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Fuzzy Set Theory :
An Overview of the Basic Principles

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Abstract

In this paper an overview is presented of the basic principles of fuzzy set theory. The fuzzy set theory was introduced by Zadeh (1965) to deal with problems in which the absence of sharply defined criteria is involved. In particular, fuzzy sets aim at mathematically representing the vagueness intrinsic in linguistic terms and approximate reasoning.

The aim is to provide the reader with a better understanding of the fundamental concepts underlying the theory of fuzzy sets and reasoning. In particular, this contribution focuses on the following aspects:

(i) relationship between crisp and fuzzy sets;
(ii) fuzzy membership functions;
(iii) relationship between probability, stochastic preference and membership function;
(iv) estimation of membership functions;
(v) fuzzy set operators;
(vi) fuzzy set hedges.

Introduction

The fuzzy set theory was introduced by Zadeh[1] (1965) to deal with problems in which the absence of sharply defined criteria is involved. In particular, fuzzy sets aim at mathematically representing the vagueness intrinsic in linguistic terms and approximate reasoning.

To illustrate the vagueness of language, take, for instance, a simple statement like "John is tall". Such a statement is abound with vague and imprecise concepts that are difficult to translate into more precise language without losing some of its semantic value. For example, the statement "John's height is 175 cm." does not explicitly state whether he is tall, and the statement "John's height is 1.2 standard deviations about the mean height for men of his age in his culture" is fraught with difficulties, and also demands further information (What is John's age? Which culture does he belong to?). And what if we would state that 190 cm. is tall, would 189 cm. not be considered tall? It is precisely such vagueness that could be an obstacle when one wants to represent knowledge in an expert system.

Note that if the aim is to design an expert system one of the major tasks is to codify the manager's decision-making process. The designer will soon learn that the manager's view of the world, despite the dependence upon numerous, empirical tests and case-studies, incorporates evaluations of facts, and relationships between then, in a "fuzzy", intuitive manner. While some of the decisions and considerations are based on objective criteria, allowing for a fuzzy system could broaden the field of decision-making tremendously.

Through the use of the fuzzy set theory, ill-defined and imprecise knowledge and concepts can be treated in an exact mathematical way (Tzafestas, 1994). It is important to note, however, that fuzziness has nothing to do with ambiguity, nor does it stem from partial or total ignorance. Fuzziness deals with the natural imprecision associated with everyday events (Cox, 1994, p. 589).

Zadeh (1965, p. 338) defined a fuzzy set as "a class of objects with a continuum of grades of membership". This fuzzy set is characterized by a *membership function* (also called truth or indicator function) which assigns to each object of the set a grade of membership ranging from zero (non-membership of the set) to one (full-membership of the set).

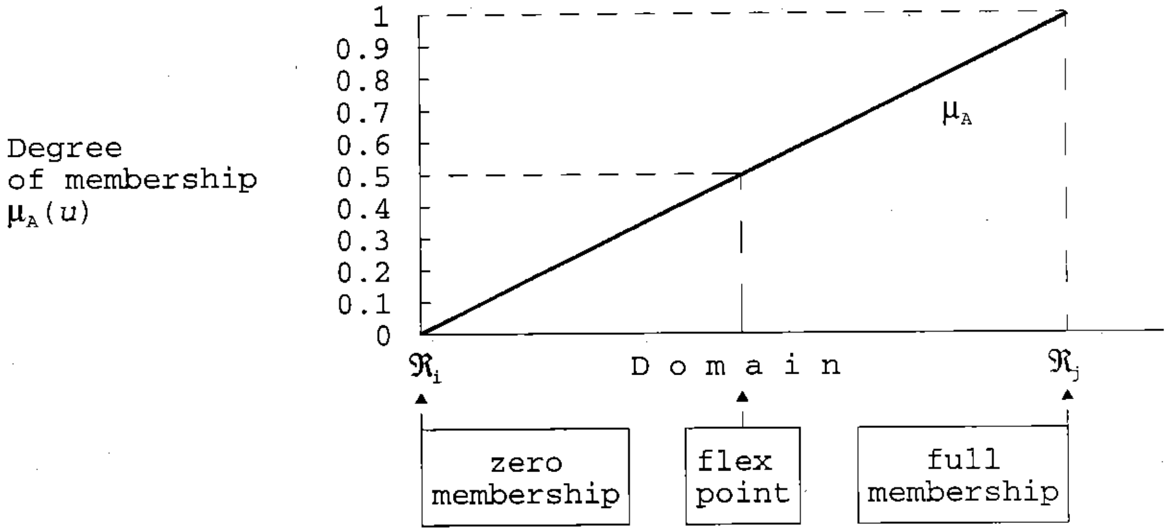
More formally, a fuzzy set is defined as follows (Zadeh, 1965, p. 339; 1975a, p. 219):

[1] For an interesting selection of papers written by Zadeh, see, e.g., Yager *et al.* (1987). For some book-length treatments on the subject of fuzzy set theory, see, e.g., Dubois and Prade (1980a), Zimmermann (1985, 1987, 1991), Kandel (1986), Lowen (1986), Klir and Folger (1988), Kerre (1993), Cox (1994) and Kosko (1994).

Definition 1: Fuzzy set
In a universe of discourse U, a fuzzy set A is given by a membership function $\mu_A: U \rightarrow [0,1]$.■

Following the above definition, a membership function associates with each element u of U a real number $\mu_A(u)$ in the interval [0,1], with $\mu_A(u)$ representing the grade of membership of u in A. Hence, $\mu_A(u)=0$ implies non-membership, $\mu_A(u)=1$ means full-membership, and $0<\mu_A(u)<1$ signifies partial membership or intermediate degrees of membership. Obviously, the nearer the value of $\mu_A(u)$ to zero (unity), the lower (higher) the grade of membership of u in A. Also, gradual transition follows from partial degree of membership of the set. The general structure of a linear increasing fuzzy set is shown in Figure 1.

Figure 1: The general structure of a linear increasing fuzzy set



Source: Cox, 1994, p. 32.

Figure 1 shows the general structure of a linear increasing fuzzy set. Three components may be distinguished: (i) a horizontal *domain* axis of monotonically increasing real numbers that make up the population of the fuzzy set (i.e., $[\Re_i,\Re_j]$ in the present context), (ii) a vertical *membership* axis ranging from zero to one indicating the degree of membership in the fuzzy set (i.e., $\mu_A(u)$), and (iii) the *surface* (or shape) of the fuzzy set itself (i.e., μ_A) that connects an element in the domain with a degree of membership in the set. In Figure 1, μ_A is a simple linear increasing curve that starts at a domain value, $\Re_i$ that has zero membership in the set and moves to the right with values that have increasing set membership. The right-hand edge of the domain, $\Re_j$ is the value with full membership. The value in the domain that has a (0.5) degree of membership is called the *flex* point (or cross-over point).

In what follows, the basic principles of the fuzzy set theory are discussed. The aim of this overview is to provide the reader with a better understanding of the fundamental concepts underlying the theory of fuzzy sets and reasoning. In particular, this contribution focuses on the following aspects:

(i) relationship between crisp and fuzzy sets;
(ii) fuzzy membership functions;
(iii) relationship between probability, stochastic preference and membership function;
(iv) estimation of membership functions;
(v) fuzzy set operators;
(vi) fuzzy set hedges.

1 Crisp sets versus fuzzy sets

At the heart of the difference between classical standard set theory and fuzzy set theory is what Aristotle called the "Law of the Excluded Middle". This law states that every proposition must either be true (not false) or false (not true). Stated in terms of classical set theory, it implies that the union of a set with its complement results in the universal set of the underlying domain. This property may be compared to the requirement of exhaustivity.

In standard set theory, an object is either a member of a set or it is not. There is no in between, no middle, no partial containment: for instance, the number 5 belongs fully to the set of odd numbers and not at all to the set of even numbers. Usually, the term *crisp* set is applied to these classical sets where membership is either equal to one (totally contained in the set) or equal to zero (totally excluded from the set). To put it succinctly, crisp sets are based on the so-called "principle of dichotomy", and avoid the contradiction that an object both is and is not a member of a set. As such, crisp sets comply with the Law of the Excluded Middle.

More formally, alterations between set inclusion or exclusion in a crisp set environment are defined as follows:

Definition 2: Crisp sets

In a universe of discourse U, a crisp set A is given by a membership function $\chi_A: U \rightarrow \{0,1\}$.■

Following Definition 2, a membership (or characteristic) function associates with each element u of U a binary number $\chi_A(u)$, with $\chi_A(u) = 1$ if $u \in A$ implying set inclusion and $\chi_A(u) = 0$ if $u \notin A$ implying set exclusion. Note that with crisp sets membership is exactly defined. Since there are only two different state values, the alteration between these states is always immediate because partial or gradual memberships are not allowed.

