University of Antwerp

FACULTY OF APPLIED ECONOMICS

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

General annuities
under truncate stochastic interest rates

Inge Koch and Ann De Schepper

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Abstract

An important part of the current financial and actuarial research deals with the investigation of present value functions in the case of a stochastic interest rate. In the present contribution, it is shown how interest rates can be restricted to meet special types of financial or actuarial constraints. Approximate but analytical expressions are given for the distribution of different types of annuities, and their accuracy is illustrated graphically.

1 Introduction

Many of the problems in the current financial and actuarial research can be reduced to the problem of finding the distribution of the present value of a cash-flow in the form

$$V(t) = \sum_{i=1}^{n} \alpha(t_i)\, e^{-X(t_i)}, \tag{1}$$

where $0 < t_1 < t_2 < \ldots < t_n = t$, where $\alpha(t_i)$ is a (positive or negative) payment at time t_i, and where $X = \{X(t)\}$ is a stochastic process with $X(t_i)$ denoting the compounded rate of return for the period $[0,t_i]$.

There exists a broad range of stochastic processes that seem to be useful to model the stochastic interest rates, which is shown by the long list of papers investigating these models. However, in many cases, the model would be more realistic if the interest rates are not completely free, but restricted to some range of acceptable values. If for example the interest rates appearing in the cash-flow are nominal interest rates, they can not become negative. If an insurance contract guarantees a minimal return, the interest rate model should be adapted in order to meet this warranty. Due to special regulations, it can also be necessary to impose an upper limit for the yield of a financial effect.

In this paper, we want to introduce a model that meets these last requirements. We show how to adapt common models to these restrictions, and we show the influence on the present value of classical actuarial functions such as annuities. Except for some special cases (concerning the restrictions and concerning the actual stochastic model), as a consequence of the adaptation, the exact distribution of the present value can no longer be calculated analytically. Therefore, we will make use of an approximation by means of convex bounds, as introduced by Goovaerts et al. [4], and generalized in Dhaene et al. [2, 3].

2 Methodology

2.1 Restrictions on the interest rates

A. Non-negative interest rates

A first and common restriction, needed in many financial applications, goes back to the fact that (if nominal rates are used) negative interest rates should be avoided. A possible solution to this problem can be reached by multiplying the compounded rate of return with the Heaviside-function $\mathcal{U}$, defined by

$$\mathcal{U}(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x \leq 0, \end{cases} \tag{2}$$

such that the discount factor in the present value becomes $e^{-X(t)\,\mathcal{U}(X(t))}$. See also [1]. With this adjustment, the compounded interest rate is kept equal to zero as long as the value of $X(t)$ is negative.

B. Truncate interest rates with fixed floor and cap

A more general solution consists of a truncate interest rate, by defining a cap and a floor for the interest rate – with the previous restriction as a special case. This can be done by mapping $X(t)$ on $c \in \mathbb{R}$ whenever X(t) exceeds c, and by mapping $X(t)$ on $f \in \mathbb{R}$ whenever $X(t)$ is smaller than f.

Definition 2.1 *Let $f, c \in \mathbb{R}$ with $f < c$. The truncate function $\mathcal{S}_f^c : \mathbb{R} \to [f, c]$ then is defined by*

$$\mathcal{S}_f^c(x) = \begin{cases} f & if \quad x < f \\ x & if \quad f \leq x \leq c \\ c & if \quad c < x. \end{cases} \tag{3}$$

The left plot of figure 1 shows a possible realisation of such a truncate interest rate. With this truncate function applied on the stochastic interest rate, the discount factor in the present value becomes $e^{-\mathcal{S}_f^c(X(t))}$.

C. Truncate interest rates with linear floor and cap

Since the stochastic variable $X(t)$ corresponds to the *cumulative* interest rate for the period $[0, t]$, it seems more appropriate to use a fixed floor and cap per unit time period, or a linear floor and cap for the whole time period. This results in the following alternative definition for the truncation of the interest rates.

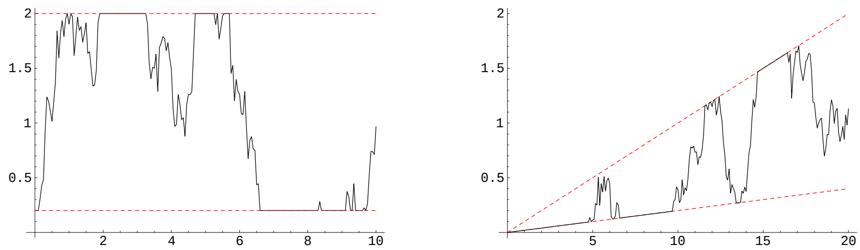


Figure 1: Example of a stochastic truncate (left) and a stochastic linear truncate (right) interest rate.

