldots, m_L, n$,

(b) $\forall i \in \{1, \ldots, L\} : \{N = m_i\}$ is independent of $X_{m_i+1}, X_{m_i+2}, \ldots$,

(c) $\forall i \in \{1, \ldots, L\} : \mathbb{P}[N = m_i \mid X_1, \ldots, X_{m_i}] = \psi_{m_i}(K_{m_i}) \prod_{j=1}^{i-1} \left[1 - \psi_{m_j}(K_{m_j})\right]$, where we denote $K_m = \sum_{i=1}^{m} X_i$ and the empty product is 1.

The above setting serves as a paradigm for a group sequential trial of random length N with outcomes $X_1, X_2, \ldots$ At each m_i, an interim analysis of the sum K_{m_i} of the outcomes is performed and, based on the generic stopping rule (c), one decides whether the trial is stopped, i.e. $N = m_i$, or continued, i.e. $N > m_i$.

Note that the product in (c) is merely the usual decomposition of the conditional probability to reach a certain sample size and the product of conditional probabilities of continuing at smaller sample sizes, given that the trial is ongoing. This is similar to decompositions encountered in longitudinal or time-series transition models, and dropout models in longitudinal studies. It follows the law of total probability.

More precisely, at the i-th interim analysis only the values of the full sums $K_{m_1}, \ldots, K_{m_i}$ have been analyzed. Therefore,

$$
\begin{aligned}
&\mathbb{P}[N = m_i \mid X_1, \ldots, X_{m_i}] \\
&\quad = \mathbb{P}[N = m_i \mid K_{m_1}, \ldots, K_{m_i}] \\
&\quad = \mathbb{P}[N = m_i, N \neq m_{i-1}, \ldots, N \neq m_1 \mid K_{m_1}, \ldots, K_{m_i}],
\end{aligned}
$$

which, by the law of total probability,

$$
\begin{aligned}
= \; & \mathbb{P}[N = m_i \mid N \neq m_{i-1}, \ldots, N \neq m_1, K_{m_1}, \ldots, K_{m_i}] \\
& \prod_{j=1}^{i-1} \mathbb{P}[N \neq m_j \mid N \neq m_{j-1}, \ldots, N \neq m_1, K_{m_1}, \ldots, K_{m_i}],
\end{aligned}
$$

which, because, given that $N \neq m_{j-1}, \ldots, N \neq m_1$, the event $\{N = m_j\}$ only depends on the analysis of the full sum K_{m_j},

$$
= \mathbb{P}[N = m_i \mid N \neq m_{i-1}, \ldots, N \neq m_1, K_{m_i}] \prod_{j=1}^{i-1} \mathbb{P}\left[N \neq m_j \mid N \neq m_{j-1}, \ldots, N \neq m_1, K_{m_j}\right],
$$

which is exactly the decomposition in (c).

We wish to highlight that the above model contains very useful settings that are extensively studied in the literature. To illustrate this, we let, for each m,

$$\psi_m(x) = 1_{\{|\cdot|\geq C_m\}}(x) = \begin{cases} 1 & \text{if} \quad |x| \geq C_m \\ 0 & \text{if} \quad |x| < C_m \end{cases},$$

with $C_m \in \mathbb{R}_0^+$ a constant. For these choices of ψ_m, expression (c) is turned into

$$\begin{aligned}
&\mathbb{P}[N = m_i \mid X_1, \ldots, X_{m_i}] \\
&= 1_{\{|\cdot|\geq C_{m_i}\}}(K_{m_i}) \prod_{j=1}^{i-1} \left[1 - 1_{\{|\cdot|\geq C_{m_j}\}}(K_{m_j})\right] \\
&= \begin{cases} 1 & \text{if} \quad |K_{m_i}| \geq C_{m_i} \text{ and } \forall j \in \{1, \ldots, i-1\} : \left|K_{m_j}\right| < C_{m_j} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

So this corresponds to a trial which is stopped either at the first m_i for which $|K_{m_i}| \geq C_{m_i}$, or at n. If, for a fixed constant $C \in \mathbb{R}_0^+$, $C_m = \sigma C\sqrt{m}$, this setting corresponds to the Pocock boundaries, studied in e.g. [S78] and [C89], and, if, for a fixed constant $C \in \mathbb{R}_0^+$, $C_m = C$, this setting corresponds to the O'Brien-Fleming boundaries, studied in e.g. [W92]. More generally, taking $\psi_m = 1_{\mathcal{S}_m}$ with $\mathcal{S}_m \subset \mathbb{R}$ a Borel measurable set, leads to the setting studied in e.g. [EF90] and [LH99]. Finally, taking $\psi_m(x) = \Phi\left(\alpha + \beta m^{-1}x\right)$ with Φ the standard normal cumulative distribution function and α, β real numbers, corresponds to the probabilistic stopping rule setting studied in e.g. [MKA14].

In this paper, we will study the ordinary sample mean $\widehat{\mu}_N = \frac{1}{N}K_N$. It is reported in the literature that in the above described group sequential trial setting, bias may occur if $\widehat{\mu}_N$ is used to estimate μ ([HP88, EF90, LH99]). However, it was shown recently in [MKA14] that if N only takes the values m and $2m$, and $\psi_m(x)$ takes the form $\Phi\left(\alpha + \beta m^{-1}x\right)$ or $\lim_{\beta\to\infty} \Phi\left(\alpha + \beta m^{-1}x\right) = 1_{\{\cdot\geq 0\}}(x)$, this bias vanishes as m tends to ∞. In this paper, we will establish explicit formulas for the expected length of the trial, the bias, and the mean squared error (MSE) in the general case, described by (a), (b), and (c). We deduce that, for fixed L, if $m_1 \to \infty$ (and hence $\forall i : m_i \to \infty$ and $n \to \infty$), the bias vanishes with rate $1/\sqrt{m_1}$, and the MSE vanishes with rate $1/m_1$. We will show that both rates are optimal. Furthermore, under a regularity condition, we will establish asymptotic normality in total variation distance for the sample mean if, for fixed L and $m_1, \ldots, m_L$,

$n \to \infty$. In some cases, this validates the use of naive confidence intervals based on the sample mean if n is large.

The paper is structured as follows. In section 2, we introduce the normal transform of a finite tuple of bounded Borel measurable maps of $\mathbb{R}$ into $\mathbb{R}$, for which we establish a recursive formula. We use the normal transform in section 3 to obtain an explicit formula for the joint density of N and K_N. We establish a fundamental result in section 4, which is used to calculate the expected length of the trial in section 5, and the bias and the MSE in section 6. It is shown that, for fixed L, if $m_1 \to \infty$ (and hence $\forall i : m_i \to \infty$ and $n \to \infty$), the bias vanishes with rate $1/\sqrt{m_1}$, and the MSE vanishes with rate $1/m_1$. Both rates are shown to be optimal. In section 7, under a regularity condition, we establish asymptotic normality in total variation distance for $\widehat{\mu}_N$ if, for fixed L and $m_1, \dots, m_L$, $n \to \infty$. We also derive a conclusion for naive confidence intervals based on $\widehat{\mu}_N$. In section 8, we show how the theory developed in this paper fits in the broader framework of likelihood theory. A simulation study, which underpins our theoretical results, is conducted in section 9. Finally, some concluding remarks are formulated in section 10.

2. The normal transform

Let ϕ be the standard normal density. For a finite tuple $B = (b_1, \dots, b_i)$ of bounded Borel measurable maps of $\mathbb{R}$ into $\mathbb{R}$, we define the *normal transform* to be the map $\mathcal{N}_{B,\mu,\sigma}$ of $]0,\infty[^{i+1} \times \mathbb{R}$ into $\mathbb{R}$ given by

$$\mathcal{N}_{B,\mu,\sigma}(x_1,\dots,x_{i+1},x) = \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{i}\frac{\phi\left(\frac{z_j-\mu x_j}{\sigma\sqrt{x_j}}\right)}{\sigma\sqrt{x_j}}\, b_j\left(\sum_{k=1}^{j} z_k\right)\frac{\phi\left(\frac{x-\sum_{k=1}^{i} z_k-\mu x_{i+1}}{\sigma\sqrt{x_{i+1}}}\right)}{\sigma\sqrt{x_{i+1}}}\, dz_i\dots dz_1}{\frac{1}{\sigma\sqrt{\sum_{k=1}^{i+1}x_k}}\,\phi\left(\frac{x-\mu\sum_{k=1}^{i+1}x_k}{\sigma\sqrt{\sum_{k=1}^{i+1}x_k}}\right)}. \tag{1}$$

We will provide a recursive formula for the normal transform in Theorem 3. We need two lemmas.

Lemma 1. *For $x_1, x_2 \in]0,\infty[$ and $x, z \in \mathbb{R}$,*

$$\phi\left(\frac{z-\mu x_1}{\sigma\sqrt{x_1}}\right)\phi\left(\frac{x-z-\mu x_2}{\sigma\sqrt{x_2}}\right) = \phi\left(\frac{x-\mu(x_1+x_2)}{\sigma\sqrt{x_1+x_2}}\right)\phi\left(\frac{\frac{x_1+x_2}{x_1}z - x}{\sigma\sqrt{\frac{x_2(x_1+x_2)}{x_1}}}\right). \tag{2}$$

Proof. This is readily verified by a straightforward calculation. □

Lemma 2. *Let ξ be a random variable with law $N(0,1)$. For a bounded Borel measurable map b of $\mathbb{R}$ into $\mathbb{R}$, $x_1, x_2 \in]0,\infty[$, and $x \in \mathbb{R}$,*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{x_1}}\phi\left(\frac{z-\mu x_1}{\sigma\sqrt{x_1}}\right) b(z) \frac{1}{\sigma\sqrt{x_2}}\phi\left(\frac{x-z-\mu x_2}{\sigma\sqrt{x_2}}\right) dz \\
&= \frac{1}{\sigma\sqrt{x_1+x_2}}\phi\left(\frac{x-\mu(x_1+x_2)}{\sigma\sqrt{x_1+x_2}}\right)\mathbb{E}\left[b\left(\frac{x_1}{x_1+x_2}x+\sigma\sqrt{\frac{x_1x_2}{x_1+x_2}}\xi\right)\right].
\end{aligned} \tag{3}$$

Proof. By (2),

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{x_1}}\phi\left(\frac{z-\mu x_1}{\sigma\sqrt{x_1}}\right) b(z) \frac{1}{\sigma\sqrt{x_2}}\phi\left(\frac{x-z-\mu x_2}{\sigma\sqrt{x_2}}\right) dz \\
&= \frac{1}{\sigma^2\sqrt{x_1x_2}}\phi\left(\frac{x-\mu(x_1+x_2)}{\sigma\sqrt{x_1+x_2}}\right)\int_{-\infty}^{\infty} b(z)\phi\left(\frac{\frac{x_1+x_2}{x_1}z-x}{\sigma\sqrt{\frac{x_2(x_1+x_2)}{x_1}}}\right)dz,
\end{aligned}$$

which is seen to coincide with the right-hand side of (3) after performing the change of variables $u = \frac{\frac{x_1+x_2}{x_1}z-x}{\sigma\sqrt{\frac{x_2(x_1+x_2)}{x_1}}}$. This finishes the proof. □

Theorem 3. *Let ξ be a random variable with law $N(0,1)$. For a bounded Borel measurable map b of $\mathbb{R}$ into $\mathbb{R}$, $x_1, x_2 \in]0,\infty[$, and $x \in \mathbb{R}$,*

$$\mathcal{N}_{b,\mu,\sigma}(x_1,x_2,x) = \mathbb{E}\left[b\left(\frac{x_1}{x_1+x_2}x+\sigma\sqrt{\frac{x_1x_2}{x_1+x_2}}\xi\right)\right]. \tag{4}$$

Furthermore, for a natural number $i \geq 2$, a tuple $(b_1,\dots,b_i)$ of bounded Borel measurable maps of $\mathbb{R}$ into $\mathbb{R}$, $x_1,\dots,x_{i+1} \in]0,\infty[$, and $x \in \mathbb{R}$,

$$\mathcal{N}_{(b_1,\dots,b_i),\mu,\sigma}(x_1,\dots,x_{i+1},x) = \mathcal{N}_{(b_1,\dots,b_{i-2},\widetilde{b}_{i-1}),\mu,\sigma}(x_1,\dots,x_{i-1},x_i+x_{i+1},x), \tag{5}$$

where

$$\widetilde{b}_{i-1}(z) = b_{i-1}(z)\mathbb{E}\left[b_i\left(\frac{x_{i+1}}{x_i+x_{i+1}}z+\frac{x_i}{x_i+x_{i+1}}x+\sigma\sqrt{\frac{x_ix_{i+1}}{x_i+x_{i+1}}}\xi\right)\right]. \tag{6}$$

Proof. Formula (4) follows directly from Lemma 2.

