The impact of lot splitting on lead times in a deterministic production line

Van Nieuwenhuyse Inneke

Research Assistant of the Fund For Scientific Research-Flanders (Belgium)

Faculty of Applied Economics UFSIA-RUCA, Antwerp, Belgium

Vandaele Nico

Faculty of Applied Economics UFSIA-RUCA, Antwerp, Belgium

Abstract: It is generally known that the mechanism of overlapping operations allows to reduce the average job lead times in a production system. However, a machine may remain idle between two consecutive sublots belonging to the same job, due to a difference between its own setup and processing times and those of the preceding machine. These idle times are referred to as "gaps". Until now, little formal results are available to calculate these "gaps" in terms of the system parameters and the transfer batch sizes used.

In this paper, we derive expressions for these gaps and the transfer batch lead times in a single-product production line with deterministic arrivals, setup and processing times. We prove that in our setting, the average transfer batch lead time is minimized when the transfer batch size equals one product unit.

Keywords: Transfer batching, Overlapping operations, Lead time management

1. Introduction

Today, it is commonly known that batching decisions have an important impact on the performance of a production system. By setting batch sizes, one can influence the amount of variability in the system, which on its turn impacts performance measures such as product cycle times and work-in-process (WIP): increasing variability leads to higher product cycle times and (given a specific throughput rate) higher WIP, whereas reducing variability improves performance. Since cycle times and WIP levels are decisive factors for the responsiveness of any production environment, setting batch sizes in a production system is an important control mechanism (e.g. Hopp and Spearman [4]).

In studying the impact of batching decisions, one should make a distinction between two types of batches: *process batches* and *transfer batches*. A process batch (also referred to as a *production batch*) is defined as the quantity of a product produced on a machine without interruption by other items, whereas a transfer batch is the size of a sublot moved after production on one machine to another operation or machine (Kropp and Smunt [5]).

In the literature, it is already widely accepted that the use of transfer batches between consecutive machines on a product's routing can reduce product lead times through the mechanism of *overlapping operations* (e.g. Goldratt [2], Hopp and Spearman [4], Çetinkaya [1], Graves and Kostreva [3], Jacobs and Bragg [5], Litchfield and Narasimhan [7]), i.e. transportation of partial batches to the next station is allowed while work proceeds at the upstream work station. This mechanism is also referred to as *"lot splitting"* (Kropp and Smunt [6], Smunt et al. [8]). Consequently, a setup can be initiated on the downstream machine as soon as this machine is idle and the first transfer batch of a particular production batch has arrived. Once a setup has been performed, the first transfer batch immediately starts processing. However, according to the definition of a production batch, the machine will have to process the entire production batch prior to switching to another product type.

This kind of policy may give rise to an interesting phenomenon: depending on the relative value of the setup and processing times on subsequent machines, it may happen that a machine remains idle between consecutive transfer batches belonging to the same production batch. We will refer to these idle times as "*gaps*". For example, in a production line with deterministic setup and processing times, gaps will occur on a machine whenever this machine has a higher production rate than its predecessor. In stochastic production environments, gaps may even occur when all machines on the line have the same production rate, due to the natural variability in the setup and processing times.

Although it is intuitively clear that using overlapping operations on a production line allows to reduce product lead times, a quantitative approach to the study of product lead times in such a system can not do without a more formal insight into the behavior of those gaps. As a starting point for the further analysis of production systems with transfer batching, this paper studies the flow of production batches through a production line with a single product type and overlapping operations, assuming that the interarrival times of production batches in front of the first machine, as well as the processing and setup times on the different machines are deterministic. Based on a flowchart of a particular example, expressions for the lead time of individual transfer batches are derived, taking into account the occurrence of gaps. The validity of the expressions has been tested by means of simulation.

The remainder of this paper will be organized as follows: in section 2, we give an overview of the assumptions and the notation used to describe the system. In section

3, expressions are derived to calculate the gaps on the different machines and the lead times of transfer batches through the system. This enables us to determine the optimal transfer batch sizing policy for a given production batch size, i.e. the transfer batch size that minimizes the average lead time of transfer batches through our system. In section 4, the expressions are applied to a fictitious example. Finally, section 5 gives an overview of the most important conclusions.

2. Assumptions and notation

It is assumed that the production line consists of M machines and one single product type. Products are processed on the consecutive machines in *production batches* of size N. Each machine m requires a setup time S_m ($m=1,\dots,M$) to be performed at the start of every production batch. The unit processing time on machine m will be denoted by X_m ($m=1,\dots,M$).