To illustrate the implications of this characteristic, take the following brief example. Suppose, a firm wants to evaluate a potential location site in respect to the supply of raw materials. The supply of raw materials may be judged "good", "medium" or "bad" depending on the outcome of only two conditions; C_1: distance (in meters) to a harbour; and, C_2: distance (in meters) to a goods station. The basic assumption being that a location site near a harbour is easy to supply, while in all other cases, the supply possibility depends on the proximity of a goods station. For both conditions, C_1 and C_2, the following condition states are specified:

C_1 : distance (X in meters) to a harbour

$$CS_{11} : X < 1000 \text{ or } X \in [0, 1000)$$

$$CS_{12} : X \geq 1000 \text{ or } X \in [1000, +\infty)$$

C_2 : distance (X in meters) to a goods station

$$CS_{21} : X < 250 \text{ or } X \in [0, 250)$$

$$CS_{22} : 250 \leq X \leq 450 \text{ or } X \in [250, 450]$$

$$CS_{23} : X > 450 \text{ or } X \in (450, +\infty)$$

Assume further, for the sake of simplicity, that condition state CS_{11} (X<1000) reflects the set of "short" distances (denoted S) and condition state CS_{12}

($X \geq 1000$) the set of "long" distances (denoted L). Then, the crisp set for the concept "*short* distance to harbour" is defined as follows:

$$\text{Short} = \{X \in U \mid CS_{11} = \text{true}\} \qquad [1]$$

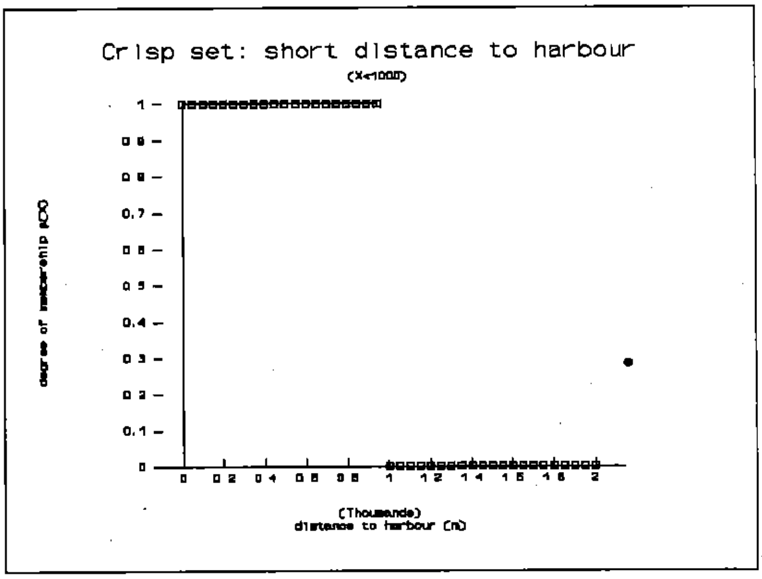
meaning that any distance *X* less than 1000 m. is a full member of the short distance set, while any distance greater than or equal to 1000 m. has a zero degree of membership to the short distance set. Analogous, the crisp set for the concept "*long* distance to harbour" is defined as follows:

$$\text{Long} = \{X \in U \mid CS_{12} = \text{true}\} \qquad [2]$$

meaning that any distance greater than or equal to 1000 m. is a full member of the long distance set, while any distance less than 1000 m. has a zero degree of membership to the long distance set. The crisp characteristic functions of both concepts are shown in Figure 2 (a) and (b).

Figure 2: The crisp sets for the concept "distance to harbour"

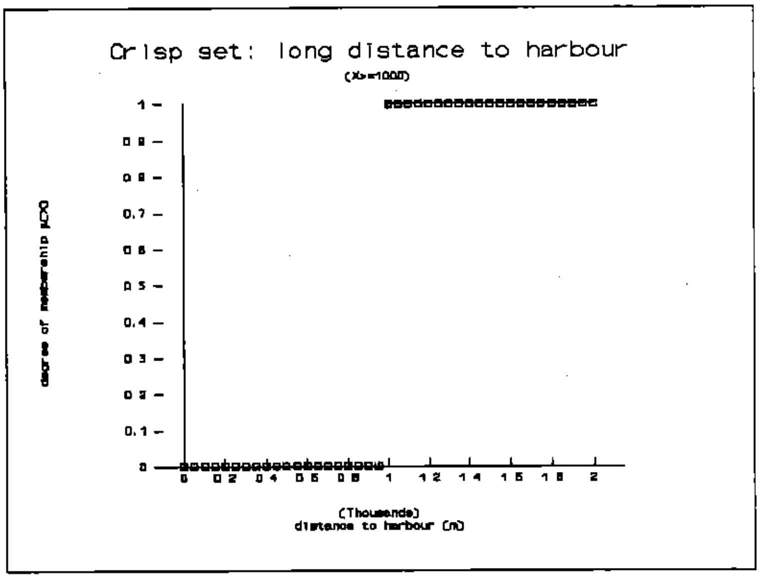
(a)



The characteristic (membership) function for the concept "*short* distance to harbour" is defined as follows:

$$\chi_{CS11} : \mathbb{R} \rightarrow [0,1], X \mapsto \begin{cases} 1 \text{ if } X < 1000, \\ 0 \text{ if } X \geq 1000 \end{cases} \quad [3]$$

(b)



The characteristic (membership) function for the concept "*long* distance to harbour" is defined as follows:

$$\chi_{CS12} : \mathbb{R} \rightarrow [0,1], X \mapsto \begin{cases} 1 \text{ if } X \geq 1000, \\ 0 \text{ if } X < 1000 \end{cases} \quad [4]$$

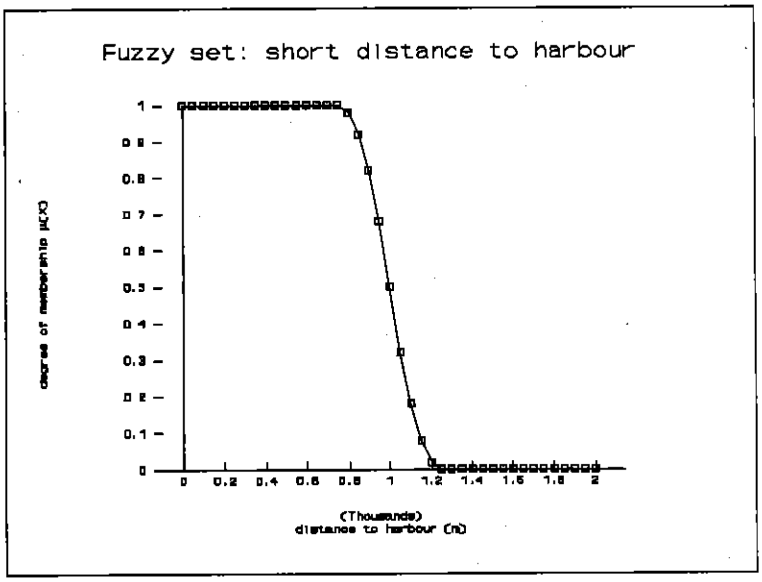
It is apparent that the characteristic functions of both crisp sets depicted in Figure 2 reflect their Boolean nature. As one moves along the distance domain, the membership of distances in the "short" ("long") set equals full (zero) membership until it jumps immediately to zero (full) membership when a distance of 1000 m. is reached. This property supports the fact that crisp sets do not allow partial or gradual condition state transitions.

Whether the membership function χ_A or μ_A can only take the values 0 and 1, or any value between 0 and 1 indicates the difference between a crisp and fuzzy representation. The former definition refers to a crisp representation: χ_A: $U \rightarrow \{0,1\}$, while the latter points to a fuzzy representation: μ_A: $U \rightarrow [0,1]$. Note that the fuzzy representation is in fact a generalization of the crisp representation. This is because a characteristic function can be extended from the binary lattice range {0,1} to the continuous range [0,1] (i.e. the so-called extension principle, see further). It is evident that in contrast to crisp sets, fuzzy sets do allow partial or gradual set memberships. The degree of set membership may equal any value in the interval [0,1]. This property is also reflected in the shape of the membership function. While the values in the domain of a fuzzy set always increase as one goes from left to right, the degree of membership follows from the surface (or shape) of the membership function of the fuzzy set. Hence, the choice and generation of the type of membership function is an important subject in fuzzy set theory.

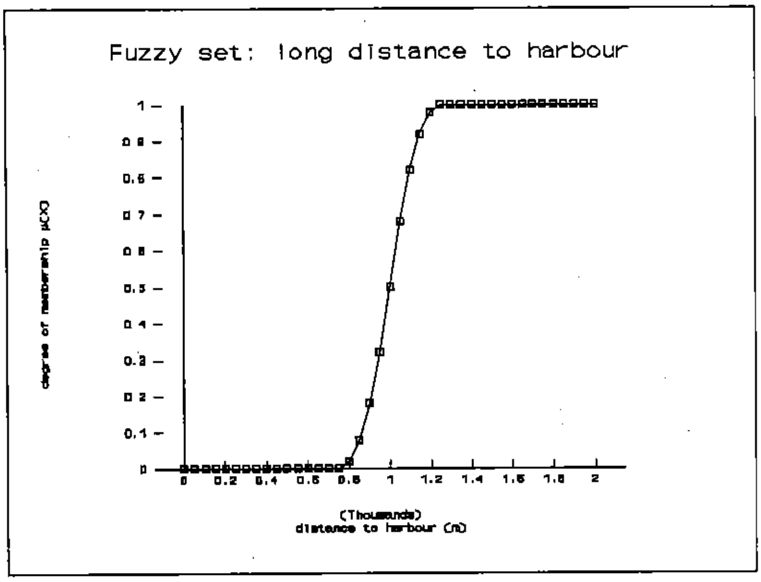
In respect to representing the concepts "short" and "long" distance, the associated fuzzy membership functions may look something like the curves depicted in Figures 3 (a) and (b).

