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HIGGS ALGEBRAS IN CLASSICAL HARMONIC ANALYSIS

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Dedicated to John Ryan

Abstract. In this paper, we will prove that the reproducing kernels $Z_k(\underline{x}, \underline{u})$ for the spaces $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of k-homogeneous harmonics can be seen as elements of an infinite-dimensional ladder operator representation for a cubic PAMA (polynomial angular momentum algebra) which is known as the Higgs algebra. This algebra will be shown to be one of two direct summands in a transvector algebra which is related to the harmonic Fischer decomposition in two vector variables.

1. Introduction

Classical Clifford analysis is often described as a higher-dimensional function theory refining harmonic analysis on $\mathbb{R}^m$, and generalising complex analysis in the plane. Historically speaking, this connection with complex analysis has served as an important source of inspiration during the first few decades. This started with quaternionic analysis, for which Fueter is often quoted as a prime instigator, but expanded into a mathematical framework in arbitrary dimension (see for instance [2, 7, 10, 11]). Whereas these more 'classical' approaches towards Clifford analysis focused on the 'classical' Dirac operator (a generalisation of the operator $\partial\!\!\!/$ on $\mathbb{R}^{1,3}$ introduced by Dirac), it has become clear in the past few decades that there are interesting extensions possible in several different directions. Without claiming completeness, we refer for instance to the Dirac operator in superspace (a combination of fermionic and bosonic variables), the so-called Hermitean and quaternionic Hermitean Dirac operators (based on a suitable reduction of the symmetry group), the higher spin Dirac operators (other conformally invariant generalisations, of which the Rarita-Schwinger operator is the most well-known example), Dunkl-Dirac operators (symmetry reductions to a finite subgroup of the orthogonal group), q-deformed Dirac operators (based on generalised commutation relations between variables and partial derivatives) and symplectic Dirac operators (switching from the orthogonal to the symplectic group).

Although these theories branch into several (seemingly unrelated) directions, there are a few common themes. Not only the results share a common ground (for instance having a Fischer decomposition is an important result in most function theories), but also the underlying techniques. The latter are usually algebraic in nature, whereby choosing a certain group or Lie (super)algebra somehow fixes the intricacies of the resulting function theory. This means that Clifford analysis can also be seen as a function theoretical framework in which models for algebraic structures are studied, an observation which allows to study these structures in terms of for instance functions belonging to the kernel of a collection of invariant operators. The contents of this paper should be understood from this point of view: we will relate reproducing kernels for spaces of harmonics to an algebraic structure known as a PAMA (short for polynomial angular momentum algebra). In full generality, these are 'ladder operator' algebras satisfying the relations

$$[K_0, K^{\pm}] = \pm 2K^{\pm} \qquad \text{and} \qquad [K^+, K^-] = \sum_{j=0}^{n} c_j K_0^j \ ,$$

whereby the classical $\mathfrak{sl}(2)$-relation $[K^+, K^-] = K_0$ is a special case. One can thus interpret a PAMA as a polynomial deformation of a classical Lie algebra. Special cases of these algebras already made their appearance in Clifford analysis: the Racah and Bannai-Ito algebras studied in for instance [6], the transvector algebra introduced in [5] and the Higgs algebra used in [9] to realise the Pizzetti formula on the (oriented) Grassmann manifold $\mathrm{Gr}_0(m, 2)$ are examples of algebras which generalise the more ubiquitous Lie algebras appearing in Clifford analysis.

In this paper we will show that the Higgs algebra is in a sense canonically connected to harmonic analysis. Not only will we relate this (cubic) algebra to the ladder operators for harmonic reproducing kernels (for the Fischer inner product), we will also revisit the transvector algebra defined in [5] and show that this is a direct sum of two Higgs algebras.

2. The Jordan-Schwinger realisation

Starting from two (not necessarily different) *commuting* realisations for the Lie algebra $\mathfrak{sl}(2)$, one can define a cubic PAMA known in the literature as the Higgs algebra. This algebra first appeared in [12] to describe symmetries for the so-called isotropic oscillator, and has since then been connected to for instance $\mathrm{SU}_q(2)$ and generalised oscillator algebras (see [13, 1]). The method described in this section is completely general (i.e. can be described in terms of abstract generators), but will come in handy later when we have concrete realisations for the Lie algebra $\mathfrak{sl}(2)$. Let $\mathsf{A}_j = \mathsf{Alg}(X_j, Y_j, H_j)$ with $j \in \{1, 2\}$ be a realisation for $\mathfrak{sl}(2)$, whereby the classical commutation relations are satisfied:

$$[H_j, X_j] = +2X_j \qquad [H_j, Y_j] = -2Y_j \qquad [X_j, Y_j] = H_j \ .$$

We can then introduce generators (for a 'new' algebraic structure) in $\mathsf{A} \otimes \mathsf{A}$ with $\mathsf{A} = \mathsf{A}_1 \oplus \mathsf{A}_2$ the vector space defined as the sum of both realisations for $\mathfrak{sl}(2)$. As the Lie-generators are supposed to be mutually commuting, we will omit the tensor product symbol and define the following elements:

$$\mathsf{K}^+ = X_1 Y_2 \qquad \mathsf{K}^- = Y_1 X_2 \qquad \mathsf{K}_0 = \frac{1}{2}(H_1 - H_2) \qquad \mathsf{C}_0 = \frac{1}{2}(H_1 + H_2) \ .$$