Definition 2.2 *Let $f, c \in \mathbb{R}$ with $f < c$. The linear truncate function $\tilde{\mathcal{S}}_f^c : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is defined by*

$$\tilde{\mathcal{S}}_f^c(t,x) = \begin{cases} f \cdot t & if \quad x < f \cdot t \\ x & if \quad f.t \leq x \leq c \cdot t \\ c \cdot t & if \quad c \cdot t < x. \end{cases} \tag{4}$$

The right plot of figure 1 shows a possible realisation of such a linear truncate interest rate. With this linear truncate function applied on the stochastic interest rate, the discount factor in the present value becomes $e^{-\tilde{\mathcal{S}}_f^c(t,X(t))}$.

2.2 Convex bounds

Since the compounded rates of return $X(t_i)$ for successive periods only differ for the last part of the period, the present value of (1) is made up as a sum of rather dependent terms. As a consequence, it is nearly impossible to derive an exact analytical expression for the distribution of such a present value. In order to solve this problem, Goovaerts et al. [4] and Dhaene et al. [2] present bounds in convexity order. Following their method, the original sum $V(t)$ is replaced by a new sum, for which the components have the same marginal distributions as the components in the original sum, but with the most “dangerous” dependence structure that is possible, and for which the calculation of the distribution is much more easy.

In this subsection, we just briefly recall definitions and most important results about this approximation method. For details, we refer to Dhaene et al. [2].

Definition 2.3 *Let X and Y be two random variables, then X is said to be* smaller *than Y in* convex order sense, *(notation $X \leq_{cx} Y$), if and only if*

$$E[v(X)] \leq E[v(Y)]$$

for all real convex functions $v : \mathbb{R} \to \mathbb{R}$, provided the expectations exist.

In fact this ordering means that the variable Y is more likely to reach extreme values than it is the case for X, or, that the variable Y is more dangerous than X. Note that for such variables it is true that $E[X] = E[Y]$ and $Var[X] \leq Var[Y]$.

Theorem 2.1 *Let $X_1, X_2, \ldots, X_n$ be random variables with marginal distribution functions known as $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, then*

$$X_1 + X_2 + \ldots + X_n \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U), \tag{5}$$

and

$$X_1 + X_2 + \ldots + X_n \geq_{cx} \mathrm{E}[X_1|\Lambda] + \mathrm{E}[X_2|\Lambda] + \ldots + \mathrm{E}[X_n|\Lambda], \tag{6}$$

with U a uniform(0,1) distributed random variable, and with Λ an arbitrary variable for which the conditional distributions of X_i given Λ are known.
The upper bound of (5) can be improved to a closer bound

$$X_1 + X_2 + \ldots + X_n \leq_{cx} F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \ldots + F_{X_n|\Lambda}^{-1}(U), \tag{7}$$

with U and Λ as before.

Note that the lower bound of (6) and the improved upper bound of (7) perform better the more Λ resembles the original sum.

If we define the inverse distribution as $F_{X_i}^{-1}(p) = \inf\{x \in \mathbb{R} : F_{X_i}(x) \geq p\}$, and $F_{X_i}^{-1+}(p) = \sup\{x \in \mathbb{R} : F_{X_i}(x) \leq p\}$, $p \in [0,1]$, the results of theorem 2.1 can be extended to functions of the variables X_i, by making use of the following lemma:

Lemma 2.2 *If ψ is a continuous real-valued function and p is any number in $]0,1[$, then if ψ is non-decreasing, $F_{\psi(X)}^{-1}(p) = \psi(F_X^{-1}(p))$, and if ψ is non-increasing, $F_{\psi(X)}^{-1}(p) = \psi(F_X^{-1+}(1-p))$.*