Figure 3: The fuzzy sets for the concept "distance to harbour"

(a)



(b)



It is apparent from the membership function specifications that items or objects (in the present context different distances) may belong (i.e. be a member of) only partially to a fuzzy set, and may also belong to more than one set at the same time. For instance, in Figure 5.3 (a) and (b), a distance of 1100 m. has a 0.18 membership value in the fuzzy set of "short" distances to a harbour and a 0.82 membership value in the fuzzy set of "long" distances to a harbour. In a crisp representation, however, 1100 m. would result in a zero membership in the set of "short" distances, and a full membership in the set of "long" distances.

The fact that an item can belong to more than one set at the same time implies that the Law of the Excluded Middle no longer holds (Laviolette *et al.*, 1995, p. 250). To illustrate the consequences of this fact, take the following ancient Greek riddle: "The liar from Crete asserts that all Cretans lie". The question now is: "Is the Cretan telling the truth or is he lying?". Note that if he lies, then he actually tells the truth and does not lie. And if he does not lie, then he tells the truth and so lies. Apparently, both cases lead to a contradiction as the Cretan seemed to lie and not lie at the same time. Clearly, faced with such a conundrum, classical logic, which states that any statement is either true or false, surrenders. In contrast, the answer in fuzzy logic is that the statement is viewed as being half true and half false. The Cretan lies fifty percent of the time and does not lie the other half (Kosko, 1994, p. 6). Such a statement implies that the Law of the Excluded Middle is violated.

Kosko and Isaka (1993, pp. 62-63) argue that the Law of the Excluded Middle only holds merely as a special case in fuzzy logic; namely, when an object or item belongs 100 percent to one group. Consequently, the only constraint on fuzzy logic is that an object's degrees of membership in complementary groups must sum to unity. Imposing this property to account for the Law of the Excluded Middle seems a rather straigthforward approach. However, in fuzzy set theory, this is by no means the case (see below). In our example, 1100 m. has a 0.18 degree membership in the fuzzy set "short" distance, and a 0.81 (1-0.18) degree membership in the fuzzy set "long" distance. Hence, the Law of the Excluded Middle holds: $\boldsymbol{\mu}_{short}(X) = (1-\boldsymbol{\mu}_{long}(X))$ and $\boldsymbol{\mu}_{long}(X) = (1-\boldsymbol{\mu}_{short}(X))$ for all $X \in U$. Stated differently, the sum of all the supports of the membership function values of the fuzzy sets involved in all fuzzy condition states should be equal to one.

Note that when all fuzzy membership grades of all elements of the space are restricted to the traditional set {0,1}, the result is again the classical, two-value crisp set. This characteristic is known as the "extension principle" (Zadeh, 1975a, p. 236). It effectively establishes that fuzzy sets are a true generalization of classical set theory. In fact, by this reasoning all crisp sets are fuzzy sets of that very special type; and there is no conflict between both methods.

2 Fuzzy membership functions

A fuzzy set is characterized by a membership function which assigns to each object of the set a degree of membership ranging from zero (non-membership of the set) to one (full-membership of the set).

Classifying membership functions is by no means an easy task. This is partly explained by the fact that the choice of a membership function is (i) context-dependent (i.e., devised for a specific, individual problem), and (ii) for a same context, dependent on the observer (different observers have different opinions).

It is not our intension here to make a complete literature coverage on membership functions. This would lead us too far, and moreover, is done elsewhere (see, e.g., Dubois and Prade, 1980a; Kandel, 1986; Dombi, 1990; Turksen, 1991; Zimmermann, 1991; Kerre, 1993; Cox, 1994; and Tzafestas, 1994). However, an effort is made to include in this paragraph the most representative standard representations of membership functions that have been advanced to represent fuzzy concepts.

In Table 1, three different classes of concepts have been distinguished: (i) increasing concepts, (ii) decreasing concepts, and (iii) the so-called "fuzzy numbers". Subject to the kind of concept that needs to be represented, potential standard membership functions are given that could be applied to represent a particular fuzzy notion.

Table 1: Summary of standard applied membership functions

kind of concept being represented	means of representing the concept	examples of standard applied fuzzy membership functions (over the fuzzy domain)
increasing concept	growth curve	* increasing linear line * sigmoid curve (*S*-curve)
decreasing concept	decline curve	* decreasing linear line * logistic curve (*L*-curve)
fuzzy number and "around" or "close to" representation	bell-shaped curve	* π-curve * Beta-curve * Gaussian-curve
	triangular curve	* triangular fuzzy set * shouldered fuzzy set * trapezoidal fuzzy set

Membership functions belonging to the first two classes of representations are frequently used to model concepts that have an intrinsic growth or decline surface. Examples of increasing concepts are 'long', 'old', 'rich', 'tall', 'heavy', etc. These concepts are best represented by a growth curve. The basic idea of a growth curve is that for a fuzzy set like 'distance', the degree of membership in the set of the concept 'long' increases as the domain values of distance augment. This follows from the logical assumption that longness is proportional to distance. It is also typical of a growth curve that it moves from no membership at its extreme left-hand side to full membership at its extreme right-hand side. In contrast, decreasing notions such as 'short', 'young', 'poor', 'low', 'light', etc. are best represented by the complement of a growth curve; that is, a decline curve. A decline curve moves from complete membership at its extreme left-hand side to zero membership at its extreme right-hand side. The growth and decline curves have in common that they are both pivoted around their inflexion point (i.e., the 0.5 membership point). Let us take a closer look at some examples of standard applied fuzzy membership functions that represent the growth and decline surfaces.

A **growth curve** may be represented by an increasing linear line or, more advanced, a sigmoid or *S*-curve. The case of the increasing linear line is rather straightforward (cf. Figure 1). An increasing linear representation assumes a direct increasing proportionality between the elements of the domain and the degree of membership of the set. Due to this specific property, linear representations fit very well the requirements of an increasing notion. However, the property of direct proportionality is sometimes a too rigid assumption to represent a concept. Therefore, as a rule, a linear fuzzy set is a good first choice when approximating an unknown or poorly understood concept that is not a fuzzy number (Cox, 1994, p. 47). In a later stage of the research, some fine tuning is needed.

A more advanced representation is a non-linear function such as the quadratic monotonic *S*-curve. An *S*-curve can be defined in terms of three parameters: the zero membership value (α), the complete membership value (γ), and the inflexion point (β). The *S*-membership function is specified as follows (Zadeh, 1975d, p. 29; Sanchez, 1986, p. 337; Hellendoorn, 1990, p. 18):

$$S\,(\cdot;\alpha,\beta,\gamma) : \mathbb{R} \rightarrow [0,1],\ u \mapsto \begin{cases} 0 & \text{if } u \leq \alpha \\ 2((u-\alpha)/(\gamma-\alpha))^2 & \text{if } \alpha < u \leq \beta \\ 1-2((u-\gamma)/(\gamma-\alpha))^2 & \text{if } \beta < u \leq \gamma \\ 1 & \text{if } u > \gamma \end{cases} \qquad [5]$$

It follows from specification [5] that only within the domain of $[\alpha,\gamma]$ partial degrees of membership are possible. This is because all values of u lower than α imply zero membership, whereas all values of u beyond γ imply full membership. The inflexion point, $\beta = (\alpha+\gamma)/2$ is the point with degree of membership equal to 0. By determining the values of α and γ for a certain concept, the S-curve is established.

A **decline curve** may be characterized by a decreasing linear line or, also more advanced, a logistic or L-curve. The L-curve is also a non-linear function. In fact, an L-curve is complemententary to an S-curve. The L-membership function is given by:

$$L\ (\cdot;\alpha,\beta,\gamma) = 1 - S\ (\cdot;\alpha,\beta,\gamma). \qquad [6]$$

It is clear that the presence of the parameters α, β and γ makes it possible to adapt the non-linear membership function to altering conditions and concepts without essentially changing the curve's general shape. The S- and L-curves allow for more flexibility and are, therefore, more commonly used in an applied context than, for instance, simple linear representations.

The S- and L-shaped membership function curves can also be defined in terms of a logistic function. Zysno (1981, p. 353), Zimmermann and Zysno (1985, p. 153) and Turksen (1991, p. 32) suggest:

$$S\ (\boldsymbol{\mu}_A(u)) = [1 + \exp\ [-a(u-b)]]^{-1} \qquad [7]$$

and

$$L\ (\boldsymbol{\mu}_A(u)) = [1 + \exp\ [a(u-b)]]^{-1} \qquad [8]$$

with a determining the slope of the curve and b determining the inflexion point.