First of all, an easy calculation quickly leads to the relations $[\mathsf{K}_0, \mathsf{K}^{\pm}] = \pm 2\mathsf{K}^{\pm}$. Moreover, it is readily verified that the element C_0 is central, as it commutes with $\mathsf{K}^{\pm}$. The third relation is more complicated and brings us outside the realm of typical Lie algebra commutation relations. Let us introduce the Casimir operator $\mathcal{C}_j \in \mathcal{U}(\mathfrak{sl}(2))$ for $j \in \{1, 2\}$, by means of

$$\mathcal{C}_j = H_j^2 + 2\{X_j, Y_j\} = H_j(H_j - 2) + 4X_jY_j = H_j(H_j + 2) + 4Y_jX_j \ .$$

Note that the subscript does not refer to the degree of the Casimir operator here (as usual in this context), but rather to the realisation for the Lie algebra $\mathfrak{sl}(2)$

we are using. A quick calculation then shows that

$$\begin{aligned}[\mathsf{K}^+,\mathsf{K}^-] &= [X_1Y_2, Y_1X_2] = H_1X_2Y_2 - H_2X_1Y_1\\ &= \frac{1}{4}H_1\big(\mathcal{C}_2 - H_2(H_2-2)\big) - \frac{1}{4}H_2\big(\mathcal{C}_1 - H_1(H_1-2)\big)\ .\end{aligned}$$

We can now use the fact that H_1 and H_2 can be rewritten in terms of K_0 and the central element C_0 to arrive at the relation

$$\begin{aligned}[\mathsf{K}^+,\mathsf{K}^-] &= \frac{1}{4}(\mathsf{K}_0+\mathsf{C}_0)\Big(\mathcal{C}_2 - (\mathsf{C}_0-\mathsf{K}_0)^2 + 2(\mathsf{C}_0-\mathsf{K}_0)\Big)\\ &\quad - \frac{1}{4}(\mathsf{C}_0-\mathsf{K}_0)\Big(\mathcal{C}_1 - (\mathsf{K}_0+\mathsf{C}_0)^2 + 2(\mathsf{K}_0+\mathsf{C}_0)\Big)\\ &= -\frac{1}{2}\mathsf{K}_0^3 + \frac{1}{2}\mathsf{K}_0\left(\mathsf{C}_0^2 + \frac{1}{2}(\mathcal{C}_1+\mathcal{C}_2)\right) - \frac{1}{4}\mathsf{C}_0(\mathcal{C}_1-\mathcal{C}_2)\ .\end{aligned}$$

What this says is that the commutator $[\mathsf{K}^+,\mathsf{K}^-]$ gives a *cubic* term plus a linear term and 'a constant' (expressed in terms of central elements).

3. The conformal Lie algebra

In this section we will consider the Lie algebra of generalised symmetries for the Laplace operator Δ_x on $\mathbb{R}^m$. It is well-known that this operator is *conformally* invariant, which means that its (generalised) symmetries realise a copy of the (real) Lie algebra $\mathfrak{so}(1,m+1)$ inside the Weyl algebra $\mathsf{Alg}(x_i,\partial_{x_j} : 1 \leq i,j \leq m)$. We briefly recall the following:

Definition 3.1. An operator φ is a generalised symmetry for a (differential) operator $\mathcal{D}$ if there exists another operator ψ such that $[\varphi,\mathcal{D}] = \psi\mathcal{D}$.

Note that for $\psi = 0$ one obtains a so-called *proper* symmetry for $\mathcal{D}$, and that although generalised symmetries do not commute with $\mathcal{D}$ one still has that $\varphi \in \mathrm{End}(\ker\mathcal{D})$ maps solutions for $\mathcal{D}$ to solutions. The first-order generalised symmetries for the Laplace operator, which generate a corresponding Lie algebra, belong to one of the following spaces:

$$\begin{aligned}\mathbb{R}^m &= \mathrm{span}\big(\xi_a := |\underline{x}|^2\partial_{x_a} - x_a(2\mathbb{E}_x + m - 2) : 1 \leq a \leq m\big)\\ \mathfrak{so}(m) &= \mathrm{Alg}\big(L_{ab} := x_a\partial_{x_b} - x_b\partial_{x_a} : 1 \leq a < b \leq m\big)\\ \mathbb{R} &= \mathrm{span}\big(\mathbb{E}_x + \frac{m}{2} - 1\big)\\ \mathbb{R}^m &= \mathrm{span}\big(\partial_{x_a} : 1 \leq a \leq m\big)\ .\end{aligned}$$

The operator $\mathbb{E}_x = \sum_a x_a\partial_{x_a}$ is the Euler operator, acting as a constant on functions (such as polynomials) which are homogeneous of a fixed degree. Note also that $\xi_a = \mathcal{I}\partial_{x_a}\mathcal{I}$, with $\mathcal{I}$ the harmonic Kelvin inversion (a conjugation of the partial derivative with the inversion). These symmetries ξ_a are the so-called special conformal transformations. The full algebra of conformal symmetries for the Laplace operator, a subalgebra of the universal enveloping algebra for $\mathfrak{so}(1,m+1)$, has been determined by Eastwood in [8]. An easy calculation leads to the following:

Lemma 3.2. *For a fixed index $1 \leq a \leq m$, we have that*

$$\mathfrak{sl}(2) \cong \mathsf{Alg}(\xi_a, \partial_{x_a}, 2\mathbb{E}_x + m - 2)\ ,$$

whereby the operators are listed in the order X, Y and H.

In a later section, we will use these Lie algebras to construct a Higgs algebra, using the realisation from the previous section. We will also consider harmonic functions (polynomials, to be more precise) in two vector variables in $\mathbb{R}^m$. This then means that we will have access to two copies of the Lie algebra $\mathfrak{so}(1, m+1)$, in the variables $\underline{x}$ and $\underline{u}$ respectively. Since both Lie algebras contain two copies of the abelian subalgebra $\mathbb{R}^m$ each, one can consider four 'mixed' operators which are orthogonally invariant: these then appear as Euclidean inner products, i.e. contractions $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. In order to have a compact notation for these four operators, we first introduce the notations $\xi_{a,x}$ and $\xi_{a,u}$ for the generalised comformal transformations (this means that an additional letter is added to the symbol ξ_a to make clear which vector variable is being considered).

