Solvability of the master equation for dichotomous flow

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We consider the one-dimensional stochastic flow $\dot{x}=f(x)+g(x)\xi(t)$, where $\xi(t)$ is a dichotomous Markov noise, and use a simple procedure to identify the conditions under which the integro-differential equation satisfied by the total probability density $P(x,t)$ of the driven variable can be reduced to a differential equation of *finite* order. This generalizes the enumeration of the "solvable" cases.

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I. INTRODUCTION

The stochastic differential equation

$$\dot{x}=f(x)+g(x)\xi(t), \tag{1}$$

where x is a scalar variable and $\xi(t)$ is a colored noise models the evolution of a wide variety of systems. In general it is a nontrivial matter to deduce the corresponding equation obeyed by the probability density $P(x,t)$ of the driven variable. Considerable progress can be made in the case in which ξ is a stationary Gaussian noise (e.g., the Ornstein-Uhlenbeck process). In particular, using the special properties of Gaussian processes, it is possible to identify the cases [i.e., the forms of $f(x)$ and $g(x)$] for which $P(x,t)$ satisfies a (partial) differential equation of second order [1]—the so-called "solvable" cases. However, it is often relevant and necessary to model ξ by a dichotomous Markov process (DMP) [2,3]. (Further, in various parameter limits the latter can be shown [4] to reduce to white shot noise, Gaussian white noise, etc.) Although the DMP is quite a simple stochastic process, the statistics of the (non-Markovian) driven process $x(t)$ then becomes remarkably complicated. In general, $P(x,t)$ satisfies a rather intractable integro-differential equation in time, and that too with a kernel that involves the exponential of the gradient operator ∂_x. While the *stationary* solution $P^{st}(x)$ to which $P(x,t)$ tends in the limit $t\to\infty$ is known under fairly general conditions [2], there are very few instances known in which the time-dependent equation can be solved. The best result to date in this regard is the exact solution obtained by Sancho [5] for the class of cases in which the functions f and g satisfy the relation $gf'-fg'=\beta g$, where β is a constant and a prime denotes the derivative with respect to x. The important special case of linear drift ($f=$ a linear function of x, $g=$ a constant) is already included in this class.

In this situation, a question that arises naturally concerns the identification of the cases in which $P(x,t)$ can be shown to satisfy a partial differential equation of *finite* order—and that, therefore, are in this sense, "solvable" in principle. In this paper, we show that a relatively simple procedure based on the algebra of the operators concerned can be used to enumerate systematically all these solvable cases. Our formulation helps place the already solved cases in the perspective of a general setting, as it pin points exactly what is needed for solvability. The physically interesting case of a linear drift in one state of the DMP, alternating with a constant drift in the other state of the DMP, is shown to lead to an equation of the *third* order for $P(x,t)$. The same is shown to remain true if the drift is proportional to x^2, but not any higher power of x. The full set of solvable cases for polynomial $f(x)$ and $g(x)$ is also identified. Finally, we revert to the second order equation satisfied by the individual probability densities $P_{\pm}(x,t)$ corresponding to a given state of the DMP, and comment on the conditions under which one may expect to obtain "closed form" solutions for these in terms of standard functions.

II. MASTER EQUATION

To avoid inessential complications, we assume that $\xi(t)$ is a symmetric DMP that flips between the values $+1$ and -1 with the same mean rate λ. Equation (1) describes the random switching between the respective flows $\dot{x}=f_{\pm}(x)\equiv f(x)\pm g(x)$. The (total) probability density $P(x,t)\equiv P_+(x,t)+P_-(x,t)$, where $P_{\pm}$ denote the individual probability densities in the two states of the DMP. For ready reference, we recapitulate very briefly the outlines of the rigorous derivation [5] of the master equation for P and define the operators,

$$D=\partial_x(\cdot),\quad A=-\partial_x(f\cdot),\quad B=-\partial_x(g\cdot). \tag{2}$$

The stochastic Liouville equation for the density $\rho(x,t)$ is given by $\dot{\rho}=A\rho+B\xi\rho$, where an overhead dot denotes the time derivative. From this, using Van Kampen's lemma $\langle\rho(x,t)\rangle=P(x,t)$ and the formula of differentiation [6] for a functional of the noise ξ, one obtains the coupled equations

$$\dot{P}=AP+BP_1,\quad \dot{P}_1=(A-2\lambda)P_1+BP, \tag{3}$$

where $P_1=\langle\xi(t)\rho[\xi(t)]\rangle$. We assume the sharp initial condition $P(x,0)=\delta(x-x_0)$. Together with the requirement $P_1(x,0)=0$, which follows from causality, these conditions make the solution of Eqs. (3) a well-posed problem. Formally, P_1 can be eliminated by integrating the second of Eqs.