A third class of membership functions deals with the so-called fuzzy numbers and the "about" or "close to" representations. Typical of this class of functions is that they represent the approximations of a central value (or plateau if truncated). Usually, fuzzy numbers are graphically visualized through the use of a **bell-shaped curve** with the most probable value for the function at the center of the bell. To illustrate, the concept "approximately 1000" or "close to

1000" is a fuzzy number and could be represented with a bell-shaped fuzzy set that is spread around the central value of 1000.

There are at least three important classes of bell-shaped curves. These are the π-, the Beta-, and the Gaussian-curve (Kerre, 1993, p. 23; Cox, 1994, p. 61). The difference between the three classes of fuzzy sets has to do with the curve's width and slope as well as the curve values at the endpoints. The width and slope indicate the degree of spreading associated with the fuzzy number.

The most preferred of the three bell-shaped representations is the *π-curve*. This is because the π-curve provides a smooth descent gradient from the central value (i.e., the concept being approximated) to a zero membership point somewhere along the domain. The fact that the endpoints of a π-curve are always bounded at a discrete and specific point, makes the function not asymptotic and most useful to approximate any fuzzy number.

The symmetric, unimodal π-membership function results from a continuous linking of an *S*-membership function and its reflected image (i.e., the *L*-curve). The π-membership function depends on two parameters: the value from the domain around which the function is centered (γ), and the bandwidth parameter (β). The bandwidth parameter is twice the distance between the inflexion point and the centered value. More formally, the π-membership function for $u \in \mathbb{R}$ is specified as follows (Hellendoorn, 1990, p. 18):

$$\Pi\,(\cdot;\beta,\gamma) : \mathbb{R} \rightarrow [0,1],\; u \mapsto \begin{cases} S\,(u;\gamma-\beta,\gamma-\beta/2,\gamma) & \text{if } u \leq \gamma \\ 1-S\,(u;\gamma,\gamma+\beta/2,\gamma+\beta) & \text{if } u \geq \gamma \end{cases} \qquad [9]$$

Note, that in the above equation of the π-curve the inflexion points ($\gamma\pm\beta/2$) are automatically defined. Also, the endpoints are determined from the specification of the π-curve's parameters as ($\gamma-\beta$) on the left-hand side and ($\gamma+\beta$) on the right-hand side. The width of the curve depends on the value of β. This is often calculated as a percentage of the fuzzy number itself (Cox, 1994, p. 67).

The *Beta-curve* differs from the π-curve in that it is a bell-shaped function without compact support and, more important, it has no bounded endpoints (asymptotic function). Like the π-curve, the Beta-curve is defined using only two parameters: the center value (γ) and the half-bandwidth parameter (β). The Beta-membership function is defined as follows:

$$\text{Beta}\ (\cdot;\beta,\gamma) : \mathbb{R} \rightarrow [0,1],\ u \mapsto \frac{1}{1 + (\frac{u-\gamma}{\beta})^2} \qquad [10]$$

Note, that the point with zero degree of membership is only attained at extremely large values for beta (β→+∞). Also, it is the half-bandwidth parameter β which plays a crucial role in the general shape of the Beta fuzzy set. The larger the β-value, the wider is the curve and, conversely, the smaller the β-value, the narrower the curve.

A third, though less popular bell-shaped curve for representing a fuzzy number is the *Gaussian-curve*. Like the Beta-curve, the Gaussian-curve is an unbounded function which is defined in terms of only two parameters: the center value (γ) and the width parameter (κ). The *G*(aussian)-membership function is given as follows:

$$G\ (\cdot;\kappa,\gamma) : \mathbb{R} \rightarrow [0,1],\ u \mapsto e^{-\kappa(\gamma-u)^2} \qquad [11]$$

Like in the case of Beta-curve, the slope of the Gaussian fuzzy set is determined by the value of the width parameter.

Aside from applying bell-shaped membership functions to depict a fuzzy variable, fuzzy numbers and "around" or "close to" notions may also be represented by means of a **triangular** form. A further distinction can be made between a simple triangular, a shouldered and a trapezoidal fuzzy set. Triangular fuzzy sets mostly originate from the field of process engineering and control.

In a *triangular* (or pyramidal) fuzzy set, the apex of the triangle is centered around the unity measure of the fuzzy number or its most representative and characteristic value of the fuzzy variable, and equals full-membership. The left and right corners of the triangle correspond with the points where set membership is equal to zero. The distance between both corner points determines the spread around the centered value. It is also typical that, when several triangular sets are applied in a sequence to represent a concept, they necessarily overlap to some extent. The overall result is a sawtooth function which is easy to specify and visualize. This is one of the reasons why triangular functions are still very popular among researchers.

In a *shouldered* fuzzy set, triangular forms are also used to represent the fuzzy variable. Graphically, this results in that the apices of left-hand and right-hand out-side triangles in a sequence are topped-off and equal full-membership. The topped-off horizontal line is called a "shoulder" or "plateau". An important advantage of using shouldered fuzzy sets is that one is able to meet the endpoints of the variable's universe of discourse. An alternative to shouldered fuzzy sets is the use of bisected triangles at the edges of the domain.

A third variety to represent a fuzzy number or an "about" or "close to" notion is by means of a *trapezoidal* (or plateau) fuzzy set. A trapezoidal set is a truncated triangular fuzzy set with a crossover plateau. All domain elements that fall within that specified crossover plateau have equal (usually complete) membership in the set. The trapezoidal membership function is defined as follows (Dombi, 1990, p. 3):

$$T(\cdot;\mathrm{a,b,c,d}): \mathbb{R} \rightarrow [0,1],\ u \mapsto \begin{cases} 0 & \text{if } u \leq \mathrm{a} \\ (u\text{-a})/(\text{b-a}) & \text{if a} < u < \text{b} \\ 1 & \text{if b} \leq u \leq \text{c} \\ (u\text{-d})/(\text{c-d}) & \text{if c} < u < \text{d} \\ 0 & \text{if } u \geq \mathrm{d} \end{cases} \qquad [12]$$

with a ≤ b ≤ c ≤ d.

Note, that in equation [12] of a trapezoidal membership function the crossover plateau is equal to [b,c] because all u that fall within that interval have full membership in the set. Only if b=c the trapezoidal membership function is transformed into a triangular fuzzy set with apex b. Examples of triangular and trapezoidal fuzzy numbers are easy to define. For instance, "approximately equal to 350 meters" can be represented by T (·;300,350,350,400), "approximately between 250 m. and 450 m." can be represented by T (·;200,250,450,500), and the non-fuzzy notion "exactly 100 m." can be represented by T (·;100,100,100,100).

It is generally accepted that fuzzy numbers are best represented by bell-shaped curves. This is because bell-shaped curves, in particular the π-curve, provide a better and more smooth way to represent individual fuzzy domain than triangular fuzzy sets. However, the use of triangular representations still remains very popular as they are easy to specify, interpret and visualize. Also, triangular fuzzy sets are not complex to defuzzify (see below).

The selection of the type of membership function that should be used to represent a particular notion is very complex and subject to criticism. Also, it is difficult to argue what a particular degree of membership for a certain object really means.

Some authors are of the opinion that these issues are not very important, because fuzzy sets are intrinsically vague and should only give an indication or tendency of the corresponding linguistic concepts (Hellendoorn, 1990, p. 19). In this respect, Dubois and Prade (1980a, p. 2) stated:

> "Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with a fuzzy set; this "ordering", when it exists, is more important than the membership values themselves".

On the other hand, Jain (1980, p. 131), for instance, has a very strict interpretation of membership function selection. He argued:

> "The first step in the application of fuzzy set theory is to select either a membership function or a fuzzy set to represent a fuzzy variable. Most papers in the field start with a given fuzzy set or a given membership function, without any mention of how and why they were chosen. (...) But fuzzy set theory wants a fuzzy set to be

unfuzzily specified. It is not clear how to represent an ill defined or imprecise variable even in a fuzzy set theoretic framework".

A somewhat more compromising position is taken by, e.g., Lakoff (1973) when discussing the notion "tallness" and using an S-membership function curve to represent the concept. In respect to choosing the membership function specification, Lakoff (1973, p. 464) wrote:

> "The curve (...) is not to be taken with great seriousness as to its exactitude. Undoubtedly the function which maps height into tallness is itself fuzzy. However, I do think that the curve (...) is not a bad approximation to my own intuitions about degrees of tallness. The curve has about the right shape. It rises continuously, as it should. It would be wrong to have a curve that falls or has several dips. It goes up from zero at about the right place and seems to hit one at about the right place. In short, there is far more right than wrong about it, which is what makes it an interesting approximation".

The fact that there exists a diversity of opinions on membership function specification indicates the controversy of the subject. As a result, fuzzy researchers and scientists have put in great efforts to produce several theories to explain and justify the choice of membership function. The most famous, and generally accepted theory is the so-called *possibility theory*.

3 Relationship between probability, stochastic preference and membership function

An important issue, which is commonly misunderstood, concerns the relationship between a probability, a stochastic preference and a fuzzy membership function (see, e.g., Cheeseman, 1986; Hisdal, 1988a; Smets and Magrez, 1988; Klir, 1989; Kosko, 1990; Thomas, 1995). The confusion arises because probability models, stochastic preference models and models using fuzzy sets all deal with uncertainty and all operate over the same numerical range (scaling between 0 and 1) to measure the uncertainty.