It is assumed that products arrive in production batches of size N in front of the first machine of the production line. The interarrival times of production batches are deterministic, as well as the setup times and the unit processing times on the different machines. Behind each machine, products are collected to form a *transfer batch* of size L ($L\beta$ N). L is supposed to be a common divisor of N, such that the single production batch is split into an integer number (T) of transfer batches ($T=N/L$). As soon as a transfer batch is complete, it is moved to the next machine (the transportation time between 2 consecutive machines is assumed to be zero). The transfer batch size used between the consecutive machines is presumed to stay the same throughout the product's routing.

As the system is completely deterministic, the mechanism of overlapping operations allows the setup on the next machine to start as soon as the first transfer batch of a given production batch has arrived.

3. Analysis

In order to clarify the analysis, we will start from a particular example. Figure I shows the flowchart of a production line consisting of four machines (*M=4*), with a production batch size equal to 6 product units (*N=6*) and a transfer batch size equal to 2 product units (*L=2)*. The setup and processing time data are given in Table I.

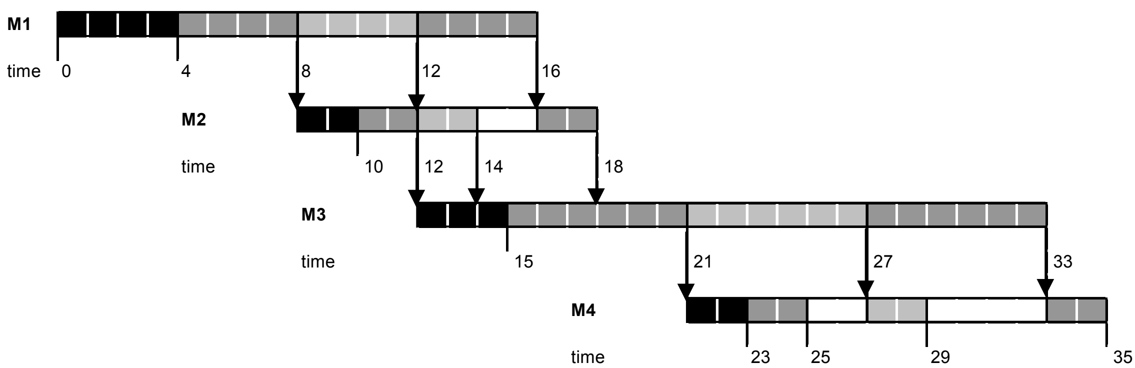


Figure I: Flowchart for the example

	M1	M2	M3	M4
Setup time *S*	4	2	3	2
Unit processing time *X*	2	1	3	1

Table I: Setup and processing time data (in time units) for the different machines

The black time blocks on the different machines refer to the setup times, the processing times for the transfer batches are indicated in dark and light grey alternately. The white time blocks indicate the gaps on the machines. As indicated by the vertical arrows between the machines, each transfer batch is moved to the next machine as soon as it has finished processing.

Figure I shows the flow of only one single production batch through the system. However, we know that in a completely deterministic and stable system, the flow pattern will be the same for each production batch going through the system: the cycle is simply repeated each time a new production batch arrives in front of the first machine in line.

In view of the analysis, the lead time of a transfer batch t $(t=1,\ldots,T)$ on an arbitrary machine m will be split up into 3 components:

- the time transfer batch t spends waiting in front of machine m because the machine is busy performing the setup ($Wait\_SU(m,t)$)
- the time transfer batch t spends waiting in front of machine m because the machine is busy processing a preceding transfer batch ($Wait\_Proc(m,t)$)
- the time transfer batch t spends in processing on machine m ($Proc(m,t)$)

As it is assumed that products arrive in production batches in front of the first machine, calculating these lead time components for an arbitrary transfer batch t on the first machine ($m=1$) is straightforward:

$$Wait\_SU(1,t)=S_1$$

$$Wait\_Proc(1,t)= (t-1)*(L*X_1)$$

$$Proc(1,t)= L*X_1$$

However, the lead time components on machine 2 to 4 are more complicated to derive. As can be seen from Figure I, both *Wait_SU(m,t)* and *Wait_Proc(m,t)* for a particular transfer batch t on a specific machine m $(m \geq 2)$ will depend on the gap that occurred in front of this transfer batch on the preceding machine ($m-1$).