We now establish formula (5). We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{i}\frac{\phi\left(\frac{z_j-\mu x_j}{\sigma\sqrt{x_j}}\right)}{\sigma\sqrt{x_j}}b_j\left(\sum_{k=1}^{j}z_k\right)\frac{\phi\left(\frac{x-\sum_{k=1}^{i}z_k-\mu x_{i+1}}{\sigma\sqrt{x_{i+1}}}\right)}{\sigma\sqrt{x_{i+1}}}dz_i\dots dz_1 \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{i-1}\frac{\phi\left(\frac{z_j-\mu x_j}{\sigma\sqrt{x_j}}\right)}{\sigma\sqrt{x_j}}b_j\left(\sum_{k=1}^{j}z_k\right) \\
&\quad\left(\int_{-\infty}^{\infty}\frac{\phi\left(\frac{z_i-\mu x_i}{\sigma\sqrt{x_i}}\right)}{\sigma\sqrt{x_i}}b_i\left(\sum_{k=1}^{i}z_k\right)\frac{\phi\left(\frac{x-\sum_{k=1}^{i}z_k-\mu x_{i+1}}{\sigma\sqrt{x_{i+1}}}\right)}{\sigma\sqrt{x_{i+1}}}dz_i\right)dz_{i-1}\dots dz_i,
\end{aligned}$$

which, applying (3) to the map $b(z) = b_i\left(\sum_{k=1}^{i-1} z_k + z\right)$ in the integration with respect to z_i, reduces to

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{i-1}\frac{\phi\left(\frac{z_j-\mu x_j}{\sigma\sqrt{x_j}}\right)}{\sigma\sqrt{x_j}}b_j\left(\sum_{k=1}^{j} z_k\right)$$
$$\mathbb{E}\left[\sum_{k=1}^{i-1} z_k + \frac{x_i}{x_i+x_{i+1}}\left(x-\sum_{k=1}^{i-1} z_k\right)+\sigma\sqrt{\frac{x_i x_{i+1}}{x_i+x_{i+1}}}\xi\right]$$
$$\frac{1}{\sigma\sqrt{x_i+x_{i+1}}}\phi\left(\frac{x-\sum_{k=1}^{i-1} z_k-\mu(x_i+x_{i+1})}{\sigma\sqrt{x_i+x_{i+1}}}\right)dz_{i-1}\ldots dz_1,$$

which, using notation (6), equals

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{i-2}\frac{\phi\left(\frac{z_j-\mu x_j}{\sigma\sqrt{x_j}}\right)}{\sigma\sqrt{x_j}}b_j\left(\sum_{k=1}^{j} z_k\right)\frac{\phi\left(\frac{z_{i-1}-\mu x_{i-1}}{\sigma\sqrt{x_{i-1}}}\right)}{\sigma\sqrt{x_{i-1}}}\widetilde{b}_{i-1}\left(\sum_{k=1}^{i-1} z_k\right)$$
$$\frac{1}{\sigma\sqrt{x_i+x_{i+1}}}\phi\left(\frac{x-\sum_{k=1}^{i-1} z_k-\mu(x_i+x_{i+1})}{\sigma\sqrt{x_i+x_{i+1}}}\right)dz_{i-1}\ldots dz_1,$$

which, by definition (1),

$$=\frac{1}{\sigma\sqrt{\sum_{k=1}^{i+1} x_k}}\phi\left(\frac{x-\mu\sum_{k=1}^{i+1} x_k}{\sigma\sqrt{\sum_{k=1}^{i+1} x_k}}\right)\mathcal{N}_{(b_1,\ldots,b_{i-2},\widetilde{b}_{i-1}),\mu,\sigma}(x_1,\ldots,x_{i-1},x_i+x_{i+1},x).$$

This finishes the proof. □

3. The joint density of N and K_N

We return to the setting of the first section. Let $f_{N,K_N}(m,x)$ be the joint density of N and K_N. Furthermore, put

$$\Delta m_1 = m_1$$

and

$$\mathcal{N}_1(x) = 1, \tag{7}$$

and, for $i \in \{2,\ldots,L\}$,

$$\Delta_{m_i} = m_i - m_{i-1}$$

and

$$\mathcal{N}_i(x) = \mathcal{N}_{\left(1-\psi_{m_1},\ldots,1-\psi_{m_{i-1}}\right),\mu,\sigma}(\Delta_{m_1},\ldots,\Delta_{m_i},x). \tag{8}$$

We first establish in Theorem 5 that each $\mathcal{N}_i$ takes values between 0 and 1. We need the following lemma.

Lemma 4. *Let $(b_1, \ldots, b_i)$ be a tuple of bounded Borel measurable maps of $\mathbb{R}$ into $\mathbb{R}$. If each b_i takes values between 0 and 1, then $\mathcal{N}_{(b_1,\ldots,b_i),\mu,\sigma}$ takes values between 0 and 1.*

Proof. Using formulas (4), (5), and (6), this follows easily by an inductive argument. □

Theorem 5. *For each $i \in \{1, \ldots, L\}$, $\mathcal{N}_i$ takes values between 0 and 1.*

Proof. Using (7), (8), and the fact that each ψ_i takes values between 0 and 1, this follows from Lemma 4. □

The importance of the normal transform is reflected by the following result, which provides a formula for the joint density of N and K_N at the places where the interim analyses are performed.

Theorem 6. *For $i \in \{1, \ldots, L\}$,*

$$f_{N,K_N}(m_i, x) = \frac{1}{\sigma\sqrt{m_i}}\phi\left(\frac{x - \mu m_i}{\sigma\sqrt{m_i}}\right)\psi_{m_i}(x)\mathcal{N}_i(x). \tag{9}$$

Proof. We first consider the case $i = 1$. We have

$$f_{N,K_N}(m_1, x) = f_{N,K_{m_1}}(m_1, x) = f_{N|K_{m_1}}(m_1 \mid x) f_{K_{m_1}}(x), \tag{10}$$

with $f_{N|K_{m_1}}(m \mid x)$ the conditional density of N given K_{m_1} and $f_{K_{m_1}}$ the density of K_{m_1}. By condition (c) in section 1, and using the discrete nature of N,

$$f_{N|K_{m_1}}(m_1 \mid x) = \psi_{m_1}(x). \tag{11}$$

Furthermore, since the X_k are independent with distribution $N(\mu, \sigma^2)$,

$$f_{K_{m_1}}(x) = \frac{1}{\sigma\sqrt{m_1}}\phi\left(\frac{x - \mu m_1}{\sigma\sqrt{m_1}}\right). \tag{12}$$

Combining (10), (11), and (12), shows that (9) holds in the case $i = 1$.

We now turn to the case $i \geq 2$. Put

$$S_{m_1} = K_{m_1}$$

and, for $j \in \{2, \ldots, L\}$,

$$S_{m_j} = K_{m_j} - K_{m_{j-1}}.$$

Let

$$f_{N,S_{m_1},\ldots,S_{m_i}}(m, x_1, \ldots, x_i)$$

be the joint density of N and $S_{m_1}, \ldots, S_{m_i}$,

$$f_{N|S_{m_1},\ldots,S_{m_i}}(m \mid x_1, \ldots, x_i)$$

the conditional density of N given $S_{m_1}, \ldots, S_{m_i}$, and

$$f_{S_{m_1},\ldots,S_{m_i}}(x_1, \ldots, x_i)$$

the joint density of $S_{m_1}, \ldots, S_{m_i}$. Then

$$\begin{aligned} & f_{N,K_N}(m_i, x) \\ = \;& f_{N,K_{m_i}}(m_i, x) \\ = \;& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{N,S_{m_1},\ldots,S_{m_i}}\left(m_i, z_1, \ldots, z_{i-1}, x - \sum_{k=1}^{i-1} z_k\right) dz_{i-1} \ldots dz_1 \\ = \;& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{N|S_{m_1},\ldots,S_{m_i}}\left(m_i \mid z_1, \ldots, z_{i-1}, x - \sum_{k=1}^{i-1} z_k\right) \\ & \qquad f_{S_{m_1},\ldots,S_{m_i}}\left(z_1, \ldots, z_{i-1}, x - \sum_{k=1}^{i-1} z_k\right) dz_{i-1} \ldots dz_1. \end{aligned} \tag{13}$$

By condition (c) in section 1, and using the discrete nature of N,

$$f_{N|S_{m_1},\ldots,S_{m_i}}\left(m_i \mid z_1, \ldots, z_{i-1}, x - \sum_{k=1}^{i-1} z_k\right) = \psi_{m_i}(x) \prod_{j=1}^{i-1}\left[1 - \psi_{m_j}\left(\sum_{k=1}^{j} z_k\right)\right]. \tag{14}$$

Furthermore, the X_k being independent with distribution $N(\mu, \sigma^2)$,

$$\begin{aligned} & f_{S_{m_1},\ldots,S_{m_i}}\left(z_1, \ldots, z_{i-1}, x - \sum_{k=1}^{i-1} z_k\right) \\ = \;& \prod_{j=1}^{i-1} f_{S_{m_j}}(z_j) f_{S_{m_i}}\left(x - \sum_{k=1}^{i-1} z_k\right) \\ = \;& \prod_{j=1}^{i-1} \frac{1}{\sigma\sqrt{\Delta m_j}} \phi\left(\frac{z_j - \mu \Delta m_j}{\sigma\sqrt{\Delta m_j}}\right) \frac{1}{\sigma\sqrt{\Delta m_i}} \phi\left(\frac{x - \sum_{k=1}^{i-1} z_k - \mu\Delta m_i}{\sigma\sqrt{\Delta m_i}}\right). \end{aligned} \tag{15}$$

Combining definition (1) with (13), (14), and (15), establishes (9). This finishes the proof. □

Finally, we will provide a formula for the joint density of N and K_N at n in Theorem 8. We need the following lemma.

Lemma 7. *Let ξ be a random variable with law $N(0,1)$. Then, for $i \in \{1, \ldots, L\}$,*

$$f_{N,K_n}(m_i, x) = \frac{1}{\sigma\sqrt{n}}\phi\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\frac{m_i}{n}x + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]. \tag{16}$$

Proof. We have

$$f_{N,K_n}(m_i, x) = \int_{-\infty}^{\infty} f_{N,K_{m_i},K_n-K_{m_i}}(m_i, z, x-z)dz,$$

which, by condition (b) in section 1,

$$= \int_{-\infty}^{\infty} f_{N,K_{m_i}}(m_i, z) f_{K_n-K_{m_i}}(x-z)dz,$$

which, by (9),

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{m_i}}\phi\left(\frac{z-\mu m_i}{\sigma\sqrt{m_i}}\right)\psi_{m_i}(z)\mathcal{N}_i(z)\frac{1}{\sigma\sqrt{n-m_i}}\phi\left(\frac{x-z-\mu\sqrt{n-m_i}}{\sigma\sqrt{n-m_i}}\right),$$

which, by (3),

$$= \frac{1}{\sigma\sqrt{n}}\phi\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\frac{m_i}{n}x + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right].$$

This finishes the proof. □

Theorem 8. *Let ξ be a random variable with law $N(0,1)$. Then*

$$f_{N,K_N}(n, x) = \frac{1}{\sigma\sqrt{n}}\phi\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\left[1 - \sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\frac{m_i}{n}x + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]\right]. \tag{17}$$

Proof. By condition (a) in section 1,

$$f_{N,K_N}(n, x) = f_{N,K_n}(n, x) = f_{K_n}(x) - \sum_{i=1}^{L} f_{N,K_n}(m_i, x),$$

which, applying (16), proves the desired result. □

4. A fundamental result

We will prove Theorem 10, which will play a fundamental role in the calculation of the expected length of the trial, the bias, and the MSE, and in the establishment of an asymptotic normality result. We need the following lemma.

Lemma 9. *Let η be a random variable with law $N(0,1)$, g a Borel measurable map of $\mathbb{R}$ into $\mathbb{R}$ with $\mathbb{E}[|g(\eta)|] < \infty$, and $m \in \mathbb{R}_0^+$. Then*

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{m}}\phi\left(\frac{x-\mu m}{\sigma\sqrt{m}}\right) g(x)dx = \mathbb{E}\left[g(\mu m + \sigma\sqrt{m}\eta)\right]. \tag{18}$$

Proof. Perform the change of variables $z = \frac{x-\mu m}{\sigma\sqrt{m}}$. □

Theorem 10. *Let ξ and η be independent random variables with law $N(0,1)$ and h a Borel measurable map of $\mathbb{R}$ into $\mathbb{R}$ with $\mathbb{E}[|h(\eta)|] < \infty$. Then, for each $i \in \{1,\ldots,L\}$,*

$$\mathbb{E}\left[h(\widehat{\mu}_N)1_{\{N=m_i\}}\right] = \mathbb{E}\left[h\left(\mu + \frac{\sigma}{\sqrt{m_i}}\xi\right)(\psi_{m_i}\mathcal{N}_i)(\mu m_i + \sigma\sqrt{m_i}\xi)\right], \tag{19}$$

and

$$\begin{aligned}&\mathbb{E}\left[h(\widehat{\mu}_N)1_{\{N=n\}}\right] \\ &= \mathbb{E}\left[h\left(\mu + \frac{\sigma}{\sqrt{n}}\eta\right)\left(1 - \sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]\right)\right].\end{aligned} \tag{20}$$

Proof. By (9), for $i \in \{1,\ldots,L\}$,

$$\mathbb{E}\left[h(\widehat{\mu}_N)1_{\{N=m_i\}}\right] = \int_{-\infty}^{\infty} h\left(\frac{1}{m_i}x\right)\frac{1}{\sigma\sqrt{m_i}}\phi\left(\frac{x-\mu m_i}{\sigma\sqrt{m_i}}\right)\psi_{m_i}(x)\mathcal{N}_i(x)dx,$$

which, using (18) with $g(x) = h\left(\frac{1}{m_i}x\right)\psi_{m_i}(x)\mathcal{N}_i(x)$ and $m = m_i$, gives (19).

Furthermore, by (17),

$$\begin{aligned}&\mathbb{E}\left[h(\widehat{\mu}_N)1_{\{N=n\}}\right] \\ &= \int_{-\infty}^{\infty} h\left(\frac{1}{n}x\right)\frac{1}{\sigma\sqrt{n}}\phi\left(\frac{x-\mu n}{\sigma\sqrt{n}}\right)\left[1 - \sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\frac{m_i}{n}x + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]\right],\end{aligned}$$

which, applying (18) with $g(x) = h\left(\frac{1}{n}x\right)\left[1 - \sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\frac{m_i}{n}x + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]\right]$ and $m = n$, and using independence of ξ and η, gives (20). □

5. The expected length of the trial

The following result provides explicit formulas for the marginal density of the actual length of the trial N.

Theorem 11. *Let ξ and η be independent random variables with law $N(0,1)$. Then, for each $i \in \{1,\ldots,L\}$,*

$$\mathbb{P}[N = m_i] = \mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)\right], \tag{21}$$

and

$$\mathbb{P}[N = n] = 1 - \sum_{i=1}^{L} \mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]. \tag{22}$$

Proof. Applying (19) with $h(x) = 1$, gives (21). Furthermore, applying (20) with $h(x) = 1$, gives (22). □

Next, we provide an explicit formula for the expected length of the trial.

Theorem 12. *Let ξ and η be independent random variables with law $N(0,1)$. Then*

$$\begin{aligned}\mathbb{E}[N] \quad = \quad & \sum_{i=1}^{L} m_i\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)\right] \\ & + n\left(1 - \sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]\right).\end{aligned} \tag{23}$$

Proof. This follows immediately from Theorem 11. □

6. The bias and the mean squared error

The following result provides an explicit formula for the bias if $\widehat{\mu}_N$ is used to estimate μ.

Theorem 13. *Let ξ and η be independent random variables with law $N(0,1)$. Then*

$$\begin{aligned}&\mathbb{E}[\widehat{\mu}_N - \mu] \\ &\quad = \sum_{i=1}^{L}\frac{\sigma}{\sqrt{m_i}}\mathbb{E}\left[\xi\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)\right] \\ &\qquad - \frac{\sigma}{\sqrt{n}}\sum_{i=1}^{L}\mathbb{E}\left[\eta\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right].\end{aligned} \tag{24}$$

Proof. We have

$$\mathbb{E}[\widehat{\mu}_N - \mu] = \sum_{i=1}^{L}\mathbb{E}[(\widehat{\mu}_N - \mu)\,1_{\{N=m_i\}}] + \mathbb{E}\left[(\widehat{\mu}_N - \mu)\,1_{\{N=n\}}\right].$$

Now (24) follows by applying (19) and (20) with $h(x) = x - \mu$. □

From Theorem 13, we derive the following universal bound for the bias.