Definition 3.3. The following differential operators are orthogonally invariant endomorphisms acting on functions $f(\underline{x}, \underline{u})$ in $\ker \Delta_x \cap \ker \Delta_u$:

$$S_x := \sum_{a=1}^m \xi_{a,x}\partial_{u_a} \quad S_u := \sum_{a=1}^m \xi_{a,u}\partial_{x_a} \quad C := \sum_{a=1}^m \xi_{x,a}\xi_{u,a} \quad A := \sum_{a=1}^m \partial_{x_a}\partial_{u_a} \ .$$

Note that these operators were used in [5] to obtain a new type of Howe duality for harmonic functions in two vector variables, whereby the classical dual partner $\mathfrak{sp}(4)$ was replaced by a transvector algebra (which made it possible to define a Pizzetti formula for the integral over the Stiefel manifold $\mathrm{St}(m, 2)$, see also [4] for a different approach). In what follows, we will consider the PAMA spanned by these operators and investigate the connection with the reproducing kernel for the space of k-homogeneous (polynomial) harmonics.

4. The harmonic reproducing kernel

In this section, we switch our attention to the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of k-homogeneous harmonics on $\mathbb{R}^m$, i.e.

$$\mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) = \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) \cap \ker \Delta_x \ ,$$

with $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) = \mathbb{C}_k[x_1, \ldots, x_m]$ the space of k-homogeneous polynomials in the variable $\underline{x} \in \mathbb{R}^m$. In order to talk about reproducing kernels, one needs an inner product and for polynomials on $\mathbb{R}^m$ this is classically done in terms of the so-called Fischer inner product:

$$\langle \cdot, \cdot \rangle_F : \mathcal{P}(\mathbb{R}^m, \mathbb{C}) \times \mathcal{P}(\mathbb{R}^m, \mathbb{C}) \to \mathbb{C} : \big(P(\underline{x}), Q(\underline{x})\big) \mapsto \left[\overline{P(\underline{\partial}_x)} Q(\underline{x})\right]_0 ,$$

which means that we let the 'dual' of $P(\underline{x})$, the operator obtained by replacing each variable by a partial derivative, act on the polynomial after which we just put $\underline{x} = \underline{0}$ to obtain a scalar. It is easy to see that homogeneous polynomials of different degree are orthogonal with respect to this inner product, but one can also show that the spaces $|\underline{x}|^{2p}\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ and $|\underline{x}|^{2q}\mathcal{H}_\ell(\mathbb{R}^m, \mathbb{C})$ are orthogonal to each other for $k \neq \ell$, even if $2p + k = 2q + \ell$ (without going into too much detail, this means that all summands in the harmonic Fischer decomposition are orthogonal to each other). With respect to this inner product, the (symmetric) polynomial

$$K_k(\underline{x}, \underline{u}) := \frac{1}{k!}\langle \underline{x}, \underline{u} \rangle^k$$

acts as a reproducing kernel for the space $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C})$, in the sense that

$$\langle K_k(\underline{x}, \underline{u}), P_k(\underline{u})\rangle_F = P_k(\underline{x})\ .$$

To obtain a reproducing kernel for the subspace of k-homogeneous harmonic polynomials it then suffices to project this kernel onto the space of harmonic polynomials (in either variable, the symmetry will automatically ensure that the result is harmonic in both $\underline{x}$ and $\underline{u} \in \mathbb{R}^m$). This reproducing kernel, which we will denote by means of $Z_k(\underline{x}, \underline{u})$ in what follows, is expressed in terms of Gegenbauer polynomials depending on the inner product $\langle \underline{x}, \underline{u}\rangle \in \mathbb{R}$.

Definition 4.1. The reproducing kernel for the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ is defined as

$$Z_k(\underline{x}, \underline{u}) := c_k |\underline{x}|^k |\underline{u}|^k C_k^{\frac{m-1}{2}}\left(\frac{\langle \underline{x}, \underline{u}\rangle}{|\underline{x}||\underline{u}|}\right)\ ,$$

where the constant c_k is given by

$$c_k = \frac{\Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m-1}{2}\right)}\ .$$

One then has that $\langle Z_k(\underline{x}, \underline{u}), H_k(\underline{u})\rangle_F = H_k(\underline{x})$, for all $H_k(\underline{u}) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$.

Explicitely expanding this reproducing kernel, hereby using the definition for the Gegenbauer polynomial, one finds that

$$|\underline{x}|^k |\underline{u}|^k C_k^{\frac{m-1}{2}}\left(\frac{\langle \underline{x}, \underline{u}\rangle}{|\underline{x}||\underline{u}|}\right) = |\underline{x}|^k |\underline{u}|^k \left(\frac{2^k \Gamma\left(k + \frac{m-1}{2}\right)}{k!\Gamma\left(\frac{m-1}{2}\right)} \frac{\langle \underline{x}, \underline{u}\rangle^k}{|\underline{x}|^k |\underline{u}|^k} + \text{L.O.T.}\right)\ ,$$

which explains the constant c_k. Indeed, to arrive at the harmonic projection our kernel must be equal to $K_k(\underline{x}, \underline{u})$ plus correction terms (expressed in terms of lower order terms in $\langle \underline{x}, \underline{u}\rangle$ multiplied with squared norm factors). As can be seen in the following lemma, this kernel can (up to a constant) be obtained in terms of the 'raising operator' C from the previous section acting on the constant 1 (note that our symbol C stands for 'creation' here).