2.3 Stochastic interest rate model

As mentioned in the introduction, there exists a long list of stochastic processes, useful to model interest rates. In the sequel we will give an elaborated example of our method for a well known and frequently used easy model, the Brownian motion with drift, defined by the stochastic differential equation

$$dX(t) = \mu dt + \sigma dW(t), \tag{8}$$

with $W = \{W(t)\}$ a standard Brownian motion.
This model benefits from the fact that it is one of the most easiest models to describe a stochastic interest rate. An advantage of this model can be found in

the appropriateness for situations with rather great variation; a disadvantage however is that for long periods, a very large value (both positive and negative) could be reached, which imposes the possibility of instability. However, by implementing a restriction as suggested in subsection 2.1, this disadvantage can be perfectly avoided.
If we use the notation $F(t,x)$ for the cumulative distribution function of the variable $X(t)$, it is well known that

$$\begin{array}{ll} F(t,x) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right), & x \in \mathbb{R} \\ F^{-1}(t,p) = \mu t + \sigma\sqrt{t}\Phi^{-1}(p), & p \in [0,1], \end{array} \tag{9}$$

with $\Phi(x)$ the standard normal cumulative distribution function.

3 Constant annuities

In this section, we first present our results without specifying the stochastic process used to model the interest rate. We provide expressions for stochastic bounds to general constant annuities. Afterwards, we show how for each of these bounds analytical results can be obtained in the case of a Brownian motion with drift.

3.1 General case

Consider a discrete annuity over the time-interval $[0,t]$, with linear truncate stochastic interest rate with floor f and cap c as defined in definition 2.2:

$$V(t) = \sum_{i=1}^{n} e^{-\tilde{S}_f^c(t_i, X(t_i))}, \tag{10}$$

where $X = \{X(t)\}$ is a stochastic process with $X(t_i)$ denoting the compounded rate of return for the period $[0,t_i]$.

Applying the methodology of convex bounds (see subsection 2.2), the following results can be obtained straightforwardly:

Theorem 3.1 *The annuity of equation (10) can be bounded in convex ordering sense as*

$$V_{low}(t) \leq_{cx} V(t) \leq_{cx} V_{imupp}(t) \leq_{cx} V_{upp}(t), \tag{11}$$

where the stochastic bounds are determined by

$$\begin{cases} V_{upp}(t) &= \sum_{i=1}^{n} e^{-\tilde{\mathcal{S}}_f^c(t_i, F^{-1+}_{X(t_i)}(1-U))} \\ \\ V_{low}(t) &= \sum_{i=1}^{n} \mathrm{E}[e^{-\tilde{\mathcal{S}}_f^c(t_i, X(t_i))}|\Lambda] \\ \\ V_{imupp}(t) &= \sum_{i=1}^{n} e^{-\tilde{\mathcal{S}}_f^c(t_i, F^{-1+}_{X(t_i)|\Lambda}(1-U))}. \end{cases} \tag{12}$$

In these expressions, U is a uniform(0,1) distributed random variable, and Λ is an arbitrary variable such that the distribution of $X(t_i)|\Lambda$ is known.

By taking limits , the case of a continuous annuity $V(t) = \int_0^t e^{-\tilde{\mathcal{S}}_f^c[\tau, X(\tau)]}\, d\tau$ can be solved in a similar way.

Remark: Note that each of the previous results remain valid when the linear truncate interest rate $\tilde{\mathcal{S}}_f^c[t, X(t)]$ is replaced by an ordinary truncate rate $\mathcal{S}_f^c[X(t)]$.

3.2 The case of a discrete annuity with a Brownian motion

Consider a discrete annuity certain as in equation (10).

Since $X(t_i)$ corresponds to the *cumulative* interest rate for the period $[0, t_i]$, it can be written as $X(t_i) = Y(t_1) + \ldots + Y(t_i)$, with $Y(t_k)$ the interest rate for the period $[t_{k-1}, t_k]$. In the Brownian model, we assume that the vector $Y = (Y(t_1), Y(t_2)), ..., Y(t_n))$ consists of independent normally distributed variables.