From the classical point of view on statistics, a probability of an event is based on the relative frequency or proportion of occurrence of that event in repeated trails of an experiment. Hence, classical probability models usually rely for operationalization on samples of responses taken at specific points in time whereby the probabilities will change during the sampling process.

While probabilities are based on the theory of chances, stochastic preferences relate to the principles of random utility theory used in decision-making models (Thurstone, 1927; Manski, 1977). In random utility modelling, it is assumed that not all factors influencing the choice processes are known. Therefore, the decision-maker's preference for a choice alternative is assumed to be stochastic. This assumption is reflected in the random utility component of the choice alternative. Depending on the assumptions made regarding the distribution of these random utility components, the actual choice model is obtained. It is then assumed that an individual draws at random from this distribution a utility function at each choice occasion.

Although there is a basic difference between a probability and a stochastic preference, both concepts relate mathematically to the techniques developed in probability theory.

In fuzzy set theory, a membership function deals with fuzziness (sometimes also referred to as "vagueness" or "imprecision"). The membership function assigns to each object in the universe of discourse a grade of set membership ranging from zero to one. Therefore, a membership function value can be considered as a measure of the feasibility or ease of attainment of an event. Unlike with probabilities and stochastic preferences, with fuzzy membership functions one cannot say unequivocally whether an event occurred or not, instead one aims at trying to model the extend (or degree) to which an event occurred.

To illustrate the fundamental difference between probability theory and fuzzy set theory, take the following example (Bezdek, 1993, p. 2). Suppose you see two bottles containing a liquid. You are told that bottle A has a 90% chance of being potable (i.e., a probability), while bottle B has a 0.90 membership function with the fuzzy set of potable ("suitable for drinking") liquids. If you had to drink one, which would you choose first? The probability that bottle A is potable is 90 percent. This means that over the long run of experiments, the contents of A is expected to be potable about 90% of the trials; so what about the other 10%? In these cases the contents will be unsavoury (indeed, possibly deadly). Bottle A could contain either water, or hydrochloric acid. On the other hand, bottle B is "mostly potable". Its membership function of 0.90 will not change. This means that the contents of bottle B is fairly similar to perfectly potable liquids (pure water). So, bottle B is something like swamp water. It would certainly not contain liquids that are poisonous or deadly. Thus, if given the choice, you better select bottle B.

To give another example to further clarify the difference between probability theory and fuzzy set theory, take Kosko's (1993, p. 4) "half-eaten apple" which goes as follows. No one would question the fact that an apple belongs to the crisp set "apples". So, if a person displays an apple and asks: "Is this an apple?", most people would answer "Yes!". If the person chews a bite from the apple and repeats the question, the answer would most likely still be "Yes!". But what happens if the person keeps on eating so that, in the end, only the core of the apple remains? Is it still an apple? Most would say: "No!". The question then becomes: When did the apple pass from being an apple (belonging to the set "apples") to a non-apple (belonging to the complementary set "not apples")? Although we at all times have all the facts about the apple — its shape, its size, its weight, etc. — and although we can measure the change in these parameters with any precision, we still cannot say exactly when it ceased to be an apple. Obviously, the problem cannot be solved using statistics, since the uncertainty involved is non-random (chance is not involved. If we instead assume that the person with the apple holds it behind his or her back, and asks: "Do I have the apple in my right or left hand?", then chance is indeed involved and we have a random uncertainty ($P = 0.5$ for either hand).

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It is well understood that probability theory is a natural tool for formalizing uncertainty in situations where class frequencies are known, where evidence is based on outcomes of large series of independent random experiments or where qualitative response variables are assumed. The random variable (being the outcome of a chance experiment or being a stochastic preference measure) is associated with a probability distribution. A minimum requirement of probabilities is additivity; that is, probabilities must add together to one. Fuzzy sets and membership functions, on the other hand, have been related to the so-called *possibility theory* (Zadeh, 1978; Yager, 1982; Dubois and Prade, 1988).

The possibility theory is applied as a tool for formalizing fuzziness that results from information that is fuzzy. In much the same way as a random variable is associated with a probability distribution, a fuzzy variable (concerning a meaning) is associated with a possibility distribution (Hellendoorn, 1990, p. 29). As such, the relationship between probabilities, stochastic preferences and membership functions (fuzzy sets) can also be viewed as a relationship between probability theory and possibility theory.

Possibility distributions have some similarities to probability distributions, but their meanings are essentially different. The mathematical relationship between the two distributions has been studied extensively in the literature (see, e.g., Zadeh, 1978, 1995; Hisdal, 1982, 1996; Cheeseman, 1986, 1988; Delgado and Moral, 1987; Dubois and Prade, 1983, 1986, 1989; Klir and Folger, 1988; Klir, 1989, 1991; Klir and Parviz, 1992). To illustrate the connection between the two theories, we overview the theories from a broader perspective of *Dempster-Shafer theory* (also known as evidence theory or theory of belief functions), under which both probability and possibility theory appear as special cases.

The Dempster-Shafer theory (Dempster, 1968; Shafer, 1976; Zadeh, 1984; Shafer and Logan, 1987; Klir and Ramer, 1990; Stein, 1993) is capable of conceptualizing two distinct types of uncertainty, one emerging from probability theory and one from possibility theory.

Let X denote a universal set under consideration, assumed here to be finite, and let $P(X)$ denote the power set of X. The power set is a set consisting of all the subsets of a particular set. Then, the Dempster-Shafer theory is based upon a function

$$m : P(X) \rightarrow [0,1] \qquad [13]$$

such that

$$m(\varnothing) = 0 \text{ and } \sum_{A \in P(X)} m(A) = 1. \qquad [14]$$

The function m resembles a probability distribution, and therefore is usually called a *basic probability assignment*. However, there is a fundamental difference between probability distribution functions and basic probability assignments. For one, probability distribution functions are defined on X, whereas basic probability assignments are defined on $P(X)$. Every set $A \in P(X)$ for which $m(A) > 0$ is called a *focal element*. As this name suggests, focal elements are subsets of X on which the available evidence focuses. The pair (F,m), where F denotes the set of all focal elements of m, is called a *body of evidence*. In the same line of thought, we are able to define *total ignorance* in terms of basic probability assignments by $m(X) = 1$ and $m(A) = 0$ for all $A \neq X$. That is, we know that the element is in the universal set, but we have no evidence about its location in any subset of X.

Associated with each basic probability assignment m is a *belief measure* (Bel) and a *plausibility measure* (Pl)[2] which are determined for all sets $A \in P(X)$ by the formulas:

$$\text{Bel} : P(X) \rightarrow [0,1],\ \text{A} \mapsto \sum_{B \subseteq A} m(B), \qquad [15]$$

$$\text{Pl} : P(X) \rightarrow [0,1],\ \text{A} \mapsto \sum_{B \cap A \neq \emptyset} m(B). \qquad [16]$$

The belief measure represents the total evidence or belief that the element belongs to A as well as to the various subsets of A. The plausibility measure represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets but also the additional evidence or belief associated with sets that overlap with A. Hence, $\text{Pl}(A) \geq \text{Bel}(A)$.

The belief and plausibility measures are connected by the equation $\text{Pl}(A) = 1 - \text{Bel}(A^c)$ or $\text{Bel}(A) = 1 - \text{Pl}(A^c)$ for all sets $A \in P(X)$, where A^c denotes the complement of A.

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[2] The belief measure and plausibility measure correspond with what Dempster (1968, p. 206) called the lower and upper probabilities associated with the proposition "$x \in A$", respectively.

The two formulas, Bel(A) and Pl(A), and the definition of the basic probability assignment, $m(A)$, form the core of the Dempster-Shafer theory. Let us now take a closer look at two specific cases.

First, when the basic assignment focuses only on singletons, belief and plausibility measures collapse into a single measure [Bel(A) = Pl(A) = $\sum_{x \in A} m(\{x\})$]. The result is the classical additive *probability measure*. As a result, for any probability measure (Prob) on a finite set X there exists a unique *probability distribution function*:

$$p : X \rightarrow [0,1] \qquad [17]$$

such that

$$\text{Prob}(A) = \sum_{x \in A} p(x) \qquad [18]$$

for all $A \subseteq X$.

From the standpoint of Dempster-Shafer theory,

$$p(x) = m(\{x\}); \qquad [19]$$

hence, it is required that

$$\sum_{x \in A} p(x) = 1. \qquad [20]$$

Second, when all of the focal elements of a body of evidence are nested (ordered by set inclusion), the belief and plausibility measures are called consonant. Thus, in a way, consonance is reflected in the fact that degrees of evidence allocated to focal elements that are nested do not conflict with each other; that is, nested focal elements are free of dissonance of evidence. In this case, the plausibility measures are called *possibility measures* (also referred to as consonant plausibility measures), and belief measures are called necessity measures. Furthermore, any possibility measure (Poss, or π) is uniquely determined by a *possibility distribution function*:

$$r : X \rightarrow [0,1] \qquad [21]$$

via the formula,

$$\text{Poss}(A) \text{ or } \pi(A) = \max_{x \in A} r(x) \qquad [22]$$

for all $A \in P(X)$. The corresponding necessity measure (Nec, or η) is determined as $\eta(A) = 1 - \pi(A^c)$. Because of this convertibility property, we are able to concentrate on only one measure. Since possibility measures are more prominent in the literature than necessity measures, we will focus on the former.