Theorem 14.

$$|\mathbb{E}[\widehat{\mu}_N - \mu]| \leq \sigma\sqrt{\frac{2}{\pi}}\left(\sum_{i=1}^{L}\frac{1}{\sqrt{m_i}} + \frac{L}{\sqrt{n}}\right). \tag{25}$$

In particular, the bias vanishes if, for fixed L, $m_1 \to \infty$ (and hence $\forall i : m_i \to \infty$ and $n \to \infty$).

Proof. Theorem 5 shows that each $\psi_{m_i}\mathcal{N}_i$ takes values between 0 and 1. Therefore, for ξ and η with law $N(0,1)$,

$$|\mathbb{E}\left[\xi\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)\right]| \leq \mathbb{E}[|\xi|] = \sqrt{\frac{2}{\pi}}$$

and

$$\left|\mathbb{E}\left[\eta\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]\right| \leq \mathbb{E}[|\eta|] = \sqrt{\frac{2}{\pi}}.$$

Now (25) follows easily from (24). □

The following result provides a formula, similar to (24), for the mean squared error (MSE).

Theorem 15. *Let ξ and η be independent random variables with law $N(0,1)$. Then*

$$\begin{aligned}
&\mathbb{E}\left[\left(\widehat{\mu}_N - \mu\right)^2\right] \qquad (26)\\
&= \sum_{i=1}^{L}\frac{\sigma^2}{m_i}\mathbb{E}\left[\xi^2\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)\right]\\
&\quad + \frac{\sigma^2}{n} - \frac{\sigma^2}{n}\sum_{i=1}^{L}\mathbb{E}\left[\eta^2\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right].
\end{aligned}$$

Proof. We have

$$\mathbb{E}\left[\left(\widehat{\mu}_N - \mu\right)^2\right] = \sum_{i=1}^{L}\mathbb{E}\left[\left(\widehat{\mu}_N - \mu\right)^2 1_{\{N=m_i\}}\right] + \mathbb{E}\left[\left(\widehat{\mu}_N - \mu\right)^2 1_{\{N=n\}}\right].$$

Now (26) follows by applying (19) and (20) with $h(x) = (x-\mu)^2$. □

Finally, from Theorem 15, we derive the following universal bound for the MSE.

Theorem 16.

$$\mathbb{E}\left[\left(\widehat{\mu}_N - \mu\right)^2\right] \leq \sigma^2\left(\sum_{i=1}^{L}\frac{1}{m_i} + \frac{L+1}{n}\right). \tag{27}$$

In particular, the MSE vanishes if, for fixed L, $m_1 \to \infty$ (and hence $\forall i : m_i \to \infty$ and $n \to \infty$).

Proof. This is derived from Theorem 15 in the same way as Theorem 14 was derived from Theorem 13. ☐

We wish to conclude this section with the following remarks:

1. The bounds (25) and (27) hold for the generic stopping rule described by (a), (b), and (c) in section 1, which contains many classical stopping rules as a special case. Therefore, both bounds have a wide range of applicability.
2. The fact that $0 < m_1 < m_2 < \ldots < m_L < n$ allows us to derive from (25) that

$$|\mathbb{E}[\widehat{\mu}_N - \mu]| \leq \frac{2\sigma L\sqrt{\frac{2}{\pi}}}{\sqrt{m_1}}.$$

That is, for fixed L, the bias converges to 0 as $m_1 \to \infty$ at least with rate $1/\sqrt{m_1}$. Moreover, this rate is optimal. Indeed, taking $\mu = 0$, $\sigma = 1$, $L = 1$, $m_1 = m$, $n = 2m$, and $\psi_m(x) = 1_{\mathbb{R}^+}$, the characteristic function of the set $\mathbb{R}^+$, leads to a trial with maximal length $2m$, in which one interim analysis is performed at m. The trial is stopped if $K_m \geq 0$, and continued otherwise. In this case, for independent ξ and η with law $N(0,1)$,

$$\mathbb{E}\left[\xi\psi_m(\sqrt{m}\xi)\right] = \int_0^\infty u\phi(u)du = \phi(0) = \frac{1}{\sqrt{2\pi}}$$

and

$$\mathbb{E}\left[\eta\psi_m\left(\sqrt{\frac{m}{2}}\xi + \sqrt{\frac{m}{2}}\eta\right)\right] = \mathbb{E}\left[\int_{-\xi}^\infty u\phi(u)du\right] = \mathbb{E}[\phi(\xi)] = \int_{-\infty}^\infty \phi^2(u)du = \frac{1}{2\sqrt{\pi}},$$

from which we deduce that (24) reduces to

$$\mathbb{E}\left[\widehat{\mu}_N - \mu\right] = \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{m}}.$$

3. The fact that $0 < m_1 < m_2 < \ldots < m_L < n$ allows us to derive from (26) that

$$\mathbb{E}\left[(\widehat{\mu}_N - \mu)^2\right] \leq \frac{\sigma^2(2L+1)}{m_1}.$$

That is, for fixed L, the MSE converges to 0 as $m_1 \to \infty$ at least with rate $1/m_1$. This rate is again optimal. Indeed, take, as in the previous remark, $\mu = 0$, $\sigma = 1$, $L = 1$, $m_1 = m$, $n = 2m$, and $\psi_m(x) = 1_{\mathbb{R}^+}$. Then, for independent ξ and η with law $N(0,1)$,

$$\mathbb{E}\left[\xi^2\psi_m(\sqrt{m}\xi)\right] = \int_0^\infty u^2\phi(u)du = \frac{1}{2}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\eta^2\psi_m\left(\sqrt{\frac{m}{2}}\xi+\sqrt{\frac{m}{2}}\eta\right)\right]\\
&= \mathbb{E}\left[\int_{-\xi}^{\infty}u^2\phi(u)du\right]=\mathbb{E}[\Phi(\xi)]+\mathbb{E}[\xi\phi(\xi)]=\int_{-\infty}^{\infty}\phi(u)\Phi(u)du+\int_{-\infty}^{\infty}u\phi^2(u)du=\frac{1}{2},
\end{aligned}$$

which shows that (26) becomes

$$\mathbb{E}\left[(\widehat{\mu}_N-\mu)^2\right]=\frac{3}{4m}.$$

4. One frequently encounters a group sequential trial in which an interim analysis is performed after every m observations, with $m \in \mathbb{N}_0$ fixed. In our setting, this corresponds to the choices $m_i = im$, where $i \in \{1, \ldots, L\}$, and $n = (L+1)m$. In this case, the bound (27) reduces to

$$\mathbb{E}\left[(\widehat{\mu}_N-\mu)^2\right]\le\frac{\sigma^2}{m}\left(1+\sum_{i=1}^{L}\frac{1}{i}\right)\le\frac{\sigma^2}{m}\left(2+\log(L)\right), \tag{28}$$

where log is the natural logarithm and the last inequality follows by

$$\sum_{i=1}^{L}\frac{1}{i}=1+\sum_{i=2}^{L}\frac{1}{i}\le 1+\int_1^L\frac{dx}{x}=1+\log(L),$$

which is seen by comparing on $[1,\infty[$ the graph of the map $y=1/x$ with the graph of the map that constantly takes the value $1/i$ on $[i-1,i]$, where $i\in\{2,3,\ldots\}$. Taking e.g. $\sigma=1$, $m=40$, and $L=9$, corresponds to a trial of maximal length 400 in which interim analyses are performed after every 40 observations. In this case, (28) gives

$$\mathbb{E}\left[(\widehat{\mu}_N-\mu)^2\right]\le\frac{1}{40}(2+\log(9))\approx 0.105.$$

5. Again in a trial in which interim analyses are performed after every m observations, the inequality

$$\mathbb{E}\left[|\widehat{\mu}_N-\mu|\right]\le\left(\mathbb{E}\left[(\widehat{\mu}_N-\mu)^2\right]\right)^{1/2}$$

allows us to derive from (28) the following bound for the bias:

$$|\mathbb{E}[\widehat{\mu}_N-\mu]|\le\sigma\sqrt{\frac{2+\log(L)}{m}}. \tag{29}$$

6. Our results show that it is beneficial to start with a sufficiently large first contingent. The bias and the MSE are then generally acceptably small, even when gauged through the uniform

bounds (25) and (27). For specific stopping rules, results may be much sharper, as will be illustrated by our simulation study in section 9. The question may arise as to whether it is ethical to expose a relatively large first contingent. However, this issue should be approached cautiously. One should consider the expected trial length; designs should be chosen by concentrating on this quantity, rather than on the minimal length. Indeed, a very small minimal length, combined with a very low probability for this to be realized in a given study, is of little value.

7. Asymptotic normality and confidence intervals

In this section, we will establish asymptotic normality in total variation distance for $\widehat{\mu}_N$. We need the following lemma.

Lemma 17. *Let ξ and η be independent random variables with law $N(0,1)$. Then, for a bounded Borel measurable map $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\left[f\left(\frac{\sqrt{N}}{\sigma}(\widehat{\mu}_N - \mu)\right)\right] = \mathbb{E}[f(\xi)] + \sum_{i=1}^{L} \mathbb{E}\left[f(\xi)\left((\psi_{m_i}\mathcal{N}_i)(\mu m_i + \sigma\sqrt{m_i}\xi) - (\psi_{m_i}\mathcal{N}_i)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right)\right]. \tag{30}$$

Proof. We have

$$\mathbb{E}\left[f\left(\frac{\sqrt{N}}{\sigma}(\widehat{\mu}_N - \mu)\right)\right] = \sum_{i=1}^{L} \mathbb{E}\left[f\left(\frac{\sqrt{m_i}}{\sigma}(\widehat{\mu}_N - \mu)\right)1_{\{N=m_i\}}\right] + \mathbb{E}\left[f\left(\frac{\sqrt{n}}{\sigma}(\widehat{\mu}_N - \mu)\right)1_{\{N=n\}}\right].$$

Now apply (19) with $h(x) = f\left(\frac{\sqrt{m_i}}{\sigma}(x-\mu)\right)$ and (20) with $h(x) = f\left(\frac{\sqrt{n}}{\sigma}(x-\mu)\right)$. This gives (30). □

Recall that the total variation distance between random variables ζ_1 and ζ_2 is given by

$$d_{TV}(\zeta_1, \zeta_2) = \sup_A |\mathbb{P}[\zeta_1 \in A] - \mathbb{P}[\zeta_2 \in A]|,$$

the supremum running over all Borel measurable sets $A \subset \mathbb{R}$. It is well known that convergence in total variation distance implies weak convergence, but that the converse generally fails to hold.

The following result provides an explicit bound for the total variation distance between the standard normal distribution and the law of the quantity $\frac{\sqrt{N}}{\sigma}(\widehat{\mu}_N - \mu)$.

Theorem 18. *Let ξ and η be independent random variables with law $N(0,1)$. Then*

$$d_{TV}\left(N(0,1), \frac{\sqrt{N}}{\sigma}\left(\widehat{\mu}_N - \mu\right)\right) \tag{31}$$

$$\leq \sum_{i=1}^{L} \mathbb{E}\left[\left|\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right) - \left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right|\right].$$

In particular, if the ψ_{m_i} are continuous, for fixed L and $m_1, \ldots, m_L$,

$$\lim_{n\to\infty} d_{TV}\left(N(0,1), \frac{\sqrt{N}}{\sigma}\left(\widehat{\mu}_N - \mu\right)\right) = 0. \tag{32}$$

Proof. Fix a Borel measurable set $A \subset \mathbb{R}$. Applying (30) with $f(x) = 1_A(x)$, gives

$$\begin{aligned}
&\left|\mathbb{E}\left[1_A\left(\frac{\sqrt{N}}{\sigma}\left(\widehat{\mu}_N - \mu\right)\right) - 1_A(\xi)\right]\right| \\
&= \left|\sum_{i=1}^{L}\mathbb{E}\left[1_A(\xi)\left(\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right) - \left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right)\right]\right| \\
&\leq \sum_{i=1}^{L}\mathbb{E}\left[\left|\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right) - \left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right|\right],
\end{aligned}$$

entailing (31).

Furthermore, suppose that the ψ_{m_i} are continuous. Using Theorem 3, it is easily checked that the $\mathcal{N}_i$ are also continuous. Hence, for fixed L and $m_1, \ldots, m_L$, $\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)_n$ tends to $\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right)$ pointwise. Thus, each $\psi_{m_i}\mathcal{N}_i$ taking values between 0 and 1 by Theorem 5, the Lebesgue dominated convergence theorem allows us to derive (32) from (31). □

From Theorem 18, we easily derive the following conclusion for naive confidence intervals based on $\widehat{\mu}_N$.

Theorem 19. *Let Φ be the standard normal cumulative distribution function, and ξ and η independent random variables with law $N(0,1)$. Then, for $x \in \mathbb{R}$,*

$$\left|2\Phi(x) - 1 - \mathbb{P}\left[\widehat{\mu}_N - \frac{\sigma}{\sqrt{N}}x \leq \mu \leq \widehat{\mu}_N + \frac{\sigma}{\sqrt{N}}x\right]\right| \tag{33}$$

$$\leq \sum_{i=1}^{L}\mathbb{E}\left[\left|\left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{m_i}\xi\right) - \left(\psi_{m_i}\mathcal{N}_i\right)\left(\mu m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right|\right],$$

which, for fixed L and $m_1, \ldots, m_L$, tends to 0 if $n \to \infty$, provided that the ψ_{m_i} are continuous.

Proof. This follows from Theorem 18 by considering the Borel set $A = [-x, x]$. □

We conclude this section with the following remarks:

1. Notice the surprising fact that the upper bound in (31) and (33) vanishes if $n \to \infty$ for all fixed choices of L and $m_1, \ldots, m_L$. In particular, contrary to the upper bounds in (25) for the bias and in (27) for the MSE, in studies with large maximal length n, the upper bound in (31) and (33) always vanishes, even if the m_i are small, i.e. if the interim analyses are performed early.
2. The bound (33) justifies the use of naive confidence intervals based on $\widehat{\mu}_N$, provided that the m_i are kept fixed, the ψ_{m_i} are continuous, and the maximal length n of the trial is large enough. We wish to point out that our conclusion for confidence intervals is less powerful than our statements for bias and MSE in the previous section. Indeed, we have not provided a rate of convergence. A deeper study of confidence intervals in a group sequential trial setting turns out to be much harder, and will be treated in subsequent work.