To take care of the gaps in our lead time calculations, we define $G(m,t)$ as the total amount of idle time (or total "gap") occurring on machine m before transfer batch t. Note that, since $G(m,t)$ refers to the *total* amount of time that machine m has been idle, it always includes the gap occurring before transfer batch $(t-1)$:

$$G(m,t) >= G(m,t-1)$$

As we have assumed that products arrive in production batches in front of the first machine, the total gap occurring in front of an arbitrary transfer batch on $M1$ will always be equal to 0: $G(1,t)= 0\ (\forall t)$. Consequently, $G(m,t)$ only needs to be defined for $m=2$ to M.

To indicate the occurrence of a gap on machine m before transfer batch t, we introduce the 0/1 variable ■(m,t) for $m \geq 2$:

$$■(m,t) = 1 \qquad \text{if } G(m-1,t)+ (t-1)*L*(X_{m-1}-X_m)-S_m >0 \qquad (1)$$
$$= 0 \qquad \text{otherwise}$$

If ■$(m,t) = 1$, this indicates that idle time occurs before transfer batch t on machine m. Note that, in our setting, ■(m,t) only needs to be defined for $m \geq 2$. On the first machine, ■$(1,t) = 0\ (\forall t)$.

Expression (1) is only valid because we have assumed that the entire system is deterministic. This implies that the first transfer batch ($t=1$) of each production batch never has to wait in front of a machine until it becomes idle, such that the setup on machines 2 to 4 can start as soon as the first transfer batch has arrived.

For ease of notation, we introduce the variable $V(m,t)$ for $m \geq 2$:

$V(m,t)=(t-1)*L*(X_{m-1}-X_m)-S_m$ (2)

such that we can rewrite (1) for $m >= 2$ as:

$$\gamma(m,t) = 1 \quad \text{if } G(m-1,t)+ V(m,t) > 0 \tag{3}$$
$$= 0 \quad \text{otherwise}$$

By means of *(2)* and *(3),* the variable $G(m,t)$ can then be expressed as (for any transfer batch t and any machine $m \geq 2$):

$$G(m,t) = \max[0, V(m,t)+G(m-1,t)]$$
$$= \gamma(m,t) * [V(m,t)+G(m-1,t)] \tag{4}$$

Solving this expression recursively finally leads to the following expression for $G(m,t)$:

$$G(m,t) = \sum_{i=2}^{m}\left[\prod_{j=i}^{m}\gamma(j,t)\right]*V(i,t) \tag{5}$$

Note that $G(m,t)$ is only strictly larger than 0 if there exists an $i \geq 2$ such that for all $j \leq m$ and larger than or equal to i the value of $\gamma(j,t)$ is equal to 1. This implies that a non-zero gap, occurring in front of transfer batch t on an upstream machine u, will only influence the gap in front of that same transfer batch on an arbitrary downstream machine d if, for all machines m between u and d, the gap in front of transfer batch t is non-zero.

To calculate the different lead time components of a transfer batch t on a machine $m \geq 2$, we define a second 0/1 variable, ● (m,t) for $m \geq 2$:

● $(m,t) = 1$ if $S_m - (t-1)* L*(X_{m-1}) > G(m-1,t)$

(6)

$=0$ otherwise

If ● (m,t) equals 1, this indicates that transfer batch t arrives at machine m before the setup on this machine has been finished. In that case, *Wait_SU(m,t)* for this transfer batch t will be strictly larger than 0. Note again that expression (6) is only valid because we have assumed that the entire system is deterministic: consequently, each machine m is always idle upon the arrival of a first transfer batch of a production batch. Definition (6) is again only valid for $m \geq 2$. On the first machine, ● (m,t) equals 1 for all t: as products arrive in production batches in front of $M1$, all transfer batches experience the setup time on $M1$.