Recall that probability measures can be represented by probability distribution functions for finite X. It turns out that possibility measures can be represented in this same way as well. Thus, given a consonant body of evidence $F = \{A_1, A_2, ..., A_n\}$ such that $A_1 \subset A_2 \subset ... \subset A_n$, the basic assignment in possibility theory is connected with the possibility distribution via the formula

$$m(A_i) = r(x_i) - r(x_{i+1}), \text{ and} \qquad [23]$$

$$r(x_i) = \sum_{k=i}^{n} m(A_k) = 1 \qquad [24]$$

for some $x_i \in A_i$, some $x_{i+1} \in A_{i+1}$, and $i = 1, 2, ..., n$, where $r(x_{n+1}) = 0$ by convention (Klir and Folger, 1988, p. 124).

Possibility theory can be formulated not only in terms of consonant bodies of evidence within the Dempster-Shafer theory, but also in terms of fuzzy sets. It was introduced in this manner by Zadeh (1978).

Zadeh (1975d; 1978, pp. 5-9) defined the concept of a possibility distribution as a fuzzy restriction which acts as an elastic constraint on the values that may be assigned to a variable. For instance, in the atomic sentence "John is young", youngness (i.e., the predicate) is viewed as a fuzzy restriction on John (i.e., the subject) because the predicate acts as an elastic constraint on the values that may be assigned on the subject.

More formally, let X be a variable taking values in a finite U, and let A act as a fuzzy restriction on X. The proposition "X is A" associates a possibility distribution, Π_X, with X. The corresponding possibility distribution function is denoted by $\pi_X(u)$. This measure (i.e., a possibility measure) may be interpreted as the possibility that a value of X belongs to A and is defined to be pointwise equal to the membership function of A, i.e.,

$$\pi_X(u) = \mu_A(u) \qquad [25]$$

for all $X \in A$. Thus, $\pi_X(u)$, the possibility that $X = u$, is postulated to be equal to $\mu_A(u)$. In this interpretation of possibility theory, focal elements are represented by so-called α-cuts of the associated fuzzy set. An α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a degree of membership in A greater than or equal to the specified value of α.

Returning to the example "John is young" (X is A), the possibility of X assuming a value u is interpreted as the degree of ease with which u may be assigned to X. In other words, suppose that an arbitrary chosen age, say u = 28, corresponds, according to a specified membership function, with a 0.70 degree of membership in the fuzzy set "young". This implies that the possibility that the variable Age (John) may take the value 28 is 0.70, with 0.70 representing the degree of ease with which 28 may be assigned to Age (John) given the elasticity of the fuzzy restriction labelled young.

The heuristic connection between possibilities and probabilities may also be stated in the form of what Zadeh (1978, p. 8) called the *possibility/probability consistency principle*. This principle establishes a basis for measuring the degree of consistency between a possibility distribution π and a probability distribution p, and is expressed by

$$C(\pi,p) = \int_{u \in X} \pi_X(u)\, p_X(u). \qquad [26]$$

When $C(\pi,p) = 1$, there are no contradictions between possibility and probability distribution transformations. Note, that it is only when the possibility distributions are normalized (i.e., $\max_{u \in X} \{\pi_X(u)\} = 1$) that the degree of consistency equals unity (Delgado and Moral, 1987, p. 312). However, following relations [14], [16] and [22], it can be seen that $\max_{u \in X} \pi_X(u) = \Pi(X) = \sum_{B \cap A \neq \emptyset} m(B) = \sum_{B \in P(X)} m(B) = 1$. An extensive overview of different types of probability-possibility transformations is given in Klir and Parviz (1992).

The fact of the matter is that fuzzy researchers have gone to great pains to distance themselves from probabilities and stochastic preferences (see, e.g., discussions in Cheeseman, 1988; Hisdal, 1988a, 1988b; Mabuchi, 1992; and Laviolette *et al.*, 1995), but often assume that the fuzzy logic related possibility distribution functions are normalized in order to be able to transform them into probability distributions, or, on a comparable level in fuzzy set theory, interpret the grades of set membership as probabilities (Ruspini, 1969; Gaines, 1975, 1978; Watanabe, 1975; Giles, 1976, 1982; Hersh and Caramazza, 1976; Kandel and Byatt, 1978; Bezdek *et al.*, 1981; Hisdal, 1982, 1988a, 1988b, 1996; Bandler and Kohout, 1985; Mabuchi, 1992; Painter, 1993; Beliakov, 1996; Kelly and Painter, 1996). Like Kandel and Byatt (1978, p. 1623) put it: "Intuitively, a similarity is felt between the concepts of fuzziness and probability. The problems in which they are used are similar (...) The fact that the assignment of a membership function of a fuzzy set is 'nonstatistical' does not mean that we cannot use probability distribution functions in assigning membership functions". Therefore, in a number of fuzzy set applications, it is assumed on methodological grounds that membership values add to one. Kosko and Isaka (1993, p. 62) argue that it is required in view of not-violating the Law of the Excluded Middle (see pp. 8-9).

Klement (1982a), Norwich and Turksen (1982d) and Turksen (1982, 1988) have tried to synthesize the relation between probabilistic and fuzzy models through the development of the notion of stochastic fuzziness. In contrast with an ordinary fuzzy set, a stochastic fuzzy set defines set membership as a random variable $\upsilon_A(u)$ having a probability density function which may be estimated instead of the usual deterministic scalar $\mu_A(u)$. As it turns out, the fuzzy domain of a probability density function appears to be smaller in size than the fuzzy domain of a membership function. Clearly, over the years, fuzzy set theory and probabilistic theory have grown closer together. Instead of being each others counterparts, much can be gained from a combination of both approaches. Stated in the words of the founding father of the fuzzy set theory: "Probability theory and fuzzy logic are complementary rather than competitive" (Zadeh, 1995, p. 271).

4 Estimation and choice of membership functions

A predominant problem in fuzzy set theory involves the estimation or choicen of membership functions. This is because estimation of $\mu_A(u)$ will eventually lead to *defuzzification* which is the ultimate aim of any fuzzy defined problem.

In his early writings, Zadeh was not really concerned with the problem of estimating membership values. The grades of membership are subjective, in the sense that their specification is a matter of definition rather than objective experimentation or analysis (Zadeh, 1972, p. 5). In spite of the fact that in the literature the estimation of $\mu_A(u)$ has not been dealt with systematically, a number of individual ideas and methods exists. Dubois and Prade (1980a, pp. 257-261), Zysno (1981), Norwich and Turksen (1982b, 1984), Bharathi Devi and Sarma (1985), Zimmermann (1985, pp. 305-339; 1986), Zimmermann and Zysno (1985), Turksen (1986b; 1991, pp. 26-33), Tzafestas (1994, pp. 238-239), Chen and Otto (1995), and Bilgic and Turksen (1996) mention among others the following techniques:

(i) **Exemplification**: In this technique, the respondents are asked to what extend a given variable (e.g., "length *l*") is considered as being a member of a pre-specified fuzzy classification (e.g., "long"). To answer the respondents have to make use of one of the following linguistic expressions: "true", "more or less true", "borderline", "more or less false", or "false". These linguistic terms are then simply translated into membership values: 1, 0.75, 0.5, 0.25, and 0, respectively.

(ii) **Distance function method** (or deformable prototypes method): In this method, first a distance $d(u)$ of an arbitrary point u from the set A under consideration is calculated. For all u that belong to the set, this distance is equal to zero; while for those elements not belonging to the set, this distance takes on some maximum value $sup(d)$. The latter value can also be interpreted as a measure of dissimilarity (or distortion) between the point u and the set A. The membership function for $\mu_A(u)$ can then be defined as:

$$1 - \frac{d(u)}{sup(d)} \ . \qquad [27]$$

(iii) **Implicit analytical definition** (or intuitive relation method): It is intuitively true that, when the belief that u belongs to the fuzzy set A strengthens, the membership value of $\mu_A(u)$ will increase, and vice versa. In other words, it is assumed that the marginal increase of a respondent's strength of belief that "$u \in A$" is proportional to the strength of belief that "$u \in A$" and the strength of belief that "$u \notin A$". Analytically, this is expressed by the relation:

$$\frac{d\mu_A(u)}{du} = k\ \mu_A(u)[1-\mu_A(u)] \qquad [28]$$

which, upon integration, gives the following membership function:

$$\mu_A(u) = \frac{1}{1 + \exp(a - bu)} \ . \qquad [29]$$

The parameters a and b of relation [29] are estimated from statistical data. Note here the similarity with the logistic S- and L-membership function definitions (see relations [7] and [8]).