8. Connections with likelihood theory

In this section, we will connect the theory developed in the previous sections to marginal and conditional maximum likelihood estimation after a group sequential trial. As a starting point for likelihood theoretic arguments, we assume that the distribution of the X_i comes from the parametric family $\{N(\theta, \sigma^2) \mid \theta \in \mathbb{R}\}$.

Since each of the X_i has distribution $N(\theta, \sigma^2)$, we observe that the joint density of the X_i gathered in the trial is given by

$$f_{X_1,\ldots,X_N}(\theta, x_1, \ldots, x_N) = \frac{1}{\sigma^N} \prod_{j=1}^{N} \phi\left(\frac{x_j - \theta}{\sigma}\right).$$

Therefore, classical likelihood theory kicks in and we conclude that the marginal maximum likelihood estimator (MLE) for μ is the ordinary sample mean $\widehat{\mu}_N$. In the previous sections, we have provided evidence of the fact that this estimator performs well in terms of bias and MSE if the first interim analysis is performed not too early, and in terms of asymptotic normality and confidence intervals if the maximal length of the study is large enough.

We now turn to conditional maximum likelihood estimation (CMLE) after a group sequential trial. More precisely, we will link the CMLE for μ, conditioned on N, to the sample average, from which it will follow that it coincides with the 'conditional bias reduction estimate', studied in [FDL00], section 3.3. We will use the following lemma, which lies at the heart of Stein's method ([CGS11]).

Lemma 20. *Let η be a random variable with law $N(0,1)$, $g : \mathbb{R} \to \mathbb{R}$ a Borel measurable map with $\mathbb{E}[|g(\eta)|] < \infty$, $A \in \mathbb{R}$, and $B \in \mathbb{R}_0$. Then the map*

$$\theta \mapsto \mathbb{E}[g(A\theta + B\eta)], \;\; \theta \in \mathbb{R},$$

is smooth. Furthermore,

$$\frac{d}{d\theta}\mathbb{E}[g(A\theta + B\eta)] = \frac{A}{B}\mathbb{E}[\eta g(A\theta + B\eta)] \tag{34}$$

and

$$\frac{d}{d\theta}\mathbb{E}[\eta g(A\theta + B\eta)] = \frac{A}{B}\mathbb{E}[(\eta^2 - 1)g(A\theta + B\eta)]. \tag{35}$$

Proof. We have

$$\frac{d}{d\theta}\mathbb{E}[g(A\theta + B\eta)] = \frac{d}{d\theta}\int_{-\infty}^{\infty} g(A\theta + Bu)\phi(u)du,$$

which, performing the change of variables $t = A\theta + Bu$,

$$\begin{aligned} = \frac{1}{B}\frac{d}{d\theta}\int_{-\infty}^{\infty} g(t)\phi\left(\frac{t - A\theta}{B}\right)dt \;\; &= \;\; \frac{1}{B}\int_{-\infty}^{\infty} g(t)\frac{d}{d\theta}\phi\left(\frac{t - A\theta}{B}\right)dt \\ &= \;\; \frac{A}{B^2}\int_{-\infty}^{\infty} g(t)\left(\frac{t - A\theta}{B}\right)\phi\left(\frac{t - A\theta}{B}\right)dt, \end{aligned}$$

which, performing the change of variables $v = \frac{t - A\theta}{B}$,

$$= \frac{A}{B}\int_{-\infty}^{\infty} vg(A\theta + Bv)\phi(v)dv = \frac{A}{B}\mathbb{E}[\eta g(A\theta + B\eta)].$$

This proves (34). The proof of (35) is analogous. □

Now let ξ and η be independent random variables with law $N(0,1)$, and suppose that, for each θ, N can take each of the values $m_1, \ldots, m_L, n$ with a strictly positive probability, i.e. the expressions (21) and (22) are nonzero if μ is replaced by θ. Then, using Bayes' Theorem, the fact that the X_i have law $N(\theta, \sigma^2)$, and plugging in (c) in section 1, and (21) and (22) with μ replaced by θ, it holds for the conditional likelihood of the X_i given N that, for $i \in \{1, \ldots, L\}$,

$$\begin{aligned} & f_{X_1,\ldots,X_{m_i}|N}(\theta, x_1, \ldots, x_{m_i} \mid m_i) \\ = \;\; & \frac{1}{\sigma^{m_i}}\prod_{j=1}^{m_i}\phi\left(\frac{x_j - \theta}{\sigma}\right)\frac{\psi_{m_i}(\sum_{k=1}^{m_i} x_k)\prod_{j=1}^{i-1}\left[1 - \psi_{m_j}\left(\sum_{k=1}^{m_j} x_k\right)\right]}{\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i + \sigma\sqrt{m_i}\xi\right)\right]}, \end{aligned} \tag{36}$$

and

$$f_{X_1,\ldots,X_n|N}(\theta, x_1, \ldots, x_n \mid n) \tag{37}$$
$$= \frac{1}{\sigma^n}\prod_{j=1}^{n}\phi\left(\frac{x_j-\theta}{\sigma}\right)\frac{1-\sum_{i=1}^{L}\psi_{m_i}(\sum_{k=1}^{m_i}x_k)\prod_{j=1}^{i-1}\left[1-\psi_{m_j}\left(\sum_{k=1}^{m_j}x_k\right)\right]}{1-\sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i+\sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi+\sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}.$$

In particular, up to an additive constant not depending on θ, the conditional log-likelihood is given by, for $i \in \{1, \ldots, L\}$,

$$\mathcal{L}(\theta, x_1, \ldots, x_{m_i} \mid m_i) = -\frac{1}{2\sigma^2}\sum_{j=1}^{m_i}(x_j-\theta)^2 - \log\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i+\sigma\sqrt{m_i}\xi\right)\right], \tag{38}$$

and

$$\mathcal{L}(\theta, x_1, \ldots, x_n \mid n) \tag{39}$$
$$= -\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j-\theta)^2 - \log\left(1-\sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i+\sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi+\sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]\right).$$

Applying (34) with $g = \psi_{m_i}\mathcal{N}_i$, $A = m_i$, and $B = \sigma\sqrt{m_i}$, shows that the partial derivative of (38) with respect to θ is

$$\frac{\partial}{\partial\theta}\mathcal{L}(\theta, x_1, \ldots, x_{m_i} \mid m_i) = \frac{1}{\sigma^2}\sum_{j=1}^{m_i}x_j - \frac{m_i}{\sigma^2}\theta - \frac{\sqrt{m_i}}{\sigma}\frac{\mathbb{E}[\xi\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i+\sigma\sqrt{m_i}\xi\right)]}{\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i+\sigma\sqrt{m_i}\xi\right)\right]}, \tag{40}$$

and, applying (34) with $g(\theta) = \mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta+\sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]$, $A = m_i$, and $B = \sigma\frac{m_i}{\sqrt{n}}$, and using independence of ξ and η, reveals that the partial derivative of (39) with respect to θ is

$$\frac{\partial}{\partial\theta}\mathcal{L}(\theta, x_1, \ldots, x_n \mid n) \tag{41}$$
$$= \frac{1}{\sigma^2}\sum_{j=1}^{n}x_j - \frac{n}{\sigma^2}\theta + \frac{\sqrt{n}}{\sigma}\frac{\sum_{i=1}^{L}\mathbb{E}\left[\eta\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i+\sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi+\sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}{1-\sum_{i=1}^{L}\mathbb{E}\left[(\psi_{m_i}\mathcal{N}_i)\left(\theta m_i+\sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi+\sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}.$$

Furthermore, using (19) with $h(x) = x - \theta$ and (21) with μ replaced by θ, (40) leads to

$$\frac{\partial}{\partial\theta}\mathcal{L}(\theta, X_1, \ldots, X_{m_i} \mid m_i) \tag{42}$$
$$= \frac{m_i}{\sigma^2}\left(\widehat{\mu}_{m_i} - \theta - \mathbb{E}_\theta\left[(\widehat{\mu}_N - \theta) \mid N = m_i\right]\right)$$
$$= \frac{m_i}{\sigma^2}\left(\widehat{\mu}_{m_i} - \mathbb{E}_\theta\left[\widehat{\mu}_N \mid N = m_i\right]\right),$$

and, using (20) with $h(x) = x - \theta$ and (22) with μ replaced by θ, (41) gives

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\mathcal{L}(\theta, X_1, \ldots, X_n \mid n) \qquad (43)\\
&= \frac{n}{\sigma^2}\left(\widehat{\mu}_n - \theta - \mathbb{E}_\theta\left[(\widehat{\mu}_N - \theta) \mid N = n\right]\right)\\
&= \frac{n}{\sigma^2}\left(\widehat{\mu}_n - \mathbb{E}_\theta\left[\widehat{\mu}_N \mid N = n\right]\right).
\end{aligned}$$

Also, applying (35) with $g = \psi_{m_i}\mathcal{N}_i$, $A = m_i$, and $B = \sigma\sqrt{m_i}$, shows that the partial derivative of (40) with respect to θ is

$$\begin{aligned}
&\frac{\partial^2}{\partial\theta^2}\mathcal{L}(\theta, x_1, \ldots, x_{m_i} \mid m_i) \qquad (44)\\
&= -\frac{m_i}{\sigma^2}\left(\frac{\mathbb{E}\left[\xi^2\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{m_i}\xi\right)\right]}{\mathbb{E}[\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{m_i}\xi\right)]} - \left(\frac{\mathbb{E}\left[\xi\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{m_i}\xi\right)\right]}{\mathbb{E}\left[\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{m_i}\xi\right)\right]}\right)^2\right),
\end{aligned}$$

and, applying (35) with $g(\theta) = \mathbb{E}\left[\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi\right)\right]$, $A = m_i$, and $B = \sigma\frac{m_i}{\sqrt{n}}$, and using independence of ξ and η, shows that the partial derivative of (41) with respect to θ is

$$\begin{aligned}
&\frac{\partial^2}{\partial\theta^2}\mathcal{L}(\theta, x_1, \ldots, x_n \mid n) \qquad (45)\\
&= \frac{n}{\sigma^2}\left(\frac{\sum_{i=1}^L \mathbb{E}\left[\eta^2\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}{1 - \sum_{i=1}^L \mathbb{E}\left[\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}\right)\\
&\quad - \frac{n}{\sigma^2}\left(\frac{\sum_{i=1}^L \mathbb{E}\left[\eta\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}{1 - \sum_{i=1}^L \mathbb{E}\left[\left(\psi_{m_i}\mathcal{N}_i\right)\left(\theta m_i + \sigma\sqrt{\frac{m_i(n-m_i)}{n}}\xi + \sigma\frac{m_i}{\sqrt{n}}\eta\right)\right]}\right)^2.
\end{aligned}$$

Furthermore, using (19), with respectively $h_1(x) = (x - \theta)^2$ and $h_2(x) = x - \theta$, and (21), each time with μ replaced by θ, transforms (44) into

$$\begin{aligned}
&\frac{\partial^2}{\partial\theta^2}\mathcal{L}(\theta, X_1, \ldots, X_{m_i} \mid m_i) \qquad (46)\\
&= -\frac{m_i^2}{\sigma^4}\left(\mathbb{E}_\theta\left[(\widehat{\mu}_N - \theta)^2 \mid N = m_i\right] - \left(\mathbb{E}_\theta\left[(\widehat{\mu}_N - \theta) \mid N = m_i\right]\right)^2\right)\\
&= -\frac{m_i^2}{\sigma^4}\mathrm{Var}_\theta[\widehat{\mu}_N \mid N = m_i],
\end{aligned}$$

and, using (20), with respectively $h_1(x) = (x - \theta)^2$ and $h_2(x) = x - \theta$, and (22), each time with μ

replaced by θ, shows that (45) gives

$$\begin{aligned}
& \frac{\partial^2}{\partial\theta^2}\mathcal{L}(\theta, X_1, \ldots, X_n \mid n) & (47)\\
= \quad & -\frac{n^2}{\sigma^4}\left(\mathbb{E}_\theta\left[(\widehat{\mu}_N-\theta)^2 \mid N=n\right] - \left(\mathbb{E}_\theta\left[(\widehat{\mu}_N-\theta) \mid N=n\right]\right)^2\right) \\
= \quad & -\frac{n^2}{\sigma^4}\mathrm{Var}_\theta\left[\widehat{\mu}_N \mid N=n\right].
\end{aligned}$$

The relations (42), (43), (46), and (47) are summarized by stating that the conditional log-likelihood satisfies the equations

$$\frac{\partial}{\partial\theta}\mathcal{L}\left(\theta, X_1, \ldots, X_N \mid N\right) = \frac{N}{\sigma^2}\left(\widehat{\mu}_N - \mathbb{E}_\theta\left[\widehat{\mu}_N \mid N\right]\right) \tag{48}$$

and

$$\frac{\partial^2}{\partial\theta^2}\mathcal{L}(\theta, X_1, \ldots, X_N \mid N) = -\frac{N^2}{\sigma^4}\mathrm{Var}_\theta\left[\widehat{\mu}_N \mid N\right]. \tag{49}$$

We conclude that the maximum likelihood estimator conditioned on N, denoted by $\widehat{\mu}_{c,N}$, satisfies the relation

$$\widehat{\mu}_N = \mathbb{E}_{\widehat{\mu}_{c,N}}[\widehat{\mu}_N \mid N]. \tag{50}$$

It follows from (50) that $\widehat{\mu}_{c,N}$ coincides with the 'conditional bias reduction estimate', studied in [FDL00], section 3.3. In section 3.4 of that paper, one provides empirical evidence of the fact that $\widehat{\mu}_{c,N}$ outperforms $\widehat{\mu}_N$ in trials that are stopped early. Combining this information with our results obtained in the previous sections, it seems plausible to recommend $\widehat{\mu}_N$ if the first interim analysis is performed not too early, and $\widehat{\mu}_{c,N}$ otherwise.