Given expression (6) and the expression for $G(m,t)$, the different lead time components for an arbitrary transfer batch t on a machine $m \geq 2$, can now be written as:

$$Wait\_SU(m,t) = ●(m,t) * [\, S_m - (t-1) * (L * X_{m-1}) - G(m-1,t)] \quad (7)$$

$$Wait\_Proc(m,t) = ●(m,t) * (t-1) * (L*X_m) + (1 - ●(m,t)) * [\, G(m,t) - G(m-1,t) - V(m,t)] \quad (8)$$

$$Proc(m,t) = L*X_m \quad (9)$$

The total lead time W_t of an individual transfer batch t can now be calculated by adding up the different lead time components over all machines m. Given our assumption that the first transfer batch of the production batch never has to queue in front of a machine, W_t can be written as:

$$W_t = \sum_{m=1}^{M} S_m + \sum_{m=1}^{M-1} (L * X_m) + t * L * X_M + G(M,t) \quad (10)$$

The average transfer batch lead time $E(W)$ is given by:

$$E(W) = \frac{\sum_{t=1}^{T} W_t}{T} = \sum_{m=1}^{M} S_m + \sum_{m=1}^{M-1} (L * X_m) + \frac{1}{2} * L * (T+1) * X_M + \frac{1}{T} * \sum_{t=1}^{T} \left[\sum_{i=1}^{M} (\prod_{j=i}^{M} \gamma(j,t)) * V(i,t) \right] \quad (11)$$

It can be proved that, in our setting, *E(W)* will be minimized when we make maximum use of overlapping operations, or, stated differently, if we split any given production batch size *N* in the largest possible number of transfer batches (i.e. *L=1*, *T=N*).

In order to prove this conjecture, we split it up in the following two conjectures which, if combined, are equivalent:

- *Conjecture 1:* In a deterministic setting, the average transfer batch lead time decreases with lot splitting;
- *Conjecture 2:* Splitting a production batch size in an ever larger number of sublots consistently leads to an extra lead time gain.

The proof of conjecture 1 is given in Appendix 1, the proof of conjecture 2 is given in Appendix 2.

4. Results for the example

In this section, we give an overview of the results for the example described in Section 3. The flowchart of the system and the setup and processing time data are given in Figure I and Table I.

	Machine 1	**Machine 2**	**Machine 3**	**Machine 4**

t=1	4	2	3	2
t=2	4	0	1	0
t=3	4	0	0	0

Table II: Overview of waiting times (in time units) during setup on the different machines, for *t=1,…,3*

The lead time components for the different transfer batches *(t=1,…,3)* on the different machines are given in Table II (*Wait_SU*), Table III (*Wait_Proc*),Table IV (*gaps*). Given the production batch size of 6 units and the transfer batch size of 2 units, the resulting average transfer batch lead time *E(W)* equals 29.67 time units.

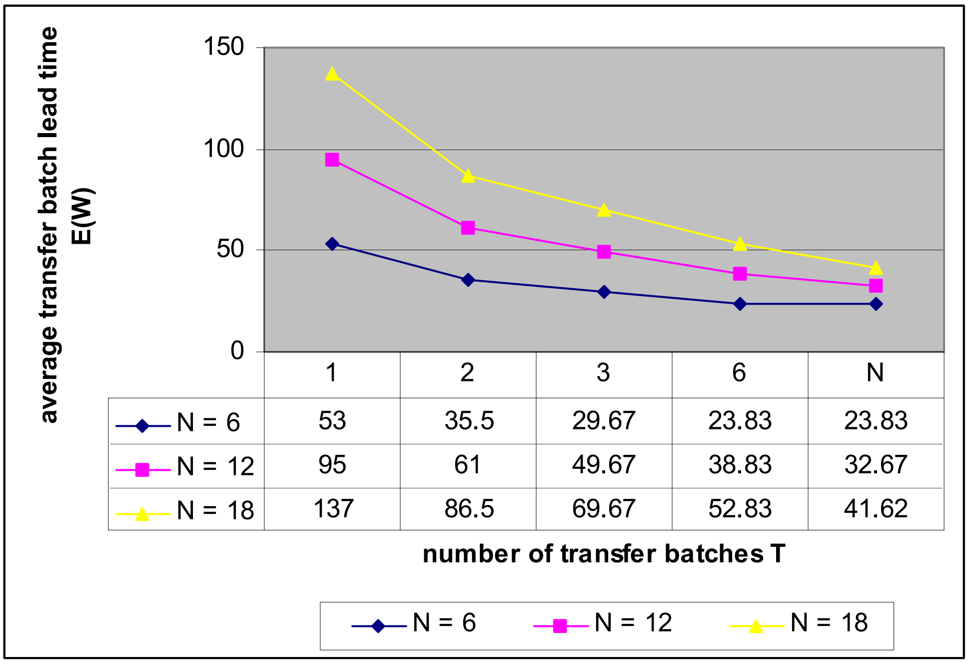
	Machine 1	**Machine 2**	**Machine 3**	**Machine 4**
t=1	0	0	0	0
t=2	4	0	6	0
t=3	8	0	9	0

