Carnot efficiency is one of the cornerstones of thermodynamics. This concept was derived by Carnot from the impossibility of a perpetuum mobile of the second kind [1]. It was used by Clausius to define thermodynamic’s most basic state function, namely, the entropy [2]. The Carnot cycle deals with the extraction, during one full cycle, of an amount of work $W$ from an amount of heat $Q$, flowing from a hot reservoir (temperature $T_h$) into a cold reservoir (temperature $T_c \leq T_h$). The efficiency $\eta$ for doing so obeys the following inequality:

$$\eta = \frac{W}{Q} \leq 1 - \frac{T_c}{T_h}. \quad (1)$$

The equality sign is reached for a reversible process, entailing overall zero entropy production. For practical applications, the inequality (1) has a limited significance. Indeed a reversible process, having no preferred direction in time, has to be infinitely slow. Hence, the corresponding power is zero (finite work divided by infinite time). This observation prompted Curzon and Alhborn [3] (see also [4]) to investigate the problem of efficiency at nonzero power. They consider a Carnot construction cycling in a finite time, in which the only irreversible steps are assumed to be the transfers of heat between reservoirs and the auxiliary work-performing system (the so-called endoreversible approximation). Assuming furthermore a (linear) Fourier law for the heat transfer, they derive the following efficiency at maximal power [3]:

$$\eta = \frac{W}{Q} \leq 1 - \sqrt{T_c / T_h}. \quad (2)$$

This formula has been shown to apply, or to be at least a good approximation, in a number of other thermodynamic machines [5], including the quantum realm [6]. Furthermore, one observes in real systems efficiencies between the Curzon-Alhborn (CA) and Carnot values, in agreement with economic considerations entailing a compromise between power and efficiency. As a result, the CA paper has triggered the development of a new field of investigation, referred to as endoreversible thermodynamics, or, more generally, finite time thermodynamics [7].

Because of the elegance of the formula, the simplicity of its derivation, and the good agreement with observed efficiencies, the CA result has also become textbook material [8]. However, the derivation is model specific and the endoreversible approximation raises the question as to the validity and generality of the bound (2). We will show below that the Curzon-Alhborn efficiency is a fundamental result that follows, without approximation, from the theory of linear irreversible thermodynamics [9].

Our starting point is a generic setup for the extraction of work from a flow of heat, see Fig. 1(a). The system performs work, $W = -Fx$, against an external force $F$ (e.g., a mechanical, chemical, or electrical force) with thermodynamically conjugate variable $x$. The corresponding thermodynamic force is $X_1 = F/T$, where $T$ is the temperature of the system. The thermodynamic flux is $J_1 = 1/T$. The dot refers to the time derivative. The power (work by the system per unit time) is thus $W = -F\dot{x} = -J_1 X_1 T$. The work is performed under the influence of a heat flux $Q$ leaving the hot reservoir at temperature $T_h$. The cold reservoir is at temperature $T_c$ (where $T_h \geq T_c$). The corresponding thermodynamic force is $X_2 = 1/T_c - 1/T_h$, and the flux is $J_2 = Q$. The temperature difference $T_0 - T_1 = \Delta T$ is assumed to be small compared to $1/T_0 \approx T_0 = T$ so that one can also write $X_2 = \Delta T/T^2$.

Linear irreversible thermodynamics is based on the assumption of local equilibrium with the following linear relationship between the fluxes and forces:

$$J_1 = L_{11} X_1 + L_{12} X_2, \quad J_2 = L_{21} X_1 + L_{22} X_2. \quad (3)$$

The positivity of the entropy production, $dS/dt = J_1 X_1 + J_2 X_2 \geq 0$, implies for the Onsager coefficients $L_{ij}$ that

$$L_{11} \geq 0, \quad L_{22} \geq 0, \quad L_{11} L_{22} - L_{12} L_{21} \geq 0. \quad (4)$$

Furthermore, the Onsager symmetry resulting from the time reversibility of the microscopic dynamics stipulates

$$L_{12} = L_{21}. \quad (5)$$

The diagonal elements $L_{11}$ and $L_{22}$ have a direct physical meaning. For $X_2 = 0$, one finds $\dot{x} = L_{11} F/T$, hence $L_{11}/T$ is the mobility of the system in response to the external force $F$. For $X_1 = 0$, we have $\dot{Q} = L_{22} \Delta T/T^2$, so that $L_{22}/T^2$ is a coefficient of thermal conductivity. The off-diagonal elements $L_{12} = L_{21}$ describe the cross coupling. These couplings have been studied in a number of

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well-documented cases, notably the Seebeck, Thomson,
and Peltier effects [8,9]. Note that their maximal (absolute)
values are limited by Eq. (4). The dimensionless coupling
strength,

\[ q = \frac{L_{12}}{\sqrt{L_{11}L_{22}}} \]  

thus obeys \(-1 \leq q \leq +1\) [10].