Lemma 4.2. *For each $k \in \mathbb{N}$, one has that*

$$Z_k(\underline{x}, \underline{u}) = \gamma_k C^k[1] = \gamma_k \left(\sum_{a=1}^m \xi_{x,a}\xi_{u,a}\right)^k [1]\ ,$$

where the constant γ_k is given by

$$\gamma_k = \frac{c_k^2}{k!} = \frac{1}{k!}\left(\frac{\Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m-1}{2}\right)}\right)^2\ .$$

Proof: first of all, we note that the action of C on 1 generates polynomials, as can be seen from the explicit definition for the conformal symmetry $\xi_a \in \mathfrak{so}(1, m+1)$. Moreover, this polynomial is clearly harmonic and symmetric in $\underline{x} \leftrightarrow \underline{u}$. It then suffices to note that it is also invariant under the action of the orthogonal group to conclude that it must be a multiple of the kernel $Z_k(\underline{x}, \underline{u})$. This is based on the (algebraic) observation that there is a unique summand $\mathbb{C} \subset \mathcal{H}_k \otimes \mathcal{H}_k$, where the spaces $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ are models for the irreducible $\mathfrak{so}(m)$-representations with highest weight $(k, 0, \ldots, 0)$. To retrieve the constant of proportionality, it suffices to note that the operator C can be written as

$$C = \langle \underline{x}, \underline{u}\rangle(2\mathbb{E}_x + m - 2)(2\mathbb{E}_u + m - 2) + \cdots\ ,$$

whereby the omitted terms will *not* contribute to the maximal power in $\langle \underline{x}, \underline{u} \rangle$. This means that under the k-fold action on $1 \in \mathbb{R}$, the leading term is given by

$$C^k[1] = [(m-2)m(m+2)\ldots(m+2k-4)]^2 \langle \underline{x}, \underline{u} \rangle^k + \text{L.O.T.} \ ,$$

which can be written as

$$C^k[1] = \left(\frac{2^k \Gamma\left(k + \frac{m-1}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \right)^2 \langle \underline{x}, \underline{u} \rangle^k + \text{L.O.T.}$$

This then leads to the constant γ_k mentioned above. □

Corollary 4.3. *If we define $\mathcal{H}(\mathbb{R}^m, \mathbb{C})$ as the space of harmonic polynomials, the direct sum of all spaces $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of a fixed degree (with $k \in \mathbb{N}$), the reproducing kernel for this space (with respect to the Fischer inner product) can formally be defined as*

$$\sum_{k=0}^{\infty} Z_k(\underline{x}, \underline{u}) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(\frac{1}{4}C)^k}{\left[\left(\frac{m-1}{2}\right)_k\right]^2}[1] \ ,$$

where $(a)_k$ stands for the Pochammer symbol $a(a+1)\ldots(a+k-1)$. This can be rewritten in terms of a particular hypergeometric function:

$$\sum_{k=0}^{\infty} Z_k(\underline{x}, \underline{u}) = {}_0F_2\left(; \frac{m-1}{2}, \frac{m-1}{2}; \frac{1}{4}C\right)[1] \ .$$

We now claim that the set $\mathcal{R}_H := \{Z_k(\underline{x}, \underline{u}) : k \in \mathbb{N}\}$ with the reproducing kernels for the spaces $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of k-homogeneous harmonic polynomials, can be seen as an infinite-dimensional representation space for a certain PAMA. For that purpose, we will prove a result which essentially tells us that the operators C and A act as raising and lowering operators between these reproducing kernels (note that the raising property clearly follows from the previous lemma). Let us first prove the following result, in which we consider the action of the so-called 'mixed Laplace-Beltrami operator' on our reproducing kernels (we adopt the notation from [3] to write this operator as $\Delta_{S,xu}$):

Lemma 4.4. *The operator $\Delta_{S,xu}$ acts as a constant on $Z_k(\underline{x}, \underline{u})$, given by*

$$\Delta_{S,xu} Z_k(\underline{x}, \underline{u}) = \left(\sum_{a<b} L_{ab}^{(x)} L_{ab}^{(u)} \right) [Z_k(\underline{x}, \underline{u})] = k(k+m-2) Z_k(\underline{x}, \underline{u}) \ .$$

Proof: the Casimir operator $\mathcal{C}_{\mathfrak{so}}$ for the regular representation on functions in $(\underline{x}, \underline{u}) \in \mathbb{R}^{2m}$, acting by means of the angular momentum operators, is given by

$$\mathcal{C}_{\mathfrak{so}} = \sum_{a<b} L_{ab}^2 = \sum_{a<b} (L_{ab}^{(x)} + L_{ab}^{(u)})^2 = \Delta_{S,x} + \Delta_{S,u} + 2\Delta_{S,xu} \ .$$

The (non-mixed) Laplace-Beltrami operator $\Delta_{S,x}$ (and $\Delta_{S,u}$), corresponding to the Casimir operator for the regular action on functions depending solely on $\underline{x}$ (resp. $\underline{u}$) can be expressed in terms of the Laplace operator:

$$\Delta_{S,x} = \sum_{a<b} \left(L_{ab}^{(x)}\right)^2 = |\underline{x}|^2 \Delta_x - \mathbb{E}_x(\mathbb{E}_x + m - 2) \ ,$$

and similarly for $\Delta_{S,u}$. This means that

$$\left(\sum_{a<b} L_{ab}^{(x)} L_{ab}^{(u)} \right) [Z_k(\underline{x}, \underline{u})] = \frac{1}{2} \left(\mathcal{C}_{\mathfrak{so}} - \Delta_{S,x} - \Delta_{S,u} \right) Z_k(\underline{x}, \underline{u}) \ .$$