Table III: Overview of waiting times (in time units) during processing on the different machines, for *t=1,…,3*

	Machine 1	**Machine 2**	**Machine 3**	**Machine 4**
t=1	0	0	0	0
t=2	0	0	0	2
t=3	0	2	0	6

Table IV: Overview of the total gaps (in time units) occurring in front of transfer batch *t* on the different machines, for $t=1,\ldots,3$

Figure II gives an overview of *E(W)* for different values of *N* (*N* = *6, 12* or *18*) and different values of *T* (*T* = *1, 2, 3, 6* or *N*). The figure illustrates that *E(W)* is indeed decreasing in *T* for a fixed production batch size *N*. Moreover, *E(W)* is increasing in *N*, a result which is already well known in the current literature.



	1	2	3	6	N
N = 6	53	35.5	29.67	23.83	23.83
N = 12	95	61	49.67	38.83	32.67
N = 18	137	86.5	69.67	52.83	41.62

Figure II: The average transfer batch lead time *E(W)* (in time units) for different values of the production batch size *N* and different number of transfer batches *T*

Consequently, two different strategies can be used to reduce lead times in deterministic production lines: if machine capacity is the main bottleneck in the system, it is often difficult to reduce production batch sizes, implying that a reduction in transfer batch sizes is the most appropriate strategy. If, on the other hand, transportation capacity is a bottleneck, very small transfer batch sizes might be infeasible, but a reduction in the production batch size may lead to improved performance.

5. Conclusions

In the preceding analysis, we have been able to derive expressions for the calculation of gaps and transfer batch lead times in a production line with deterministic setup and processing times, assuming that the machines are always idle upon the arrival of the first transfer batch of a production batch. It turns out that in this kind of system, for any arbitrary production batch size, the average transfer batch lead time is minimized when the production batch is split up in the largest possible number of transfer batches (i.e., transfer batch size equal to one product unit).

Although the mechanism of lot splitting and overlapping operations may induce idle times on the machines, it at the same time ensures that, whenever a transfer batch has finished processing on a machine, it can start processing as early as possible on the next machine. So, although the occurrence of gaps seems to introduce inefficiencies in the system, it does not have a negative impact on the lead times in our system.

However, it is important to note that this observation probably no longer holds when we change the layout of the production system from a production line to a job-shop layout and introduce multiple product types, each with its own product routing, in the system. In that case, gaps occurring on a particular machine between consecutive transfer batches of a particular product type will have an impact on the lead time of the other product types competing for capacity on that same machine, for during these gaps the machine is not available although it is idle. Consequently, there will be a trade-off between the lead time gain obtained by means of overlapping operations from the viewpoint of one product type, and the possible lead time losses that this implies for other product types.

Further research will focus on assessing the usefulness of transfer batching and overlapping operations for lead time reduction in this kind of systems. It is however obvious that the methodology used above is no longer appropriate for this type of analysis, such that we will have to resort to a different approach.

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Appendix 1: Proof of conjecture 1

Conjecture 1: *In a deterministic setting, the average transfer batch lead time decreases with lot splitting.*

In order to prove conjecture 1, we first rewrite expression (11) in terms of *N* and *T*:

$$\begin{aligned} &E(W) \\ &= \sum_{m=1}^{M} S_m + \sum_{m=1}^{M-1} (\frac{N}{T} * X_m) + \frac{1}{2} * N * (1 + \frac{1}{T}) * X_M + \frac{1}{T} * \sum_{t=1}^{T} G(M,t) \end{aligned} \quad (12)$$

In the absence of lot splitting, *T=1* and the transfer batch lead time is equal to the production batch lead time. Expression (12) then reduces to:

$$E(W)\big|_{T=1} = \sum_{m=1}^{M} S_m + \sum_{m=1}^{M} N * X_m \quad (13)$$

The gain in lead time obtained by the use of lot splitting is given by:

$$\begin{aligned} \Delta E(W) &= E(W)\big|_{T=1} - E(W) \\ &= N * (1 - \frac{1}{T}) * \sum_{m=1}^{M-1} X_m + \frac{1}{2} * (1 - \frac{1}{T}) * N * X_M - \frac{1}{T} * \sum_{t=1}^{T} G(M,t) \end{aligned} \quad (14)$$