Next, define Λ as a lineair combination of the variables $Y(t_k)$, or

$$\Lambda = \sum_{i=1}^{n} a_i Y(t_i), \qquad a_i \in \mathbb{R}, \tag{13}$$

such that the distribution function of Λ can be written as

$$F_\Lambda(\lambda) = \Phi\left(\frac{\lambda - \mu t \sum a_i}{\sqrt{\sigma^2 t \sum a_i^2}}\right). \tag{14}$$

Since Λ and each variable $X(t_i)$ $(i = 1, ..., n)$ are combinations of the components of Y, it follows that $X(t_i)|\Lambda$ is also normally distributed with mean and variance given by

$$\begin{cases} \bar{\mu}_i(\Lambda) = \mathrm{E}[X(t_i)] + corr[X(t_i), \Lambda]\frac{\sigma_{X(t_i)}}{\sigma_\Lambda}(\Lambda - \mathrm{E}[\Lambda]) \\ \bar{\sigma}_i^2 = \sigma^2_{X(t_i)}(1 - corr[X(t_i), \Lambda]^2). \end{cases} \tag{15}$$

The following results hold :

Theorem 3.2 *In the Brownian case, the discrete annuity certain as in equation (10) can be bounded by*

$$V_{low}(t) \leq_{cx} V(t) \leq_{cx} V_{imupp}(t) \leq_{cx} V_{upp}(t), \tag{16}$$

where

$$V_{upp}(t) = \sum_{i=1}^{n} e^{-\tilde{S}_f^c[(\mu t_i + \sigma\sqrt{t_i}\Phi^{-1}(1-U))]}, \tag{17}$$

$$\begin{aligned} V_{low}(t) = \sum_{i=1}^{n} \Bigg(& e^{-f\cdot t_i}\Phi\left(\frac{f\cdot t_i - \bar{\mu}_i(\Lambda)}{\bar{\sigma}_i}\right) + e^{-c\cdot t_i}\Phi\left(\frac{\bar{\mu}_i(\Lambda) - c\cdot t_i}{\bar{\sigma}_i}\right) \\ & + e^{-\bar{\mu}_i(\Lambda) + \frac{1}{2}\bar{\sigma}_i^2} \cdot \left(\Phi\left(\frac{c\cdot t_i - \bar{\mu}_i(\Lambda) + \bar{\sigma}_i^2}{\bar{\sigma}_i}\right) - \Phi\left(\frac{f\cdot t_i - \bar{\mu}_i(\Lambda) + \bar{\sigma}_i^2}{\bar{\sigma}_i}\right)\right)\Bigg), \end{aligned} \tag{18}$$

and

$$V_{imupp} = \sum_{i=1}^{n} G_i(U, \Lambda), \tag{19}$$

where the functions $G_i : [0,1] \times \mathbb{R} \to \mathbb{R}^+ : (p, \lambda) \mapsto G_i(p, \lambda)$ *are defined by*

$$G_i(p, \lambda) = \begin{cases} e^{-c\cdot t_i} & \textit{if } p \in [0, p_i^{(2)}(\lambda)[\\ e^{-f\cdot t_i} & \textit{if } p \in [1 - p_i^{(1)}(\lambda), 1] \\ e^{\bar{\sigma}_i\Phi^{-1}(p) - \bar{\mu}_i(\lambda)} & \textit{if } p \in [p_i^{(2)}(\lambda), 1 - p_i^{(1)}(\lambda)[\end{cases} \tag{20}$$

with $p_i^{(1)}(\lambda) = \Phi(\frac{f\cdot t_i - \bar{\mu}_i((\lambda))}{\bar{\sigma}_i})$ *and* $p_i^{(2)}(\lambda) = \Phi(\frac{-c\cdot t_i + \bar{\mu}_i((\lambda))}{\bar{\sigma}_i})$.

Proof. This follows after a few calculations when the methodology explained in subsection 2.2 is applied. □

Concerning the distributions of these bounds, the results are summarized in the following theorem, where the notation F_Z is used as notation for the cumulative distribution function of the variable Z, or $F_Z(x) = \text{Prob}\,(Z \leq x)$.

Theorem 3.3 *The cumulative distribution functions of the convex bounds of theorem 3.2 can be calculated as follows:*

$$\begin{cases} F_{V_{upp}}(x) &= 1 - \Phi(\nu_x), \\ F_{V_{low}}(x) &= 1 - \Phi\left(\frac{\lambda_x - \mu t \sum a_i}{\sqrt{\sigma^2 t \sum a_i^2}}\right), \\ F_{V_{imupp}}(x) &= \int_{-\infty}^{+\infty} \kappa(\lambda, x) dF_\Lambda(\lambda), \end{cases} \tag{21}$$

with ν_x, λ_x defined implicitly and $\kappa(\lambda, x)$ defined explicitly as

$$\begin{cases} \sum_{i=1}^n e^{-\tilde{\mathcal{S}}_f^c[t_i, \mu t_i + \sigma\sqrt{t_i}\nu_x]} = x, \\ \\ V_{low}(t)|_{\Lambda=\lambda_x} = x, \\ \\ \kappa(\lambda, x) = \sup\{p \in [0,1] | \sum_{i=1}^n G_i(p, \Lambda = \lambda) \leq x\}. \end{cases} \tag{22}$$