9. Simulations

To illustrate our findings, a simulation study was conducted to investigate the speed of convergence. Two different cases were considered: continuous normal, $X_1, X_2, \ldots, X_n$ i.i.d. $\sim N(\mu, 1)$, and discrete Bernoulli, $X_1, X_2, \ldots, X_n$ i.i.d. $\sim B(\pi)$, with different choices for the parameter values. For each case, the following design choices were made: 1000 random samples were generated, each of size n. To every sample, several stopping conditions were applied: (1) no stopping; (2) one stopping occasion (m_1) using the K criterion; (3) 1 stopping occasion (m_1) using the probit criterion; (4) 3 stopping occasions (m_1, m_2, or m_3) using the K criterion; and (5) 3 stopping occasions (m_1, m_2, or m_3) using the probit criterion. All stopping criteria were applied to the sample mean

$K_{m_i} = \frac{1}{m_i}\sum_{i=1}^{m_i} X_i$ in the following way. For the K criterion, stop at m_i, if $K_{m_i} < 0$. For the probit criterion, first, the probability $\Phi(\alpha + \beta K_{m_i})$ was calculated and a random uniform vector $U \sim$ Uniform (0,1) was generated; if $U \leq \Phi(\alpha + \beta K_{m_i})$, then stop at m_i, otherwise continue to m_{i+1} (or n). All calculations were performed with the R statistical software (R version 3.3.3).

The parameter choice was made as follows: total sample size $n = 400$, for probit $\alpha = 0$ (kept fixed) and $\beta = -2, -1, 0, 1, 2$. For 1 stopping occasion, $m_1 = 200$, for 3 stopping occasions, different scenarios were considered: (a) 3 'late' stopping occasions with $m_1 = 100$, $m_2 = 200$, $m_3 = 300$; (b) 3 'early' stopping occasions with $m_1 = 50$, $m_2 = 100$, $m_3 = 150$ and $m_1 = 25$, $m_2 = 50$, $m_3 = 75$; (c) 3 'extremely early' stopping occasions with $m_1 = 10$, $m_2 = 20$, $m_3 = 30$; $m_1 = 5$, $m_2 = 10$, $m_3 = 15$ and $m_1 = 2$, $m_2 = 4$, $m_3 = 6$. For the distribution parameters, the following choice was made. For the normal case, $\mu = -2, -1, 0, 1, 2$ and the standard deviation is kept fixed $\sigma = 1$; for the Bernoulli case, $\pi = 0.1, 0.3, 0.5, 0.7, 0.9$ and also some 'extreme' values of π such as $0.001, 0.01$.

For each generated sample, the following statistics were calculated: bias as $\frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta} - \theta)$, relative bias as $\frac{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}-\theta)}{\theta}$, mean square error (MSE) as $\frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta} - \theta)^2$, 95% confidence interval as an average 95% confidence interval over all 1000 generated samples, true coverage probability, and average sample size for all 1000 generated samples. All results are summarized in the accompanying supplementary material.

We focused in this study primarily on the behavior of the bias and the MSE. The simulations conducted confirm the theoretical results for both generic stopping rules in the case of normal target distributions: bias of the sample mean converges to zero with speed $\frac{1}{\sqrt{m_1}}$ and the MSE of the sample mean converges to zero with speed $\frac{1}{m_1}$. In addition, we examined the behavior of the 95% confidence interval and the finite sample size, and we noted that the coverage probabilities in some cases are not 95%. This is compatible with our results, taking into account the second remark accompanying Theorem 19. A deeper study of confidence intervals turns out to be much harder, and will be treated in subsequent work.

10. Concluding remarks

In this paper, we have studied the theory of estimation after a group sequential trial with independent and identically distributed normal outcomes $X_1, X_2, \ldots$ with mean μ and variance σ^2. We have denoted the maximal length of the trial as n, the places at which the interim analyses of the sum of the outcomes are performed as $0 < m_1 < m_2 < \ldots < m_L < n$, and the actual length of the trial, which is a random variable, as N. At each m_i, one decides whether the trial is stopped, i.e. $N = m_i$, or continued, i.e. $N > m_i$. We have based this decision on the generic stopping rule given by (a), (b), and (c) in Section 1, which was shown to contain many classical stopping rules from the literature. Therefore, our setting has a wide range of applicability.

The main goal of this paper is to gain an understanding of the quality of the sample mean $\widehat{\mu}_N = \frac{1}{N} K_N$, where $K_N = \sum_{i=1}^{N} X_i$, as an estimator for μ in the above group sequential trial setting. To this end, we have used the normal transform, defined by (1), as an auxiliary analytic tool to establish the explicit expressions (9) and (17) for the joint density of N and K_N. These expressions were used to obtain formula (23) for the expected length of the trial, formula (24) for the bias, and formula (26) for the MSE. We have derived the upper bound (25) for the bias, which shows that, for fixed L, the bias vanishes as $m_1 \to \infty$ at least with rate $1/\sqrt{m_1}$, and the upper bound (27) for the MSE, entailing that, for fixed L, the MSE vanishes as $m_1 \to \infty$ at least with rate $1/m_1$. Both rates were shown to be optimal. Also, for trials in which an interim analysis is performed after every m observations, we have obtained the bound (28) for the MSE and the bound (29) for the bias. Furthermore, we have obtained the upper bound (31) for the total variation distance between $N(0, 1)$ and the law of the quantity $\frac{\sqrt{N}}{\sigma} (\widehat{\mu}_N - \mu)$, which, under a regularity condition, was shown to vanish if, for fixed L and $m_1, \ldots, m_L$, $n \to \infty$. This has also led to the bound (33), which, in some cases, justifies the use of naive confidence intervals based on $\widehat{\mu}_N$ if n is large. It is quite surprising that, contrary to the upper bounds in (25) for the bias and in (27) for the MSE, the upper bound in (31) and (33) always vanishes if n is large, even if the m_i are small, i.e. if the interim analyses are performed early. Finally, the theory developed in this paper for the sample mean was shown to fit naturally in the broader framework of maximum likelihood estimation after a group sequential trial. More precisely, the marginal MLE coincides with $\widehat{\mu}_N$, and the conditional MLE $\widehat{\mu}_{c,N}$ satisfies equation (50), from which we derived that it coincides with the 'conditional bias reduction estimate'. Our theoretical findings were illustrated by several simulations.

Based on the obtained results, we suggest that in many realistic cases it is safe to use the ordinary sample mean as a reliable estimator after a group sequential trial.

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On the sample mean after a group sequential trial

(Supplementary material)

Table 1: *Simulation results for the normal case with the number of simulated i.i.d. draws per sample 400, the number of simulated samples 1000. Values for standard deviation and α for the probit rule are kept fixed: $\sigma = 1$, $\alpha = 0$. CL: 95% confidence limit, Cov.Prob.: coverage probability.*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
				No Stopping				
-2		0.000040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400
			1 Stopping, $m = 200$, K criterion					
-2		0.00148	-0.00074	0.00475	-2.13705	-1.86000	0.961	200
			1 Stopping, $m = 200$, probit criterion					
-2	-2	0.00148	-0.00074	0.00475	-2.13705	-1.86000	0.961	200
-2	-1	0.00128	-0.00064	0.00471	-2.13624	-1.86121	0.962	205
-2	0	0.00077	-0.00039	0.00361	-2.11833	-1.88012	0.961	296
-2	1	0.00017	-0.00009	0.00238	-2.09861	-1.90104	0.955	396
-2	2	0.00040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400
			3 Stoppings, $m = 100, 200, 300$, K criterion					
-2		-0.00049	0.00024	0.00946	-2.19627	-1.8047	0.955	100
			3 Stoppings, $m = 100, 200, 300$, probit-criterion					
-2	-2	-0.00049	0.00024	0.00946	-2.19627	-1.80470	0.955	100
-2	-1	-0.00037	0.00018	0.00943	-2.19473	-1.80601	0.954	102
-2	0	0.00141	-0.00070	0.00604	-2.15872	-1.83846	0.963	185
-2	1	0.00037	-0.00019	0.00251	-2.10101	-1.89825	0.958	387
-2	2	0.00040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400
				No Stopping				
-1		0.0004	-0.0004	0.00235	-1.09764	-0.90157	0.955	400

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Table 1 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
			1 Stopping, $m = 200$, K criterion					
-1		0.00148	-0.00148	0.00475	-1.13705	-0.86000	0.961	200
			1 Stopping, $m = 200$, probit criterion					
-1	-2	0.00118	-0.00118	0.00469	-1.13621	-0.86142	0.962	205
-1	-1	0.00000	0.00000	0.00433	-1.13168	-0.86831	0.966	234
-1	0	0.00077	-0.00077	0.00361	-1.11833	-0.88012	0.961	296
-1	1	0.00092	-0.00092	0.00267	-1.10346	-0.89471	0.959	369
-1	2	0.00036	-0.00036	0.00239	-1.09851	-0.90076	0.956	396
			3 Stoppings, $m = 100, 200, 300$, K criterion					
-1		-0.00049	0.00049	0.00946	-1.19627	-0.8047	0.955	100
			3 Stoppings, $m = 100, 200, 300$, probit criterion					
-1	-2	-0.00112	0.00112	0.00934	-1.19529	-0.80695	0.955	103
-1	-1	-0.00244	0.00244	0.00841	-1.18774	-0.81715	0.954	120
-1	0	0.00141	-0.00141	0.00604	-1.15872	-0.83846	0.963	185
-1	1	0.00335	-0.00335	0.00371	-1.11750	-0.87580	0.955	314
-1	2	0.00071	-0.00071	0.00253	-1.10069	-0.89789	0.958	387
			No Stopping					
0		0.00040	–	0.00235	-0.09764	0.09843	0.955	400
			1 Stopping, $m = 200$, K criterion					
0		-0.01285	–	0.00348	-0.13064	0.10493	0.961	303
			1 Stopping, $m = 200$, probit criterion					
0	-2	-0.00094	–	0.00347	-0.11946	0.11758	0.964	299
0	-1	0.00011	–	0.00356	-0.11885	0.11908	0.963	297
0	0	0.00077	–	0.00361	-0.11833	0.11988	0.961	296
0	1	0.00196	–	0.00368	-0.11744	0.12136	0.962	295
0	2	0.00294	–	0.00362	-0.11650	0.12238	0.963	294

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Table 1 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
			3 Stoppings, $m = 100, 200, 300$, K criterion					
0		-0.03133	–	0.00596	-0.18477	0.1221	0.958	219
			3 Stoppings, $m = 100, 200, 300$, probit criterion					
0	-2	-0.00470	–	0.00631	-0.16439	0.15500	0.958	186
0	-1	-0.00151	–	0.00617	-0.16167	0.15865	0.959	185
0	0	0.00141	–	0.00604	-0.15872	0.16154	0.963	185
0	1	0.00408	–	0.00616	-0.15596	0.16413	0.960	185
0	2	0.00648	–	0.00606	-0.15330	0.16626	0.965	185
			No Stopping					
1		0.00040	0.00040	0.00235	0.90236	1.09843	0.955	400
			1 Stopping, $m = 200$, K criterion					
1		0.00040	0.00040	0.00235	0.90236	1.09843	0.955	400
			1 Stopping, $m = 200$, probit criterion					
1	-2	0.00009	0.00009	0.00238	0.90130	1.09887	0.955	396
1	-1	-0.00016	-0.00016	0.00268	0.89550	1.10419	0.958	369
1	0	0.00077	0.00077	0.00361	0.88167	1.11988	0.961	296
1	1	0.00128	0.00128	0.00443	0.86926	1.13329	0.966	232
1	2	0.00149	0.00149	0.00473	0.86388	1.13910	0.961	204
			3 Stoppings, $m = 100, 200, 300$, K criterion					
1		0.00040	0.00040	0.00235	0.90236	1.09843	0.955	400
			3 Stoppings, $m = 100, 200, 300$, probit criterion					
1	-2	-0.00057	-0.00057	0.00252	0.89818	1.10067	0.958	387
1	-1	-0.00066	-0.00066	0.00360	0.87856	1.12011	0.963	315
1	0	0.00141	0.00141	0.00604	0.84128	1.16154	0.963	185
1	1	0.00067	0.00067	0.00826	0.81548	1.18585	0.957	120
1	2	0.00036	0.00036	0.00935	0.80628	1.19444	0.955	103
			No Stopping					

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Table 1 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
1 Stopping, $m = 200$, K criterion								
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
1 Stopping, $m = 200$, probit criterion								
2	-2	0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
2	-1	0.00009	0.00004	0.00238	1.90130	2.09887	0.955	396
2	0	0.00077	0.00039	0.00361	1.88167	2.11988	0.961	296
2	1	0.00125	0.00063	0.00470	1.86373	2.13877	0.962	205
2	2	0.00152	0.00076	0.00474	1.86303	2.14001	0.961	200
3 Stoppings, $m = 100, 200, 300$, K criterion								
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
3 Stoppings, $m = 100, 200, 300$, probit criterion								
2	-2	0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
2	-1	-0.00031	-0.00016	0.00250	1.89846	2.10092	0.957	387
2	0	0.00141	0.00070	0.00604	1.84128	2.16154	0.963	185
2	1	0.00009	0.00005	0.00941	1.80584	2.19435	0.954	103
2	2	-0.00049	-0.00024	0.00946	1.80373	2.19530	0.955	100

Table 2: *Simulation results for the normal case with the number of simulated i.i.d. draws per sample 400, the number of simulated samples 1000. Different scenarios for the three stopping occasions. Values for standard deviation and α for the probit rule are kept fixed: $\sigma = 1$, $\alpha = 0$. CL: 95% confidence limit, Cov.Prob.: coverage probability.*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
			3 Stoppings, $m = 50, 100, 150$, K criterion					
-2		-0.00284	0.00142	0.02007	-2.27878	-1.7269	0.939	50
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
-2	-2	-0.00284	0.00142	0.02007	-2.27878	-1.72690	0.939	50
-2	-1	-0.00278	0.00139	0.01988	-2.27687	-1.72869	0.940	51
-2	0	0.00014	-0.00007	0.01324	-2.22014	-1.77959	0.952	118
-2	1	0.00189	-0.00095	0.00303	-2.10414	-1.89208	0.959	379
-2	2	0.00040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400
			3 Stoppings, $m = 50, 100, 150$, K criterion					
-1		-0.00284	0.00284	0.02007	-1.27878	-0.7269	0.939	50
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
-1	-2	-0.00332	0.00332	0.01972	-1.27710	-0.72955	0.939	51
-1	-1	-0.00349	0.00349	0.01824	-1.26446	-0.74251	0.943	62
-1	0	0.00014	-0.00014	0.01324	-1.22014	-0.77959	0.952	118
-1	1	0.00604	-0.00604	0.00658	-1.14064	-0.84728	0.952	276
-1	2	0.00356	-0.00356	0.00323	-1.10266	-0.89023	0.958	378
			3 Stoppings, $m = 50, 100, 150$, K criterion					
0		-0.05706	–	0.01184	-0.26121	0.14709	0.956	171
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
0	-2	-0.01417	–	0.01256	-0.23159	0.20325	0.955	122
0	-1	-0.00827	–	0.01279	-0.22817	0.21164	0.955	119