It then suffices to observe that the Casimir operator $\mathcal{C}_{\mathfrak{so}}$ will act trivially (since $Z_k(\underline{x},\underline{u})$ comes from the action of C^k on $1 \in \mathbb{R}$, with $[\mathcal{C}_{\mathfrak{so}}, C] = 0$ and $\mathcal{C}_{\mathfrak{so}}[1] = 0$), and that for instance

$$\Delta_{S,x} Z_k(\underline{x},\underline{u}) = \big(|\underline{x}|^2\Delta_x - \mathbb{E}_x(\mathbb{E}_x + m - 2)\big) Z_k(\underline{x},\underline{u}) = -k(k+m-2) Z_k(\underline{x},\underline{u}) \ ,$$

because $Z_k(\underline{x},\underline{u})$ is harmonic in $\underline{x}$ (and $\underline{u}$). This leads to the desired result. □

Corollary 4.5. *The action of the mixed Laplace-Beltrami operator on $Z_k(\underline{x},\underline{u})$ can be written as*

$$\left(\sum_{a<b} L_{ab}^2\right)[Z_k(\underline{x},\underline{u})] = \frac{1}{4}(\mathbb{E}_x + \mathbb{E}_u)(\mathbb{E}_x + \mathbb{E}_u + 2m - 4)[Z_k(\underline{x},\underline{u})] \ .$$

Note that one can also omit the restriction that the action is only considered on the reproducing kernels $Z_k(\underline{x},\underline{u})$. This essentially means that $C_{\mathfrak{so}}$ is taken into account in the calculations above (it can be omitted when acting on the trivial representation, like in the lemma above).

Proposition 4.6. *The algebra generated by the operators $C = K^+$ and $A = K^-$, considered as ladder operators on harmonic functions in $(\underline{x},\underline{u}) \in \mathbb{R}^{2m}$ with equal degrees in $\underline{x}$ and $\underline{u}$, is a (cubic) PAMA defined by the following relations:*

$$[K_0, K^{\pm}] = \pm 2K^{\pm} \quad \text{and} \quad [K^+, K^-] = -2K_0^3 + (m-2)(m-4)K_0 \ ,$$

with $K_0 = \mathbb{E}_x + \mathbb{E}_u + m - 2$.

Proof: the relation $[K_0, K^{\pm}] = \pm 2K^{\pm}$ is trivially verified, so that we are left with the commutator:

$$[K^+, K^-] = [C, A] = \sum_{a=1}^{m}\sum_{b=1}^{m} [\xi_{a,x}\xi_{a,u}, \partial_{x_b}\partial_{u_b}] \ .$$

We will consider the cases $a = b$ and $a \neq b$ separately. First of all, for $a < b$ we invoke the relation $[\partial_{x_a}, \xi_{x,b}] = 2L_{ab}^{(x)}$ to calculate

$$\begin{aligned}
[\xi_{a,x}\xi_{a,u}, \partial_{x_b}\partial_{u_b}] &= \xi_{a,x}\partial_{x_b}[\xi_{a,u}, \partial_{u_b}] + [\xi_{a,x}, \partial_{x_b}]\partial_{u_b}\xi_{a,u} \\
&= 2\xi_{a,x}\partial_{x_b}L_{ab}^{(u)} + 2L_{ab}^{(x)}\partial_{u_b}\xi_{a,u} \\
&= 2\xi_{a,x}\partial_{x_b}L_{ab}^{(u)} + 2L_{ab}^{(x)}\xi_{a,u}\partial_{u,b} - 4L_{ab}^{(x)}L_{ab}^{(u)} \ .
\end{aligned}$$

To this expression, we then add the expression corresponding to the case $a > b$ (which corresponds to the same formula, but where a and b are swapped), which gives (still for $a \neq b$)

$$\begin{aligned}
&[\xi_{a,x}\xi_{a,u}, \partial_{x_b}\partial_{u_b}] + [\xi_{b,x}\xi_{b,u}, \partial_{x_a}\partial_{u_a}] \\
=& -8L_{ab}^{(x)}L_{ab}^{(u)} + 2\big(\xi_{a,x}\partial_{x_b} - \xi_{x,b}\partial_{x_a}\big)L_{ab}^{(u)} + 2\big(\xi_{a,u}\partial_{u_b} - \xi_{u,b}\partial_{u_a}\big)L_{ab}^{(x)} \ .
\end{aligned}$$

Using the definition for the conformal symmetry ξ_a, this can be reduced:

$$[\xi_{a,x}\xi_{a,u}, \partial_{x_b}\partial_{u_b}] + [\xi_{b,x}\xi_{b,u}, \partial_{x_a}\partial_{u_a}] = -4(\mathbb{E}_x + \mathbb{E}_u + m - 2)L_{ab}^{(x)}L_{ab}^{(u)} \ .$$

Summing over all $a < b$, the previous lemma tells us that the resulting operator will act as the constant $\mu_k := -4k(k+m-2)(2k+m-2)$ on $Z_k(\underline{x},\underline{u})$. Note that we hereby used the fact that this polynomial has equal degree in $\underline{x}$ and $\underline{u}$. Note that this expression is *cubic* in k. As a matter of fact, in terms of the operator K_0 we thus have that

$$\sum_{a\neq b}[\xi_{a,x}\xi_{a,u},\partial_{x_b}\partial_{u_b}] = -K_0\big(K_0^2-(m-2)^2\big)\ . \tag{4.1}$$