Proof. In order to prove these statements, use can be made of the results mentioned in subsection 2.2 Note that the values reached by the functions $G_i : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ in fact can be written as

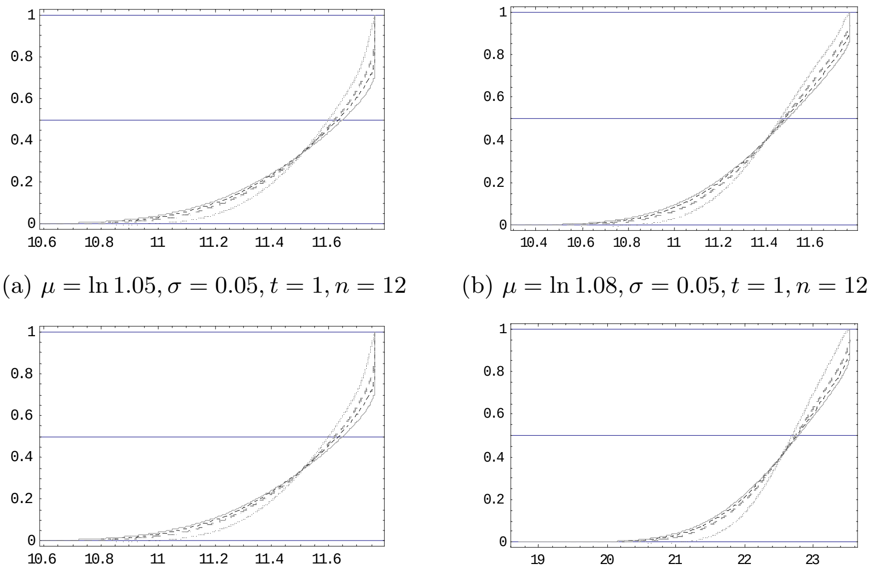
$$G_i(p, \lambda) = F^{-1}_{e^{-\tilde{\mathcal{S}}_f^c(t_i, X(t_i))}|\Lambda=\lambda}(p) = \text{Prob}\left(e^{-\tilde{\mathcal{S}}_f^c(t_i, X(t_i))} \leq p | \Lambda = \lambda\right). \tag{23}$$

□

Remark: Note that –in analogy with the previous subsection– each of the previous results can be reformulated easily when the linear truncate interest rate $\tilde{\mathcal{S}}_f^c[t, X(t)]$ is replaced by an ordinary truncate rate $\mathcal{S}_f^c[X(t)]$.

Some numerical examples of these convex bounds are shown in figures 2 and 3, for different choices of the parameters. Both figures consist of four plots of the distribution function of the original discrete annuity of (10)(simulated by means of a Monte-Carlo procedure) and the distribution functions for the three convex bounds as obtained in theorem 3.3. Figure 2 deals with the case of an ordinary truncate stochastic interest rate, while in figure 3 the plots are made for linear truncate stochastic interest rates.

upper (__), lower (__), improved upper(- -), simulated (- -) bound



(a) $\mu = \ln 1.05, \sigma = 0.05, t = 1, n = 12$ (b) $\mu = \ln 1.08, \sigma = 0.05, t = 1, n = 12$