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(3) and substituting the result in the first. One then obtains the (integro-differential) master equation for P mentioned in Sec. I, namely,

$$\partial_t P(x,t) = AP(x,t) + B\int_0^t dt_1 \exp[(A-2\lambda)\times(t-t_1)]BP(x,t_1). \tag{4}$$

III. DIFFERENTIAL EQUATION FOR *P*

Instead of using Eq. (4) to proceed with the analysis, we shall use a much simpler algebraic method directly on the coupled set of equations (3), following the work of [5]. Differentiating the first of Eqs. (3) and using the second equation in the result yields

$$\ddot{P} = 2(A-\lambda)\dot{P} + (2\lambda A + B^2 - A^2)P - C_1 P_1, \tag{5}$$

where

$$C_1 = [A,B] = -\partial_x((f'g - g'f)\cdot). \tag{6}$$

Some classes of particular cases can be analyzed immediately, to exhibit where the known solved cases fit in and to lead us to the general result to follow.

Case 1. The most obvious one is

$$C_1 = 0, \quad \text{i.e.,} \quad f'g - g'f = 0, \tag{7}$$

which is solved by $g(x) = kf(x)$, where k is a constant. Then $B = kA$, so that Eq. (5) becomes a closed equation for P:

$$\ddot{P} = 2(A-\lambda)\dot{P} + 2\lambda AP + (k^2-1)A^2P. \tag{8}$$

This can also be solved explicitly: we note that Eq. (1) is, in this case, just $\dot{x} = (1+k\xi)f(x)$. Setting $q = \int^x du/f(u)$, this is $\dot{q} = 1 + k\xi$. Therefore, the process $y = q - t$ satisfies $\dot{y} = k\xi$, which is pure dichotomous diffusion for which the solution is well known (see, e.g., Ref. [7]).

(i) A subcase of interest is $k=1$ or $g(x) = f(x)$, so that $f_-(x) = 0$. This corresponds to "*delayed evolution*" in which the deterministic dynamics governed by the flow $\dot{x} = f_+(x) = 2f(x)$ is interrupted at random instants and x remains frozen at its current value, till the noise switches back to its $+1$ state and the evolution is resumed.

(ii) Another subcase of special interest corresponds to $f = 0$, so that $f_-(x) = -f_+(x)$. Equation (1) becomes $\dot{x} = g(x)\xi$, which reduces to dichotomous diffusion given by $\dot{Q} = \xi$ for the variable $Q = \int^x du/g(u)$. Physically, this case is of interest (for instance) in problems involving the *exchange of stability* between two different fixed points in the two states of the DMP. An explicit example is $f_\pm(x) = \pm(1-x^2)^{1/2}$, $x \in [-1,1]$.

Case 2. Next comes the case

$$C_1 = [A,B] = -\beta B, \quad \text{or} \quad gf' - g'f = \beta g, \tag{9}$$

where β is a constant. As stated in Sec. I, this is the class fully solved by Sancho [5]. The master equation obtained for $P(x,t)$ is

$$\ddot{P} = (2A - 2\lambda + \beta)\dot{P} + (2\lambda+\beta)AP + (B^2 - A^2)P. \tag{10}$$

If $g(x)$ is set equal to unity, it follows from Eq. (9) that f must be a linear function of x in order to be included in this class. The solution for $P(x,t)$ turns out to be expressible in terms of hypergeometric functions. More generally [5], using Eq. (9) in Eq. (1) leads to the stochastic differential equation $\dot{Q} = \beta Q + \xi + K$ (K = constant) for the variable $Q = \int^x du/g(u)$. Therefore, this reduces to the linear case, and can be solved explicitly. The Höngler model [8] with $f(x) = -\tanh x$, $g(x) = \operatorname{sech} x$ falls in this category.