Let us then consider the 'missing' indices $a=b$ in our commutator $[A,C]$. We will use the Jordan-Schwinger realisation here, where our two commuting copies of $\mathfrak{sl}(2)$ are defined as follows (the index $1\leq a\leq m$ is fixed here, so we are in fact defining $2m$ Lie algebras in total):

$$\begin{aligned}\mathfrak{sl}_1(2) &= \mathsf{Alg}(X_1,Y_1,H_1) = \mathsf{Alg}\big(\xi_{a,x},\partial_{x_a},2\mathbb{E}_x+m-2\big)\\ \mathfrak{sl}_2(2) &= \mathsf{Alg}(X_2,Y_2,H_2) = \mathsf{Alg}\big(\partial_{u_a},\xi_{a,u},-2\mathbb{E}_u-(m-2)\big)\ .\end{aligned}$$

Note that the role played by the special conformal symmetry and the partial derivative is not the same for these realisations. The reason for this is that we can now define the operators $\mathsf{K}^+ = X_1Y_2 = \xi_{a,x}\xi_{a,u}$ and $\mathsf{K}^- = Y_1X_2 = \partial_{x_a}\partial_{u_a}$, which means that

$$[\xi_{a,x}\xi_{a,u},\partial_{x_a}\partial_{u_a}] = -\frac{1}{2}\mathsf{K}_0^3+\frac{1}{2}\mathsf{K}_0\left(\mathsf{C}_0^2+\frac{1}{2}(\mathcal{C}_1^{(a)}+\mathcal{C}_2^{(a)})\right)-\frac{1}{4}\mathsf{C}_0(\mathcal{C}_1^{(a)}-\mathcal{C}_2^{(a)})\ ,$$

where the correct identifications are still to be made:

$$\begin{aligned}\mathsf{K}_0 &= \frac{1}{2}(H_1-H_2) = \mathbb{E}_x+\mathbb{E}_u+m-2 = K_0\\ \mathsf{C}_0 &= \frac{1}{2}(H_1+H_2) = \mathbb{E}_x-\mathbb{E}_u = C_0\ .\end{aligned}$$

Note that we use two different fonts here: K and so on for the $\mathfrak{sl}(2)$-realisations from above, and italic capitals such as K for the operators we are interested in (i.e. defined in terms of A and C). Note also that C_0 acts trivially on $Z_k(\underline{x},\underline{u})$, as these kernels have equal degrees in $\underline{x}$ and $\underline{u}$. Finally, these Casimir operators are still to be determined. Note that these still depend on the index a (in contrast to the operators K_0 and C_0), but since we will eventually sum over this index we determine the following:

$$\begin{aligned}\sum_{a=1}^m\mathcal{C}_1^{(a)} &= mH_1(H_1-2)+4\sum_{a=1}^m(|\underline{x}|^2\partial_{x_a}-x_a(2\mathbb{E}_x+m-2))\partial_{x_a}\\ &= mH_1^2-2mH_1-4(2\mathbb{E}_x+m-4)\mathbb{E}_x\\ \sum_{a=1}^m\mathcal{C}_2^{(a)} &= mH_2(H_2+2)+4\sum_{a=1}^m(|\underline{u}|^2\partial_{u_a}-x_a(2\mathbb{E}_u+m-2))\partial_{u_a}\\ &= mH_2^2+2mH_2-4(2\mathbb{E}_u+m-4)\mathbb{E}_u\ ,\end{aligned}$$

where we have used that our operators will act on harmonic polynomials in $\underline{x}$ and $\underline{u}$ to omit the Laplace operators Δ_x and Δ_u. First of all, we have that

$$m(H_1^2+H_2^2)-2m(H_1-H_2) = m\big((\mathsf{K}_0+\mathsf{C}_0)^2+(\mathsf{K}_0-\mathsf{C}_0)^2\big)-4m\mathsf{K}_0\ .$$

Again invoking the fact that we restrict our attention to polynomials of equal degree in $\underline{x}$ and $\underline{u}$, which means that $\mathbb{E}_x$ and $\mathbb{E}_u$ can be used interchangeably, we thus also have that

$$-4(2\mathbb{E}_x+m-4)\mathbb{E}_x-4(2\mathbb{E}_u+m-4)\mathbb{E}_u = -4(\mathsf{K}_0-2)\big(\mathsf{K}_0-(m-2)\big)\ .$$

This means that we finally arrive at the following:

$$\begin{aligned}\sum_{a=1}^{m}[\xi_{a,x}\xi_{a,u},\partial_{x_a}\partial_{u_a}] &= -\frac{m}{2}\mathsf{K}_0^3+\frac{1}{4}\mathsf{K}_0\sum_{a=1}^{m}(\mathcal{C}_1^{(a)}+\mathcal{C}_2^{(a)})\\ &= -\mathsf{K}_0^3-2(m-2)\mathsf{K}_0\ .\end{aligned}$$

We can then add the previous result to (4.1) to arrive at the desired conclusion. □

As a simple test, we will let this commutator act on $Z_1(\underline{x},\underline{u}) = (m-2)^2\langle\underline{x},\underline{u}\rangle$ (which follows fom letting $C = K^+$ act on the constant $1 \in \mathbb{R}$). It is immediately seen that

$$K^- Z_1(\underline{x},\underline{u}) = (m-2)^2\langle\underline{\partial}_x,\underline{\partial}_u\rangle\langle\underline{x},\underline{u}\rangle = m(m-2)^2\ .$$

On the other hand, a direct calculation tells us that

$$K_0[1] = -2(m-2)^3+(m-4)(m-2)^2 = -m(m-2)^2\ .$$

It then suffices to see that $K^-Z_1 = K^-K^+(1) = [K^-,K^+](1) = -K_0(1)$.