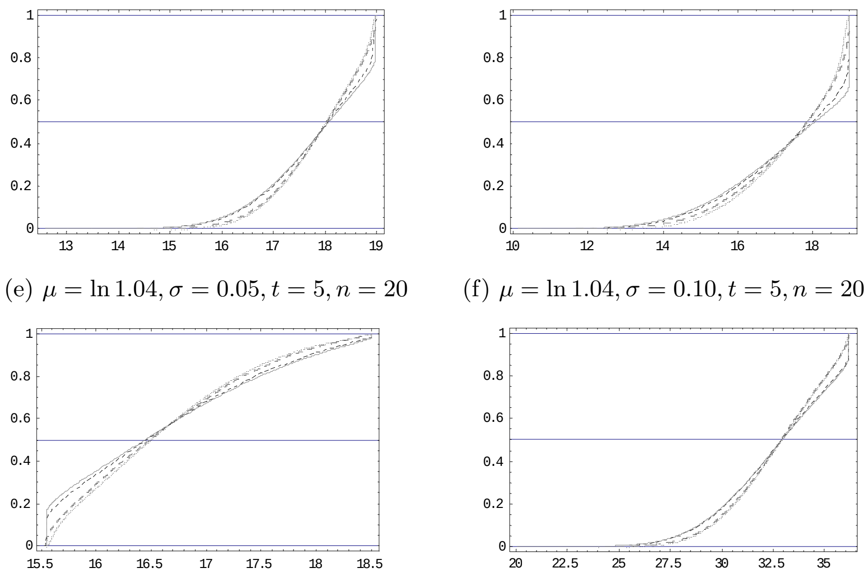
(c) $\mu = \ln 1.05, \sigma = 0.10, t = 1, n = 12$ (d) $\mu = \ln 1.05, \sigma = 0.05, t = 2, n = 24$

Figure 2: Examples of annuities with Brownian motion, truncate interest rate

The four plots in figure 2 are considered in a Brownian context, where we change in each plot the values for one of the parameters μ, σ, t and n. The conditioning variable Λ is defined by its coefficients $a_i = 1 + \frac{i}{24}$; for the floor and cap the parameter values are $f = 0.02$ and $c = 0.3$. It can be seen that the improved upper bound and the lower bound are close to the simulation of the distribution. The bounds are more accurate the lower the volatility σ. Note in each plot the kink in the distribution functions, the position of which is proportional to the probability that the stochastic interest rate is smaller than f (in the case of a kink on the right) or greater than c (in the case of a kink on the left).

Similar results about the performances of the bounds can be observed in the plots of figure 3, where we used a lineair tuncate interest rate and a longer time horizon. The conditioning variable Λ here is defined by its coefficients $a_i = 20 - i/2$, with $i = 1, .., 20$ for plots (e),(f) and (g) and $i = 1, .., 40$ for plot (h). For the floor and cap, the values are $f = 0.02$ and $c = 0.3$ for the plots (e), (f), (h) and $f = 0.03$ and $c = 0.1$ for plot (g).

upper (__), lower (__), improved upper(- -), simulated (- -) bound



(e) $\mu = \ln 1.04, \sigma = 0.05, t = 5, n = 20$ (f) $\mu = \ln 1.04, \sigma = 0.10, t = 5, n = 20$

(g) $\mu = \ln 1.08, \sigma = 0.05, t = 5, n = 20$ (h) $\mu = \ln 1.04, \sigma = 0.05, t = 10, n = 40$

Figure 3: Examples of annuities with Brownian motion, lineair truncate interest rate

4 Applications and extensions

Consider a general discrete annuity over the time-interval $[0, t]$, with a linear truncate stochastic interest rate with floor f and cap c as defined in definition 2.2:

$$V^*(t) = \sum_{i=1}^{n} \alpha(t_i)\, e^{-\tilde{S}_f^c(t_i, X(t_i))}, \tag{24}$$

where $X = \{X(t)\}$ is a stochastic process with $X(t_i)$ denoting the compounded rate of return for the period $[0, t_i]$. Theorem 3.1 can be extended as follows:

Theorem 4.1 *The annuity of equation (24) can be bounded in convex ordering sense as*

$$V^*_{low}(t) \leq_{cx} V^*(t) \leq_{cx} V^*_{imupp}(t) \leq_{cx} V^*_{upp}(t), \tag{25}$$

where the stochastic bounds are determined by

$$\begin{cases} V^*_{upp}(t) & = & \sum_{i=1}^n \max(0,\alpha(t_i))\ e^{-\tilde{\mathcal{S}}^c_f(t_i,F^{-1+}_{X(t_i)}(1-U))} \\ & & \quad -\sum_{i=1}^n \min(0,-\alpha(t_i))\ e^{-\tilde{\mathcal{S}}^c_f(t_i,F^{-1}_{X(t_i)}(U))} \\ & & \\ V^*_{low}(t) & = & \sum_{i=1}^n \alpha(t_i)\ \mathrm{E}[e^{-\tilde{\mathcal{S}}^c_f(t_i,X(t_i))}|\Lambda] \\ & & \\ V^*_{imupp}(t) & = & \sum_{i=1}^n \max(0,\alpha(t_i))\ e^{-\tilde{\mathcal{S}}^c_f(t_i,F^{-1+}_{X(t_i)|\Lambda}(1-U))} \\ & & \quad -\sum_{i=1}^n \max(0,-\alpha(t_i))\ e^{-\tilde{\mathcal{S}}^c_f(t_i,F^{-1}_{X(t_i)|\Lambda}(U))}, \end{cases} \tag{26}$$

where U is a uniform(0,1) distributed random variable, and where Λ is an arbitrary variable such that the distribution of $X(t_i)|\Lambda$ is known.