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0	0	0.00014	–	0.01324	-0.22014	0.22041	0.952	118
0	1	0.00587	–	0.01330	-0.21403	0.22577	0.951	118
0	2	0.01370	–	0.01322	-0.20658	0.23397	0.952	118
			3 Stoppings, $m = 50, 100, 150$, K criterion					
1		0.00040	0.00040	0.00235	0.90236	1.09843	0.955	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
1	-2	-0.00321	-0.00321	0.00346	0.89025	1.10333	0.954	378
1	-1	-0.00534	-0.00534	0.00659	0.84740	1.14192	0.960	275
1	0	0.00014	0.00014	0.01324	0.77986	1.22041	0.952	118
1	1	0.00148	0.00148	0.01783	0.73998	1.26298	0.946	62
1	2	-0.00085	-0.00085	0.01955	0.72588	1.27242	0.940	52
			3 Stoppings, $m = 50, 100, 150$, K criterion					
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
2	-2	0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
2	-1	-0.00106	-0.00053	0.00297	1.89342	2.10445	0.956	380
2	0	0.00014	0.00007	0.01324	1.77986	2.22041	0.952	118
2	1	-0.00175	-0.00087	0.01979	1.72455	2.27195	0.938	51
2	2	-0.00264	-0.00132	0.02002	1.72146	2.27327	0.939	50
			3 Stoppings, $m = 25, 50, 75$, K criterion					
-2		-0.00254	0.00127	0.04125	-2.39018	-1.61489	0.938	25
			3 Stoppings, $m = 25, 50, 75$, probit criterion					
-2	-2	-0.00254	0.00127	0.04125	-2.39018	-1.61489	0.938	25
-2	-1	-0.00441	0.00221	0.04033	-2.38904	-1.61979	0.938	26
-2	0	-0.00207	0.00104	0.02729	-2.30824	-1.69591	0.942	85
-2	1	0.00400	-0.00200	0.00377	-2.10888	-1.88312	0.958	374
-2	2	0.00040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
3 Stoppings, $m = 25, 50, 75$, K criterion								
-1		-0.00254	0.00254	0.04125	-1.39018	-0.61489	0.938	25
3 Stoppings, $m = 25, 50, 75$, probit criterion								
-1	-2	-0.00691	0.00691	0.03961	-1.39115	-0.62266	0.941	26
-1	-1	-0.00679	0.00679	0.03667	-1.37363	-0.63995	0.942	33
-1	0	-0.00207	0.00207	0.02729	-1.30824	-0.69591	0.942	85
-1	1	0.01088	-0.01088	0.01202	-1.17330	-0.80495	0.956	254
-1	2	0.00757	-0.00757	0.00440	-1.10696	-0.87791	0.956	372
3 Stoppings, $m = 25, 50, 75$, K criterion								
0		-0.08579	–	0.02305	-0.36318	0.19161	0.953	143
3 Stoppings, $m = 25, 50, 75$, probit criterion								
0	-2	-0.02926	–	0.02560	-0.33039	0.27187	0.948	90
0	-1	-0.01517	–	0.02600	-0.31956	0.28922	0.947	86
0	0	-0.00207	–	0.02729	-0.30824	0.30409	0.942	85
0	1	0.00996	–	0.02800	-0.29806	0.31799	0.946	82
0	2	0.02320	–	0.02695	-0.28038	0.32679	0.950	87
3 Stoppings, $m = 25, 50, 75$, K criterion								
1		0.00040	0.00040	0.00235	0.90236	1.09843	0.955	400
3 Stoppings, $m = 25, 50, 75$, probit criterion								
1	-2	-0.01063	-0.01063	0.00645	0.87354	1.10521	0.950	371
1	-1	-0.01469	-0.01469	0.01315	0.80079	1.16982	0.953	257
1	0	-0.00207	-0.00207	0.02729	0.69176	1.30409	0.942	85
1	1	0.00496	0.00496	0.03672	0.63731	1.37261	0.942	33
1	2	-0.00027	-0.00027	0.03978	0.61566	1.38380	0.940	26
3 Stoppings, $m = 25, 50, 75$, K criterion								
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
3 Stoppings, $m = 25, 50, 75$, probit criterion								

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
2	-2	0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
2	-1	-0.00356	-0.00178	0.00457	1.88412	2.10876	0.956	375
2	0	-0.00207	-0.00104	0.02729	1.69176	2.30409	0.942	85
2	1	-0.00126	-0.00063	0.04022	1.61407	2.38341	0.940	26
2	2	-0.00254	-0.00127	0.04125	1.60982	2.38511	0.938	25
3 Stoppings, $m = 10, 20, 30$, K criterion								
-2		0.00158	-0.00079	0.10434	-2.5983	-1.39855	0.911	10
3 Stoppings, $m = 10, 20, 30$, probit criterion								
-2	-2	0.00094	-0.00047	0.10323	-2.59878	-1.39934	0.911	10
-2	-1	-0.00185	0.00093	0.10176	-2.59677	-1.40694	0.911	10
-2	0	0.00189	-0.00095	0.06994	-2.46607	-1.53014	0.929	65
-2	1	0.01088	-0.00544	0.00774	-2.11596	-1.86227	0.954	371
-2	2	0.00040	-0.00020	0.00235	-2.09764	-1.90157	0.955	400
3 Stoppings, $m = 10, 20, 30$, K criterion								
-1		0.00011	-0.00011	0.10225	-1.59953	-0.40025	0.912	10
3 Stoppings, $m = 10, 20, 30$, probit criterion								
-1	-2	-0.00858	0.00858	0.09600	-1.59957	-0.41759	0.914	11
-1	-1	-0.01666	0.01666	0.09109	-1.58457	-0.44875	0.918	15
-1	0	0.00189	-0.00189	0.06994	-1.46607	-0.53014	0.929	65
-1	1	0.03379	-0.03379	0.03176	-1.22465	-0.70777	0.947	239
-1	2	0.02406	-0.02406	0.01362	-1.11470	-0.83719	0.950	360
3 Stoppings, $m = 10, 20, 30$, K criterion								
0		-0.14051	–	0.05513	-0.54643	0.26542	0.936	136
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0	-2	-0.06390	–	0.06283	-0.51695	0.38916	0.935	78
0	-1	-0.03391	–	0.06667	-0.49646	0.42864	0.934	69
0	0	0.00189	–	0.06994	-0.46607	0.46986	0.929	65

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0	1	0.03043	–	0.07240	-0.43543	0.49629	0.924	66
0	2	0.06657	–	0.06925	-0.38770	0.52085	0.934	78
3 Stoppings, $m = 10, 20, 30$, K criterion								
1		-0.00057	-0.00057	0.00337	0.90082	1.09804	0.954	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
1	-2	-0.03090	-0.03090	0.01923	0.82879	1.10941	0.938	359
1	-1	-0.03707	-0.03707	0.03385	0.71194	1.21392	0.934	247
1	0	0.00189	0.00189	0.06994	0.53393	1.46986	0.929	65
1	1	0.01775	0.01775	0.09253	0.45157	1.58392	0.922	15
1	2	0.01001	0.01001	0.09775	0.41860	1.60143	0.919	11
3 Stoppings, $m = 10, 20, 30$, K criterion								
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
2	-2	0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
2	-1	-0.01437	-0.00719	0.01043	1.85870	2.11256	0.951	371
2	0	0.00189	0.00095	0.06994	1.53393	2.46986	0.929	65
2	1	0.00666	0.00333	0.09977	1.41215	2.60117	0.916	10
2	2	0.00179	0.00089	0.10434	1.40209	2.60148	0.911	10
3 Stoppings, $m = 5, 10, 15$, K criterion								
-2		0.00499	-0.0025	0.20374	-2.8349	-1.15511	0.884	5
3 Stoppings, $m = 5, 10, 15$, probit criterion								
-2	-2	0.00422	-0.00211	0.20275	-2.83500	-1.15655	0.884	5
-2	-1	-0.00218	0.00109	0.19784	-2.83348	-1.17088	0.890	5
-2	0	0.00014	-0.00007	0.13493	-2.65100	-1.34873	0.903	58
-2	1	0.01727	-0.00863	0.01399	-2.12791	-1.83755	0.952	369
-2	2	0.00137	-0.00068	0.00338	-2.09703	-1.90023	0.954	400
3 Stoppings, $m = 5, 10, 15$, K criterion								

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
-1		-0.00094	0.00094	0.19386	-1.83806	-0.16382	0.884	5
3 Stoppings, $m = 5, 10, 15$, probit criterion								
-1	-2	-0.01799	0.01799	0.17915	-1.84115	-0.19482	0.896	6
-1	-1	-0.03255	0.03255	0.17104	-1.81522	-0.24988	0.896	10
-1	0	0.00014	-0.00014	0.13493	-1.65100	-0.34873	0.903	58
-1	1	0.07150	-0.07150	0.06649	-1.26519	-0.59181	0.921	235
-1	2	0.05333	-0.05333	0.03940	-1.12594	-0.76739	0.936	348
3 Stoppings, $m = 5, 10, 15$, K criterion								
0		-0.19706	–	0.10483	-0.75579	0.36167	0.928	133
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0	-2	-0.11859	–	0.12012	-0.74316	0.50599	0.915	76
0	-1	-0.06873	–	0.12762	-0.71105	0.57359	0.912	64
0	0	0.00014	–	0.13493	-0.65100	0.65127	0.903	58
0	1	0.06384	–	0.13592	-0.57304	0.70072	0.914	65
0	2	0.11084	–	0.12544	-0.50580	0.72748	0.925	78
3 Stoppings, $m = 5, 10, 15$, K criterion								
1		-0.01086	-0.01086	0.01554	0.88354	1.09475	0.946	396
3 Stoppings, $m = 5, 10, 15$, probit criterion								
1	-2	-0.06428	-0.06428	0.04773	0.75278	1.11865	0.929	346
1	-1	-0.07541	-0.07541	0.07071	0.58046	1.26873	0.926	234
1	0	0.00014	0.00014	0.13493	0.34900	1.65127	0.903	58
1	1	0.02523	0.02523	0.17868	0.23621	1.81426	0.892	9
1	2	0.02334	0.02334	0.18438	0.20168	1.84500	0.888	6
3 Stoppings, $m = 5, 10, 15$, K criterion								
2		0.00040	0.00020	0.00235	1.90236	2.09843	0.955	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
2	-2	-0.00037	-0.00018	0.00302	1.90071	2.09855	0.955	400

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
2	-1	-0.02672	-0.01336	0.02218	1.82580	2.12076	0.945	367
2	0	0.00014	0.00007	0.13493	1.34900	2.65127	0.903	58
2	1	0.01010	0.00505	0.19822	1.17678	2.84343	0.887	5
2	2	0.00579	0.00290	0.20214	1.16567	2.84592	0.885	5
3 Stoppings, $m = 2, 4, 6$, K criterion								
-2		-0.00268	0.00134	0.51563	-3.10141	-0.90394	0.687	2
3 Stoppings, $m = 2, 4, 6$, probit criterion								
-2	-2	-0.01054	0.00527	0.49998	-3.11239	-0.90870	0.694	2
-2	-1	-0.02893	0.01446	0.48272	-3.12770	-0.93016	0.704	2
-2	0	0.00501	-0.00250	0.36711	-2.88776	-1.10223	0.778	54
-2	1	0.06251	-0.03126	0.06741	-2.12864	-1.74633	0.932	360
-2	2	0.01476	-0.00738	0.02292	-2.09431	-1.87616	0.949	396
3 Stoppings, $m = 2, 4, 6$, K criterion								
-1		-0.0634	0.0634	0.40771	-2.15826	0.03146	0.730	3
3 Stoppings, $m = 2, 4, 6$, probit criterion								
-1	-2	-0.07022	0.07022	0.42042	-2.15092	0.01048	0.726	3
-1	-1	-0.06700	0.06700	0.42843	-2.11614	-0.01786	0.733	8
-1	0	0.00501	-0.00501	0.36711	-1.88776	-0.10223	0.778	54
-1	1	0.17484	-0.17484	0.19682	-1.32545	-0.32487	0.869	222
-1	2	0.16560	-0.16560	0.17259	-1.13567	-0.53313	0.883	314
3 Stoppings, $m = 2, 4, 6$, K criterion								
0		-0.32619	–	0.27966	-1.07851	0.42612	0.831	122
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0	-2	-0.24247	–	0.30000	-1.04447	0.55953	0.817	94
0	-1	-0.15688	–	0.32610	-0.99637	0.68262	0.799	72
0	0	0.00501	–	0.36711	-0.88776	0.89777	0.778	54
0	1	0.17018	–	0.30724	-0.68867	1.02903	0.809	73

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Table 2 – *Continued from previous page*

μ	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0	2	0.25107	–	0.28894	-0.57418	1.07632	0.824	92
				3 Stoppings, $m = 2, 4, 6$, K criterion				
1		-0.11654	-0.11654	0.17114	0.69811	1.0688	0.902	365
				3 Stoppings, $m = 2, 4, 6$, probit criterion				
1	-2	-0.17063	-0.17063	0.19322	0.53865	1.12009	0.882	313
1	-1	-0.16427	-0.16427	0.21376	0.35156	1.31990	0.878	225
1	0	0.00501	0.00501	0.36711	0.11224	1.89777	0.778	54
1	1	0.07527	0.07527	0.41361	0.02334	2.12721	0.734	10
1	2	0.08223	0.08223	0.38976	0.00611	2.15834	0.732	6
				3 Stoppings, $m = 2, 4, 6$, K criterion				
2		-0.01081	-0.0054	0.0279	1.88572	2.09267	0.951	398
				3 Stoppings, $m = 2, 4, 6$, probit criterion				
2	-2	-0.02488	-0.01244	0.04915	1.86658	2.08366	0.943	394
2	-1	-0.08119	-0.04059	0.09972	1.72882	2.10880	0.916	355
2	0	0.00501	0.00250	0.36711	1.11224	2.89777	0.778	54
2	1	0.02065	0.01032	0.47933	0.92699	3.11431	0.697	2
2	2	0.00802	0.00401	0.49502	0.90976	3.10627	0.691	2

Table 3: *Simulation results for the Bernoulli case with the number of simulated i.i.d. draws per sample 400, the number of simulated samples 1000. Value of α for the probit rule is kept fixed: $\alpha = 0$. CL: 95% confidence limit, Cov.Prob.: coverage probability.*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
				No Stopping				
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400