5. The harmonic transvector algebra revisited

Note that one can also drop the condition that the degree in $\underline{x}$ and $\underline{u} \in \mathbb{R}^m$ must be equal. This will still lead to a (cubic) PAMA, with a slight modification. As a matter of fact, the only thing that changes is that the Casimir operator $\mathcal{C}_{\mathfrak{so}}$ will still appear in the commutation relation between K^+ and K^-. It is quite remarkable that one can still ignore $C_0 = \mathbb{E}_x - \mathbb{E}_u$, despite the fact that de degrees in $\underline{x}$ and $\underline{u}$ are not necessarily equal. The reason for this is the following: the operators $K^{\pm}$ are symmetric in $(\underline{x},\underline{u})$, whereas C_0 is not. Hence, $[K^+,K^-]$ is still symmetric and thus cannot contain C_0.

Theorem 5.1. *The algebra spanned by K^+ and $K^- \in End\big(\ker\Delta_x \cap \ker\Delta_u\big)$ is a Higgs algebra (i.e. a cubic PAMA), defined in terms of the following relations:*

$$[K_0,K^{\pm}] = \pm 2K^{\pm} \quad \text{and} \quad [K^+,K^-] = -2K_0^3+\big((m-2)(m-4)-2\mathcal{C}_{\mathfrak{so}}\big)K_0\ ,$$

with $K_0 = \mathbb{E}_x + \mathbb{E}_u + m - 2$ and $\mathcal{C}_{\mathfrak{so}}$ the Casimir operator for $\mathfrak{so}(m)$ defined in terms of the regular representation on functions in two vector variables.

We can again consider a simple example here, whereby we will let K^+ act on the (harmonic) polynomial x_1. Note that since

$$\mathcal{C}_{\mathfrak{so}}\Big|_{\mathcal{H}_k} = -k(k+m-2)\mathrm{Id}\Big|_{\mathcal{H}_k}\ ,$$

it is immediately clear that $[K^+,K^-]x_1 = -(m-1)(m-2)(m+2)x_1$. On the other hand, we have that $K^+K^-x_1 = 0$ (because $K^- = \langle\underline{\partial}_x,\underline{\partial}_u\rangle$ acts trivially) and

$$\begin{aligned}K^+x_1 &= \langle\underline{\partial}_x,\underline{\partial}_u\rangle\sum_{a=1}^{m}\xi_{a,u}\xi_{a,x}[x_1]\\ &= \xi_{1,u}|\underline{x}|^2 - mx_1\sum_{a=1}^{m}\xi_{a,u}x_a\\ &= -(m-2)u_1|\underline{x}|^2 + m(m-2)x_1\langle\underline{x},\underline{u}\rangle\ ,\end{aligned}$$

which is a harmonic polynomial of degree $(2,1)$ in $(\underline{x},\underline{u})$. Letting the operator K^- act on this polynomial, we get

$$\langle\underline{\partial}_x,\underline{\partial}_u\rangle\Big(-(m-2)u_1|\underline{x}|^2+m(m-2)x_1\langle\underline{x},\underline{u}\rangle\Big)=(m-1)(m-2)(m+2)x_1\ .$$

This means that we again obtain $[K^+,K^-]x_1=-(m-1)(m-2)(m+2)x_1$.

Now that we have the algebra generated by $K^\pm\in\text{End}\big(\ker\Delta_x\cap\ker\Delta_u\big)$, we will look at the transvector algebra $Z(\mathfrak{sp}(4),\mathfrak{so}(4))$ considered in [5]. This means that instead of considering the operators $K^\pm$ only (something we did above, in view of the connection with harmonic reproducing kernels), we will again add the remaining operators S_x and S_u from section 3.

Theorem 5.2. *The algebra generated by the operators A,C,S_x and S_u is semi-simple, and decomposes into a direct sum of commuting subalgebras:*

$$\mathsf{Alg}(A,C,S_x,S_u)=\mathsf{Alg}(A,C)\oplus\mathsf{Alg}(S_x,S_u)\ .$$

Proof: first of all, we have that

$$\begin{aligned}[A,S_x]&=\sum_{a,b}[\partial_{x_a}\partial_{u_a},\xi_{b,x}\partial_{u_b}]=\sum_{a,b}[\partial_{x_a},\xi_{b,x}]\partial_{u_a}\partial_{u_b}\\&=\sum_{a,b}\big(2L_{ab}^{(x)}-\delta_{ab}(2\mathbb{E}_x+m-2)\big)\partial_{u_a}\partial_{u_b}\ .\end{aligned}$$

The second term is zero when acting on harmonics, whereas the first term is a (trivial) contraction between a symmetric and an anti-symmetric tensor. This means that $[A,S_x]=0$. As for the other operator, we note that

$$\begin{aligned}[C,S_x]&=\sum_{a,b}[\xi_{a,x}\xi_{a,u},\xi_{b,x}\partial_{u_b}]=\sum_{a,b}[\xi_{a,u},\partial_{u_b}]\xi_{a,x}\xi_{b,x}\\&=\sum_{a,b}\big(\delta_{ab}(2\mathbb{E}_u+m-2)+2L_{ab}^{(u)}\big)R_x^{(a)}R_x^{(b)}\ .\end{aligned}$$

The last term again vanishes as the (trivial) contraction of a symmetric and an anti-symmetric tensor. In order to explain why also the first term disappears, we observe that

$$\begin{aligned}\sum_{a=1}^m\xi_{a,x}^2&=\sum_{a=1}^m\big(|\underline{x}|^2\partial_{x_a}-x_a(2\mathbb{E}_x+m-2)\big)^2\\&=|\underline{x}|^4\Delta_x+2|\underline{x}|^2\mathbb{E}_x-|\underline{x}|^2(m+\mathbb{E}_x)(2\mathbb{E}_x+m-2)\\&-|\underline{x}|^2\mathbb{E}_x(2\mathbb{E}_x+m)+|\underline{x}|^2(2\mathbb{E}_x+m)(2\mathbb{E}_x+m-2)\ ,\end{aligned}$$

which is indeed zero on harmonics. This proves the result. □

The first algebra (generated by $K^\pm$) as identified as a Higgs algebra (see above), and somewhat surprisingly the same thing can be said about the second algebra (generated by S_x and S_u).