(iv) **Binary polling method** (or use of statistics): In this method, the respondents are faced with the question: "Does u belong to the fuzzy set A?", and are only allowed a binary reply (yes or no). The value of $\mu_A(u)$ is then estimated as follows for finite U:

$$\mu_A(u) = \frac{\textit{number of positive replies}}{\textit{total number of replies}} \ . \qquad [30]$$

(v) **Comparison of subsets**: Suppose A is a fuzzy set of U with membership function μ_A. A fuzzy subset A' on $\mathscr{F}(U)$ from A can be introduced as follows:

$$\mu_{A'}(\{u_1,u_2,...,u_k\}) = \frac{1}{k}\sum_{i=1}^{k} \mu_A(u_i) \ . \qquad [31]$$

This specification could be interpreted as an "average membership" of $\{u_1,u_2,...,u_k\}$ in A. Next, a preference relation ($\succ$) is introduced in $\mathscr{F}(U)$ such that

$$\forall\, S_1, S_2 \in \mathscr{F}(U), S_1 \succ S_2 \qquad \text{iff} \qquad \mu_{A'}(S_1) \succ \mu_{A'}(S_2) \qquad [32]$$

The preference relation of $S_1 \succ S_2$ signifies "S_1 matches A better than S_2". Resulting from this relation are preference data between subsets of U, that can be translated into a system of inequalities among membership values for $\mu_A(u_i)$. This method is only applicable if $\mathscr{F}(U)$ is limited in size, and the usefulness of the method is determined by the quality of the subset induction process.

(vi) **Filter function method**: The filter function $F\ (u;IP,w)$ is characterized by two parameters: (i) the inflexion point IP (i.e., the point along the domain of u with 0.5 membership value), and (ii) the width $2w$ of the transition between non-membership (where $u \leq IP - w$) and full-membership (where $u \geq IP + w$). Next, a specification is needed for the fuzzy domain $(IP-w,IP+w)$ which is associated with the concept that needs to be represented.

It is assumed that the fuzzy concept can be related to a given population for which a normal probability distribution applies. If $\bar{u}$ and σ are the parameters of this distribution, the membership function of $\mu_A(u)$ is then estimated by the following filter function: $F(u; \bar{u} + \alpha\sigma, \beta\sigma)$ where α and β have to be determined experimentally.

(vii) **Direct rating** (or magnitude or point estimation): Randomly selected elements $u \in U$ are presented to a respondent in a random manner. The subject is asked to respond to a question of the following form: "How A is u?" (i.e., How *long* is *1000 m.*?) where A represents a fuzzy set. The subject's response (a rating) is a value $r(u) \in [\mu_L, \mu_U]$ where μ_L and μ_U are arbitrary membership values indicating a lower and upper bound, respectively. The respondent is also asked to identify u_{min}, corresponding to μ_L, and u_{max}, corresponding to μ_U, in repeated experiments in order to identify the range $[U_{min}, U_{max}]$ of the referential set (i.e., the support or domain of the base variable). Note that the respondent always gives his/her rating while bearing in mind that μ_L represents total agreement that u is not A and μ_U represents total agreement that u is A. This is equivalent to asking that $\frac{r(u) - \mu_L}{\mu_U - \mu_L}$ represent the ratio of the increase with respect to being A from u_{min} to u, to that of the increase from u_{min} to u_{max}. Because the ratings μ_L and μ_U represent the membership in A of u_{min} and u_{max} respectively, the value $r(u)$ is interpreted as the membership of u in A (i.e., $\mu_A(u)$). Hence, the membership ratio of differences is equal to:

$$\frac{\mu_A(u) - \mu_A(u_{min})}{\mu_A(u_{max}) - \mu_A(u_{min})} . \quad [33]$$

The subject's responses are recorded as ratings $r(u)$ — being the observed values of $\mu_A(u)$ — for given $u \in U$, i.e., $r(u)|u$. Thus, in every direct rating experiment the recorded values $r(u)|u$ generate a distribution function. Let $\phi_{R(u)|u}(r(u)|u)$ be the probability density function of the distribution function $\Phi_{R(u)|u}(r(u)|u)$ of the random variable $R(u)$ generated in a direct rating experiment at a given value $u \in U$. According to Turksen (1991, p. 27), these are error distributions, chosen as Beta distributions near the boundaries μ_L and μ_U but as Gaussian $N(\mu_A(u), \sigma^2)$ in the rest of the range $[\mu_L, \mu_U]$, with mean $\mu_{R(u)|u} = E(R(u)|u)$ and variance $\sigma^2_{R(u)|u} = Var(R(u)|u)$. These error distributions capture the subjectiveness in human perception in identifying the grades of membership $\mu_A(u) \in [\mu_L,\mu_U]$. Both mean and variance are unknown parameters. Unbiased estimates of these parameters can be obtained in the usual

manner: $$\bar{r}(u)|u = \frac{1}{n} \sum [r_i(u)|u] \text{ and } \quad [34]$$

$$s^2_{R(u)|u} = \frac{1}{n-1} \sum [r_i(u)|u - \bar{r}(u)|u]^2 . \quad [35]$$

(viii) **Reverse rating**: Randomly selected ratings $r(u) \in [\mu_L,\mu_U]$ are presented to a subject in a random manner. The subject is asked to respond to the following question "Identify or choose a $u \in U$ that possesses or best corresponds to the $r(u)$-th degree (grade) of membership in the fuzzy set A". The same degree of membership or rating is presented to the respondent a reasonable number of times in between other random presentations of $r(u)$ in order to avoid memorization. The subject's responses are recorded as observed values of u for given $r(u)$, i.e., $u\,|\,r(u)$. The distributions in this case turn out to be Gaussian error distributions $N(\mu_A(u),\sigma^2)$ with $\mu_{U|r(u)} = E(U\,|\,r(u))$ and variance $\sigma^2_{U|r(u)} = Var(U\,|\,r(u))$. Again, these unknown parameters can be estimated by $\bar{u}|r(u)$ and $s^2_{U|r(u)}$ (see relations [34] and [35]).

Having examined some standard membership function representations and methods to construct and estimate membership functions, we now turn our attention to the issue of fuzzy set operators.

5 Fuzzy set operators

Fuzzy set operators enables one to combine and modify fuzzy sets. Originally, Zadeh (1965, pp. 340-342; 1975a, p. 225) advanced three basic type of operations on fuzzy sets. These so-called standard Zadeh min/max operations are the (i) intersection, (ii) union, and (iii) complement operators.

The intersection operator is the equivalent of the logical AND-operation. This means that the intersection (or conjunction) of two sets yields a new set containing only those elements that are common to both sets. In conventional fuzzy set theory, the AND-operator is supported by taking the minimum of the membership functions for the two intersected sets. In contrast, the union (or disjunction) of two sets yields a new set that contains all elements that are in either of the sets. Its equivalent is the logical OR-operation which, in conventional fuzzy set theory, is supported by taking the maximum of the membership functions for the two combined sets. The complement (or negation) of a set contains all the elements that are not in the original set. Its equivalent is the NOT-operator (Rödder, 1975, pp. 2-3).

More explicitly, the intersection, union and complement operators are defined as follows (Zadeh, 1965):

Definition 3: Intersection
In a universe of discourse U, the intersection of two fuzzy sets A and B of U is defined as $\mu_{A \cap B} : U \rightarrow [0,1],\ u \mapsto \min \{\mu_A(u), \mu_B(u)\}$.■

Definition 4: Union
In a universe of discourse U, the union of two fuzzy sets A and B of U is defined as $\mu_{A \cup B} : U \rightarrow [0,1],\ u \mapsto \max \{\mu_A(u), \mu_B(u)\}$.■

Definition 5: Complement
In a universe of discourse U, the complement of a fuzzy set A of U is defined as $\mu_{\neg A} : U \rightarrow [0,1],\ u \mapsto 1 - \mu_A(u)$.■

Note that operations on fuzzy sets always take place at membership function level (degree of membership), and that any operation on a fuzzy set results in the creation of a new fuzzy set.

Besides utilizing the classical Zadeh-type operations on fuzzy sets, there exists other specifications of operators. These alternative forms of the AND, OR and NOT operations are termed *compensatory operators* because they tend to compensate for the strict minimum, maximum, and complement of the Zadeh operators (Turksen, 1992). An example of such a compensatory operator is the product operator (see Definition 6). Bellman and Zadeh (1970, p. 145) referred to this operator as the 'soft AND' operator.

The classical Zadeh intersection/union operators also do not satisfy the Excluded Middle Laws. Therefore, the product operator for the intersection of two fuzzy sets is often advanced as alternative (Bellman and Zadeh, 1970, p. 145; Thole *et al.*, 1979, Dubois and Prade, 1980a, p. 262; 1980b, p. 63; Zimmermann and Zysno, 1980; Francioni and Kandel, 1988).