IV. GENERALIZATION

A. Simplest extension

Cases 1 and 2 above exhaust the ones in which P satisfies a second order partial differential equation. It is clear from the foregoing that the origin of the problem lies in the noncommutativity of A and B, which precludes a direct solution [9] using the properties of the Poisson process governing the DMP $\xi(t)$. But it does suggest that it is the algebra of these operators and their successive commutators that decides the question of solvability.

It is evident that an immediate generalization of Eq. (9) is

$$C_1 = [A,B] = -\alpha A - \beta B, \tag{11}$$

where α and β are constants. Thus, the algebra of commutators still closes at the level of C_1 itself. However, owing to the presence of the term involving A in C_1, the elimination of P_1 requires an additional differentiation with respect to t. Simplification then yields the *third* order equation

$$\partial_t^3 P = (3A - 4\lambda + \beta)\ddot{P} + (B^2 - 3A^2 + (6\lambda+\beta)A + 2\lambda(\beta - 2\lambda))\dot{P} + (A^3 - AB^2 - (4\lambda+\beta)A^2 + \alpha AB + 2\lambda B^2 + 2\lambda(2\lambda+\beta)A)P. \tag{12}$$

An important illustration of this case is provided by the dichotomous flow that corresponds to a *uniform* drift to the right ($f_+ = c$) randomly alternating with a linear flow to a stable fixed point at the origin ($f_- = -\gamma x$). The stochastic differential equation is

$$\dot{x} = \frac{(c-\gamma x)}{2} + \frac{(c+\gamma x)}{2}\xi(t). \tag{13}$$

It is easily checked that $C_1 = -\gamma/2(A+B)$ in this case, so that Eq. (11) applies.

B. Further extension

From the procedure used to analyze the cases above, it is straightforward to deduce the following general result. Define the multiple commutator C_n recursively by

$$C_n=[A,C_{n-1}] \tag{14}$$

with $C_0 \equiv B$. It is clear from Eqs. (3) and (5) that $C_0 P_1$ is determined in terms of $\dot{P}$ and P; $C_1 P_1$ is determined in terms of $\ddot{P}$, $\dot{P}$, and P; and so on. Therefore, if C_n turns out to be a linear combination of the preceding C_i, that is, if

$$C_n=-\sum_{i=0}^{n-1} \beta_i C_i, \tag{15}$$

and n is the smallest integer for which this happens, then $P(x,t)$ satisfies a partial differential equation of order $n+1$. On the other hand, if the linear combination involves A as well, i.e., if

$$C_n=-\alpha A-\sum_{i=0}^{n-1} \beta_i C_i, \tag{16}$$

then $P(x,t)$ satisfies a partial differential equation of order $n+2$.

An interesting illustration is provided once again by the case of a uniform drift ($f_+=c$) alternating with flow to a stable fixed point as in Eq. (13) above, but with a *quadratic* drift $f_-=-\gamma x^2$ (we consider $x \geqslant 0$). Rescaling x and t to $(\gamma/c)^{1/2}x$ and $(\gamma c)^{1/2}t$, respectively, the stochastic flow is now given by

$$\dot{x}=\frac{(1-x^2)}{2}+\frac{(1+x^2)}{2}\xi(t). \tag{17}$$

(This model has a special symmetry: under the interchange $x \leftrightarrow 1/x$, the flows f_+ and f_- exchange roles, as do the fixed points at 0 and ∞.) We find that $C_2=[A,C_1]=C_0$ itself. Therefore, $P(x,t)$ is guaranteed to satisfy an equation of order three in this case too.

A natural question to ask is whether this continues to be so for the general case $f_+=c$, $f_-=-\gamma x^n$, when $n \geqslant 3$. The answer is that *it does not*: it is easily checked that C_2 now has terms proportional to $x^{2n-2}D$ and x^{2n-3} (where $D=\partial_x$ as already defined), which then lead to even higher powers of x in C_3, and so on. For all $n \geqslant 3$, therefore, P does not satisfy an equation of finite order.

C. Polynomial $f(x)$ and $g(x)$

This brings us to the question: If f and g are generic polynomials in x, under what conditions does the probability density P satisfy a partial differential equation of finite order? We exclude the case $g(x)=kf(x)$, leading at once to $C_1=0$, which has already been dealt with. Note also that if f_+ and f_- are any two polynomials, then f and g become polynomials of the *same* order, which is a special case of the more general one in which f and g are arbitrary polynomials.