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Table 3 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
			1 Stopping, $m = 200$, K criterion					
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
			1 Stopping, $m = 200$, probit criterion					
0.001	-2	0.00002	0.02500	0.00000	-0.00079	0.00284	0.257	296
0.001	-1	0.00003	0.02750	0.00000	-0.00079	0.00285	0.257	296
0.001	0	0.00003	0.02750	0.00000	-0.00079	0.00285	0.257	296
0.001	1	0.00004	0.03500	0.00000	-0.00080	0.00287	0.257	296
0.001	2	0.00004	0.03500	0.00000	-0.00080	0.00287	0.257	296
			3 Stoppings, $m = 100, 200, 300$, K criterion					
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
			3 Stoppings, $m = 100, 200, 300$, probit criterion					
0.001	-2	-0.00001	-0.01333	0.00001	-0.00085	0.00283	0.169	185
0.001	-1	-0.00001	-0.01167	0.00001	-0.00086	0.00283	0.169	185
0.001	0	-0.00001	-0.01167	0.00001	-0.00086	0.00283	0.169	185
0.001	1	-0.00001	-0.00667	0.00001	-0.00086	0.00285	0.169	185
0.001	2	-0.00001	-0.00667	0.00001	-0.00086	0.00285	0.169	185
			No Stopping					
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
			1 Stopping, $m = 200$, K criterion					
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
			1 Stopping, $m = 200$, proit criterion					
0.010	-2	-0.00016	-0.01550	0.00004	-0.00087	0.02056	0.875	298
0.010	-1	-0.00014	-0.01375	0.00004	-0.00088	0.02060	0.875	297
0.010	0	-0.00012	-0.01200	0.00004	-0.00089	0.02065	0.876	296
0.010	1	-0.00008	-0.00800	0.00004	-0.00090	0.02074	0.876	295
0.010	2	-0.00007	-0.00725	0.00004	-0.00092	0.02077	0.876	294
			3 Stoppings, $m = 100, 200, 300$, K criterion					

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Table 3 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
		3 Stoppings, $m = 100, 200, 300$, probit criterion						
0.010	-2	-0.00019	-0.01900	0.00007	-0.00308	0.02270	0.733	186
0.010	-1	-0.00018	-0.01825	0.00007	-0.00311	0.02275	0.732	186
0.010	0	-0.00011	-0.01142	0.00007	-0.00311	0.02288	0.731	185
0.010	1	-0.00011	-0.01075	0.00007	-0.00316	0.02295	0.731	183
0.010	2	-0.00008	-0.00783	0.00007	-0.00319	0.02303	0.731	183
		No Stopping						
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
		1 Stopping, $m = 200$, K criterion						
0.100		-0.00038	-0.00380	.00022	0.07037	0.12887	0.950	400
		1 Stopping, $m = 200$, probit criterion						
0.100	-2	-0.00035	-0.00350	0.00030	0.06529	0.13401	0.944	315
0.100	-1	-0.00039	-0.00385	0.00032	0.06474	0.13449	0.941	306
0.100	0	-0.00027	-0.00268	0.00032	0.06424	0.13522	0.941	296
0.100	1	-0.00038	-0.00378	0.00033	0.06368	0.13556	0.939	288
0.100	2	-0.00050	-0.00500	0.00034	0.06327	0.13573	0.938	283
		3 Stoppings, $m = 100, 200, 300$, K criterion						
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
		3 Stoppings, $m = 100, 200, 300$, probit criterion						
0.10	-2	-0.00079	-0.00793	0.00060	0.05446	0.14395	0.935	210
0.10	-1	-0.00034	-0.00336	0.00064	0.05344	0.14589	0.939	196
0.10	0	-0.00005	-0.00053	0.00066	0.05241	0.14749	0.939	185
0.10	1	-0.00003	-0.00029	0.00068	0.05130	0.14864	0.939	174
0.10	2	0.00034	0.00341	0.00071	0.05071	0.14997	0.935	165
		No Stopping						
0.30		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400

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Table 3 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
1 Stopping, $m = 200$, K criterion								
0.30		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
1 Stopping, $m = 200$, probit criterion								
0.30	-2	0.00066	0.00221	0.00069	0.25089	0.35043	0.945	347
0.30	-1	0.00084	0.00278	0.00076	0.24888	0.35279	0.945	323
0.30	0	0.00069	0.00230	0.00080	0.24620	0.35518	0.947	296
0.30	1	0.00038	0.00128	0.00083	0.24408	0.35668	0.947	276
0.30	2	0.00088	0.00294	0.00090	0.24250	0.35927	0.943	254
3 Stoppings, $m = 100, 200, 300$, K criterion								
0.30		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 100, 200, 300$, probit criterion								
0.30	-2	-0.00028	-0.00092	0.00117	0.23847	0.36098	0.935	265
0.30	-1	-0.00008	-0.00027	0.00140	0.23261	0.36722	0.944	222
0.30	0	0.00068	0.00227	0.00161	0.22751	0.37385	0.939	185
0.30	1	0.00108	0.00359	0.00177	0.22372	0.37844	0.939	158
0.30	2	0.00201	0.00668	0.00193	0.22084	0.38317	0.941	138
No Stopping								
0.50		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
1 Stopping, $m = 200$, K criterion								
0.50		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
1 Stopping, $m = 200$, probit criterion								
0.50	-2	0.00056	0.00112	0.00071	0.44841	0.55271	0.944	368
0.50	-1	0.00069	0.00137	0.00078	0.44571	0.55567	0.948	340
0.50	0	0.00055	0.00110	0.00088	0.44111	0.56000	0.954	296
0.50	1	-0.00002	-0.00003	0.00098	0.43719	0.56278	0.955	263
0.50	2	0.00005	0.00009	0.00109	0.43408	0.56601	0.952	231
3 Stoppings, $m = 100, 200, 300$, K criterion								

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Table 3 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.50		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 100, 200, 300$, probit criterion								
0.50	-2	-0.00144	-0.00288	0.00109	0.43808	0.55904	0.944	312
0.50	-1	-0.00132	-0.00264	0.00141	0.42961	0.56775	0.944	251
0.50	0	-0.00031	-0.00063	0.00183	0.41983	0.57954	0.943	185
0.50	1	0.00011	0.00023	0.00217	0.41298	0.58724	0.938	144
0.50	2	-0.00031	-0.00062	0.00240	0.40715	0.59223	0.938	120
No Stopping								
0.70	-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400	
1 Stopping, $m = 200$, K criterion								
0.70		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
1 Stopping, $m = 200$, probit criterion								
0.70	-2	-0.00095	-0.00136	0.00059	0.65269	0.74541	0.946	384
0.70	-1	-0.00116	-0.00165	0.00068	0.64957	0.74812	0.947	353
0.70	0	-0.00069	-0.00099	0.00080	0.64482	0.75380	0.947	296
0.70	1	-0.00033	-0.00047	0.00093	0.64074	0.75859	0.942	248
0.70	2	-0.00017	-0.00025	0.00102	0.63822	0.76143	0.941	219
3 Stoppings, $m = 100, 200, 300$, K criterion								
0.70		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 100, 200, 300$, probit criterion								
0.70	-2	-0.00135	-0.00192	0.00079	0.64838	0.74893	0.945	355
0.70	-1	-0.00168	-0.00240	0.00110	0.63835	0.75828	0.940	277
0.70	0	-0.00068	-0.00097	0.00161	0.62615	0.77249	0.939	185
0.70	1	-0.00061	-0.00087	0.00193	0.61733	0.78145	0.944	133
0.70	2	-0.00070	-0.00100	0.00217	0.61249	0.78612	0.938	110
No Stopping								
0.90		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400

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Table 3 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
1 Stopping, $m = 200$, K criterion								
0.90		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
1 Stopping, $m = 200$, probit criterion								
0.90	-2	0.00027	0.00030	0.00022	0.87067	0.92988	0.950	394
0.90	-1	0.00013	0.00014	0.00025	0.86863	0.93163	0.949	363
0.90	0	0.00027	0.00030	0.00032	0.86478	0.93576	0.941	296
0.90	1	0.00068	0.00076	0.00041	0.86176	0.93960	0.929	237
0.90	2	0.00039	0.00043	0.00046	0.85964	0.94114	0.926	207
3 Stoppings, $m = 100, 200, 300$, K criterion								
0.90		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
3 Stoppings, $m = 100, 200, 300$, probit criterion								
0.90	-2	0.00011	0.00013	0.00025	0.86929	0.93094	0.949	379
0.90	-1	-0.00009	-0.00010	0.00038	0.86284	0.93698	0.947	301
0.90	0	0.00005	0.00006	0.00066	0.85251	0.94759	0.939	185
0.90	1	0.00037	0.00041	0.00083	0.84600	0.95473	0.933	124
0.90	2	0.00049	0.00054	0.00090	0.84333	0.95765	0.932	104

Table 4: *Simulation results for the Bernoulli case with the number of simulated i.i.d. draws per sample 400, the number of simulated samples 1000. Different scenarios for the three stopping occasions. Value of α for the probit rule is kept fixed: $\alpha = 0$. CL: 95% confidence limit, Cov.Prob.: coverage probability.*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
			3 Stoppings, $m = 50, 100, 150$, K criterion					
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
0.001	-2	0.00002	0.01833	0.00001	-0.00091	0.00295	0.109	118
0.001	-1	0.00002	0.01833	0.00001	-0.00091	0.00295	0.109	118
0.001	0	0.00002	0.01833	0.00001	-0.00091	0.00295	0.109	118
0.001	1	0.00002	0.01833	0.00001	-0.00091	0.00295	0.109	118
0.001	2	0.00002	0.01833	0.00001	-0.00091	0.00295	0.109	118
			3 Stoppings, $m = 50, 100, 150$, K criterion					
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
0.010	-2	-0.00010	-0.00958	0.00013	-0.00503	0.02484	0.557	119
0.010	-1	-0.00004	-0.00408	0.00013	-0.00508	0.02500	0.557	118
0.010	0	-0.00003	-0.00342	0.00013	-0.00515	0.02508	0.557	118
0.010	1	0.00004	0.00367	0.00013	-0.00524	0.02531	0.557	117
0.010	2	0.00013	0.01267	0.00014	-0.00530	0.02556	0.557	116
			3 Stoppings, $m = 50, 100, 150$, K criterion					
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
0.100	-2	-0.00091	-0.00906	0.00111	0.03865	0.15953	0.911	142
0.100	-1	0.00002	0.00023	0.00120	0.03713	0.16292	0.905	128
0.100	0	0.00042	0.00417	0.00127	0.03538	0.16546	0.902	118

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.100	1	0.00080	0.00799	0.00134	0.03395	0.16765	0.900	108
0.100	2	0.00132	0.01317	0.00145	0.03256	0.17007	0.897	97
3 Stoppings, $m = 50, 100, 150$, K criterion								
0.300		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 50, 100, 150$, probit criterion								
0.300	-2	-0.00147	-0.00489	0.00212	0.21895	0.37811	0.929	209
0.300	-1	-0.00129	-0.00429	0.00259	0.20843	0.38899	0.936	158
0.300	0	0.00014	0.00047	0.00306	0.19961	0.40067	0.933	118
0.300	1	0.00114	0.00380	0.00341	0.19329	0.40899	0.933	90
0.300	2	0.00099	0.00331	0.00360	0.18751	0.41447	0.934	75
3 Stoppings, $m = 50, 100, 150$, K criterion								
0.500		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 50, 100, 150$, probit criterion								
0.500	-2	-0.00452	-0.00904	0.00192	0.42227	0.56869	0.940	276
0.500	-1	-0.00447	-0.00893	0.00272	0.40515	0.58591	0.937	194
0.500	0	-0.00258	-0.00515	0.00353	0.38753	0.60732	0.933	118
0.500	1	-0.00164	-0.00327	0.00432	0.37636	0.62037	0.926	79
0.500	2	-0.00220	-0.00441	0.00486	0.36781	0.62778	0.925	61
3 Stoppings, $m = 50, 100, 150$, K criterion								
0.700		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 50, 100, 150$, probit criterion								
0.700	-2	-0.00319	-0.00456	0.00110	0.64010	0.75352	0.944	334
0.700	-1	-0.00308	-0.00441	0.00195	0.61998	0.77385	0.937	224
0.700	0	-0.00014	-0.00020	0.00306	0.59933	0.80039	0.933	118
0.700	1	0.00178	0.00254	0.00370	0.58693	0.81662	0.933	70
0.700	2	0.00148	0.00211	0.00407	0.57985	0.82311	0.933	56
3 Stoppings, $m = 50, 100, 150$, K criterion								

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.900		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
			3 Stoppings, $m = 50, 100, 150$, probit criterion					
0.900	-2	-0.00015	-0.00017	0.00030	0.86703	0.93266	0.948	369
0.900	-1	-0.00076	-0.00084	0.00064	0.85332	0.94517	0.940	258
0.900	0	-0.00042	-0.00046	0.00127	0.83454	0.96462	0.902	118
0.900	1	-0.00016	-0.00018	0.00174	0.82424	0.97544	0.878	64
0.900	2	-0.00066	-0.00073	0.00191	0.81949	0.97919	0.872	52
			3 Stoppings, $m = 25, 50, 75$, K criterion					
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
			3 Stoppings, $m = 25, 50, 75$, probit criterion					
0.001	-2	0.00006	0.06083	0.00003	-0.00094	0.00306	0.076	85
0.001	-1	0.00008	0.07833	0.00003	-0.00096	0.00311	0.076	85
0.001	0	0.00008	0.07833	0.00003	-0.00096	0.00311	0.076	85
0.001	1	0.00008	0.07833	0.00003	-0.00096	0.00311	0.076	85
0.001	2	0.00011	0.11167	0.00003	-0.00099	0.00321	0.076	85
			3 Stoppings, $m = 25, 50, 75$, K criterion					
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
			3 Stoppings, $m = 25, 50, 75$, probit criterion					
0.010	-2	-0.00036	-0.03617	0.00026	-0.00592	0.02520	0.379	86
0.010	-1	-0.00022	-0.02200	0.00027	-0.00605	0.02561	0.379	85
0.010	0	-0.00007	-0.00675	0.00028	-0.00615	0.02602	0.379	85
0.010	1	-0.00007	-0.00675	0.00028	-0.00618	0.02604	0.379	85
0.010	2	0.00005	0.00483	0.00029	-0.00633	0.02642	0.379	84
			3 Stoppings, $m = 25, 50, 75$, K criterion					
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
			3 Stoppings, $m = 25, 50, 75$, probit criterion					
0.100	-2	-0.00236	-0.02358	0.00215	0.01869	0.17660	0.915	108