Theorem 5.3. *The algebra generated by the operators S_x and S_u satisfies the following commutation relations:*

$$\begin{aligned}
[S_x, S_u] &= -2C_0^3 + \big((m-2)(m-4) - 2\mathcal{C}_2^{\mathfrak{so}}\big)C_0 \\
[C_0, S_x] &= +2S_x \\
[C_0, S_u] &= -2S_u \ .
\end{aligned}$$

The operator C_0 hereby stands for $C_0 = \mathbb{E}_x - \mathbb{E}_u$.

Proof: this time, we get that

$$[S_x, S_u] = \sum_{a,b}[\partial_{u_a}, \xi_{b,u}]\xi_{a,x}\partial_{x_b} - \sum_{a,b}[\partial_{x_b}, \xi_{a,x}]\xi_{b,u}\partial_{u_a} \ .$$

The first summation above reduces to

$$\begin{aligned}
&\sum_{a,b}\big(2L_{ab}^{(u)} - \delta_{ab}(2\mathbb{E}_u + m - 2)\big)\xi_{a,x}\partial_{x_b} \\
&= \mathbb{E}_x(2\mathbb{E}_x + m - 4)(2\mathbb{E}_u + m - 2) - 2(2\mathbb{E}_x + m - 4)\sum_{a<b} L_{ab}^{(x)}L_{ab}^{(u)} \ .
\end{aligned}$$

Similarly, the second summation gives

$$\begin{aligned}
&-\sum_{a,b}\big(2L_{ba}^{(x)} - \delta_{ab}(2\mathbb{E}_x + m - 2)\big)R_u^{(b)}\partial_{u_a} \\
&= 2(2\mathbb{E}_u + m - 4)\sum_{a<b} L_{ab}^{(x)}L_{ab}^{(u)} - \mathbb{E}_u(2\mathbb{E}_x + m - 2)(2\mathbb{E}_u + m - 4) \ .
\end{aligned}$$

Bringing all these expressions together, we find that

$$[S_x, S_u] = -4C_0\sum_{a<b} L_{ab}^{(x)}L_{ab}^{(u)} + \big(C_0K_0^2 - C_0^2 - 2(m-2)C_0\big) \ .$$

In a sense, this is a weird intermediate result, since the operator K_0 appears here. Indeed, since $[S_x, S_u]$ changes sign when $\underline{x}$ and $\underline{u}$ are swapped, we do not expect the (symmetric) operator K_0 to appear here. However, plugging in the expression for the mixed Casimir operator, which is equal to

$$\sum_{a<b} L_{ab}^{(x)}L_{ab}^{(u)} = \frac{1}{2}\mathcal{C}_{\mathfrak{so}} + \frac{1}{4}\big(C_0^2 + K_0^2 - (m-2)^2\big) \ ,$$

it is easy to see that the term involving K_0 indeed cancels, and that we are left with the desired expression. □

Corollary 5.4. *The transvector algebra $Z(\mathfrak{sp}(4), \mathfrak{so}(4))$ can actually be seen as a direct copy of two mutually commuting Higgs algebras:*

$$Z(\mathfrak{sp}(4), \mathfrak{so}(4)) = \mathcal{H}_3 \oplus \mathcal{H}_3 \ ,$$

where $\mathcal{H}_3$ is a shorthand notation for the (cubic) Higgs algebra. As a matter of fact, defining $\mathcal{H}_3 := \mathsf{Alg}(\mathsf{K}^+, \mathsf{K}^-)$ with

$$[\mathsf{K}^+, \mathsf{K}^-] = -2\mathsf{K}_0^3 + \alpha\mathsf{K}_0$$

and $[\mathsf{K}_0, \mathsf{K}^\pm] = \pm 2\mathsf{K}^\pm$, we have two copies for

$(\mathsf{K}^+, \mathsf{K}^-, \mathsf{K}_0) = (C, A, \mathbb{E}_x + \mathbb{E}_u + m - 2)$ *and* $(\mathsf{K}^+, \mathsf{K}^-, \mathsf{K}_0) = (S_x, S_u, \mathbb{E}_x - \mathbb{E}_u)$

respectively, with $\alpha = (m-2)(m-4) - 2\mathcal{C}_{\mathfrak{so}}$*. The Casimir operator* $\mathcal{C}_{\mathfrak{so}}$ *can hereby be seen as a central element, which gives a constant when acting on a (simplicial) harmonic polynomial.*

Finally, we note that the set $\mathcal{R}_H := \{Z_k(\underline{x}, \underline{u}) : k \in \mathbb{N}\}$ we have introduced earlier, containing the reproducing kernels for the spaces of harmonic polynomials, can be seen not just as an irreducible representation for the Higgs algebra $\mathcal{H}_3$ but also for the transvector algebra as a whole. For that purpose it suffices to see that $S_x Z_k(\underline{x}, \underline{u}) = S_u Z_k(\underline{x}, \underline{u}) = 0$, which follows from the fact that $[S_x, C]$ and $[S_u, C]$ are both trivial, and the property that our kernels $Z_k(\underline{x}, \underline{u})$ are up to a constant equal to $C^k 1]$.

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