After these preliminaries, the question of efficiency at
maximal power can be easily addressed. One immediately
concludes that the power, \( W = -J_1X_1T = -(L_{11}X_1^2 +
L_{12}X_1X_2)T \), is maximal when applying a force \( X_1^{\text{max}} \) equal
to half the stopping force, namely,

\[ X_1^{\text{max}} = \frac{X_1^{\text{stop}}}{2} \quad \text{with} \quad X_1^{\text{stop}} = \frac{L_{12}X_2}{L_{11}}. \]  

The stopping force corresponds to a value of the external
force \( F = X_1^{\text{stop}} T \) for which the motion of the system halts,
i.e., \( J_1 = 0 \). The efficiency, which is the output power
over input heat flux, can be conveniently written in terms of
the variable \( \kappa = X_1/X_2 \) as follows:

\[ \eta = \frac{W}{Q} = -\frac{\Delta T}{T} \frac{J_1X_1}{J_2X_2} = -\frac{\Delta T}{T} \kappa \frac{L_{11}\kappa + L_{12}}{L_{22}\kappa + L_{22}}. \]  

The efficiency evaluated at maximal power, \( \kappa^{\text{max}} = X_1^{\text{max}}/X_2 = -L_{12}/(2L_{11}) \), is then found to be:

\[ \eta = \frac{1}{2} \frac{\Delta T}{T} \frac{q^2}{2 - q^2}. \]  

This efficiency is thus equal to half of the Carnot effi-
ciency, \( \Delta T/T \), times a factor that is a function only of the
coupling strength \(|q|\) defined in Eq. (6). In particular, it is
independent of the overall time scale (i.e., the result is
invariant upon rescaling the time). In the optimal limit of
perfectly coupled systems, \(|q| \to 1\), the efficiency is ex-
actly half of the Carnot efficiency [11]. This result is valid
to the lowest order in \( \Delta T/T \), in which limit it actually
coincides with the CA bound, Eq. (2) \([1 - \sqrt{T_1/\Delta T}]\).

To go beyond the linear approximation in \( \Delta T/T \), while
staying within the framework of linear irreversible ther-
modynamics, we consider a cascade construction as in
Fig. 1(b) [12]. We introduce, between the hot reservoir at
\( T_0 \), and the cold one at \( T_1 \), a set of auxiliary heat reser-
voirs, labeled by the index \( y = j/N \), \( j = 1, \ldots, N-1 \) at
decreasing temperatures \( T(y) \) \([T(0) = T_0, T(1) = T_1]\).
These reservoirs will play a role akin to a catalyst, serv-
ing merely as temporary repositories of energy. We
furthermore assume that we have at our disposal a set of
\( N \) identical copies of the auxiliary system, oper-
ating between the successive pairs of reservoirs. For sim-
plicity, we assume from the onset that these machines
operate at maximum power with coupling strength
\(|q| = 1\); i.e., each of them transforms heat flux into power
at half of the Carnot efficiency. Finally, we take the con-
tinuum limit \( N \to \infty \). Each system and temperature reser-
voir is now characterized by the continuous index \( y = j/N \), with the step size \( dy = 1/N \) tending to zero.
The heat flux traversing the reservoir located at \( y \) at tem-
perature \( T(y) \), will be denoted by \( \dot{Q}(y) \) \([\dot{Q}(y = 0) = \dot{Q} \textbf{being the heat flux leaving the hot reservoir}]. The incremental power delivered by the system located between \( y \) and \( y + dy \) is denoted by \( d\dot{W}(y) \). Since the power is
derived solely from the transfer of the heat, and not from
the internal energy of the system, conservation of en-
ergy implies that \( \dot{Q}(y + dy) = \dot{Q}(y) + d\dot{W}(y) \), whence:
functions at CA efficiency between the intermediate temperature heat bath (temperature $T'$, heat flux $Q'$) and the low temperature heat bath $T_1$, producing a power $W'' = \eta(T_1/T')Q'$. A short calculation shows that the overall efficiency is given by $(W' + W'')/Q = \eta(T_1/T_0)$, i.e., the CA efficiency reproduces itself upon concatenation. This feature is also a property of machines operating at Carnot efficiency. In fact, one easily verifies that the most general function $\eta$ that reproduces this invariance (with boundary condition $\eta(1) = 0$) is $\eta(t) = 1 - t^\rho$. The Carnot case corresponds to $\alpha = 1$ while the CA bound corresponds to $a = 1/2$ or more generally $a = \rho/2$. 