The answer is found by examining the commutator $[Dx^m,Dx^n]$ ($m,n=$ natural numbers): using the basic commutator $Dx-xD=1$, this works out to

$$[Dx^m,Dx^n]=(n-m)Dx^{n+m-1}. \tag{18}$$

It is immediately obvious that if m and n do not exceed unity, higher powers x are not generated: indeed, it is easily shown that, when f and g are arbitrary linear functions of x, C_2 is a constant multiple of C_1. Therefore, in this case P is guaranteed to satisfy an equation of order no higher than three (which, as we have seen, may reduce to two in special cases). When either f or g is a polynomial of order $\geqslant 2$, P does not *in general* satisfy an equation of finite order—although it may do so in special cases when there are additional relations between the coefficients of the polynomials concerned, as in the example of Eq. (17) above. As a corollary, we recover the following (presumably known) fact: Even in the case when the noise is *additive*, i.e., $g(x)=k=$constant, P does not satisfy a finite order equation if there is any nonlinearity in the drift f. In the present formalism, this follows at once from the fact that for $f(x)=cx^m$, $g(x)=k$, we have $C_0=-kD$, $C_1=-ckmDx^{m-1}$, and

$$C_n=(-c)^n kmDx^{n(m-1)}\prod_{j=0}^{n-2}(jm-j-1) \quad (n \geqslant 2). \tag{19}$$

It is evident that the sequence $\{C_n\}$ involves higher and higher powers of x for all $m \geqslant 2$.

D. Solvability of the equations for $P_\pm$

The question of solvability discussed in the foregoing can also be viewed, in part, from another angle, that helps augment our understanding of the problem. We conclude with a brief remark on this aspect.

As is well known, the individual probabilities $P_\pm(x,t)$ obey the coupled equations

$$\partial_t P_+=-\partial_x(f_+P_+)+\lambda(P_--P_+),$$
$$\partial_t P_-=-\partial_x(f_-P_-)+\lambda(P_+-P_-). \tag{20}$$

It is quite straightforward to eliminate either one of the two unknowns $P_\pm$, to obtain a second order equation for the remaining one. Defining $p_\pm(x,t)=e^{\lambda t}P_\pm(x,t)$, we find

$$\partial_t^2 p_\pm+2\partial_x(f_\pm\partial_t p_\pm)+\partial_x[f_\mp\partial_x(f_\pm p_\pm)]-\lambda^2 p_\pm=0. \tag{21}$$

The case of physical interest generally corresponds to oppositely directed flows, so that $f_+(x)f_-(x)<0$ in the region concerned. The discriminant of Eq. (21), $f_\pm^2-f_\pm f_\mp$, is then positive so that the equation is hyperbolic. One may then attempt to apply the standard method of characteristics [10], whose equations are given formally by

$$\zeta_\pm=t-\int\frac{dx}{f_\pm+(f_\pm^2-f_\pm f_\mp)^{1/2}},$$
$$\eta_\pm=t-\int\frac{dx}{f_\pm-(f_\pm^2-f_\pm f_\mp)^{1/2}}. \tag{22}$$

The basic problem manifests itself here in the fact that, in general, the required antiderivatives do not exist in closed form. When they do, the next step is to invert these equations to express x and t in terms of $\zeta_\pm$ and $\eta_\pm$, and then reduce Eq. (21) to canonical form. Here again, this inversion may not be possible in terms of standard functions. For instance, consider the flow described by Eq. (13) above, in which P has been shown to satisfy an equation of the third order. The characteristics can be found in closed form in this case. Inverting these, we find (for p_+, for instance) the rather involved expressions

$$x=\frac{c}{\gamma}\left(\phi^2\left(\frac{\zeta_+}{\eta_+}\right)-1\right),\quad t=\frac{1}{\gamma}\ln\left(\frac{\zeta_+\eta_+}{\phi^2\left(\frac{\zeta_+}{\eta_+}\right)-1}\right),\quad (23)$$

where ϕ stands for the inverse of the function $w=[(z-1)/(z+1)]e^{2z}$, i.e., $z=\phi(w)$. And finally, even after reduction to canonical form, the analytic solution of the equations in terms of standard functions is, of course, possible only in a very small number of cases.

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