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.100	-1	-0.00098	-0.00976	0.00239	0.01628	0.18177	0.917	95
0.100	0	0.00102	0.01020	0.00258	0.01410	0.18794	0.918	85
0.100	1	0.00223	0.02227	0.00276	0.01243	0.19203	0.917	77
0.100	2	0.00322	0.03220	0.00297	0.00998	0.19646	0.916	64
3 Stoppings, $m = 25, 50, 75$, K criterion								
0.300		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 25, 50, 75$, probit criterion								
0.300	-2	-0.00569	-0.01897	0.00391	0.19044	0.39817	0.941	181
0.300	-1	-0.00287	-0.00957	0.00474	0.17523	0.41903	0.949	125
0.300	0	0.00137	0.00456	0.00582	0.16280	0.43994	0.947	85
0.300	1	0.00272	0.00906	0.00658	0.15221	0.45322	0.948	56
0.300	2	0.00534	0.01781	0.00753	0.14640	0.46429	0.942	42
3 Stoppings, $m = 25, 50, 75$, K criterion								
0.500		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 25, 50, 75$, probit criterion								
0.500	-2	-0.00724	-0.01448	0.00359	0.40141	0.58412	0.942	255
0.500	-1	-0.00596	-0.01193	0.00513	0.37358	0.61449	0.944	164
0.500	0	-0.00163	-0.00325	0.00676	0.34676	0.64999	0.957	85
0.500	1	0.00063	0.00126	0.00843	0.33018	0.67107	0.954	46
0.500	2	0.00054	0.00107	0.00953	0.31942	0.68165	0.952	32
3 Stoppings, $m = 25, 50, 75$, K criterion								
0.700		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 25, 50, 75$, probit criterion								
0.700	-2	-0.00514	-0.00734	0.00175	0.62967	0.76005	0.946	325
0.700	-1	-0.00251	-0.00359	0.00360	0.59786	0.79711	0.940	200
0.700	0	-0.00137	-0.00195	0.00582	0.56006	0.83720	0.947	85
0.700	1	0.00076	0.00109	0.00744	0.54079	0.86074	0.942	39

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.700	2	0.00010	0.00014	0.00833	0.52994	0.87026	0.939	28
			3 Stoppings, $m = 25, 50, 75$, K criterion					
0.900		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
			3 Stoppings, $m = 25, 50, 75$, probit criterion					
0.900	-2	-0.00067	-0.00074	0.00041	0.86389	0.93478	0.948	364
0.900	-1	-0.00090	-0.00100	0.00128	0.84251	0.95570	0.938	239
0.900	0	-0.00102	-0.00113	0.00258	0.81206	0.98590	0.918	85
0.900	1	-0.00002	-0.00003	0.00346	0.79898	1.00098	0.905	34
0.900	2	0.00007	0.00007	0.00383	0.79412	1.00601	0.903	26
			3 Stoppings, $m = 10, 20, 30$, K criterion					
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
			3 Stoppings, $m = 10, 20, 30$, probit criterion					
0.001	-2	0.00010	0.09667	0.00007	-0.00091	0.00310	0.056	65
0.001	-1	0.00019	0.19417	0.00008	-0.00099	0.00338	0.056	65
0.001	0	0.00022	0.22500	0.00008	-0.00102	0.00347	0.056	65
0.001	1	0.00024	0.24167	0.00008	-0.00104	0.00352	0.056	65
0.001	2	0.00024	0.24167	0.00008	-0.00104	0.00352	0.056	65
			3 Stoppings, $m = 10, 20, 30$, K criterion					
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
			3 Stoppings, $m = 10, 20, 30$, probit criterion					
0.010	-2	-0.00107	-0.10683	0.00056	-0.00634	0.02420	0.234	67
0.010	-1	-0.00074	-0.07408	0.00059	-0.00664	0.02516	0.234	66
0.010	0	-0.00053	-0.05308	0.00061	-0.00686	0.02580	0.234	65
0.010	1	-0.00010	-0.00992	0.00067	-0.00714	0.02694	0.234	63
0.010	2	0.00014	0.01358	0.00069	-0.00728	0.02755	0.234	63
			3 Stoppings, $m = 10, 20, 30$, K criterion					
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0.100	-2	-0.00644	-0.06438	0.00460	-0.00696	0.19408	0.763	93
0.100	-1	-0.00254	-0.02539	0.00521	-0.01238	0.20730	0.768	75
0.100	0	0.00142	0.01417	0.00597	-0.01610	0.21894	0.767	65
0.100	1	0.00542	0.05422	0.00666	-0.01876	0.22960	0.767	57
0.100	2	0.00797	0.07967	0.00723	-0.02260	0.23853	0.769	46
3 Stoppings, $m = 10, 20, 30$, K criterion								
0.300		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0.300	-2	-0.01617	-0.05390	0.00949	0.13942	0.42824	0.890	163
0.300	-1	-0.00842	-0.02806	0.01231	0.11491	0.46825	0.887	105
0.300	0	-0.00036	-0.00119	0.01509	0.09422	0.50506	0.884	65
0.300	1	0.00506	0.01687	0.01720	0.07927	0.53085	0.883	35
0.300	2	0.00837	0.02790	0.01885	0.06856	0.54818	0.878	24
3 Stoppings, $m = 10, 20, 30$, K criterion								
0.500		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0.500	-2	-0.01862	-0.03725	0.00892	0.35580	0.60696	0.935	242
0.500	-1	-0.00997	-0.01994	0.01279	0.31347	0.66660	0.932	145
0.500	0	-0.00023	-0.00046	0.01778	0.27093	0.72861	0.917	65
0.500	1	0.00439	0.00878	0.02133	0.24596	0.76283	0.902	27
0.500	2	0.00444	0.00888	0.02342	0.22892	0.77996	0.894	15
3 Stoppings, $m = 10, 20, 30$, K criterion								
0.700		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0.700	-2	-0.01218	-0.01740	0.00455	0.60216	0.77347	0.943	311
0.700	-1	-0.00841	-0.01201	0.00936	0.54889	0.83429	0.917	183

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.700	0	0.00036	0.00051	0.01509	0.49494	0.90578	0.884	65
0.700	1	0.00595	0.00850	0.01902	0.46825	0.94365	0.848	19
0.700	2	0.00223	0.00319	0.01998	0.44891	0.95555	0.846	11
3 Stoppings, $m = 10, 20, 30$, K criterion								
0.900		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
3 Stoppings, $m = 10, 20, 30$, probit criterion								
0.900	-2	-0.00263	-0.00292	0.00099	0.85565	0.93909	0.942	357
0.900	-1	-0.00269	-0.00299	0.00273	0.82267	0.97195	0.879	224
0.900	0	-0.00142	-0.00157	0.00597	0.78106	1.01610	0.767	65
0.900	1	-0.00016	-0.00018	0.00792	0.76663	1.03305	0.681	16
0.900	2	-0.00187	-0.00207	0.00885	0.75822	1.03805	0.657	10
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.001	-2	0.00036	0.36083	0.00015	-0.00112	0.00384	0.051	58
0.001	-1	0.00046	0.46083	0.00018	-0.00118	0.00410	0.051	58
0.001	0	0.00053	0.52500	0.00019	-0.00124	0.00429	0.051	58
0.001	1	0.00056	0.55833	0.00019	-0.00126	0.00438	0.051	58
0.001	2	0.00072	0.72500	0.00023	-0.00138	0.00483	0.051	58
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.010	-2	-0.00133	-0.13267	0.00098	-0.00582	0.02317	0.185	61
0.010	-1	-0.00040	-0.03975	0.00118	-0.00653	0.02573	0.185	59
0.010	0	0.00045	0.04525	0.00134	-0.00722	0.02812	0.185	58
0.010	1	0.00138	0.13842	0.00154	-0.00792	0.03069	0.185	57
0.010	2	0.00201	0.20108	0.00167	-0.00849	0.03251	0.185	55

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.100	-2	-0.01102	-0.11019	0.00829	-0.01726	0.19522	0.590	85
0.100	-1	-0.00372	-0.03716	0.01003	-0.02464	0.21721	0.590	70
0.100	0	0.00370	0.03700	0.01172	-0.03124	0.23864	0.591	58
0.100	1	0.00841	0.08406	0.01284	-0.03569	0.25250	0.592	50
0.100	2	0.01413	0.14126	0.01508	-0.04107	0.26932	0.587	40
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.300		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.300	-2	-0.03043	-0.10143	0.01704	0.09311	0.44603	0.850	154
0.300	-1	-0.01489	-0.04962	0.02187	0.06332	0.50690	0.852	98
0.300	0	0.00108	0.00358	0.02818	0.03722	0.56493	0.845	58
0.300	1	0.01274	0.04246	0.03262	0.01832	0.60715	0.847	30
0.300	2	0.01986	0.06621	0.03594	0.00686	0.63287	0.842	18
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.500		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.500	-2	-0.03272	-0.06544	0.01516	0.30543	0.62913	0.937	236
0.500	-1	-0.02032	-0.04063	0.02252	0.24597	0.71340	0.926	135
0.500	0	-0.00118	-0.00236	0.03350	0.19676	0.80088	0.922	58
0.500	1	0.01307	0.02614	0.03974	0.17034	0.85580	0.934	19
0.500	2	0.01661	0.03321	0.04331	0.15396	0.87925	0.940	9
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.700		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.700	-2	-0.02273	-0.03247	0.00991	0.56915	0.78539	0.929	305
0.700	-1	-0.01782	-0.02546	0.01795	0.49359	0.87076	0.897	175
0.700	0	-0.00107	-0.00154	0.02818	0.43507	0.96278	0.845	58
0.700	1	0.00747	0.01067	0.03536	0.41027	1.00466	0.810	15
0.700	2	0.00779	0.01113	0.03871	0.39544	1.02014	0.799	6
3 Stoppings, $m = 5, 10, 15$, K criterion								
0.900		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
3 Stoppings, $m = 5, 10, 15$, probit criterion								
0.900	-2	-0.00509	-0.00566	0.00193	0.84798	0.94184	0.931	359
0.900	-1	-0.00977	-0.01086	0.00622	0.79625	0.98420	0.812	216
0.900	0	-0.00370	-0.00411	0.01172	0.76136	1.03124	0.591	58
0.900	1	-0.00048	-0.00054	0.01465	0.75066	1.04837	0.473	10
0.900	2	-0.00050	-0.00056	0.01632	0.74697	1.05203	0.430	5
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.001		0.00004	0.04000	0.00000	-0.00076	0.00284	0.335	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.001	-2	0.00006	0.06167	0.00034	-0.00059	0.00272	0.046	56
0.001	-1	0.00072	0.72333	0.00062	-0.00091	0.00436	0.046	55
0.001	0	0.00138	1.37500	0.00077	-0.00139	0.00614	0.046	54
0.001	1	0.00138	1.37500	0.00077	-0.00139	0.00614	0.046	54
0.001	2	0.00162	1.62500	0.00095	-0.00141	0.00666	0.046	54
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.010		-0.00009	-0.00925	0.00003	0.00060	0.01922	0.906	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.010	-2	-0.00500	-0.49992	0.00093	-0.00236	0.01236	0.142	59
0.010	-1	-0.00214	-0.21400	0.00238	-0.00343	0.01915	0.142	57
0.010	0	0.00067	0.06692	0.00366	-0.00473	0.02607	0.143	54

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.010	1	0.00272	0.27175	0.00478	-0.00539	0.03083	0.143	53
0.010	2	0.00330	0.33008	0.00518	-0.00547	0.03208	0.143	53
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.100		-0.00038	-0.00380	0.00022	0.07037	0.12887	0.950	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.100	-2	-0.03106	-0.31057	0.01499	-0.00712	0.14500	0.365	90
0.100	-1	-0.01085	-0.10845	0.02380	-0.01570	0.19401	0.364	69
0.100	0	0.00608	0.06083	0.03219	-0.02219	0.23435	0.362	54
0.100	1	0.01843	0.18427	0.04088	-0.01967	0.25652	0.354	47
0.100	2	0.02697	0.26972	0.04452	-0.02527	0.27921	0.357	42
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.300		0.00058	0.00192	0.00054	0.25571	0.34544	0.946	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.300	-2	-0.07020	-0.23401	0.03773	0.07350	0.38610	0.645	154
0.300	-1	-0.03432	-0.11439	0.05279	0.04818	0.48318	0.634	99
0.300	0	0.00908	0.03025	0.07342	0.03309	0.58506	0.601	54
0.300	1	0.03216	0.10719	0.08456	0.02594	0.63837	0.589	31
0.300	2	0.04043	0.13477	0.08908	0.01169	0.66917	0.588	19
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.500		0.00089	0.00177	0.00063	0.45195	0.54983	0.943	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.500	-2	-0.08033	-0.16066	0.04400	0.25455	0.58479	0.801	233
0.500	-1	-0.04041	-0.08082	0.06181	0.19978	0.71940	0.748	130
0.500	0	0.00142	0.00284	0.08358	0.17317	0.82967	0.684	54
0.500	1	0.02771	0.05541	0.09759	0.17129	0.88412	0.631	18
0.500	2	0.03284	0.06567	0.10155	0.15880	0.90687	0.609	8
3 Stoppings, $m = 2, 4, 6$, K criterion								

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Table 4 – *Continued from previous page*

π	β	Bias	Relative Bias	MSE	Lower CL	Upper CL	Cov.Prob.	Aver.Size
0.700		-0.00057	-0.00082	0.00054	0.65456	0.74429	0.946	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.700	-2	-0.06777	-0.09682	0.03800	0.49359	0.77086	0.874	289
0.700	-1	-0.05260	-0.07515	0.05167	0.43196	0.86283	0.784	177
0.700	0	-0.00907	-0.01296	0.07342	0.41494	0.96691	0.601	54
0.700	1	0.01732	0.02474	0.08664	0.43616	0.99848	0.471	11
0.700	2	0.02245	0.03207	0.08951	0.43130	1.01360	0.441	3
3 Stoppings, $m = 2, 4, 6$, K criterion								
0.900		0.00038	0.00042	0.00022	0.87113	0.92963	0.950	400
3 Stoppings, $m = 2, 4, 6$, probit criterion								
0.900	-2	-0.02239	-0.02487	0.01270	0.81194	0.94329	0.913	350
0.900	-1	-0.02509	-0.02788	0.02241	0.76524	0.98457	0.703	216
0.900	0	-0.00608	-0.00676	0.03219	0.76565	1.02219	0.362	54
0.900	1	0.00709	0.00787	0.03638	0.78855	1.02562	0.217	8
0.900	2	0.00467	0.00519	0.03988	0.78210	1.02724	0.192	2