To prove conjecture 1, it suffices to show that this expression is always positive for *T* larger than or equal to *2*, or stated differently:

$$N * (1 - \frac{1}{T}) * \sum_{m=1}^{M-1} X_m + \frac{1}{2} * (1 - \frac{1}{T}) * N * X_M \geq \frac{1}{T} * \sum_{t=1}^{T} G(M,t) \quad (15)$$

We also know that, for $\forall t < T$, $G(M,t) \leq G(M,T)$ such that:

$$\frac{1}{T} * \sum_{t=1}^{T} G(M,t) \leq G(M,T)$$

Consequently, expression (15) can be replaced by the following, more severe inequality:

$$N * (1 - \frac{1}{T}) * \sum_{m=1}^{M-1} X_m + \frac{1}{2} * (1 - \frac{1}{T}) * N * X_M \geq G(M,T) \qquad (16)$$

In expression (16), $G(M,T)$ is equal to:

$$G(M,T) = \max\left[0, (T-1) * \frac{N}{T}(X_{M-1} - X_M) - S_M + G(M-1,T)\right]$$

with $\forall m,\ 2 \leq m \leq M$:

$$G(m,T) = \max\left[0, (T-1) * \frac{N}{T}(X_{m-1} - X_m) - S_m + G(m-1,T)\right]$$

and

$$G(1,T) = 0$$

If $G(M,T)$ is equal to 0, inequality (16) is obviously true. If $G(M,T)>0$, it can be rewritten as:

$$G(M,T) = \sum_{m=z}^{M} (T-1) * \frac{N}{T} * \left(X_{m-1} - X_m\right) - \sum_{m=z}^{M} S_m \qquad (17)$$

in which case machine $z-1$ $(1 \leq z-1 \leq T-1)$ is the last machine in line for which $G(z-1,T)$ is equal to 0. Expression (16) then reduces to:

$$N * (1 - \frac{1}{T}) * \sum_{m=1}^{M-1} X_m + \frac{1}{2} * (1 - \frac{1}{T}) * N * X_M \geq N * (1 - \frac{1}{T}) * (X_{z-1} - X_M) - \sum_{m=z}^{T} S_m$$

or

$$N * (1 - \frac{1}{T}) * (\sum_{m=1}^{z-2} X_m + \sum_{m=z}^{M-1} X_m) + \frac{3}{2} * (1 - \frac{1}{T}) * N * X_M + \sum_{m=z}^{T} S_m \geq 0 \qquad (18)$$

This inequality is always true, as each of the terms is strictly larger than 0. Consequently, we can conclude that expression (16) is always true, which proves conjecture 1.

Appendix 2: Proof of conjecture 2

Conjecture 2: Splitting a production batch size in an ever larger number of sublots consistently leads to an extra lead time gain.

To prove conjecture 2, it suffices to show that the lead time gain obtained by splitting a production batch in P sublots $(X < P \leq N)$ is larger than the lead time gain obtained by splitting a production batch in X sublots $(2 \leq X < N)$.

We refer to expression (14), which yields:

$$\Delta E(W)_{T=P} = E(W)\big|_{T=P} - E(W)$$
$$= N*(1-\frac{1}{P})*\sum_{m=1}^{M-1} X_m + \frac{1}{2}*(1-\frac{1}{P})*N*X_M - \frac{1}{P}*\sum_{t=1}^{P} G(M,t)$$
$$\Delta E(W)_{T=X} = E(W)\big|_{T=X} - E(W)$$
$$= N*(1-\frac{1}{X})*\sum_{m=1}^{M-1} X_m + \frac{1}{2}*(1-\frac{1}{X})*N*X_M - \frac{1}{X}*\sum_{t=1}^{X} G(M,t)$$

Consequently, the incremental lead time gain obtained by splitting the production batch in a larger number of sublots is equal to:

$$\Delta E(W)_{T=P} - \Delta E(W)_{T=X}$$
$$= \left[\frac{1}{X} - \frac{1}{P}\right]*\left[N*\sum_{m=1}^{M-1} X_m + \frac{1}{2}*N*X_M + \sum_{t=1}^{P} G(M,t) + \sum_{t=1}^{X} G(M,t)\right] \tag{19}$$

As $X < P$, the first factor is always strictly larger than 0. The second factor is also strictly larger than 0, as the first two terms are strictly larger than 0 and the last two terms are larger than or equal to 0.

Consequently, we can conclude that for $P > X$, $\Delta E(W)_{T=P} - \Delta E(W)_{T=X}$ is always >0, which proves conjecture 2.