Definition 6: Product

In a universe of discourse U, the product of two fuzzy sets A and B of U is defined as $\mu_{A \otimes B} : U \rightarrow [0,1],\ u \mapsto \mu_A(u).\mu_B(u)$.■

The product operator has obviously some intuitive appeal; it is simple, compatible to the conjunction of standard logic, and has a much more softer interpretation of the notion "and" than the classical Zadeh intersection operator (Bellman and Zadeh, 1970; Turksen, 1992). The product operator belongs to the so-called class of triangular norms (or briefly, t-norms). Such a t-norm is a mapping or function T: $[0,1] \times [0,1] \rightarrow [0,1]$ fulfilling the following four conditions (Klement, 1981, p. 219; 1982, p. 223; Zimmermann, 1985, p. 32):

1. $T(0,0) = 0;\ T(x,1) = T(1,x) = x$ for all $x \in [0,1]$, [36]

2. if $0 \leq x \leq p \leq 1$ and $0 \leq y \leq q \leq 1$ then $T(x,y) \leq T(p,q)$, [37]

3. $\forall_{x,y \in [0,1]}\ T(x,y) = T(y,x)$, [38]

4. $\forall_{x,y,z \in [0,1]}\ T(T(x,y),z) = T(x,T(y,z))$. [39]

The selection of the appropriate operator for a particular model is generally done heuristically. However, as a rule, one usually starts with the conventional Zadeh operators and evaluates the model's result. If the result is deemed unsatisfactory, a number of relatively simple algebraic transformations could bring some relief. Examples of such algebraic compensatory operations involve taking the mean or average, the squared mean, the product, and the bounded sum of the membership functions involved (Dubois and Prade, 1980a, p. 262). If the result is still unacceptable, a move to more complex compensatory operators might be advisable. Examples of such operators are the Yager-operators, the Zimmermann and Zysno-operators and the Dubois and Prade-operators (see Dubois and Prade, 1980b; Dombi, 1982; Klement, 1981, 1982b; Klir and Folger, 1988, pp. 37-64; Cox, 1994, pp. 143-152; Tzafestas, 1994, p. 233).

6 Fuzzy set hedges

Apart from applying fuzzy set operators to transform fuzzy sets, modification of sets can also be accomplished by means of using *hedges* (Zadeh, 1972; Lakoff, 1973).

A hedge, also sometimes referred to as a linguistic variable, is a linguistic term that modifies the shape characteristics of a fuzzy set. Hedges have a typical adjectival or adverbial relationship with a fuzzy set. In a sense, they can be compared to operators because they also transform one fuzzy set into another, new fuzzy set. The newly acquired set is in fact a rescaled version of the original set. In all, four important categories of hedges may be distinguished: (i) concentrators, (ii) dilators, (iii) contrast hedges, and (iv) approximation hedges.

First, an example of a hedge that has received particular attention in the literature (see Zadeh, 1972, pp. 22-25; MacVicar-Whelan, 1978), is the "very" hedge. The idea behind the linguistic term "very" is rather self-evident: one aims at intensifying a notion. The consequence of this intensification operation is that the surface of the fuzzy set will be rescaled. In the case of the "very" hedge, rescaling implies reducing the membership function for each value of the domain except at the fuzzy set extremes (i.e., the points representing absolute set membership and set exclusion). To illustrate this further, in considering the fuzzy set "long" and its intensified set "*very* long", we would expect that a representative distance from the intensified fuzzy set is rated at least as true as the same distance in the base fuzzy set, but the reciprocal is not true. Thus, a distance that is considered "very long" would also generally be rated as "simple long", but a simply long distance would not be categorized as very long. From this, it is clear that the effect of the "very" hedge — in Zadeh's words, a *concentrator* — is to reduce or concentrate the overall fuzzy domain (support) of a concept. Consequently, class membership becomes more restrictive.

Originally, Zadeh (1972, p. 23; 1975a, p. 226; 1975b, p. 322) heuristically conjectured that a good estimate for the "very" hedges would be squaring the membership function at each point in the set (or, $\mu_{very\ A}(u) = \mu_A(u)^2$). More recently, a generalization of the Zadeh concentrator hedge replaces the exponent of the intensification function with any real positive number greater than one. The concentrator has the following construct:

$$\mu_{concentrator\ (A)} : U \rightarrow [0,1],\ u \mapsto \mu_A(u)^n;\ \text{where } n > 1 \qquad [40]$$

Other examples of concentrator hedges or intensification transformers are "extremely", "pretty", "slightly", "a little", etc. Cox (1994, p. 174) argues that n will most likely fall in the interval [1,4] whereby fractional exponents are also allowed. This is because exponents outside the interval will push the membership function quickly toward zero.

Second, the complement of the concentrator hedge is the hedge group represented by such linguistic terms like "somewhat", "rather", "quite", "sort of", "more or less", etc. These hedges, all basic synonyms for each other, have the capacity to dilute a notion. In the case of the "somewhat" hedge — in Zadeh's terminology, a *dilator* — rescaling the original fuzzy set implies increasing the membership function for each domain value except at the fuzzy set extremes. This is explained by the fact that by diluting a notion, its class membership becomes less restrictive; hence, the surface of the membership function is expanded. Zadeh's estimate for the "somewhat" hedge was simple to take the square root of the membership function at each point along the set (or, $\mu_{somewhat\ A}(u) = \mu_A(u)^{0.5}$). However, by replacing the exponent with any real number less than one, a generalization of the Zadeh dilator hedge is obtained, and has the following construct:

$$\mu_{dilator\ (A)} : U \rightarrow [0,1],\ u \mapsto \mu_A(u)^{1/n};\ \text{where } n > 1 \qquad [41]$$

In practical use, n will fall within the interval [1,8]. This is because exponents outside this interval tend to push the membership function quickly toward one.

A third category of hedges is termed *contrast* hedges. A contrast hedge changes the nature of the fuzzy domain by either making the region less fuzzy (in which case, the hedge is called a contrast intensification hedge) or more fuzzy (in which case, the hedge is termed a contrast diffusion hedge). Examples of contrast intensification modifiers are "positively", "absolutely", "precisely", "definitely", etc., while examples of contrast diffusion modifiers are, for instance, "generally", and "usually".

The effect of a contrast intensification hedge is to make a region less fuzzy. This is accomplished by systematically increasing all the membership function values above 0.5 and diminishing all the membership function values below 0. The overall effect of this process is a shift in the membership values either closer to one (above 0.5) or closer to zero (below 0.5); hence, reducing the overall fuzziness of the region. Conversely, the effect of a contrast diffusion hedge is to make a region more fuzzy. This is achieved by systematically reducing all the membership function values above 0.5 and increasing all the membership function values below 0. The overall effect of this process is a move of all membership values toward the inflexion point (i.e., the 0.5 membership point); hence, increasing the overall fuzziness of the region.

The contrast hedge can be formalized as follows (Zimmermann, 1987, p. 238):

$$\mu_{\text{contrast (A)}} : U \rightarrow [0,1],\ u \mapsto \begin{cases} n(\mu_A(u))^n & \text{if } \mu_A(u) \in [0,0.5] \\ 1\text{-}n(1\text{-}\ \mu_A(u))^n & \text{otherwise} \end{cases} \qquad [42]$$

where $n > 1$ implies intensification, and
$n < 1$ implies diffusion.

A fourth and final category of hedges is the *approximation* hedge group. To this important category belong such representations as "about", "around", "in the vicinity of", "close to", "near", "roughly", etc. It should be clear that the result of applying an approximation hedge is a fuzzy number. As this subject has already been dealt with earlier, we will not repeat it here.

It is important to note that all hedges described until now owe their origin to a purely mathematical operation; that is, the degree of membership is simply raised to an arbitrary power. These hedges are therefore referred to as *powered hedges*. An alternative to applying powered hedges is the use of so-called *shifted hedges* (Hellendoorn, 1990). In a shifted hedge approach, the shape of the membership function is altered by changing the parameters of the membership function. Thus, instead of raising the degrees of membership to a power, a shift in membership values is obtained by changing the parameters, and with it, the overall shape of the fuzzy set. For example, suppose a notion like "old" is represented by a sigmoid membership function, $S(\cdot;\alpha,\beta,\gamma)$ with $\alpha = 70$, $\beta = 75$, and $\gamma = 80$, then the notion "very old" would be represented as a shifted version of that sigmoid membership function, e.g., $S(\cdot;\alpha,\beta,\gamma)$ with $\alpha = 80$, $\beta = 85$, and $\gamma = 90$.

7 Conclusions

In this paper, we focused on the theory of fuzzy sets as initially introduced by Zadeh (1965).

The fuzzy set theory aims at mathematically representing the vagueness intrinsic in linguistic terms and approximate reasoning. Through the use of the fuzzy sets, ill-defined and imprecise knowledge and concepts can be treated in an exact mathematical way. Fuzzy sets allow for partial or gradual set memberships. This property is reflected in the shape of the membership function.

The overview presented concentrated on six key issues of fuzzy set theory. These are: (i) relationship between crisp and fuzzy sets; (ii) fuzzy membership functions; (iii) relationship between probability, stochastic preference and membership function; (iv) estimation of membership functions; (v) fuzzy set operators; and (vi) fuzzy set hedges.

Given that this paper is an introductory paper on fuzzy set theory, a number of elements have not been dealt with. In this respect, we would like to mention such items as fuzzy reasoning, fuzzy relations, defuzzification processes, and also applications of fuzzy set theory (approximate reasoning, fuzzy control and expert systems, pattern recognition, decision-making, operational research, etc.).

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