Applications of this more general result are obvious, e.g.

- for an indexed payment, use can be made of $\alpha(t) = (1+d_t)^t$, with d_t the indexing factor for the period $[t_{i-1}, t_i]$;
- for a life annuity, $\alpha(t) = {}_tp_x$, where ${}_tp_x$ is the classical notation used for the probability of a person of age x to be still alive after t years;
- for an indexed life annuity: $\alpha(t) = (1+d_t)^t \cdot_t p_x$;
- for a life assurance policy: $\alpha(t) = {}_tp_x \cdot \mu_{x+t}$, where μ_x is the mortality intensity at age x.

In figure 4, we illustrate the possibilities of these applications. The four plots deal with the distribution function of the present value of

(i) an indexed payment, yearly 3%, 24 payments;

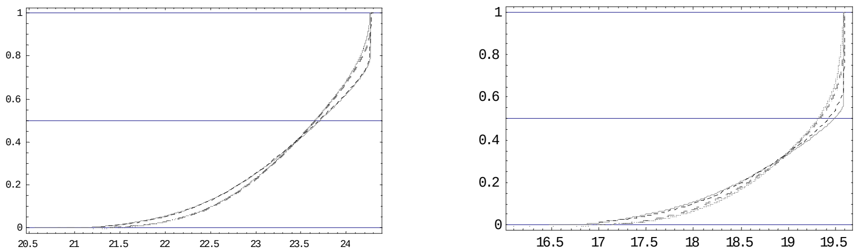
(j) a life annuity, age 35, 20 payments;

(k) an indexed life annuity, yearly 1.5%, age 30, 10 payments;

(l) a life assurance policy, age 40, duration of 20 years.

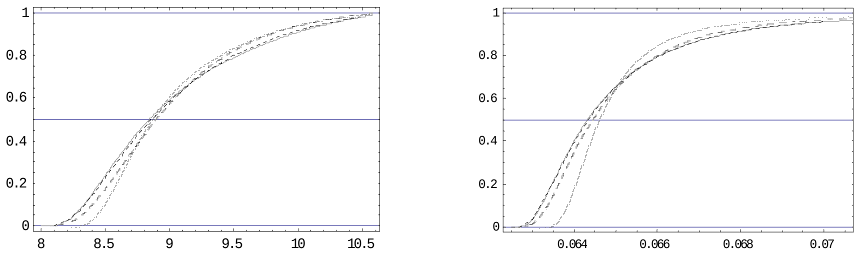
In order to conclude, we would like to mention that these results can be nicely extended, mainly in two directions. Firstly, the underlying stochastic process used to model the interest rates, can be modified. The use of a Vasicek or Ho-Lee model e.g. instead of a Brownian motion, seems to be more realistic. Secondly, also the function $\tilde{\mathcal{S}}^c_f$ can be altered, in order to deal with specific economic prerequisites, e.g. certain amortization schemes. Results about these and similar generalizations will be presented in forthcoming papers.

upper (—), lower (—), improved upper(- -), simulated (- -) bound



(i) indexed payment
$\mu = \ln 1.04, \sigma = 0.05, t = 20,$
$f = 0.02, c = 0.3$

(j) life annuity
$\mu = \ln 1.04, \sigma = 0.05, t = 2,$
$f = 0.02, c = 0.2$



(k) indexed life annuity
$\mu = \ln 1.04, \sigma = 0.1, t = 1,$
$f = 0.02, c = 0.2$

(l) life assurance policy
$\mu = \ln 1.04, \sigma = 0.05, t = 10,$
$f = 0.02, c = 0.3$

Figure 4: Examples of applications

Corresponding author

Inge Koch
Department of Applied Economics
Prinsstraat 13
B-2000 Antwerp
Belgium
tel : +32-3-220.41.33
fax : +32-3-220.48.17
email: inge.koch@ua.ac.be

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